SIX DIMENSIONAL ALMOST COMPLEX MANIFOLDS
DEFINED BY MEANS OF THREE-FOLD
VECTOR CROSS PRODUCTS

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1. Introduction. Every orientable 6-dimensional immersed submanifold M of R^8 possesses an almost complex structure [7]. In fact R^8 (as a vector space) possesses two nonisomorphic 3-fold vector cross products ; each of these induces an almost complex structure on M. In general the two almost complex structures are distinct, and the manifolds thus obtained are not Kählerian. However, the almost complex structures do have some properties that are similar to, but more complicated than, those of Kähler manifolds.

In this paper we investigate the topology and differential geometry of three of the most important types of these almost Hermitian manifolds, namely those which are nearly Kählerian, Hermitian, and quasi-Kählerian. We assume throughout this paper, unless stated otherwise, that M is an orientable 6-dimensional submanifold of R^8. The almost complex structure is defined by means of a 3-fold vector cross product [2], [7], and the induced metric from R^8.

In §2 we discuss nearly Kähler manifolds. The canonical example of a non-Kähler nearly Kähler manifold is S^6. However, according to [7], the nearly Kähler structure of S^6 is not unique. Nonetheless, it seems plausible that every compact nearly Kähler manifold M obtained by means of a 3-fold vector cross product is isometric to S^6. We show that this is the case if M is Einstein and has positive sectional curvature.

If the almost complex structure of M is integrable, so that M is Hermitian, then M is a minimal variety of R^8 [7]. This implies that M is noncompact. We give more detailed information about the homotopy type of M in §3. Furthermore we show that the curvature operator of M satisfies certain identities which are satisfied by Kähler manifolds, but not by all Hermitian manifolds.

In §4 we investigate principal distributions defined in [5] on quasi-Kähler manifolds and generalize some results of [7].

2. Nearly Kähler manifolds. Let M be a C^∞ almost Hermitian manifold with metric tensor $< , >$, Riemannian connection $\bigtriangledown$, and almost complex structure J. Denote by $\mathfrak{F}(M)$ the real valued C^∞ functions on M and by $\mathfrak{X}(M)$ the C^∞

vector fields of M. Then M is said to be a *nearly Kähler manifold* provided $\nabla_x(J)(X) = 0$ for all $X \in \mathfrak{X}(M)$.

The following notions will also be useful. Let M be any almost Hermitian manifold, and for $m \in M$ denote by M_m the tangent space to M at m. Then M is said to be of *constant type* at $m \in M$ provided that for all $x \in M_m$ we have $\|\nabla_x(J)(y)\| = \|\nabla_x(J)(z)\|$ whenever $<x,y> = <Jx, y> = <x, z> = <Jx, z> = 0$ and $\|y\| = \|z\|$. If this holds for all $m \in M$ we say that M has *pointwise constant type*. If M has pointwise constant type and for $X, Y \in \mathfrak{X}(M)$ with $<X, Y> = <JX, Y> = 0$ the function $\|\nabla_x(J)(Y)\|$ is constant whenever $\|X\| = \|Y\| = 1$, then we say that M has *global constant type*.

PROPOSITION 2.1. *Let M be a nearly Kähler manifold. Then M has pointwise constant type if and only if there exists $\alpha \in \mathfrak{F}(M)$ such that*

$$\|\nabla_W(J)(X)\|^2 = \alpha\{\|W\|^2\|X\|^2 - <W, X>^2 - <W, JX>^2\}$$

for all $W, X \in \mathfrak{X}(M)$. Furthermore M has global constant type if and only if this equation holds with a constant function α.

The proof of proposition 2.1 is easy, and so we omit it. We agree to call α the *constant type* of M.

Now let M be a 6-dimensional orientable immersed submanifold of R^8. Denote by P the 3-fold vector cross product on R^8; then P determines an almost complex structure on M by the formula

$$JA = P(N, JN, A)$$

for $A \in \mathfrak{X}(M)$. Here N and JN are unit normal vector fields with $<N, JN> = 0$ defined locally on M (see [7]).

THEOREM 2.2. *If M is nearly Kählerian, then M has pointwise constant type.*

PROOF. Let T denote the configuration tensor of M in R^8 [4]. According to [7, theorem 6.13] there exists a 1-form β defined on the normal bundle of M such that

(1) $$T_A JN \pm JT_A N = \beta(JN)A \pm \beta(N)JA$$

for all $A \in \mathfrak{X}(M)$. (Here $+$ or $-$ is determined by the isomorphism class of the

3-fold vector cross product P.) Furthermore by [7, theorem 6.4] we have

$$<\nabla_A(J)(B), C> \ = \ <P(N, T_AJN \pm JT_AN, B), C>$$

for $A, B, C \in \mathfrak{X}(M)$. Therefore

$$\nabla_A(J)(B) = P(N, T_AJN \pm JT_AN, B) + <T_AJN \pm JT_AN, JB> JN.$$

Hence

$$
\begin{aligned}
(2) \quad \|\nabla_A(J)(B)\|^2 &= \|P(N, T_AJN \pm JT_AN, B)\|^2 - <T_AJN \pm JT_AN, JB>^2 \\
&= \|T_AJN \pm JT_AN\|^2 \|B\|^2 - <T_AJN \pm JT_AN, B>^2 \\
&\quad - <T_AJN \pm JT_AN, JB>^2 \\
&= \|\beta(JN)A \pm \beta(N)JA\|^2 - <\beta(JN)A \pm \beta(N)JA, B>^2 \\
&\quad - <\beta(JN)A \pm \beta(N)JA, JB>^2 \\
&= \{\beta(JN)^2 + \beta(N)^2\} \{\|A\|^2 - <A, B>^2 - <JA, B>^2\} \\
&= \|\beta\|^2 \{\|A\|^2 \|B\|^2 - <A, B>^2 - <JA, B>^2\} \ .
\end{aligned}
$$

Hence the theorem follows.

We remark that the homogeneous space $F_4/A_2 \times A_2$ has a nearly Kähler structure which is not of constant type.

THEOREM 2.3. *Suppose the hypotheses of theorem 2.2 hold. In order that M have global constant type, it is necessary and sufficient that the mean curvature vector of M in R^8 have constant length.*

PROOF. The mean curvature of M in R^8 is defined by $H = \Sigma_{i=1}^6 T_{E_i}E_i$, where $\{E_1, \cdots, E_6\}$ is any local orthonormal frame field on M. It is not hard to see that

$$<T_AA + T_{JA}JA, N> \ = \ -2\beta(N)\|A\|^2 \qquad \text{for } A \in \mathfrak{X}(M),$$

and so $<H, N> \ = \ -6\beta(N)$. Hence $\|H\|^2 = 36\|\beta\|^2$. Now theorem 2.3 follows from theorem 2.2.

We now give sufficient conditions that a nearly Kähler manifold be isometric to a sphere.

THEOREM 2.4. *Let M be a 6-dimensional orientable immersed submanifold of R^8, and assume that M has the induced metric, and the almost complex structure derived from a 3-fold vector cross product on R^8. In addition assume that M is a compact nearly Kähler Einstein manifold with positive sectional curvature. Then M is isometric to a 6-dimensional sphere.*

PROOF. According to Theorem 2.2 M has pointwise constant type. The theorem is now a consequence of [8, Theorem 8.1].

3. Hermitian manifolds. In this section we assume that the almost complex structure on M defined by a 3-fold vector cross product on R^8 is integrable. We shall need the following lemma, which is proved in [7].

LEMMA 3.1 *For all $X, Y \in \mathfrak{X}(M)$ we have*

$$T_X Y + T_{JX} JY = 0.$$

As an immediate consequence, we have the following theorem.

THEOREM 3.2. *M has the homotopy type of a CW-complex with no cells of dimension greater than 3.*

PROOF. Lemma 3.1 implies that for each $m \in M$ and each $z \in M_m^\perp$, at least 3 of the eigenvalues of $x \to T_x z$ are nonpositive. Hence by [6, Lemma 3.2], the theorem follows.

Next we prove that the curvature operator $R_{XY}(X, Y) \in \mathfrak{X}(M)$ satisfies certian identities. Also let k, R, K and B denote the Ricci curvature, Ricci scalar curvature, sectional curvature, and holomorphic bisectional curvature of M. The last is defined by $B_{XY}\|X\|^2\|Y\|^2 = <R_{XJX}Y, JY>$ for $X, Y \in \mathfrak{X}(M)$, whenever X and Y are non zero. (See [3], [8]).

THEOREM 3.3. *We have*

(i) $B_{XY} = K_{XY} + K_{XJY} = - \|T_X Y\|^2 - \|T_X JY\|^2 \leqq 0 ,$

for $X, Y \in \mathfrak{X}(M)$ whenever $\|X\| = \|Y\| = 1, <X, Y> = 0$:

(ii) $<R_{XY}X, Y> = <R_{XY}JX, JY> - <R_{XJY}JX, Y> - <R_{XJY}X, JY> ,$

for $X, Y \in \mathfrak{X}(M)$:

(iii) $<R_{WX}Y, Z> = <R_{JWJX}JY, JZ>$,

for $W, X, Y, Z \in \mathfrak{X}(M)$:

(iv) $k(X, X) = \sum_{i=1}^{3} B_{X E_i} = - \sum_{i=1}^{3} \{\|T_X E_i\|^2 + \|T_X J E_i\|^2\} \leqq 0$,

for $X \in \mathfrak{X}(M)$ with $\|X\| = 1$, where $\{E_1, E_2, E_3, JE_1, JE_2, JE_3\}$ is any local frame field on M:

(vi) $\dfrac{1}{V(S^5)} \displaystyle\int_{S_m} B_{xx} dx = R,$

where S_m denotes the unit sphere in M_m for any $m \in M$, dx is the canonical measure on S_m, and $V(S^5)$ denotes the volume of the unit 5-dimensional sphere.

PROOF. The Gauss equation [4] states that $<R_{WX}Y, Z> = <T_WY, T_XZ>$ $- <T_WZ, T_XY>$ for $W, X, Y, Z \in \mathfrak{X}(M)$. In particular for $X, Y \in \mathfrak{X}(M)$,

$$<R_{XY}X, Y> = <T_XX, T_YY> - \|T_XY\|^2 .$$

$$<R_{XJY}X, JY> = <T_XX, T_{JY}JY> - \|T_XJY\|^2$$

$$<R_{XJX}Y, JY> = <T_XY, T_{JX}JY> - <T_XJY, T_{JX}Y> .$$

Now (i) follows from Lemma 3.1 and these equations. Similar applications of Lemma 3.1 and the Gauss equation yield (ii). Then (iii), (iv), and (v) follow from (i). For (vi) we note that Berger [1] has proved exactly the same formula for Kähler manifolds. An examination of Berger's proof shows that (i) is all that is needed to prove (vi) for the case we are considering.

As an immediate consequence of Theorem 3.3 and [7] we obtian the following result.

THEOREM 3.4. *In order that the Hermitian manifold M be Kählerian and totally geodesic, it is necessary and sufficient that any of the following vanish on M: the sectional curvature, holomorphic sectional curvature, holomorphic bisectional curvature, Ricci curvature, Ricci scalar curvature.*

4. Quasi-Kähler manifolds. Recall that an almost Hermitian manifold M is *quasi-Kählerian* provided $\bigtriangledown_X(J)(Y) + \bigtriangledown_{JX}(J)(JY) = 0$ for all $X, Y \in \mathfrak{X}(M)$ (see [6]). A nearly Kähler manifold is quasi-Kählerian [4].

We shall also need some results of [5]. Let M be a Riemannian submanifold of a Riemannian manifold $\bar{M}$, and let M_m denote the tangent space at $m \in M$. We call a subspace $C(m) \subset M_m$ a *principal subspace* if dim $C(m) \geqq 1$ and there exists a 1-form γ on the normal bundle of M such that $T_x z = \gamma(z)x$ for $x \in C(m)$, $z \in M_m^\perp$. M is said to be *principally reducible* provided each tangent space M_m is the direct sum of principal subspaces. The distribution $m \to C(m)$ is said to be *parallel* provided $\bar{\nabla}_X(\gamma)(Z) = 0$ for $X \in \mathfrak{X}(M)$, $Z \in \mathfrak{X}(M)$, where $\bar{\nabla}$ denotes the Riemannian connection of $\bar{M}$.

Now we resume our consideration of 6-dimensional orientable submanifolds of R^8.

THEOREM 4. 1. *Suppose M is quasi-Kählerian and principally reducible. Then*

(i) *each principal subspace is closed under J;*

(ii) *$T_A B = T_{JA} JB$ for all $A, B \in \mathfrak{X}(M)$;*

(iii) *$<R_{AB}C, D> = <R_{JAJB}JC, JD>$ for all $A, B, C, D \in \mathfrak{X}(M)$;*

(iv) *$B_{AB}\|A\|^2\|B\|^2 = \|T_A B\|^2 + \|T_A JB\|^2 \geqq 0$ whenever $A, B \in \mathfrak{X}(M)$ are nonzero;*

(v) *$\nabla_A(J)(A) = 0$ if A always lies in a principal subspace.*

PROOF. According to [7], M is quasi-Kählerian if and only if

$$(3) \qquad J(T_A A - T_{JA} JA) \pm 2 T_A JA = 0$$

for all $A \in \mathfrak{X}(M)$. Now assume that A is in a principal distribution $m \to C(m)$. Then there exists a 1-form γ on the normal bundle of M such that for $N \in \mathfrak{X}(M)^\perp$,

$$(4) \qquad T_A N = \gamma(N)A .$$

From (4) it follows easily that $T_A JA = 0$. Thus, since M is principally reducible, (ii) follows from (3). Furthermore (i) is a conequence of (ii). Also (iii) and (iv) follow from (ii) and the Gauss equation [4]. Finally (v) is an easy calculation from (2).

Next we combine Theorem 4.1 with a result of [5]. This generalizes a result of [7].

THEOREM 4. 2 *Suppose M is quasi-Kählerian and principally reducible. Also, assume that each principal distribution is parallel and has the same dimension on all of M. Then each principal distribution is integrable and there are at most three of them. Furthermore each of the integral manifolds is a totally geodesic quasi-Kähler submanifold of M.*

PROOF. That the principal distributions are integrable and totally geodesic follows from [5]. The rest is a consequence of Theorem 4.1.

Finally we state some further results. The proofs are similar to those of Theorems 4.1 and 4.2.

THEOREM 4.3. (i) *Assume that M is principally reducible. Then M is Kählerian if and only if M is totally geodesic.*

(ii) *Suppose M is nearly Kählerian and principally reducible. Then there is exactly one principal distribution. This distribution is parallel if and only if (a) the mean curvature vector of M has constant length, or (b) M has global constant type.*

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