

SINGULARITY SUBTRACTION INTEGRAL FORMULAE FOR SURFACE INTEGRAL EQUATIONS WITH RWG, ROOFTOP AND HYBRID BASIS FUNCTIONS

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Abstract—Numerical solution of electromagnetic scattering problems by the surface integral methods leads to numerical integration of singular integrals in the Method of Moments. The heavy numerical cost of a straightforward numerical treatment of these integrals can be avoided by a more efficient and accurate approach based on the singularity subtraction method. In the literature the information of the closed form integral formulae required by the singularity subtraction method is quite fragmented. In this paper we give a uniform presentation of the singularity subtraction method for planar surface elements with RWG, $\hat{\mathbf{n}} \times \text{RWG}$, rooftop, and $\hat{\mathbf{n}} \times \text{rooftop}$ basis functions, the latter three cases being novel applications. We also discuss the hybrid use of these functions. The singularity subtraction formulas are derived recursively and can be used to subtract more than one term in the Taylor series of the Green's function.

1. INTRODUCTION

Numerical solutions of three-dimensional electromagnetic scattering problems are usually based on surface integral equation formulations, for example electric and magnetic field integral equations (EFIE and MFIE, respectively) or the combined field integral equation (CFIE). The Method of Moments (MoM) is the most common numerical method for solving these equations. In the MoM solution the induced electric and magnetic currents are unknowns, and the surface is usually subdivided into small planar patches of simple shapes where these currents are approximated by the appropriate basis functions.

The most common shapes used for this subdivision are triangular and rectangular patches. If the characteristic size of the patches is sufficiently small, the induced surface currents can be approximated by triangular rooftop functions, also known as Rao-Wilton-Glisson (RWG) functions [1], or by rectangular rooftop functions. Application of these functions with the Method of Moments to solve the surface integral equations leads to the evaluation of double integrals with singular kernels.

Several different methods have been proposed to efficiently and accurately compute these singular integrals, for example the so-called singularity cancellation methods that include the polar transformation technique, Duffy's transformation [2], and the recent Khayat-Wilton method [3]. The most successful approach has been based on the singularity subtraction technique with closed-form integral presentations [4–8]. The information regarding the closed-form integration formulae required by the singularity subtraction technique is quite fragmented in the literature, and the articles on these methods usually do not include the derivation of these formulae.

In this paper, we introduce the ideas behind the singularity subtraction method and derive the related closed-form integrals for planar surface elements in a clear and self-contained way. We give a uniform presentation of the closed-form integrals for the RWG, the $\hat{n}\times$ RWG, the rooftop, and the $\hat{n}\times$ rooftop basis functions, the latter three cases being novel applications, and also for the hybrid use of these functions. The formulas are derived in a recursive way and they are suitable for subtracting more than one term in the Taylor series of the Green's function. For convenience, a summary of these formulae is presented in the Appendix B.

2. SINGULARITY SUBTRACTION

Scattering and radar cross section computations very often use the Method of Moments formulation to solve the involved field integral equations in a matrix form. Unfortunately, the integral equations contain singularities, and all matrix elements are not easy to compute in any straightforward way. In order to modify these equations in such a way that the matrix elements can be numerically and efficiently computed, we first need to consider the integral operators involved.

Let us consider a scattering body D of a homogeneous dielectric material and of an arbitrary shape which is enveloped by a closed surface S . Let the exterior of D be characterised by constant permittivity ϵ_1 and permeability μ_1 and the interior by ϵ_2 and μ_2 . The electromagnetic field scattered by D can be obtained by finding

the induced electric and magnetic surface current distributions $\mathbf{J}(\mathbf{r})$ and $\mathbf{M}(\mathbf{r})$ on the surface S , when the primary (i.e., incident) electric field $\mathbf{E}^p(\mathbf{r})$ and magnetic field $\mathbf{H}^p(\mathbf{r})$ on it are known. The electric and magnetic surface currents are defined as $\mathbf{J}(\mathbf{r}) = \hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})$, and $\mathbf{M}(\mathbf{r}) = -\hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{E}(\mathbf{r})$. The electric and magnetic field integral equations outside D (EFIE₁ and MFIE₁) are given by, respectively,

$$\begin{aligned} & -\frac{1}{i\omega\epsilon_1} \hat{\mathbf{t}}(\mathbf{r}) \cdot (D_1 \mathbf{J})(\mathbf{r}) - \hat{\mathbf{t}}(\mathbf{r}) \cdot (K_1 \mathbf{M})(\mathbf{r}) - \frac{1}{2} \hat{\mathbf{t}}(\mathbf{r}) \cdot \hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{M}(\mathbf{r}) \\ = & -\hat{\mathbf{t}}(\mathbf{r}) \cdot \mathbf{E}^p(\mathbf{r}), \end{aligned} \quad (1)$$

and

$$\begin{aligned} & -\frac{1}{i\omega\mu_1} \hat{\mathbf{t}}(\mathbf{r}) \cdot (D_1 \mathbf{M})(\mathbf{r}) + \hat{\mathbf{t}}(\mathbf{r}) \cdot (K_1 \mathbf{J})(\mathbf{r}) + \frac{1}{2} \hat{\mathbf{t}}(\mathbf{r}) \cdot \hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{J}(\mathbf{r}) \\ = & -\hat{\mathbf{t}}(\mathbf{r}) \cdot \mathbf{H}^p(\mathbf{r}), \end{aligned} \quad (2)$$

where $\hat{\mathbf{t}}(\mathbf{r})$ is the unit tangential vector of S , $\hat{\mathbf{n}}(\mathbf{r})$ is the outward directed normal vector of D on S , and the integral operators $(K\mathbf{F})(\mathbf{r})$ and $(D\mathbf{F})(\mathbf{r})$ on the surface S are defined as

$$(K_n \mathbf{F})(\mathbf{r}) = \int_S \nabla G_n(\mathbf{r}, \mathbf{r}') \times \mathbf{F}(\mathbf{r}') dS' = - \int_S \nabla' G_n(\mathbf{r}, \mathbf{r}') \times \mathbf{F}(\mathbf{r}') dS', \quad (3)$$

and

$$(D_n \mathbf{F})(\mathbf{r}) = (\nabla \nabla \cdot + k_n^2) \int_S G_n(\mathbf{r}, \mathbf{r}') \mathbf{F}(\mathbf{r}') dS', \quad (4)$$

for $\mathbf{r} \in S$, and $G_n(\mathbf{r}, \mathbf{r}')$ is the Green's function given by

$$G_n(\mathbf{r}, \mathbf{r}') = \frac{e^{ik_n |\mathbf{r} - \mathbf{r}'|}}{4\pi |\mathbf{r} - \mathbf{r}'|}, \quad (5)$$

which is a singular function for $\mathbf{r} = \mathbf{r}'$. The integral operator $(D_n \mathbf{F})(\mathbf{r})$ can also be written as

$$\begin{aligned} & (D_n \mathbf{F})(\mathbf{r}) \\ = & p.v. \nabla \int_S \nabla \cdot (G_n(\mathbf{r}, \mathbf{r}') \mathbf{F}(\mathbf{r}')) dS' + k_h^2 \int_S G_n(\mathbf{r}, \mathbf{r}') \mathbf{F}(\mathbf{r}') dS' \\ = & p.v. \nabla \int_S (\nabla G_n(\mathbf{r}, \mathbf{r}')) \cdot \mathbf{F}(\mathbf{r}') dS' + k_n^2 \int_S G_n(\mathbf{r}, \mathbf{r}') \mathbf{F}(\mathbf{r}') dS' \\ = & -p.v. \nabla \int_S (\nabla' G_n(\mathbf{r}, \mathbf{r}')) \cdot \mathbf{F}(\mathbf{r}') dS' + k_n^2 \int_S G_n(\mathbf{r}, \mathbf{r}') \mathbf{F}(\mathbf{r}') dS'. \end{aligned} \quad (6)$$

Using the product rule, we can further write this as

$$(D_n \mathbf{F})(\mathbf{r}) = p.v. \nabla \int_S G_n(\mathbf{r}, \mathbf{r}') \nabla'_s \cdot \mathbf{F}(\mathbf{r}') \\ - p.v. \nabla \int_S \nabla' \cdot (G_n(\mathbf{r}, \mathbf{r}') \mathbf{F}(\mathbf{r}')) dS' + k_n^2 \int_S G_n(\mathbf{r}, \mathbf{r}') \mathbf{F}(\mathbf{r}') dS'. \quad (7)$$

We can similarly define the electric and magnetic field integral equations inside the scatterer D (EFIE₂ and MFIE₂). For a perfect electric conductor (PEC), the electric and magnetic field integral equations have otherwise the same forms as above, but since there are no magnetic surface currents, the terms containing the magnetic surface current $\mathbf{M}(\mathbf{r})$ are removed.

We now have four equations for two unknowns $\mathbf{J}(\mathbf{r})$ and $\mathbf{M}(\mathbf{r})$. In principle, only two equations would suffice to find out the two unknowns, but at the resonant frequencies of the cavity formed by the closed surface S , the solutions given by both the EFIE and the MFIE are not unique. Often the combined field integral equations (CFIE) are used to avoid these internal Maxwell resonances. These equations are a linear combination of the electric and magnetic field integral equations as follows,

$$a \text{EFIE}_1 + b \hat{\mathbf{n}} \times \text{MFIE}_1, \quad (8)$$

$$a \text{EFIE}_2 + b \hat{\mathbf{n}} \times \text{MFIE}_2, \quad (9)$$

where $a = \alpha$, $b = (1 - \alpha)\eta_1$, $\eta_1 = \sqrt{\mu_1/\epsilon_1}$, and the parameter α is chosen as $0 < \alpha < 1$. It can be shown that the CFIE have a unique solution even for the internal resonance frequencies.

In the numerical integration of the Green's function (5) and its first partial derivatives with respect to \mathbf{r}' we meet difficulties if the point \mathbf{r} is in the domain of integration or very close to it. To overcome the difficulties in such a singular case, we expand the exponential part as a Taylor series at $R = |\mathbf{r} - \mathbf{r}'| = 0$,

$$e^{ikR} = \sum_{q=0}^{\infty} \frac{(ikR)^q}{q!} = 1 + ikR - \frac{k^2 R^2}{2} + \frac{ik^3 R^3}{6} - \dots \quad (10)$$

The Green's function then becomes

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi} \sum_{q=-1}^{\infty} \frac{(ik)^{q+1} R^q}{(q+1)!} = \frac{1}{4\pi} \left(\frac{1}{R} + ik - \frac{k^2 R}{2} + \frac{ik^2 R^2}{6} - \dots \right) \quad (11)$$

The problematic terms in the numerical integration are the odd terms ($q = -1, 1, 3, \dots$) since they are not smooth. The even terms

on the other hand are smooth and they pose no problems in the numerical integration. The first impression might lead one to suggest that only the first, i.e., the singular, term in the series needs to be subtracted, but even with the singular term removed, the integrals can not be accurately evaluated since the derivate of Green's function is discontinuous at $R = 0$. So more odd terms than one need to be removed for an accurate evaluation of the integrals [8].

Fortunately, we don't need to compute all the odd terms analytically (or to be more precise, semi-analytically) since we need to use the series expansion only for the singular case where the product kR is small. With a small kR , the higher powers of kR diminish very quickly, so in practise only the first two odd terms ($q = -1$ and $q = 1$) are subtracted. In practical implementations the singular integrals are then split into smooth and non-smooth parts, e.g.,

$$\begin{aligned} & \int_S G(\mathbf{r}, \mathbf{r}') \mathbf{F}(\mathbf{r}') dS' \\ &= \int_S G_s(\mathbf{r}, \mathbf{r}') \mathbf{F}(\mathbf{r}') dS' + \frac{1}{4\pi} \int_S \frac{1}{R} \mathbf{F}(\mathbf{r}') dS' - \frac{k^2}{8\pi} \int_S R \mathbf{F}(\mathbf{r}') dS', \end{aligned} \quad (12)$$

where the "smooth" Green's function $G_s(\mathbf{r}, \mathbf{r}')$ is the original Green's function (5) with the first two non-smooth terms removed,

$$G_s(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi} \left(\frac{e^{ikR} - 1}{R} + \frac{k^2 R}{2} \right). \quad (13)$$

More odd terms may be removed, if needed. The smooth part can be integrated in a straightforward way by using the usual efficient numerical integration routines, like the Gaussian quadrature. The non-smooth parts can be integrated exactly by using the closed-form integral presentations like the one given in this paper. In a practical Method of Moments formulation the outer testing integral may still have a logarithmic singularity, but for most applications this is not a serious problem. By changing the order of the integrations, we are able to accurately and efficiently compute also these integrals, see Appendix A.

3. BASIS FUNCTIONS AND RELATED INTEGRALS

When the Green's function in the integral operator $(D\mathbf{F})(\mathbf{r})$ (7) is replaced by the series expansion (11), the odd terms of the resulting

series of integrals take the form (without the leading coefficients)

$$\begin{aligned} (D\mathbf{F})^q(\mathbf{r}) &= \nabla \int_S R^q \nabla'_s \cdot \mathbf{F}(\mathbf{r}') dS' - \nabla \int_S \nabla' \cdot (R^q \mathbf{F}(\mathbf{r}')) dS' \\ &\quad + k^2 \int_S R^q \mathbf{F}(\mathbf{r}') dS', \end{aligned} \quad (14)$$

where $q = -1, 1, 3, \dots$. Similarly, replacing the Green's function in the integral operator $(K\mathbf{F})(\mathbf{r})$ (3) by the series expansion gives us the terms

$$(K\mathbf{F})^q(\mathbf{r}) = \int_S \nabla' R^q \times \mathbf{F}(\mathbf{r}') dS'. \quad (15)$$

In numerical computations the unknown surface current distribution $\mathbf{F}(\mathbf{r}')$ is replaced by an expansion in the basis functions,

$$\mathbf{F}(\mathbf{r}') \simeq \sum_{n=1}^N x_n \mathbf{f}_n(\mathbf{r}'). \quad (16)$$

The most common basis functions are triangular rooftop functions, also called Rao-Wilton Glisson (RWG) functions [1], and rectangular rooftop functions, both of which are defined below.

For a triangular mesh of a given surface the RWG functions $\mathbf{f}_n(\mathbf{r})$ are defined as follows: for each pair of adjacent triangular surface elements T^+ and T^- , with a common edge I_n , the basis function $\mathbf{f}_n(\mathbf{r})$ is defined by

$$\mathbf{f}_n(\mathbf{r}) = \begin{cases} \frac{L}{2A^+}(\mathbf{r} - \mathbf{p}^+), & \text{if } \mathbf{r} \in T^+, \\ -\frac{L}{2A^-}(\mathbf{r} - \mathbf{p}^-), & \text{if } \mathbf{r} \in T^-, \\ 0, & \text{otherwise,} \end{cases} \quad (17)$$

where L is the length of the edge I_n , A^+ and A^- are the areas of T^+ and T^- , respectively, and \mathbf{p}^+ and \mathbf{p}^- are the vertices of T^+ and T^- opposite to edge I_n (see Fig. 1).

The rooftop functions are defined for pairs of adjacent rectangles of a rectangular discretisation mesh as follows. For a pair of rectangles P^+ and P^- , which share a common edge I_n , the shape function $s_n(\mathbf{r})$ is defined by

$$s_n(\mathbf{r}) = \begin{cases} \frac{L}{A^+}(\mathbf{r} - \mathbf{p}^+) \cdot \hat{\mathbf{u}}^+, & \text{if } \mathbf{r} \in P^+, \\ -\frac{L}{A^-}(\mathbf{r} - \mathbf{p}^-) \cdot \hat{\mathbf{u}}^-, & \text{if } \mathbf{r} \in P^-, \\ 0, & \text{otherwise,} \end{cases} \quad (18)$$

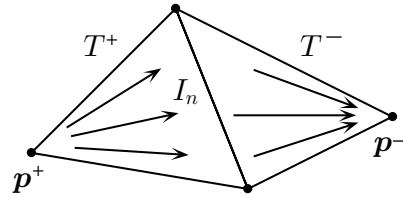


Figure 1. RWG functions.

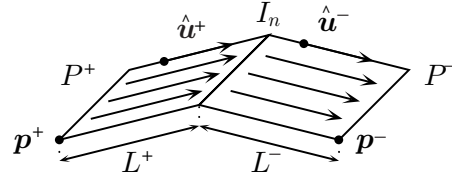


Figure 2. Rooftop functions.

where L is the length of the edge I_n , A^+ and A^- are the areas of P^+ and P^- , respectively, and \mathbf{p}^+ , \mathbf{p}^- and the unit vectors $\hat{\mathbf{u}}^+$, $\hat{\mathbf{u}}^-$ are as in Fig. 2. The constants L/A^+ and L/A^- can also be written as $L/A^+ = 1/L^+$ and $L/A^- = 1/L^-$. The basis functions are defined with the help of the shape functions,

$$\mathbf{f}_n(\mathbf{r}) = s_n(\mathbf{r})\hat{\mathbf{u}}, \tag{19}$$

where $\hat{\mathbf{u}} = \hat{\mathbf{u}}^+$ for $\mathbf{r} \in P^+$ and $\hat{\mathbf{u}} = \hat{\mathbf{u}}^-$ for $\mathbf{r} \in P^-$.

Replacing the surface current distribution $\mathbf{F}(\mathbf{r}')$ in the equation (14) by the expansion (16) shows us that the second integral on the first line of (14) is zero for RWG and rooftop basis functions. Using Gauss' theorem we are able to write the integral as

$$\nabla \int_{\text{spt}(\mathbf{f}_n)} \nabla' \cdot (R^q \mathbf{f}_n(\mathbf{r}')) dS' = \nabla \int_{\partial \text{spt}(\mathbf{f}_n)} \hat{\mathbf{m}}(\mathbf{r}') \cdot \mathbf{f}_n(\mathbf{r}') R^q dl' = 0, \tag{20}$$

where $\hat{\mathbf{m}}(\mathbf{r}')$ is the outward directed normal vector of the boundary $\partial \text{spt}(\mathbf{f}_n)$, and $\text{spt}(\mathbf{f}_n)$ is the support of the basis function $\mathbf{f}_n(\mathbf{r}')$. The dot product of the outward normal with RWG or rooftop basis functions is zero, since these basis functions are parallel to the boundary on the outer edges and their normal components are continuous across the common edge. So finally, the integral operator $(D\mathbf{F})^q(\mathbf{r})$ (14) takes the form,

$$(D\mathbf{F})^q(\mathbf{r}) = \nabla \int_S R^q \nabla'_s \cdot \mathbf{F}(\mathbf{r}') dS' + k^2 \int_S R^q \mathbf{F}(\mathbf{r}') dS'. \tag{21}$$

The gradient operator in front of the first integral in (21) is still quite problematic, especially the normal part. However, if in the Method of Moments formulation RWG or rooftop functions, or some other divergence conforming functions, are used as testing functions, we can move the gradient operator to operate on the testing function only, which leads to a computationally stable result. In practise the closed-form integral formulae needed in (21) and (15) have to be derived separately for each kind of basis function used, unless they can be expressed, for example, by the polynomial shape functions for which the formulae can be derived in a more general way [9, 10].

In general, we need three main types of integrals, which for RWG-basis and rooftop-basis functions are as follows,

$$K_1^q(\text{spt}(\mathbf{f}_n)) = \int_{\text{spt}(\mathbf{f}_n)} R^q \nabla' \cdot \mathbf{f}_n(\mathbf{r}') dS', \quad (22)$$

$$\mathbf{K}_2^q(\text{spt}(\mathbf{f}_n)) = \int_{\text{spt}(\mathbf{f}_n)} R^q \mathbf{f}_n(\mathbf{r}') dS', \quad (23)$$

$$\mathbf{K}_4^q(\text{spt}(\mathbf{f}_n)) = \int_{\text{spt}(\mathbf{f}_n)} (\nabla' R^q) \times \mathbf{f}_n(\mathbf{r}') dS'. \quad (24)$$

Because RWG and rooftop functions are planar, the integral \mathbf{K}_4^q is zero when \mathbf{r} and \mathbf{r}' are on the same integration element. For convenience we introduce an intermediate type of integral \mathbf{K}_3^q , which is used in the computing of (24), and which has the following form for RWG functions

$$\mathbf{K}_3^q(\text{spt}(\mathbf{f}_n)) = \int_{\text{spt}(\mathbf{f}_n)} \nabla' R^q dS', \quad (25)$$

and for rooftop functions

$$\mathbf{K}_3^q(\text{spt}(\mathbf{f}_n)) = \int_{\text{spt}(\mathbf{f}_n)} (\nabla' R^q) s_n(\mathbf{r}') dS', \quad (26)$$

where $s_n(\mathbf{r}')$ is the shape function of the corresponding rooftop function $\mathbf{f}_n(\mathbf{r}')$.

In some cases we must use $\hat{\mathbf{n}}(\mathbf{r}') \times \mathbf{f}_n(\mathbf{r}')$ functions as basis functions, so the surface current distribution $\mathbf{F}(\mathbf{r}')$ is replaced by the expansion

$$\mathbf{F}(\mathbf{r}') \simeq \sum_{n=1}^N x_n \hat{\mathbf{n}}(\mathbf{r}') \times \mathbf{f}_n(\mathbf{r}'). \quad (27)$$

Then in the equation (14) the first integral on the first line becomes zero, and the second integral may be written as a line integral over

the boundary of the integration element with the help of the Gauss' theorem,

$$\begin{aligned}
& \nabla \int_{\text{spt}(\mathbf{f}_n)} R^q \nabla' \cdot (\hat{\mathbf{n}}(\mathbf{r}') \times \mathbf{f}_n(\mathbf{r}')) dS' \\
& - \nabla \int_{\text{spt}(\mathbf{f}_n)} \nabla' \cdot (R^q (\hat{\mathbf{n}}(\mathbf{r}') \times \mathbf{f}_n(\mathbf{r}'))) dS' \\
= & - \nabla \int_{\partial \text{spt}(\mathbf{f}_n)} \hat{\mathbf{m}}(\mathbf{r}') \cdot (\hat{\mathbf{n}} \times \mathbf{f}_n(\mathbf{r}')) R^q dl' \\
= & \nabla \int_{\partial \text{spt}(\mathbf{f}_n)} \hat{\mathbf{s}}(\mathbf{r}') \cdot \mathbf{f}_n(\mathbf{r}') R^q dl', \tag{28}
\end{aligned}$$

where $\hat{\mathbf{s}}(\mathbf{r}') = -\hat{\mathbf{m}}(\mathbf{r}') \times \hat{\mathbf{n}}(\mathbf{r}')$ is the unit tangential vector of the boundary $\partial \text{spt}(\mathbf{f}_n)$. Again, the gradient operator creates problems, unless RWG or rooftop functions are used as testing functions, in which case the gradient operator can be moved to operate on the testing function. The $(K\mathbf{F})^q(\mathbf{r})$ -operator (15) with $\hat{\mathbf{n}} \times \mathbf{f}_n(\mathbf{r}')$ -basis functions can be written,

$$\begin{aligned}
& \int_{\text{spt}(\mathbf{f}_n)} \nabla R^q \times (\hat{\mathbf{n}}(\mathbf{r}') \times \mathbf{f}_n(\mathbf{r}')) dS' \\
= & - \int_{\text{spt}(\mathbf{f}_n)} \nabla' R^q \times (\hat{\mathbf{n}}(\mathbf{r}') \times \mathbf{f}_n(\mathbf{r}')) dS' \\
= & - \int_{\text{spt}(\mathbf{f}_n)} \hat{\mathbf{n}}(\mathbf{r}') (\nabla' R^q \cdot \mathbf{f}_n(\mathbf{r}')) dS' \\
& + \int_{\text{spt}(\mathbf{f}_n)} \mathbf{f}_n(\mathbf{r}') (\hat{\mathbf{n}}(\mathbf{r}') \cdot \nabla' R^q) dS'. \tag{29}
\end{aligned}$$

Thus we have three additional integrals for $\hat{\mathbf{n}}(\mathbf{r}') \times \mathbf{f}_n(\mathbf{r}')$ basis functions,

$$K_5^q(\text{spt}(\mathbf{f}_n)) = \int_{\partial \text{spt}(\mathbf{f}_n)} \hat{\mathbf{s}}(\mathbf{r}') \cdot \mathbf{f}_n(\mathbf{r}') R^q dl', \tag{30}$$

$$\mathbf{K}_6^q(\text{spt}(\mathbf{f}_n)) = \int_{\text{spt}(\mathbf{f}_n)} \hat{\mathbf{n}}(\mathbf{r}') (\nabla' R^q \cdot \mathbf{f}_n(\mathbf{r}')) dS', \tag{31}$$

$$\mathbf{K}_7^q(\text{spt}(\mathbf{f}_n)) = \int_{\text{spt}(\mathbf{f}_n)} \mathbf{f}_n(\mathbf{r}') (\hat{\mathbf{n}}(\mathbf{r}') \cdot \nabla' R^q) dS'. \tag{32}$$

Because RWG and rooftop functions are defined on triangular or rectangular element pairs it is more efficient to compute the Method of Moments system matrix entries by an “element by element” way than by “basis function by basis function”. Therefore, it suffices to

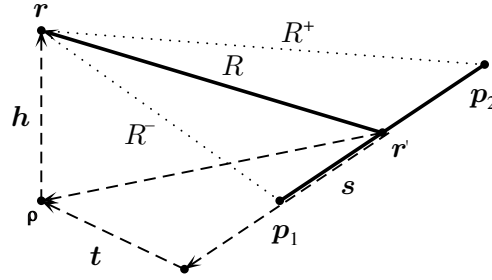


Figure 3. The vector $\mathbf{r} - \mathbf{r}'$, where \mathbf{r}' resides on the segment ΔL of the line L between \mathbf{p}_1 to \mathbf{p}_2 , and \mathbf{r} does not reside on the line L , can be divided into three local orthogonal components \mathbf{s} , \mathbf{t} and \mathbf{h} , which are also either orthogonal or parallel to the line L .

consider the evaluation of the above integrals over single triangular or rectangular elements only. We will next proceed to show how the integrals (22)–(26) and (30)–(32) can be computed for these two types of basis functions.

4. TWO BASIC INTEGRAL RESULTS

Computation of the integrals (22)–(26) and (30)–(32) will eventually be reduced to the computation of a series of line and surface integrals. In the end, through the use of recursive formulae [8], we will basically only need to compute two simple integrals, which will be introduced in the following. These two integrals can be evaluated in closed forms. The results shown here and in the next sections are exact and do not contain any approximations. Let us first define the line integral $I_q^L(\Delta L)$ over a line segment ΔL and the surface integral $I_q^S(S)$ over a surface S , with the integrand $R^q = |\mathbf{r} - \mathbf{r}'|^q$, as follows,

$$I_q^L(\Delta L) = \int_{\Delta L} R^q dl', \quad (33)$$

and

$$I_q^S(S) = \int_S R^q dS', \quad (34)$$

where in both integrals, the integrals are taken over the dotted variable \mathbf{r}' . Let L be a straight line through the points \mathbf{p}_1 and \mathbf{p}_2 and let \mathbf{r}' reside on the segment ΔL of the line L which is between \mathbf{p}_1 and \mathbf{p}_2 . Then the line integral from \mathbf{p}_1 to \mathbf{p}_2 of $1/R = 1/|\mathbf{r} - \mathbf{r}'|$, with $\mathbf{r} - \mathbf{r}'$

presented in a local orthogonal coordinate system as $\mathbf{r} - \mathbf{r}' = \mathbf{s} + \mathbf{t} + \mathbf{h}$ (see Fig. 3), can be expressed as

$$I_{-1}^L(\Delta L) = \int_{\Delta L} \frac{1}{R} dl' = \int_{s^-}^{s^+} \frac{1}{\sqrt{s^2 + R_0^2}} ds, \quad (35)$$

where $R_0^2 = t^2 + h^2 > 0$, i.e., the point \mathbf{r} can not reside on the line L . We will see later that in cases when \mathbf{r} resides on the line, this integral is fortunately cancelled out and so this condition will not affect the computation of the system matrix elements. The constants s^+ and s^- are defined as

$$s^+ = (\mathbf{p}_2 - \mathbf{r}) \cdot \hat{\mathbf{s}}, \quad s^- = (\mathbf{p}_1 - \mathbf{r}) \cdot \hat{\mathbf{s}}, \quad (36)$$

and $\hat{\mathbf{s}} = (\mathbf{p}_2 - \mathbf{p}_1)/|\mathbf{p}_2 - \mathbf{p}_1|$. The value of the integral on the right-hand side of (35) in a closed form can be found in e.g., [11],

$$\int_{s^-}^{s^+} \frac{1}{\sqrt{s^2 + R_0^2}} ds = \ln \left(s^+ + \sqrt{(s^+)^2 + R_0^2} \right) - \ln \left(s^- + \sqrt{(s^-)^2 + R_0^2} \right). \quad (37)$$

Defining two new constants, R^+ and R^- as

$$R^+ = \sqrt{(s^+)^2 + R_0^2} = |\mathbf{r} - \mathbf{p}_2|, \quad (38)$$

$$R^- = \sqrt{(s^-)^2 + R_0^2} = |\mathbf{r} - \mathbf{p}_1|, \quad (39)$$

we can rewrite the Equation (35) in a closed form,

$$\boxed{I_{-1}^L(\Delta L) = \ln \left(\frac{R^+ + s^+}{R^- + s^-} \right)}. \quad (40)$$

The distance of point \mathbf{r} from the line L is given by

$$0 < R_0^2 = t^2 + h^2 = (R^+)^2 - (s^+)^2 = (R^-)^2 - (s^-)^2, \quad (41)$$

from which we get

$$(R^+ + s^+)(R^+ - s^+) = (R^- + s^-)(R^- - s^-), \quad (42)$$

that is,

$$\frac{R^+ + s^+}{R^- + s^-} = \frac{R^- - s^-}{R^+ - s^+}, \quad (43)$$

which gives us an alternate form for the expression inside the logarithm in (40). Of the expressions in (43) it is more advantageous to use the

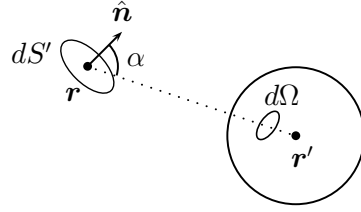


Figure 4. The solid angle $d\Omega$ spanned by dS' as seen from \mathbf{r} depends on the angle α between the right-handed normal vector of dS' and the vector $\mathbf{r} - \mathbf{r}'$.

one for which the absolute value of the divisor is greater. This way, when computing the value of (40), the numerical accuracy is better.

Next we compute the surface integral I_{-3}^S of $1/R^3$ over a flat polygon P . We do this in the terms of the solid angle Ω spanned by P as seen from the point \mathbf{r} and of the quantity $h = \hat{\mathbf{n}}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')$, where $\hat{\mathbf{n}}(\mathbf{r}')$ is the right-handed unit normal vector of P and $|h|$ is the distance from \mathbf{r} to the plane of P . By multiplying I_{-3}^S by h , we get

$$\begin{aligned} hI_{-3}^S(P) &= h \int_P \frac{1}{R^3} dS' = \int_P \frac{\hat{\mathbf{n}}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{R^3} dS' \\ &= - \int_P \hat{\mathbf{n}}(\mathbf{r}') \cdot \nabla' \frac{1}{R} dS'. \end{aligned} \quad (44)$$

Because $\hat{\mathbf{n}}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')/R = \cos(\alpha)$, where α is the angle between the normal vector $\hat{\mathbf{n}}(\mathbf{r}')$ and the vector $\mathbf{r} - \mathbf{r}'$, the area of the orthogonal projection of the surface element dS' on P to the plane perpendicular to the vector $\mathbf{r} - \mathbf{r}'$ is $|\cos(\alpha)|dS'$. Therefore,

$$d\Omega = \frac{\cos(\alpha)}{R^2} dS' = \frac{\hat{\mathbf{n}}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{R^3} dS' = \frac{h}{R^3} dS', \quad (45)$$

is the solid angle spanned by dS' as seen from \mathbf{r} with the usual sign convention, i.e. the solid angle is positive if $\hat{\mathbf{n}}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}') > 0$ and negative if $\hat{\mathbf{n}}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}') < 0$. By integrating $d\Omega$ over P we get, with (44) and (45),

$$\Omega = \int_P d\Omega = \int_P \frac{h}{R^3} dS' = hI_{-3}^S(P), \quad (46)$$

and the surface integral of $1/R^3$ over P can be written simply as

$$\boxed{I_{-3}^S(P) = \frac{1}{h}\Omega, \quad h \neq 0.} \quad (47)$$

In all the cases we need this integral, the $1/h$ will be cancelled out and so the equation (47) can be used also when $h = 0$. The solid angle of any polygon surface can be computed using Girard's spherical excess formula [12],

$$|\Omega| = \sum_{n=1}^m \alpha_n - (m-2)\pi, \quad (48)$$

where α_n is the n :th angle of the m -sided spherical polygon which is the projection of the polygon P to the surface of the unit sphere centered at \mathbf{r} . In principle, it is thus possible to compute the solid angle directly for any polygon P using this general theorem. However, computing the angles α_n for any random polygon is a somewhat elaborate task. It is often simpler to just subdivide the polygon into triangles and apply the formula for the triangles. For a triangle T , using the Euler-Eriksson's formula [13], we get another presentation for the solid angle Ω ,

$$|\Omega| = 2 \arctan(y/x) \quad (49)$$

where the branch of \arctan must be chosen in such a way that $-\pi \leq \arctan(y/x) \leq \pi$ for $x, y \in \mathbb{R}$ (for example, by using the `Matlab`-function `atan2(y, x)`), with

$$x = 1 + \mathbf{a}_1 \cdot \mathbf{a}_2 + \mathbf{a}_1 \cdot \mathbf{a}_3 + \mathbf{a}_2 \cdot \mathbf{a}_3, \quad y = |\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)|, \quad (50)$$

and

$$\mathbf{a}_n = \frac{\mathbf{p}_n - \mathbf{r}}{|\mathbf{p}_n - \mathbf{r}|}, \quad n = 1, 2, 3, \quad (51)$$

where $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are the vertices of T listed in the positive rotation order with respect to \mathbf{n} , i.e.,

$$\mathbf{n} = \frac{\mathbf{b}}{|\mathbf{b}|}, \quad \mathbf{b} = (\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1). \quad (52)$$

Since any polygon can be represented as a sum of triangles, we can compute the solid angle of any polygon by repeatedly applying (49). Thus we are able to simply evaluate the integral (47) for any type of polygon without the need to resort to numerical integration rules.

5. RECURSIVE FORMULAE FOR LINE AND SURFACE INTEGRALS

In the previous section we derived the basic formulae to compute the line integral $I_1^L(\Delta L)$ and the surface integral $I_3^S(P)$. When computing the integrals (22)–(26), we need these integrals with the

odd order q of R^q so that $q \geq -1$ for the line integrals and $q \geq -3$ for the surface integrals. Fortunately, all the higher order integrals are easily reduced to the integrals of (40) and (47) through calculus and vector algebra. We begin by deriving a recursive rule for the line integral $I_q^L(\Delta L)$ over the segment of the line ΔL . Let q be odd and $q \geq 1$, and we have

$$R^q = R^{q-2}R^2 = R^{q-2}(s^2 + R_0^2). \quad (53)$$

Then the line integral along the segment ΔL from \mathbf{p}_1 to \mathbf{p}_2 can be written

$$\int_{\Delta L} R^q dl' = R_0^2 \int_{\Delta L} R^{q-2} dl' + \int_{\Delta L} s^2 R^{q-2} dl'. \quad (54)$$

Since

$$\frac{\partial}{\partial s} R^q = qsR^{q-2}, \quad (55)$$

we can use integration by parts on the latter integral on the right hand side of (54), and get

$$\frac{1}{q} \int_{\Delta L} qs^2 R^{q-2} dl' = \frac{1}{q} \int_{\Delta L} s(qsR^{q-2}) dl' = \frac{1}{q} [sR^q]_{s=s^-}^{s=s^+} - \frac{1}{q} \int_{\Delta L} R^q dl'. \quad (56)$$

Now, (54) and (56) yield

$$I_q^L(\Delta L) = \int_{\Delta L} R^q dl' = \frac{qR_0^2}{q+1} \int_{\Delta L} R^{q-2} dl' + \frac{1}{q+1} [sR^q]_{s=s^-}^{s=s^+}, \quad (57)$$

or

$$\boxed{I_q^L(\Delta L) = \frac{qR_0^2}{q+1} I_{q-2}^L(\Delta L) + \frac{1}{q+1} (s^+(R^+)^q - s^-(R^-)^q).} \quad (58)$$

In the case $q = 1$ the original restriction $R_0^2 > 0$ for the equation (40) would normally come into play, but as we also multiply the equation (40) by R_0^2 , this term will be cancelled out when $R_0^2 = 0$. In practical implementations, one has to check that if $R_0^2 = 0$, then I_{q-2}^L in (58) is not computed. The original line integral of order q is now reducible to the case $q = -1$ through repeatedly applying (58), without the need to use any numerical integration rules.

The recursive rule for the surface integral is more involved than for the line integral and requires a bit more complicated manipulation. Let again q be odd and $q \geq -1$. To compute the surface integral over a flat polygon P , we first begin by computing the surface divergence

$$\nabla'_s \cdot ((\mathbf{r} - \mathbf{r}')R^q) = (\nabla'_s \cdot (\mathbf{r} - \mathbf{r}')) R^q + (\mathbf{r} - \mathbf{r}') \cdot (\nabla'_s R^q)$$

$$\begin{aligned}
&= -2R^q + q(\mathbf{r} - \mathbf{r}') \cdot (\mathbf{r}' - \boldsymbol{\rho}) R^{q-2} \\
&= -2R^q + q(\mathbf{r} - \mathbf{r}') \cdot ((\mathbf{r}' - \mathbf{r}) + (\mathbf{r} - \boldsymbol{\rho})) R^{q-2} \\
&= -2R^q - qR^q + qR^{q-2}(\mathbf{r} - \mathbf{r}') \cdot (\mathbf{r} - \boldsymbol{\rho}) \\
&= -2R^q - qR^q + qR^{q-2}h^2, \tag{59}
\end{aligned}$$

where $\boldsymbol{\rho} = \mathbf{r} - \hat{\mathbf{n}}(\mathbf{r}')(\hat{\mathbf{n}}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}'))$. Thus we can write the integrand R^q as

$$R^q = \frac{q}{q+2} h^2 R^{q-2} - \frac{1}{q+2} \nabla'_s \cdot ((\mathbf{r} - \mathbf{r}') R^q), \tag{60}$$

and so the surface integral over P can be written as

$$\int_P R^q dS' = \frac{qh^2}{q+2} \int_P R^{q-2} dS' - \frac{1}{q+2} \int_P \nabla'_s \cdot ((\mathbf{r} - \mathbf{r}') R^q) dS'. \tag{61}$$

By Gauss' theorem, the second integral on the right hand side of (61) can be expressed as a line integral over the boundary of P ,

$$\int_P \nabla'_s \cdot ((\mathbf{r} - \mathbf{r}') R^q) dS' = \int_{\partial P} \hat{\mathbf{m}}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}') R^q dl' = \sum_{i=1}^m t_i \int_{\partial P_i} R^q dl', \tag{62}$$

where $\hat{\mathbf{m}}(\mathbf{r}')$ is the outward directed unit normal vector of the boundary ∂P (and is thus in the plane of P) and $t_i = \hat{\mathbf{m}}_i \cdot (\mathbf{r} - \mathbf{r}')$ remains constant for all \mathbf{r}' in the i :th side ∂P_i of P , as is easily seen by geometry. This gives us the recursive formula for the surface integral I_q^S ,

$$\boxed{I_q^S(P) = \frac{qh^2}{q+2} I_{q-2}^S(P) - \frac{1}{q+2} \sum_{i=1}^m t_i I_q^L(\partial P_i)}, \tag{63}$$

which again reduces to the computation of the basic integrals (40) and (47). In the case $q = -1$, we should normally verify that in the integral I_{-3}^S the term $h \neq 0$, but since we also multiply the integral I_{q-2}^S by h^2 , the integral is cancelled out if $h = 0$. In practical implementations, one has to check that if $h = 0$, then the integral term I_{q-2}^S is not computed. We now have all the required tools to compute the integrals (22)–(26), and we proceed to these integrals with various basis functions.

6. RWG FUNCTIONS

The integrals (22)–(26) are computed element-wise by reducing them to a series of surface and line integrals which can be evaluated in closed

forms. Since the surface divergence of the RWG functions is piecewise constant, the first integral K_1^q (22) is easily computed by

$$\int_{T^\pm} R^q (\nabla'_s \cdot \mathbf{f}_n(\mathbf{r}')) dS' = \pm \frac{L}{A^\pm} \int_{T^\pm} R^q dS', \quad (64)$$

or

$$\boxed{K_1^q(T^\pm) = \pm \frac{L}{A^\pm} I_q^S(T^\pm)}. \quad (65)$$

To compute the second integral \mathbf{K}_2^q (23) we begin by writing

$$\mathbf{r}' - \mathbf{p}^\pm = (\mathbf{r}' - \boldsymbol{\rho}) + (\boldsymbol{\rho} - \mathbf{p}^\pm), \quad (66)$$

and computing surface gradient of R^{q+2} in the form

$$\nabla'_s R^{q+2} = (q+2)(\mathbf{r}' - \boldsymbol{\rho}) R^q. \quad (67)$$

Applying Gauss' theorem

$$\int_{T^\pm} \nabla'_s R^q dS' = \int_{\partial T^\pm} \hat{\mathbf{m}}(\mathbf{r}) R^q dl', \quad (68)$$

and using the previous formulae, we arrive at the following form for the second integral

$$\begin{aligned} \mathbf{K}_2^q(T^\pm) &= \int_{T^\pm} R^q \mathbf{f}_n(\mathbf{r}') dS' = \pm \frac{L}{2A^\pm} \int_{T^\pm} R^q (\mathbf{r}' - \mathbf{p}^\pm) dS' \\ &= \pm \frac{L}{2A^\pm} \left(\frac{1}{q+2} \int_{T^\pm} \nabla'_s R^{q+2} dS' + (\boldsymbol{\rho} - \mathbf{p}^\pm) \int_{T^\pm} R^q dS' \right) \\ &= \pm \frac{L}{2A^\pm} \left(\frac{1}{q+2} \sum_{i=1}^3 \hat{\mathbf{m}}_i \int_{\partial T_i^\pm} R^{q+2} dl' + (\boldsymbol{\rho} - \mathbf{p}^\pm) \int_{T^\pm} R^q dS' \right), \end{aligned} \quad (69)$$

where we can recognise the surface and line integrals derived previously, and obtain

$$\boxed{\mathbf{K}_2^q(T^\pm) = \pm \frac{L}{2A^\pm} \left(\frac{1}{q+2} \sum_{i=1}^3 \hat{\mathbf{m}}_i I_{q+2}^L(\partial T_i^\pm) + (\boldsymbol{\rho} - \mathbf{p}^\pm) I_q^S(T^\pm) \right)}. \quad (70)$$

The integral \mathbf{K}_3^q (25) can be computed in the following way: Start by dividing the gradient into surface and normal parts,

$$\int_{T^\pm} \nabla' R^q dS' = \int_{T^\pm} \nabla'_s R^q dS' + \int_{T^\pm} \nabla'_n R^q dS', \quad (71)$$

where ∇'_s and ∇'_n are the surface and normal gradient, respectively. The normal gradient can be written

$$\begin{aligned}\nabla'_n R^q &= \hat{\mathbf{n}}(\mathbf{r}')(\hat{\mathbf{n}}(\mathbf{r}') \cdot \nabla')R^q = -\hat{\mathbf{n}}(\mathbf{r}')(\hat{\mathbf{n}}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}'))qR^{q-2} \\ &= -\hat{\mathbf{n}}(\mathbf{r}')hqR^{q-2}.\end{aligned}\quad (72)$$

The integral of the surface gradient can again be expressed as a line integral with the help of Gauss' theorem (68), and so the integral (71) can be written

$$\int_{T^\pm} \nabla' R^q dS' = \sum_{i=1}^3 \hat{\mathbf{m}}_i \int_{\partial T_i^\pm} R^q dl' - hq\hat{\mathbf{n}} \int_{T^\pm} R^{q-2} ds, \quad (73)$$

or equivalently,

$$\boxed{\mathbf{K}_3^q(T^\pm) = \sum_{i=1}^3 \hat{\mathbf{m}}_i I_q^L(\partial T_i^\pm) - hq\hat{\mathbf{n}} I_{q-2}^S(T^\pm).} \quad (74)$$

Again, for the case $q = -1$, we normally should verify that in the integral I_3^S the term $h \neq 0$, but since we also multiply the integral by h , the singularity is cancelled out. In practical implementations, one has to check that if $q > -1$ and if $h = 0$, then the integral I_{q-2}^S is not computed. The surface gradient term still contains a logarithmic singularity, which however can be removed, see e.g., Appendix A.

The integral \mathbf{K}_4^q (24) can now be given in terms of the integral \mathbf{K}_3^q (25) as follows,

$$\begin{aligned}&\int_{T^\pm} \nabla' R^q \times (\mathbf{r}' - \mathbf{p}^\pm) dS' \\ &= \int_{T^\pm} \nabla' R^q \times ((\mathbf{r}' - \mathbf{r}) + (\mathbf{r} - \mathbf{p}^\pm)) dS' \\ &= \int_{T^\pm} \nabla' R^q \times (\mathbf{r}' - \mathbf{r}) dS' - (\mathbf{r} - \mathbf{p}^\pm) \times \int_{T^\pm} \nabla' R^q dS',\end{aligned}\quad (75)$$

where $\nabla' R^q \times (\mathbf{r}' - \mathbf{r}) = 0$. Applying (75), we reduce the integral (24) to

$$\begin{aligned}\mathbf{K}_4^q(T^\pm) &= \pm \frac{L}{2A^\pm} \int_{T^\pm} \nabla' R^q \times (\mathbf{r}' - \mathbf{p}^\pm) dS' \\ &= \mp \frac{L}{2A^\pm} (\mathbf{r} - \mathbf{p}^\pm) \times \int_{T^\pm} \nabla' R^q dS',\end{aligned}\quad (76)$$

and so

$$\boxed{\mathbf{K}_4^q(T^\pm) = \mp \frac{L}{2A^\pm} (\mathbf{r} - \mathbf{p}^\pm) \times \mathbf{K}_3^q(T^\pm).} \quad (77)$$

The integral \mathbf{K}_4^q is zero when the integration elements of the outer testing integral and the inner integral are the same.

To compute the integral K_5^q (30) we begin by writing the integrand as

$$\begin{aligned} (\mathbf{r}' - \mathbf{p}^\pm) \cdot \hat{\mathbf{s}}(\mathbf{r}') &= ((\mathbf{r}' - \boldsymbol{\rho}) + (\boldsymbol{\rho} - \mathbf{p}^\pm)) \cdot \hat{\mathbf{s}}(\mathbf{r}') \\ &= (\mathbf{r}' - \boldsymbol{\rho}) \cdot \hat{\mathbf{s}}(\mathbf{r}') + s_0^\pm, \end{aligned} \quad (78)$$

where $s_0^\pm = \hat{\mathbf{s}}(\mathbf{r}') \cdot (\boldsymbol{\rho} - \mathbf{p}^\pm)$. Inserting this into the integral K_5^q and using similar notation as in Section 4, so that $\mathbf{r}' - \boldsymbol{\rho} = \mathbf{s} + \mathbf{t}$, allows us to write the integral as

$$\begin{aligned} &\int_{\partial T^\pm} \hat{\mathbf{s}}(\mathbf{r}') \cdot \mathbf{f}_n(\mathbf{r}') R^q dl' \\ &= \pm \frac{L}{2A^\pm} \int_{\partial T^\pm} \hat{\mathbf{s}}(\mathbf{r}') \cdot (\mathbf{s} + \mathbf{t}) R^q dl' \pm \frac{L}{2A^\pm} \int_{\partial T^\pm} s_0^\pm R^q dl' \\ &= \pm \frac{L}{2A^\pm} \sum_{i=1}^3 \int_{s_i^-}^{s_i^+} s (s^2 + t^2 + h^2)^{q/2} ds \pm \frac{L}{2A^\pm} \int_{\partial T^\pm} s_0^\pm R^q dl', \end{aligned} \quad (79)$$

where the sum is taken over the three sides of the triangle T^\pm . The value of the first integral is easily computed by

$$\int_{s_i^-}^{s_i^+} s (s^2 + t^2 + h^2)^{q/2} ds = \frac{1}{q+2} \left((R_i^+)^{q+2} - (R_i^-)^{q+2} \right), \quad (80)$$

so finally the integral K_q^5 can be written

$$\boxed{K_q^5(T^\pm) = \pm \frac{L}{2A^\pm} \left(\frac{1}{(q+2)} \sum_{i=1}^3 \left((R_i^+)^{q+2} - (R_i^-)^{q+2} \right) + \sum_{i=1}^3 s_{0,i}^\pm I_q^L(\partial T_i^\pm) \sum_{i=1}^3 \right)}. \quad (81)$$

To compute the integral \mathbf{K}_q^6 (31) we begin by computing the gradient of R^q in the integrand, and by writing $\mathbf{r}' - \mathbf{p}^\pm = (\mathbf{r}' - \mathbf{r}) + (\mathbf{r} - \mathbf{p}^\pm)$,

$$\begin{aligned} \hat{\mathbf{n}} (\nabla' R^q \cdot (\mathbf{r}' - \mathbf{p}^\pm)) &= -q \hat{\mathbf{n}} R^{q-2} (\mathbf{r} - \mathbf{r}') \cdot (\mathbf{r}' - \mathbf{p}^\pm) \\ &= -q \hat{\mathbf{n}} R^{q-2} (\mathbf{r} - \mathbf{r}') \cdot ((\mathbf{r}' - \mathbf{r}) + (\mathbf{r} - \mathbf{p}^\pm)). \end{aligned} \quad (82)$$

We can take the term $(\mathbf{r} - \mathbf{r}') \cdot (\mathbf{r}' - \mathbf{r}) = -R^2$ outside the parentheses, which leaves us only one vector term. Next we write $\mathbf{r} - \mathbf{r}' = (\mathbf{r} - \boldsymbol{\rho}) + (\boldsymbol{\rho} - \mathbf{r}')$. Inserting this into (82) gives us

$$\begin{aligned} & q\hat{\mathbf{n}}R^q - q\hat{\mathbf{n}}R^{q-2}(\mathbf{r} - \mathbf{r}') \cdot (\mathbf{r} - \mathbf{p}^\pm) \\ &= q\hat{\mathbf{n}}R^q - q\hat{\mathbf{n}}R^{q-2}((\mathbf{r} - \boldsymbol{\rho}) + (\boldsymbol{\rho} - \mathbf{r}')) \cdot (\mathbf{r} - \mathbf{p}^\pm) \\ &= q\hat{\mathbf{n}}R^q - qh^2\hat{\mathbf{n}}R^{q-2} - q\hat{\mathbf{n}}R^{q-2}(\boldsymbol{\rho} - \mathbf{r}') \cdot (\mathbf{r} - \mathbf{p}^\pm), \end{aligned} \quad (83)$$

since $(\mathbf{r} - \boldsymbol{\rho}) \cdot (\mathbf{r} - \mathbf{p}^\pm) = h^2$. The surface gradient of R^q is $-q(\boldsymbol{\rho} - \mathbf{r}')R^q$, and so we can write (83) as

$$q\hat{\mathbf{r}}R^q - qh^2\hat{\mathbf{r}}R^{q-2} + \hat{\mathbf{r}}(\mathbf{r} - \mathbf{p}^\pm) \cdot \nabla'_s R^q. \quad (84)$$

Inserting this into (31) and using Gauss' theorem gives us (omitting constant $\pm L/2A^\pm$)

$$\begin{aligned} & \int_{S'} \hat{\mathbf{n}}(\nabla'_s R^q \cdot (\mathbf{r}' - \mathbf{p}^\pm)) dS' \\ &= q\hat{\mathbf{n}} \int_{S'} R^q dS' - qh^2\hat{\mathbf{n}} \int_{S'} R^{q-2} dS' + \hat{\mathbf{n}}(\mathbf{r} - \mathbf{p}^\pm) \cdot \int_{S'} \nabla'_s R^q dS' \\ &= q\hat{\mathbf{n}} \int_{S'} R^q dS' - qh^2\hat{\mathbf{n}} \int_{S'} R^{q-2} dS' + \hat{\mathbf{n}}(\mathbf{r} - \mathbf{p}^\pm) \cdot \int_{\partial S'} \hat{\mathbf{m}}R^q dl'. \end{aligned} \quad (85)$$

The integral \mathbf{K}_6^q (31) can thus be written

$$\boxed{\mathbf{K}_6^q(T^\pm) = \pm \frac{L}{2A^\pm} \hat{\mathbf{n}} \left(\sum_{i=1}^3 qI_q^S(T^\pm) - qh^2 I_{q-2}^S(T^\pm) + (\mathbf{r} - \mathbf{p}^\pm) \cdot \sum_{i=1}^3 \hat{\mathbf{m}}'_i I_q^L(\partial T_i^\pm) \right)}. \quad (86)$$

The integral \mathbf{K}_7^q (32) can be written as (omitting constant $\pm L/2A^\pm$)

$$\begin{aligned} \int_{S'} (\mathbf{r}' - \mathbf{p}^\pm)(\hat{\mathbf{n}} \cdot \nabla'_s R^q) dS' &= -q \int_{S'} R^{q-2} \hat{\mathbf{n}} \cdot (\mathbf{r} - \mathbf{r}')(\mathbf{r} - \mathbf{p}^\pm) dS' \\ &= -qh \int_{S'} R^{q-2}(\mathbf{r}' - \mathbf{p}^\pm) dS', \end{aligned} \quad (87)$$

since $h = \hat{\mathbf{n}} \cdot (\mathbf{r} - \mathbf{r}')$. Replacing $\mathbf{r}' - \mathbf{p}^\pm$ by $(\mathbf{r}' - \boldsymbol{\rho}) + (\boldsymbol{\rho} - \mathbf{p}^\pm)$ and using Gauss' theorem gives us

$$-qh \int_{S'} R^{q-2} ((\mathbf{r}' - \boldsymbol{\rho}) + (\boldsymbol{\rho} - \mathbf{p}^\pm)) dS'$$

$$\begin{aligned}
&= -h \int_{S'} \nabla'_s R^q dS' - qh(\boldsymbol{\rho} - \mathbf{p}^\pm) \int_{S'} R^{q-2} dS' \\
&= -h \int_{\partial S'} \hat{\mathbf{m}} R^q dl' - qh(\boldsymbol{\rho} - \mathbf{p}^\pm) \int_{S'} R^{q-2} dS'. \quad (88)
\end{aligned}$$

Thus the integral \mathbf{K}_q^7 can be computed by

$$\boxed{\mathbf{K}_q^7(T^\pm) = \mp \frac{Lh}{2A^\pm} \left(\sum_{i=1}^3 \hat{\mathbf{m}}'_i I_q^L(\partial T_i^\pm) + q(\boldsymbol{\rho} - \mathbf{p}^\pm) I_{q-2}^S(T^\pm) dS' \right)}. \quad (89)$$

7. ROOFTOP FUNCTIONS

Rooftop functions are, in principle, treated the same way as the RWG functions, though some details need extra work. All the integrals involving the basis functions are easily reduced to integrals involving only the shape functions. The divergences of the rooftop functions are piecewise constants, so the integral K_1^q (22) now becomes

$$\int_{P^\pm} (\nabla' \cdot \mathbf{f}_n(\mathbf{r}')) R^q dS' = \pm \frac{1}{L^\pm} \int_{P^\pm} R^q dS', \quad (90)$$

or

$$\boxed{K_1^q(P^\pm) = \pm \frac{1}{L^\pm} I_q^S(P^\pm)}. \quad (91)$$

The integral \mathbf{K}_2^q (23) over P^\pm takes the form

$$\mathbf{K}_2^q(P^\pm) = \int_{P^\pm} \mathbf{f}_n(\mathbf{r}') R^q dS' = \hat{\mathbf{u}}^\pm \int_{P^\pm} s_n(\mathbf{r}') R^q dS'. \quad (92)$$

Writing $\mathbf{r}' - \mathbf{p}^\pm = (\mathbf{r}' - \boldsymbol{\rho}) + (\boldsymbol{\rho} - \mathbf{p}^\pm)$, the integral (92) can be written

$$\begin{aligned}
\int_{P^\pm} s_n(\mathbf{r}') R^q dS' &= \pm \frac{1}{L^\pm} \hat{\mathbf{u}}^\pm \cdot \int_{P^\pm} (\mathbf{r}' - \boldsymbol{\rho}) R^q dS' \\
&\quad \pm \frac{1}{L^\pm} \hat{\mathbf{u}}^\pm \cdot (\boldsymbol{\rho} - \mathbf{p}^\pm) \int_{P^\pm} R^q dS'. \quad (93)
\end{aligned}$$

Using the previous Equations (67) and (68), we can further write (93) as

$$\begin{aligned}
&\int_{P^\pm} s_n(\mathbf{r}') R^q dS' \\
&= \pm \frac{1}{L^\pm(q+2)} \sum_{i=1}^4 \hat{\mathbf{u}}^\pm \cdot \hat{\mathbf{m}}_i \int_{\partial P_i^\pm} R^{q+2} dl' \pm \frac{u_0^\pm}{L^\pm} \int_{P^\pm} R^q dS', \quad (94)
\end{aligned}$$

where $u_0^\pm = \hat{\mathbf{u}}^\pm \cdot (\boldsymbol{\rho} - \mathbf{p}^\pm)$, so finally the integral (23) can be written with the earlier derived line and surface integrals as

$$\boxed{\mathbf{K}_2^q(P^\pm) = \pm \frac{\hat{\mathbf{u}}^\pm}{L^\pm} \left(\frac{1}{q+2} \sum_{i=1}^4 \hat{\mathbf{u}}^\pm \cdot \hat{\mathbf{m}}_i I_{q+2}^L(\partial P_i^\pm) + u_0^\pm I_q^S(P^\pm) \right)}. \quad (95)$$

The integral \mathbf{K}_3^q (26) is the most elaborate one to compute because we have to take into account several different possibilities. We begin, as for the RWG functions, by dividing the gradient into surface and normal parts,

$$\begin{aligned} \mathbf{K}_3^q(P^\pm) &= \int_{P^\pm} s_n(\mathbf{r}') \nabla' R^q dS' \\ &= \int_{P^\pm} s_n(\mathbf{r}') \nabla'_s R^q dS' + \int_{P^\pm} s_n(\mathbf{r}') \nabla'_n R^q dS' \\ &= \int_{P^\pm} s_n(\mathbf{r}') \nabla'_s R^q dS' - \int_{P^\pm} s_n(\mathbf{r}') \hat{\mathbf{n}} h q R^{q-2} dS', \end{aligned} \quad (96)$$

which can then be written by integration by parts as

$$\int_{P^\pm} \nabla'_s (s_n(\mathbf{r}') R^q) dS' - \int_{P^\pm} (\nabla'_s s_n(\mathbf{r}')) R^q - \hat{\mathbf{n}} h q \int_{P^\pm} s_n(\mathbf{r}') R^{q-2} dS'. \quad (97)$$

The surface gradient of the shape function $s_n(\mathbf{r}')$ is piecewise constant. This can be easily seen by writing the shape function in a local coordinate system (u, v, n) so that $(\mathbf{r} - \mathbf{p}^\pm) \cdot \hat{\mathbf{u}}^\pm = u$, $u = [0, L^\pm]$. Then $\nabla'_s s_n(u) = \pm 1/L^\pm (\hat{\mathbf{u}} \partial_u + \hat{\mathbf{v}} \partial_v) u = \pm \hat{\mathbf{u}}/L^\pm$. The first integral can be further written by Gauss' theorem (71) as a line integral, so we get

$$\begin{aligned} &\int_{\partial P^\pm} \hat{\mathbf{m}}(\mathbf{r}') s_n(\mathbf{r}') R^q dl' \mp \frac{\hat{\mathbf{u}}^\pm}{L^\pm} \int_{P^\pm} R^q dS' - \hat{\mathbf{n}} h q \int_{P^\pm} s_n(\mathbf{r}') R^{q-2} dS' \\ &= \sum_{i=1}^4 \hat{\mathbf{m}}_i \int_{\partial P_i^\pm} s_n(\mathbf{r}') R^q dl' \mp \frac{\hat{\mathbf{u}}^\pm}{L^\pm} I_q^S(P^\pm) - \hat{\mathbf{n}} h q K_2^{q-2}(P^\pm), \end{aligned} \quad (98)$$

where K_2^q is the integral \mathbf{K}_2^q (92) with the vector part removed,

$$K_2^q = \int_{P^\pm} s_n(\mathbf{r}') R^q dS'. \quad (99)$$

The first integral on the second line of (98) is handled as follows. The integral can be written, with same definitions as for (94),

$$\int_{\partial P_i^\pm} s_n(\mathbf{r}') R^q dl' = \pm \frac{1}{L^\pm} \int_{\partial P_i^\pm} \hat{\mathbf{u}}^\pm \cdot (\mathbf{r}' - \boldsymbol{\rho}) R^q dl' \pm \frac{u_0^\pm}{L^\pm} \int_{\partial P_i^\pm} R^q dl', \quad (100)$$

which must now be divided into several different cases according to the edge along which the integral is to be computed. For the rectangle P^+ the integral can further be written (using similar notation as in Section 4, so that $\mathbf{r}' - \boldsymbol{\rho} = -\mathbf{s} - \mathbf{t}$), when the line ∂P_i^+ is parallel to $\hat{\mathbf{u}}^+$ as

$$\begin{aligned}
& \pm \frac{1}{L^+} \int_{\partial P_i^+} \hat{\mathbf{u}}^+ \cdot (-\mathbf{s} - \mathbf{t}) R^q dl' + \frac{u_0^+}{L^+} I_q^L(\partial P_i^+) \\
&= \pm \frac{1}{L^+} \int_{s_i^-}^{s_i^+} (-s) \left(\sqrt{(-s)^2 + (-t)^2 + (-h)^2} \right)^q ds + \frac{u_0^+}{L^+} I_q^L(\partial P_i^+) \\
&= \pm \frac{1}{L^+(q+2)} \left[\left(\sqrt{s^2 + t^2 + h^2} \right)^{q+2} \right]_{s=s_i^-}^{s=s_i^+} + \frac{u_0^+}{L^+} I_q^L(\partial P_i^+) \\
&= \pm \frac{1}{L^+(q+2)} \left[(R_i^+)^{q+2} - (R_i^-)^{q+2} \right] + \frac{u_0^+}{L^+} I_q^L(\partial P_i^+), \tag{101}
\end{aligned}$$

where the sign of the integral is positive if the line integral is taken in the direction of $\hat{\mathbf{u}}^+$ and negative otherwise. Nota bene: the \pm -sign in the Equation (100) is not related to the \pm -sign in the Equation (101), so for the rectangle P^- the signs in the equation (101) must be reversed (i.e., \pm becomes \mp). When the integral is taken along the edge I_n , the integral is

$$\int_{I_n} s_n(\mathbf{r}') R^q dl' = \int_{I_n} R^q dl' = I_q^L(I_n), \tag{102}$$

since the shape function equals one on the line I_n , and along the edge opposite to I_n the integral is zero,

$$\int_{\partial P_i^\pm} s_n(\mathbf{r}') R^q dl' = 0, \tag{103}$$

since the shape function equals zero. The integral (26) can now be written

$$\boxed{\mathbf{K}_3^q(P^\pm) = \sum_{i=1}^4 \hat{\mathbf{m}}_i \int_{\partial P_i^\pm} s_n(\mathbf{r}') R^q dl' \mp \frac{\hat{\mathbf{u}}^\pm}{L^\pm} I_q^S(P^\pm) - \hat{\mathbf{n}} h q K_2^{q-2}(P^\pm)}. \tag{104}$$

The integral \mathbf{K}_4^q (24) is again easily get from (104),

$$\mathbf{K}_4^q(P^\pm) = \int_{P^\pm} \mathbf{f}_n(\mathbf{r}') \times \nabla' R^q dS' = \hat{\mathbf{u}}^\pm \times \int_{P^\pm} s_n(\mathbf{r}') \nabla' R^q dS', \tag{105}$$

or

$$\boxed{\mathbf{K}_4^q(P^\pm) = \hat{\mathbf{u}}^\pm \times \mathbf{K}_3^q(P^\pm)}. \tag{106}$$

The integral K_5^q (30) can be computed similarly as in the Section 6. Begin by writing

$$\begin{aligned} \hat{\mathbf{s}}(\mathbf{r}') \cdot \hat{\mathbf{u}}^\pm \hat{\mathbf{u}}^\pm \cdot (\mathbf{r}' - \mathbf{p}^\pm) &= \pm \hat{\mathbf{u}}^\pm \cdot ((\mathbf{r}' - \boldsymbol{\rho}) + (\boldsymbol{\rho} - \mathbf{p}^\pm)) \\ &= \pm \hat{\mathbf{u}}^\pm \cdot (\mathbf{r}' - \boldsymbol{\rho}) \pm u_0^\pm, \end{aligned} \quad (107)$$

where the dot product $\hat{\mathbf{s}}(\mathbf{r}') \cdot \hat{\mathbf{u}}^\pm = 1$ on the side where $\hat{\mathbf{s}}(\mathbf{r}') = \hat{\mathbf{u}}^\pm$, $\hat{\mathbf{s}}(\mathbf{r}') \cdot \hat{\mathbf{u}}^\pm = -1$ on the side where $\hat{\mathbf{s}}(\mathbf{r}') = -\hat{\mathbf{u}}^\pm$, and zero otherwise. Inserting (107) into the integral K_5^q , and using the results from Section 6, allows us to write the integral as

$$\begin{aligned} &\int_{\partial P^\pm} \hat{\mathbf{s}}(\mathbf{r}') \cdot \mathbf{f}_n(\mathbf{r}') R^q dl' \\ &= \pm \frac{1}{L^\pm} \int_{\partial P^\pm} \hat{\mathbf{s}}(\mathbf{r}') \cdot \hat{\mathbf{u}}^\pm \hat{\mathbf{u}}^\pm \cdot (\mathbf{s} + \mathbf{t}) R^q dl' \pm \frac{1}{L^\pm} \int_{\partial P^\pm} \hat{\mathbf{s}}(\mathbf{r}') \cdot \hat{\mathbf{u}}^\pm u_0^\pm R^q dl' \\ &= \pm \frac{1}{L^\pm} \int_{\partial P^\pm} \hat{\mathbf{s}}(\mathbf{r}') \cdot (\mathbf{s} + \mathbf{t}) R^q dl' \pm \frac{1}{L^\pm} \int_{\partial P^\pm} \hat{\mathbf{s}}(\mathbf{r}') \cdot \hat{\mathbf{u}}^\pm u_0^\pm R^q dl', \end{aligned} \quad (108)$$

where the first integral on the last line is evaluated only on the sides where the dot product $\hat{\mathbf{s}}(\mathbf{r}') \cdot \hat{\mathbf{u}}^\pm$ is non-zero. The value of the first integral is easily computed by (80) so finally the integral K_q^5 can be written

$$\boxed{K_q^5(P^\pm) = \pm \frac{1}{L^\pm} \left(\frac{1}{(q+2)} \sum_{i=1}^4 |\hat{\mathbf{s}}_i \cdot \hat{\mathbf{u}}^\pm| \left((R_i^+)^{q+2} - (R_i^-)^{q+2} \right) + \sum_{i=1}^4 \hat{\mathbf{s}}_i \cdot \hat{\mathbf{u}}^\pm u_0^\pm I_q^L(\partial T_i^\pm) \right)}, \quad (109)$$

where the summation is over the sides of the rectangle P^\pm .

Let us now consider the integral \mathbf{K}_6^q (31). Since the rooftop function is tangential to the surface, the gradient of the Green's function can be considered as the surface gradient, so the integral (31) can be written as

$$\begin{aligned} \hat{\mathbf{n}} \int_{P^\pm} \nabla' R^q \cdot \mathbf{f}_n(\mathbf{r}') dS' &= \hat{\mathbf{n}} \int_{P^\pm} \nabla'_s R^q \cdot \mathbf{f}_n(\mathbf{r}') dS' \\ &= \hat{\mathbf{n}} \hat{\mathbf{u}}^\pm \cdot \int_{P^\pm} (\nabla'_s R^q) s_n(\mathbf{r}') dS' \end{aligned} \quad (110)$$

Using the product rule, we can write (110) as

$$\hat{\mathbf{n}} \hat{\mathbf{u}}^\pm \cdot \int_{P^\pm} (\nabla'_s R^q) s_n(\mathbf{r}') dS'$$

$$= \hat{\mathbf{n}}\hat{\mathbf{u}}^\pm \cdot \int_{P^\pm} \nabla'_s (R^q s_n(\mathbf{r}')) dS' - \hat{\mathbf{n}}\hat{\mathbf{u}}^\pm \cdot \int_{P^\pm} R^q \nabla'_s s_n(\mathbf{r}') dS' \quad (111)$$

By Gauss' theorem, the first integral on the last line of (111) can be transformed into a line integral over the boundary of the integration element P^\pm . The surface gradient of the shape function is a constant, see explanation on page 263 for the integral \mathbf{K}_3^q . Then the Equation (111) becomes

$$\begin{aligned} \hat{\mathbf{n}}\hat{\mathbf{u}}^\pm \cdot \int_{\partial P^\pm} \hat{\mathbf{m}}(\mathbf{r}') R^q s_n(\mathbf{r}') dl' \mp \frac{1}{L^\pm} \hat{\mathbf{n}}\hat{\mathbf{u}}^\pm \cdot \hat{\mathbf{u}}^\pm \int_{P^\pm} R^q dS' \\ = \hat{\mathbf{n}} \int_{I_n} R^q dl' \mp \frac{1}{L^\pm} \hat{\mathbf{n}} \int_{P^\pm} R^q dS', \end{aligned} \quad (112)$$

since the dot product of the line normal is parallel to the unit vector $\hat{\mathbf{u}}^\pm$ only on the sides where the shape function equals either one or zero. The integral \mathbf{K}_6^q (31) can now be written

$$\boxed{\mathbf{K}_6^q(P^\pm) = \hat{\mathbf{n}} \left(I_q^L(I_n) \mp \frac{1}{L^\pm} I_q^S(P^\pm) \right)}. \quad (113)$$

For the integral \mathbf{K}_7^q (32) we begin by computing the gradient $\nabla' R^q$,

$$\begin{aligned} \hat{\mathbf{u}}^\pm \int_{P^\pm} (\hat{\mathbf{u}}^\pm \cdot (\mathbf{r}' - \mathbf{p}^\pm)) (\hat{\mathbf{n}} \cdot \nabla' R^q) dS' \\ = -q\hat{\mathbf{u}}^\pm \int_{P^\pm} (\hat{\mathbf{u}}^\pm \cdot (\mathbf{r}' - \mathbf{p}^\pm)) (\hat{\mathbf{n}} \cdot (\mathbf{r} - \mathbf{r}') R^{q-2}) dS'. \end{aligned} \quad (114)$$

Next, we write the vector $\mathbf{r}' - \mathbf{p}^\pm$ as $(\mathbf{r}' - \boldsymbol{\rho}) + (\boldsymbol{\rho} - \mathbf{p}^\pm)$, which gives us

$$\begin{aligned} -qh\hat{\mathbf{u}}^\pm \int_{P^\pm} R^{q-2} \hat{\mathbf{u}}^\pm \cdot ((\mathbf{r}' - \boldsymbol{\rho}) + (\boldsymbol{\rho} - \mathbf{p}^\pm)) dS' \\ = qh\hat{\mathbf{u}}^\pm \int_{P^\pm} R^{q-2} \hat{\mathbf{u}}^\pm \cdot (\boldsymbol{\rho} - \mathbf{r}') dS' - qhu_0^\pm \hat{\mathbf{u}}^\pm \int_{P^\pm} R^{q-2} dS'. \end{aligned} \quad (115)$$

Since $\nabla'_s R^q = -q(\boldsymbol{\rho} - \mathbf{r}') R^{q-2}$, we can write (115) using Gauss' theorem as

$$\begin{aligned} -h\hat{\mathbf{u}}^\pm \left(\hat{\mathbf{u}}^\pm \cdot \int_{P^\pm} \nabla'_s R^q dS' \right) - qhu_0^\pm \hat{\mathbf{u}}^\pm \int_{P^\pm} R^{q-2} dS' \\ = -h\hat{\mathbf{u}}^\pm \left(\hat{\mathbf{u}}^\pm \cdot \int_{\partial P^\pm} \hat{\mathbf{m}}(\mathbf{r}') R^q dl' \right) - qhu_0^\pm \hat{\mathbf{u}}^\pm \int_{P^\pm} R^{q-2} dS'. \end{aligned} \quad (116)$$

So finally, the integral \mathbf{K}_7^q (32) can be written as

$$\boxed{\mathbf{K}_7^q(P^\pm) = \mp \frac{h}{L^\pm} \hat{\mathbf{u}}^\pm \left(\hat{\mathbf{u}}^\pm \cdot \sum \hat{\mathbf{m}}_i I_q^L(\partial P_i^\pm) + qu_0^\pm I_{q-2}^S(P^\pm) \right)}. \quad (117)$$

8. HYBRID FUNCTIONS

In some applications it is useful to use both RWG and rooftop functions. For instance, if we have an (electrically large) flat surface with non-rectangular boundaries, we can improve the boundary approximation by using RWG functions close to it, and use rooftop functions in the interior in order to reduce the number of unknowns. For combining these two grids we need the RWG-rooftop hybrid functions.

For an element pair consisting of a triangle T^+ and a rectangle P^- , which share a common edge I_n , the hybrid basis function associated with them can be written as

$$\mathbf{f}_n(\mathbf{r}') = \begin{cases} \frac{L}{A^+}(\mathbf{r}' - \mathbf{p}^+), & \text{if } \mathbf{r}' \in T^+, \\ -\frac{L}{A^-}((\mathbf{r}' - \mathbf{p}^-) \cdot \hat{\mathbf{u}}) \hat{\mathbf{u}}, & \text{if } \mathbf{r}' \in P^-, \\ 0, & \text{otherwise,} \end{cases} \quad (118)$$

where

$$\hat{\mathbf{u}} = \frac{\mathbf{p}^- - \mathbf{p}^*}{|\mathbf{p}^- - \mathbf{p}^*|}, \quad (119)$$

and L is the length of the edge I_n , A^+ and A^- are the areas of T^+ and P^- , respectively, and \mathbf{p}^+ , \mathbf{p}^- , and \mathbf{p}^* are as in Fig. 5.

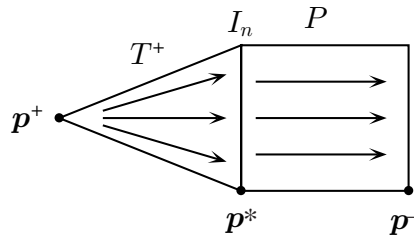


Figure 5. Hybrid functions are defined on an element pair consisting of one triangle and of one rectangle.

We already have all the needed integration formulae for closed-form integration of the hybrid functions, since the hybrid functions are

a combination of RWG and rooftop functions, and since in compiling the system matrix it is more cost effective to use “element by element” than “basis function by basis function” type integration, so that when integrating the hybrid functions, all that is needed is to check whether to use RWG or rooftop integration formulae in a given element.

9. NUMERICAL EXAMPLES

In order to verify the accuracy of the presented integration formulas, we apply them to the following examples. Since the validity of the majority of the presented integration formulas has been demonstrated for RWG functions, see e.g., [8], we will concentrate on the rooftop functions. We will first consider the integral

$$\int_P G(\mathbf{r}, \mathbf{r}') \mathbf{f}(\mathbf{r}') dS', \quad (120)$$

where P is the square defined by vertices $(0, 0, 0)$, $(\lambda, 0, 0)$, $(\lambda, \lambda, 0)$ and $(0, \lambda, 0)$, where λ is the wavelength. As for the rooftop function, we will simply use the function $f(\mathbf{r}) = x\hat{\mathbf{x}}$. The integral was computed for four different field points \mathbf{r} . First point is at the center of P , i.e. $\mathbf{r}_1 = 0.5\lambda(\hat{\mathbf{x}} + \hat{\mathbf{y}})$, second just above the center, $\mathbf{r}_2 = 0.5\lambda(\hat{\mathbf{x}} + \hat{\mathbf{y}}) + 0.1\lambda\hat{\mathbf{z}}$, and two points above the first corner of P , i.e., $\mathbf{r}_3 = 0.1\lambda\hat{\mathbf{z}}$ and $\mathbf{r}_4 = 0.01\lambda\hat{\mathbf{z}}$.

The integral was computed using three methods: the singularity subtraction method using the integration formula (95) for the first two non-smooth terms of the Taylor series in (12) and Gaussian quadrature for the remaining smooth part (13), the singularity cancellation method using Duffy’s transformation for the first term of the Taylor series (i.e., the $1/R$ -term) [2] and Gaussian quadrature for the remaining part, and lastly by the straightforward Gaussian quadrature. The results are presented in the Fig. 6–7. In the figures, we have plotted the relative errors of the different methods against the accurate value, obtained by the singularity subtraction method with a sufficiently (i.e., very) large number of integration points. The accurate value was considered to be achieved when increasing the number of integration points did not improve the results in a significant way. The Duffy’s method and the Gaussian quadrature also converge towards this accurate value, but they require even a higher number of integration points.

As can be seen, the singularity subtraction method and the Duffy’s method converge quickly towards the same value for field point in the center of P . The straightforward Gaussian quadrature takes much longer to converge. On the other hand, if we choose the field point to be above the center or above the first corner of P , then the singularity subtraction method still gives correct results with a fairly small number

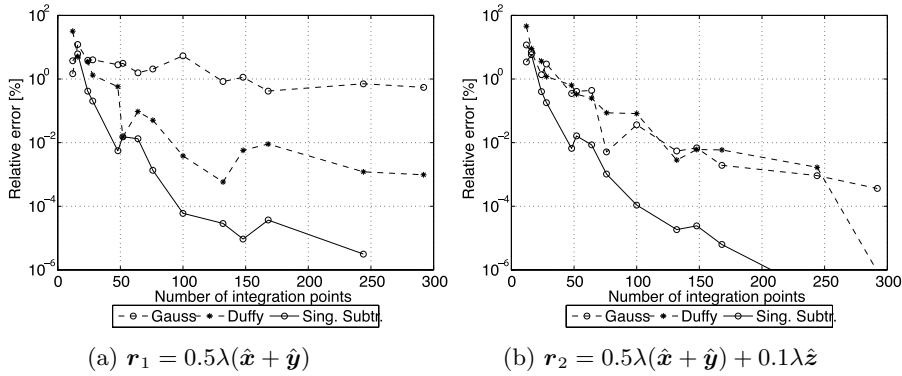


Figure 6. Relative error of the x -component of the integral (120) as a function of the integration points with the field point in the center (a) or above the center (b) of the integration element.

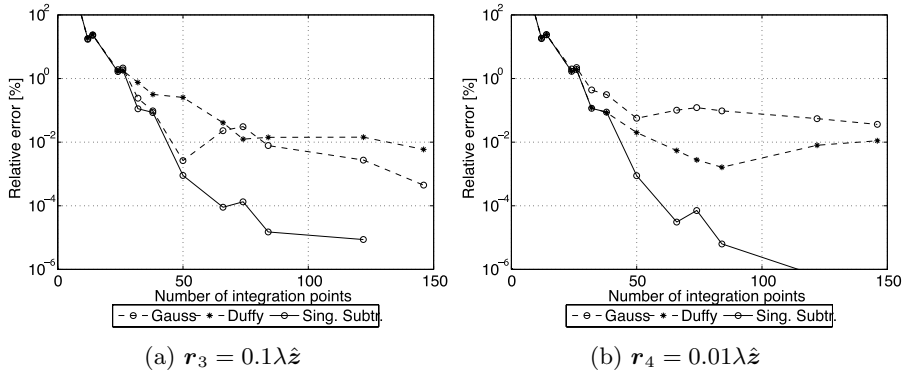


Figure 7. Relative error of the x -component of the integral (120) as a function of the integration points with with the field point above a corner of the integration element.

of integration points, but the advantage of the Duffy's transformation above the Gaussian quadrature is diminished. For the field point above the first corner of P , the relative errors of the Duffy's method and Gaussian quadrature do not seem to diminish as quickly as for the singularity subtraction method as the number of integration points is increased.

Next we consider the integral

$$\int_P (\nabla'G(\mathbf{r}, \mathbf{r}')) \times \mathbf{f}(\mathbf{r}') dS', \tag{121}$$

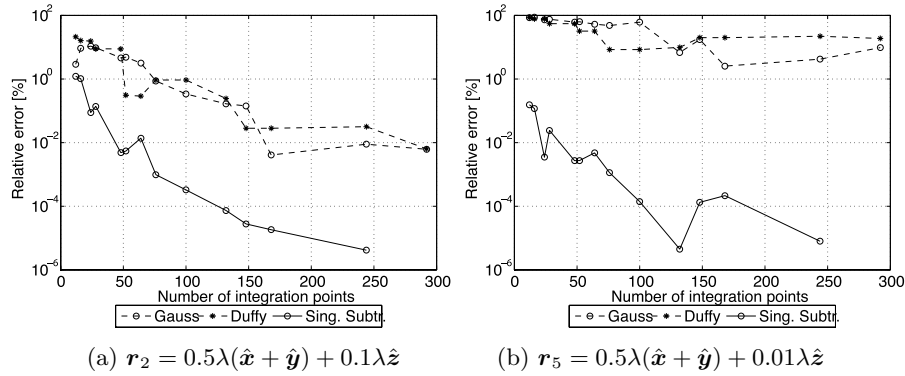


Figure 8. Relative error of the y -component of the integral (121) as a function of the integration points with the field point above the center of the integration element.

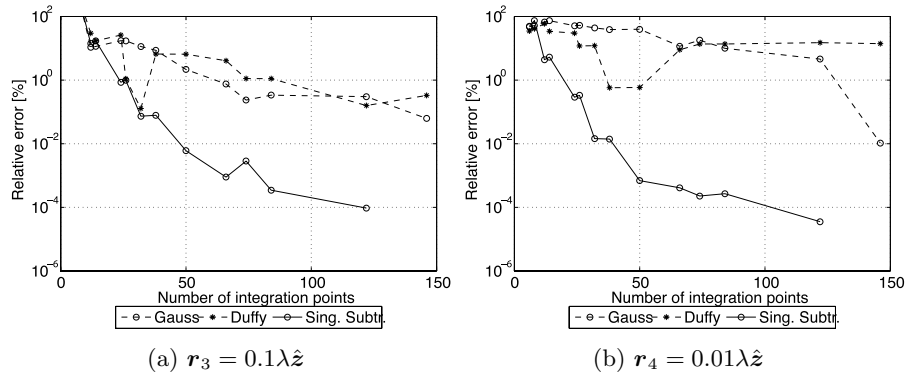


Figure 9. Relative error of the y -component of the integral (121) as a function of the integration points with the field point above a corner of the integration element.

where P and $\mathbf{f}(\mathbf{r})$ are defined as in the integral (120). We compute the value of the integral for the field points \mathbf{r}_2 , \mathbf{r}_3 , \mathbf{r}_4 , defined above, and for the field point $\mathbf{r}_5 = 0.5\lambda(\hat{\mathbf{x}} + \hat{\mathbf{y}}) + 0.01\lambda\hat{\mathbf{z}}$. Similarly as before, we compute the integral using the singularity subtraction method with the Equation (106), Duffy's method and straightforward Gaussian quadrature. The results are shown in Fig. 8 and Fig. 9. In the figures, the relative errors of the y -components of the results are plotted. The singularity subtraction method arrives to the correct result fairly quickly, whereas Duffy's method and the straightforward Gaussian

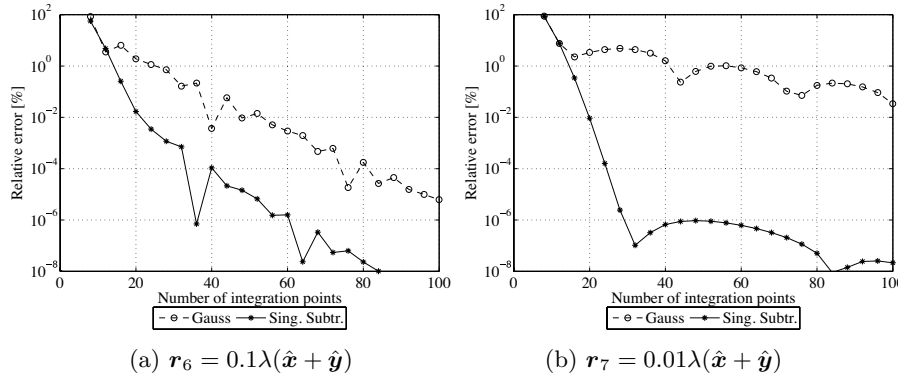


Figure 10. The relative error of the integral (122) as a function of the integration points with the field point close to the first corner of P .

quadrature do not seem to converge very well, i.e., the relative errors of these two methods remain quite large even as the number of the integration points is increased. The closer the field point is to the integration element P , the slower is the convergence of the Duffy’s method and of the Gaussian quadrature.

Lastly, we consider the integral

$$\int_{\partial P} \hat{s}(\mathbf{r}') \cdot \mathbf{f}(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') dl', \quad (122)$$

with the same definitions as above. We compute the value of the integral for two field points $\mathbf{r}_6 = 0.1\lambda(\hat{x} + \hat{y})$, and $\mathbf{r}_7 = 0.01\lambda(\hat{x} + \hat{y})$, using the singularity subtraction method with (109), and the Gaussian quadrature. The integral can also be computed fully analytically using e.g., Maple or Mathematica. The relative errors of the singularity subtraction method and the Gaussian quadrature compared to the analytic result obtained with Maple are presented in the Fig. 10. Again, the Gaussian quadrature requires quite many integration points before the relative error can be considered very small.

10. CONCLUSION

We have presented all the necessary formulae to compute the non-smooth parts of the integrals (3) and (7) in closed forms for the RWG, $\hat{n} \times \text{RWG}$, rooftop and $\hat{n} \times \text{rooftop}$ basis functions, the latter three cases being novel applications. Also the hybrid use of these functions has been discussed. The hybrid functions are especially suitable for

electrically large structures with non-rectangular boundaries. The heavy numerical cost of a straightforward numerical treatment of these integrals can thus be avoided without loss of accuracy. For convenience, a summary of these formulae is presented in the Appendix B.

The numerical implementation of these formulae is a straightforward but somewhat elaborate procedure. The resulting codes are easy to test because the formulae are also valid in the smooth cases, i.e., for $R > 0$ everywhere in the integration, where a direct numerical integration yields accurate results, too. However, when integrating the Green's function (5) with large values of R , the accuracy of the computation suffers, since the Taylor series used in the singularity subtraction was taken at $R = 0$, so it is valid only for sufficiently small values of R . For large values of R , the accuracy can be improved by including more terms from the Taylor series, or by deriving the Taylor series about a suitable point R .

APPENDIX A. INTEGRAL K_4^q AND LOGARITHMIC SINGULARITY

In the Method of Moments implementation the integral term K_4^q

$$\mathbf{K}_4^q = \int_{\text{spt}(\mathbf{f}_n)} \nabla' R^q \times \mathbf{f}_n(\mathbf{r}') dS', \quad (\text{A1})$$

with $R = |\mathbf{r} - \mathbf{r}'|$, and $q = -1$, may still lead to a logarithmic singularity in the outer (i.e., testing) integral when the outer integration element and the inner integration element are not in the same plane but have common points [6]. If the elements share an edge, the integrand is singular on the whole edge, so the integral is difficult to evaluate. If the outer integration element and the inner integration element are the same, then the integral K_4^q is zero. Fortunately, this singularity can also be removed for RWG-functions [8], as is shown here. This method can also be used for rooftop functions with some modifications.

The term in the MoM matrix element equation which contains the integral K_4^q , in the case when the basis function $\mathbf{f}_n(\mathbf{r}')$ is also chosen as the testing function $\mathbf{f}_m(\mathbf{r})$, i.e., by using the Galerkin method, is of the form (again, we use “element by element” type integration)

$$-C_{nm} \int_{T_m} (\mathbf{r} - \mathbf{p}_m) \cdot \left(\int_{T_n} \nabla' R^{-1} \times (\mathbf{r}' - \mathbf{p}_n) dS' \right) dS, \quad (\text{A2})$$

where \mathbf{p}_m and \mathbf{p}_n are the appropriate vertices of the triangles T^m and T^n , and the coefficients of the RWG functions are grouped together in $C_{mn} = \pm L_n L_m / 8\pi A_n A_m$, with the sign chosen appropriately. By

replacing $\mathbf{r}' - \mathbf{p}_n$ with $(\mathbf{r}' - \mathbf{r}) + (\mathbf{r} - \mathbf{p}_m) + (\mathbf{p}_m - \mathbf{p}_n)$, we can write (A2) as

$$C_{nm} \int_{T_m} (\mathbf{r} - \mathbf{p}_m) \cdot \left((\mathbf{p}_m - \mathbf{p}_n) \times \int_{T_n} \nabla' R^{-1} dS' \right) dS, \quad (\text{A3})$$

since $\nabla' R^{-1} \times (\mathbf{r}' - \mathbf{r}) = 0$. Next, we divide the gradient of R^{-1} into the normal and surface components,

$$C_{nm} \int_{T_m} (\mathbf{r} - \mathbf{p}_m) \cdot (\mathbf{p}_m - \mathbf{p}_n) \times \left(\int_{T_n} \nabla'_n R^{-1} + \nabla'_s R^{-1} dS' \right) dS, \quad (\text{A4})$$

where the integral of the normal gradient is identified with the integral hI_{-3}^S in (44). By applying Gauss' theorem to the surface gradient component in (A4), we get

$$C_{nm} \int_{T_m} (\mathbf{r} - \mathbf{p}_m) \cdot (\mathbf{p}_m - \mathbf{p}_n) \times \left(\int_{\partial T_n} \hat{\mathbf{m}}(\mathbf{r}') R^{-1} dl' \right) dS. \quad (\text{A5})$$

Finally, by changing the order of integration, we can write (A5) in the form

$$C_{nm} \int_{\partial T_n} (\mathbf{p}_m - \mathbf{p}_n) \times \hat{\mathbf{m}}(\mathbf{r}') \cdot \left(\int_{T_m} R^{-1} (\mathbf{r} - \mathbf{p}_m) dS \right) dl', \quad (\text{A6})$$

where the inner integral is now of the form of (23) and is thus analytically integrable.

If the testing function is chosen as $\hat{\mathbf{n}}(\mathbf{r}) \times \mathbf{f}_m(\mathbf{r})$, then we are able to use the same ideas as above, even though the derivation is somewhat more complicated. The term in the MoM matrix element equation which contains the integral K_4^q is now written as

$$-C_{nm} \int_{T_m} \hat{\mathbf{n}}(\mathbf{r}) \times (\mathbf{r} - \mathbf{p}_m) \cdot \left(\int_{T_n} \nabla' R^{-1} \times (\mathbf{r}' - \mathbf{p}_n) dS' \right) dS, \quad (\text{A7})$$

with the same definitions as for (A2). By replacing $\mathbf{r}' - \mathbf{p}_n$ with $(\mathbf{r}' - \mathbf{r}) + (\mathbf{r} - \mathbf{p}_n)$, we can write (A7) as

$$C_{nm} \int_{T_m} \hat{\mathbf{n}}(\mathbf{r}) \times (\mathbf{r} - \mathbf{p}_m) \cdot \left((\mathbf{r} - \mathbf{p}_n) \times \int_{T_n} \nabla' R^{-1} dS' \right) dS. \quad (\text{A8})$$

Next, we divide the gradient of R^{-1} into the normal and surface components,

$$C_{nm} \int_{T_m} \hat{\mathbf{n}}(\mathbf{r}) \times (\mathbf{r} - \mathbf{p}_m) \cdot \left((\mathbf{r} - \mathbf{p}_n) \times \int_{T_n} \nabla'_n R^{-1} dS' \right. \\ \left. + (\mathbf{r} - \mathbf{p}_n) \times \int_{T_n} \nabla'_s R^{-1} dS' \right) dS, \quad (\text{A9})$$

where the integral of the normal gradient is again easily identified with the integral hI_{-3}^S (44). By applying Gauss' theorem to the surface gradient component, we get

$$C_{nm} \int_{T_m} \hat{\mathbf{n}}(\mathbf{r}) \times (\mathbf{r} - \mathbf{p}_m) \cdot \left((\mathbf{r} - \mathbf{p}_n) \times \int_{\partial T_n} \hat{\mathbf{m}}(\mathbf{r}') R^{-1} dl' \right) dS. \quad (\text{A10})$$

Next we replace $\mathbf{r} - \mathbf{p}_n$ with $(\mathbf{r} - \mathbf{p}_m) + (\mathbf{p}_m - \mathbf{p}_n)$ in the inner integral and apply vector algebra, and get

$$\begin{aligned} & C_{nm} \int_{T_m} \hat{\mathbf{n}}(\mathbf{r}) \times (\mathbf{r} - \mathbf{p}_m) \cdot \left(((\mathbf{r} - \mathbf{p}_m) + (\mathbf{p}_m - \mathbf{p}_n)) \right. \\ & \quad \left. \times \int_{\partial T_n} \hat{\mathbf{m}}(\mathbf{r}') R^{-1} dl' \right) dS \\ = & C_{nm} \int_{T_m} \hat{\mathbf{n}}(\mathbf{r}) \times (\mathbf{r} - \mathbf{p}_m) \cdot \left(\int_{\partial T_n} (\mathbf{p}_m - \mathbf{p}_n) \times \hat{\mathbf{m}}(\mathbf{r}') R^{-1} dl' \right) dS \\ & - C_{nm} \int_{T_m} |\mathbf{r} - \mathbf{p}_m|^2 \hat{\mathbf{n}}(\mathbf{r}) \cdot \left(\int_{\partial T_n} \hat{\mathbf{m}}(\mathbf{r}') R^{-1} dl' \right) dS. \quad (\text{A11}) \end{aligned}$$

Finally, by changing the order of integration, we can write (A11) in the form

$$\begin{aligned} & C_{nm} \int_{\partial T_n} (\mathbf{p}_m - \mathbf{p}_n) \times \hat{\mathbf{m}}(\mathbf{r}') \cdot \left(\hat{\mathbf{n}}(\mathbf{r}) \times \int_{T_m} \frac{\mathbf{r} - \mathbf{p}_m}{R} dS \right) dl' \\ & - C_{nm} \int_{\partial T_n} \hat{\mathbf{m}}(\mathbf{r}') \cdot \hat{\mathbf{n}}(\mathbf{r}) \left(\int_{T_m} \frac{|\mathbf{r} - \mathbf{p}_m|^2}{R} dS \right) dl'. \quad (\text{A12}) \end{aligned}$$

The first inner integral in (A12) can be computed analytically by using (23). Also the second inner integral can be computed analytically, which can be seen by writing the integrand as

$$\frac{|\mathbf{r} - \mathbf{p}_m|^2}{R} = R + 2(\mathbf{r}' - \mathbf{p}_m) \cdot \frac{(\mathbf{r} - \mathbf{p}_m)}{R} - |\mathbf{r}' - \mathbf{p}_m|^2 \frac{1}{R}, \quad (\text{A13})$$

and by writing $\mathbf{r} - \mathbf{p}_m = (\mathbf{r} - \mathbf{r}') + (\mathbf{r}' - \mathbf{p}_m)$.

By considering the normal and surface components of the gradients separately, and by changing the order of integration, we are thus able to avoid the logarithmic singularity in the outer integral. This treatment, however, is more laborious. In most cases, the regular closed form formulae give accurate results, if the integration points are not very close to the common edges.

APPENDIX B. SUMMARY OF FORMULAE

B.1. Line and Surface Integrals

The line integral of $1/R$ (where $R = |\mathbf{r} - \mathbf{r}'|$) over the segment of the line ΔL defined by \mathbf{p}_1 and \mathbf{p}_2 , and the surface integral of $1/R^3$ over a polygon P can be computed with

$$I_{-1}^L(\Delta L) = \ln \left(\frac{R^+ + s^+}{R^- + s^-} \right) = \ln \left(\frac{R^- - s^-}{R^+ - s^+} \right), \quad I_{-3}^S(P) = \frac{1}{h} \Omega.$$

where

$$\begin{aligned} R^+ &= |\mathbf{r} - \mathbf{p}_2|, & R^- &= |\mathbf{r} - \mathbf{p}_1|, \\ s^+ &= (\mathbf{p}_2 - \mathbf{r}) \cdot \hat{\mathbf{s}}, & s^- &= (\mathbf{p}_1 - \mathbf{r}) \cdot \hat{\mathbf{s}}, \end{aligned}$$

and

$$h = \hat{\mathbf{n}} \cdot (\mathbf{r} - \mathbf{r}').$$

The unit edge vectors are defined as

$$\hat{\mathbf{s}} = (\mathbf{p}_2 - \mathbf{p}_1) / |\mathbf{p}_2 - \mathbf{p}_1|,$$

and $\hat{\mathbf{n}}$ is the unit normal vector to the surface P , and $\hat{\mathbf{m}}$ is the outward directed unit normal vector to the line L , when L is a boundary line of P . Then we can also write

$$\mathbf{r} - \mathbf{r}' = \mathbf{s} + \mathbf{t} + \mathbf{h} = s\hat{\mathbf{s}} + t\hat{\mathbf{m}} + h\hat{\mathbf{n}}.$$

Recursive formula for the line integral of R^q for odd powers of q and $q \geq 1$ over the line segment ΔL is defined as

$$I_q^L(\Delta L) = \frac{qR_0^2}{q+1} I_{q-2}^L + \frac{1}{q+1} (s^+(R^+)^q - s^-(R^-)^q),$$

where

$$R_0 = t^2 + h^2,$$

and the surface integral of R^q for odd powers of q and $q \geq -1$ over the surface P is

$$I_q^S(P) = \frac{qh^2}{q+2} I_{q-2}^S(P) - \frac{1}{q+2} \sum_{i=1}^M t_i I_q^L(\partial P_i).$$

B.2. RWG Functions

$$\begin{aligned}
K_1^q(T^\pm) &= \pm \frac{L}{A^\pm} I_q^S(T^\pm), \\
\mathbf{K}_2^q(T^\pm) &= \pm \frac{L}{2A^\pm} \left(\frac{1}{q+2} \sum_{i=1}^3 \hat{\mathbf{m}}_i I_{q+2}^L(\partial T_i^\pm) + (\boldsymbol{\rho} - \mathbf{p}^\pm) I_q^S(T^\pm) \right), \\
\mathbf{K}_3^q(T^\pm) &= \sum_{i=1}^M \hat{\mathbf{m}}_i I_q^L(\partial T_i^\pm) - hq \hat{\mathbf{n}} I_{q-2}^S(T^\pm), \\
\mathbf{K}_4^q(T^\pm) &= \mp \frac{L}{2A^\pm} (\mathbf{r} - \mathbf{p}^\pm) \times \mathbf{K}_3^q(T^\pm), \\
\mathbf{K}_q^5(T^\pm) &= \pm \frac{L}{2A^\pm} \left(\frac{1}{(q+2)} \sum_{i=1}^3 \left((R_i^+)^{q+2} - (R_i^-)^{q+2} \right) \right. \\
&\quad \left. + \sum_{i=1}^3 s_{0,i}^\pm I_q^L(\partial T_i^\pm) \right), \\
\mathbf{K}_6^q(T^\pm) &= \pm \frac{L}{2A^\pm} \hat{\mathbf{n}} \left(q I_q^S(T^\pm) - qh^2 I_{q-2}^S(T^\pm) \right. \\
&\quad \left. + (\mathbf{r} - \mathbf{p}^\pm) \cdot \sum_{i=1}^3 \hat{\mathbf{m}}'_i I_q^L(\partial T_i^\pm) \right), \\
\mathbf{K}_q^7(T^\pm) &= \mp \frac{Lh}{2A^\pm} \left(\sum_{i=1}^3 \hat{\mathbf{m}}'_i I_q^L(\partial T_i^\pm) + q(\boldsymbol{\rho} - \mathbf{p}^\pm) I_{q-2}^S(T^\pm) dS' \right), \\
\boldsymbol{\rho} &= \mathbf{r} - h\hat{\mathbf{n}}, \quad s_{0,i}^\pm = \hat{\mathbf{s}}_i \cdot (\boldsymbol{\rho} - \mathbf{p}^\pm), \quad h = \hat{\mathbf{n}} \cdot (\mathbf{r} - \mathbf{r}'),
\end{aligned}$$

and $\hat{\mathbf{m}}_i$ is the outward directed unit vector of the i :th side ∂T_i^\pm of the triangle T^\pm . For integrals \mathbf{K}_3^q and \mathbf{K}_4^q , special care must be exercised for the case $h = 0$.

B.3. Rooftop Functions

$$\begin{aligned}
K_1^q(P^\pm) &= \pm \frac{1}{L^\pm} I_q^S(P^\pm), \\
\mathbf{K}_2^q(P^\pm) &= \pm \frac{\hat{\mathbf{u}}^\pm}{L^\pm} \left(\frac{1}{q+2} \sum_{i=1}^4 \hat{\mathbf{u}}^\pm \cdot \hat{\mathbf{m}}_i I_{q+2}^L(\partial P_i^\pm) + u_0^\pm I_q^S(P^\pm) \right), \\
\mathbf{K}_3^q(P^\pm) &= \sum_{i=1}^4 \hat{\mathbf{m}}_i \int_{\partial P_i^\pm} s_n(\mathbf{r}') R^q dl' \mp \frac{\hat{\mathbf{u}}^\pm}{L^\pm} I_q^S(P^\pm) - \hat{\mathbf{n}} h q K_2^{q-2}(P^\pm),
\end{aligned}$$

$$\begin{aligned}
\mathbf{K}_4^q(P^\pm) &= \hat{\mathbf{u}}^\pm \times \mathbf{K}_3^q(P^\pm), \\
K_q^5(P^\pm) &= \pm \frac{1}{L^\pm} \left(\frac{1}{(q+2)} \sum_{i=1}^4 |\hat{\mathbf{s}}_i \cdot \hat{\mathbf{u}}^\pm| \left((R_i^+)^{q+2} - (R_i^-)^{q+2} \right) \right. \\
&\quad \left. + \sum_{i=1}^4 \hat{\mathbf{s}}_i \cdot \hat{\mathbf{u}}^\pm u_0^\pm I_q^L(\partial T_i^\pm) \right), \\
\mathbf{K}_6^q(P^\pm) &= \hat{\mathbf{n}} \left(I_q^L(I_n) \mp \frac{1}{L^\pm} I_q^S(P^\pm) \right), \\
\mathbf{K}_7^q(P^\pm) &= \mp \frac{h}{L^\pm} \hat{\mathbf{u}}^\pm \left(\hat{\mathbf{u}}^\pm \cdot \sum \hat{\mathbf{m}}_i I_q^L(\partial P_i^\pm) + q u_0^\pm I_{q-2}^S(P^\pm) \right), \\
\rho &= \mathbf{r} - h \hat{\mathbf{n}}, \quad u_0^\pm = \hat{\mathbf{u}}^\pm \cdot (\rho - \mathbf{p}^\pm), \quad h = \hat{\mathbf{n}} \cdot (\mathbf{r} - \mathbf{r}')
\end{aligned}$$

and $\hat{\mathbf{m}}_i$ is the outward directed unit vector of the i :th side ∂P_i^\pm of the rectangle P^\pm . For integrals \mathbf{K}_3^q and \mathbf{K}_4^q , special care must be exercised for the case $h = 0$.

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