Six-dimensional D_N theory and four-dimensional SO–USp quivers

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ABSTRACT: We realize four-dimensional $\mathcal{N} = 2$ superconformal quiver gauge theories with alternating SO and USp gauge groups as compactifications of the six-dimensional D_N theory with defects. The construction can be used to analyze infinitely strongly-coupled limits and S-dualities of such gauge theories, resulting in a new class of isolated four-dimensional $\mathcal{N} = 2$ superconformal field theories with $SO(2N)^3$ flavor symmetry.

KEYWORDS: Supersymmetry, S-duality, Quiver gauge theory .

Contents

1. Introduction

A method to systematically understand and construct a large class of four-dimensional $\mathcal{N} = 2$ superconformal field theories (SCFTs) was recently presented by Gaiotto in [1]. By means of a clever rewriting of the known Seiberg-Witten curves for quiver theories based on SU gauge groups [2], Gaiotto showed that such a theory arises as a compactification on a Riemann surface of the six-dimensional A_{N-1} theory with $(2,0)$ supersymmetry, with punctures associated to defect operators. The marginal couplings of a quiver theory are encoded in the moduli of the punctured Riemann surface, and both weakly-coupled and strongly-coupled limits were shown to correspond to degenerations of the Riemann surface. This approach gave a unified perspective on the S-dualities of SU(2) gauge theory with four flavors [3], which involved the triality of

SO(8) flavor symmetry, and of SU(3) gauge theory with six flavors, the strongly-coupled limit of which is dual to the mysterious isolated SCFT with E_6 flavor symmetry [4] coupled to an SU(2) gauge group with one flavor [5]. It also predicted a whole new family of SCFTs with $SU(N)^3$ flavor symmetry which are isolated, i.e. have no marginal couplings. The holographic description of these theories was found and discussed in [6].

We call this Riemann surface with punctures the *G-curve* of the theory, in order to avoid confusion and to distinguish it from the Seiberg-Witten curve, which is, roughly speaking, an N-sheeted cover of the G-curve. In this framework, vacuum expectation values (vevs) of dimension-d Coulomb branch operators are encoded in a degree-d differential on the Riemann surface, which is allowed to have poles at each of the punctures. These differentials are the scalar fields of the six-dimensional A_{N-1} theory. Therefore, from the six-dimensional viewpoint, each puncture corresponds to a defect operator that introduces singularities to fields, much like 't Hooft loops or surface operators do in four-dimensional gauge theory.

It was observed in [1] that the punctures of the A_{N-1} theory are naturally labeled by Young tableaux with N boxes, which also specify embeddings of SU(2) into SU(N). It was also found that there is a natural mapping between tableaux and tails of conformal quivers. In other words, we can obtain information about the elusive six-dimensional conformal field theory from the study of the quiver theories with SU gauge groups.

The main objective of this paper is to repeat Gaiotto's analysis for the quivers with USp and SO gauge groups, by realizing them using M5-branes at an M-theory orientifold. Recall that six-dimensional A_{N-1} theory is realized as the low-energy effective theory on the N coincident M5-branes; one can then introduce M-theoretic orientifolding, which flips five directions of the spacetime. $2N$ M5-branes on top of the orientifold singularity realize in the low energy limit the six-dimensional D_N theory.

We first show that the Seiberg-Witten curves of these quivers can be recast into a form which makes manifest their correspondence to compactifications of the D_N theory on Riemann surfaces with punctures. We construct new isolated SCFTs with $SO(2N)^3$ flavor symmetry, which appears when the Riemann surface on which the D_N theory is compactified degenerates and develops a sphere with three necks attached.

We then study defects of the D_N theory via an analysis of the tails of superconformal quivers. We find that defects are naturally labeled by embeddings of SU(2) into either SO($2N$) or USp($2N-2$).

We will see, along the way, that the compactification of the A_3 theory and of the D_3 theory on the same surface gives the same four-dimensional theory. The way it works is rather nontrivial: in the four-dimensional description the A_3 theory involves hypermultiplets in the **4** of SU(4) while the D_3 theory contains multiplets in the **6** of SO(6). In the M-theory description, the A_3 theory is defined on a stack of four M5-

branes, whereas the D_3 theory is realized by six M5-branes on the M-theory orientifold. We will find that subtle properties of the M-theory orientifold [7,8] play crucial roles in this equivalence. All these facts support non-perturbative equivalence of the A_3 theory and the D_3 theory as six-dimensional superconformal theories.

Finally we will see that the isolated SCFT with E_7 flavor symmetry [9] arises from a strongly-coupled limit of a particular quiver with a USp(4) factor, as anticipated in [5].

The paper is organized as follows: we start by reviewing the analysis of SU(N) quivers and their relation to the A_{N-1} theory in Sec. 2. We then analyze the SO–USp quivers and their relation to the D_N theory in Sec. 3. We conclude with some discussion in Sec. 4. There are two appendices: Appendix A is an analysis of SO(4)–USp(2) quivers in our framework. Appendix B contains a detailed derivation of the G-curve of SO–USp quivers from the corresponding Seiberg-Witten curve.

2. Review: 6d A_{N-1} theory and SU(N) quivers

2.1 Superconformal quivers, G-curve and Young tableaux

Let us start by considering an $\mathcal{N} = 2$ supersymmetric linear quiver gauge theory with a chain of SU groups

$$\mathrm{SU}(d_1) \times \mathrm{SU}(d_2) \times \cdots \times \mathrm{SU}(d_{n-1}) \times \mathrm{SU}(d_n), \tag{2.1}$$

a bifundamental hypermultiplet between each pair of consecutive gauge groups $\mathrm{SU}(d_a) \times \mathrm{SU}(d_{a+1})$, and k_a extra fundamental hypermultiplets for $\mathrm{SU}(d_a)$. To make every gauge coupling constant marginal, we require

$$k_a = d_{a-1} + d_{a+1} - 2d_a = (d_{a-1} - d_a) - (d_a - d_{a+1}), \tag{2.2}$$

where we defined $d_0 = d_{n-1} = 0$. Since k_a is non-negative, we have

$$d_1 < d_2 < \cdots < d_l = \cdots = d_r > d_{r+1} > \cdots > d_n. \tag{2.3}$$

Let us denote $N = d_l = \cdots = d_r$; we refer to the parts to the right of d_r and to the left of d_l as the two tails of this superconformal quiver. Consider the tail on the right hand side of (2.3),

$$N = d_r > d_{r+1} > \cdots > d_n. \tag{2.4}$$

$d_a - d_{a+1}$ is monotonically non-decreasing because $k_a \geq 0$; therefore we can associate naturally a Young tableau to the tail by requiring that it has a row of width $d_a - d_{a+1}$ for each $a \geq r$. For illustration, the possible types of tails for $N = 4$ are shown in

Tableau	Alias	Flavor (p_2, p_3, p_4)	poles	Quiver	G-curve
(one row of 4)	$\odot$	SU(4)	$(1,2,3)$	4–④–4	
		SU(2) × U(1)	$(1,2,2)$	4–④–③–2, 1	
		SU(2)	$(1,1,2)$	4–④–2, 2	
	$\bullet$	U(1)	$(1,1,1)$	4–④–③–②–1, 1	
		none	$(0,0,0)$	4–④–③–②–①, 1	

Table 1: Young tableaux with N boxes label the punctures of the six-dimensional A_{N-1} theory. The case $N = 4$ is shown here. Each tableau has its associated flavor symmetry, and worldvolume fields ϕ_k are allowed to have poles of degree p_k at the puncture. A puncture whose tableau consists of one row of width N is also known as a *full puncture* and marked by $\odot$. A puncture whose tableau consists of one column of height $N - 1$ and another of height 1 is the same as a *simple puncture* marked by $\bullet$. For each tableau, a four-dimensional quiver gauge theory with the corresponding tail is shown. 'SU(1)' gauge groups need to be understood as a shorthand for the brane construction, as explained in the text.

Table 1. As is customary, a circle or a box with n inside stands for an $SU(n)$ gauge group or flavor symmetry respectively, and a line connecting two objects stands for a bifundamental hypermultiplet.

The Seiberg-Witten curves for these quivers were originally found in [2], and rewritten into the following form in [1]: We start from a Riemann surface Σ, in this case $\Sigma = \mathbb{CP}^1$, with several punctures on it. Then the Seiberg-Witten curve is realized as a subspace of the total space $T^*\Sigma$ of the bundle of holomorphic differential on Σ, given as follows:

$$0 = x^N + x^{N-2}\phi_2 + x^{N-3}\phi_3 + \cdots + \phi_N \tag{2.5}$$

where x is a holomorphic differential on the Riemann surface Σ, and ϕ_d is a degree-d

differential with poles at the punctures, encoding vevs of Coulomb branch operators of dimension d. Then the Seiberg-Witten differential is x itself. This form makes the superconformal property of the theory manifest: one can assign the scaling dimension of the various fields to be equal to the degree of the corresponding differentials. One finds that, for the general quiver (2.3), one has $n + 1$ punctures of the same type, which we call the *simple punctures* and denote by $\bullet$, and two extra punctures each of which encodes the type of the tails. We label these two punctures by the Young tableaux associated to the corresponding tails. In Table 1, the curves with punctures are shown along with the corresponding quiver gauge theories. The puncture whose tableau consists of one row of width N is called the *full puncture* and labeled by $\odot$.

As explained in [1], this system can be thought of as a compactification of the six-dimensional A_{N-1} (2, 0)-theory on Σ with defects at the punctures. Recall that the A_{N-1} theory is the low-energy limit of the theory of N coincident M5-branes, or of the compactification of type IIB strings on the four-dimensional asymptotically locally Euclidean (ALE) space of type A_{N-1}; this theory has operators of dimension $2, 3, \ldots, N$. Compactifications on a Riemann surface which preserves the supersymmetry involve twisting as usual, which turns an operator of dimension d into a meromorphic differential of degree d. Another way to understand this twisting is to recall that the space in which the M5-branes are embedded needs to be hyperkähler to preserve $\mathcal{N} = 2$ supersymmetry in four dimensions. The neighborhood of Σ in such a space can be approximated by $T^*\Sigma$, which is exactly the space used in (2.5). Then N solutions of (2.5) determine the position of N M5-branes in the fiber direction x, at each point of the base Σ.

This construction generalizes the realization of Seiberg-Witten curves as compactifications of the A_{N-1} theory discussed in [10,11]: the ϕ_d are now allowed to have poles at a finite number of punctures on Σ. These describe conformal defects of the A_{N-1} theory.

At a simple puncture ϕ_d is allowed to have a simple pole. The orders of poles ϕ_d at a general puncture can be determined from the Seiberg-Witten curve of the corresponding superconformal quiver, and can be easily read off from the associated tableau. Given a tableau with rows of width $w_1 \geq w_2 \geq w_3 \cdots$, we define a sequence of integers ν_i as follows

$$(\nu_1, \nu_2, \ldots) = (\underbrace{1, \ldots, 1}_{w_1}, \underbrace{2, \ldots, 2}_{w_2}, \cdots,). \tag{2.6}$$

Then, ϕ_d is allowed to have poles of order $p_d = d - \nu_d$. The set of orders p_d define the *pole structure* of the puncture. Again for illustration, the tableaux and the corresponding pole structures (p_2, p_3, p_4) are listed in Table 1 for the case $N = 4$.
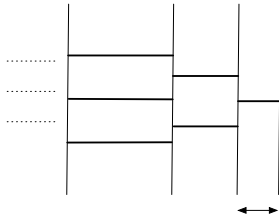
Figure 1: Brane configuration involving "SU(1)" part. Vertical lines stand for NS5 branes extended along $x^{0,1,2,3,4,5}$, and horizontal lines are D4-branes extended along $x^{0,1,2,3,6}$ where the extent x^6 is bounded by two NS5-branes. It would correspond to a quiver with gauge groups $\cdots \times SU(3) \times SU(2) \times$ "SU(1)".

One notable property is that the puncture associated to the tail whose tableau consists of one column of height $N - 1$ and another of height 1 has the same pole structure as the simple puncture. Furthermore, the rightmost gauge group of the tail is SU(2) coupled effectively to four flavors, and the S-duality of this gauge group exchanges the puncture associated to the tail with the simple puncture. We therefore identify the simple puncture with the puncture associated to this tableau.

We call this set consisting of a Riemann surface Σ and punctures marked by Young tableaux the *G-curve* of the system, to distinguish it from the Seiberg-Witten curve. Given pole structures at the punctures, the number of moduli in ϕ_d is the dimension of the space of the holomorphic differentials of degree d with prescribed singularities, given by the formula

$$\# \text{ moduli in } \phi_d = (\sum_{\text{punctures}} p_d^{(i)}) - (2d - 1) \tag{2.7}$$

where $(p_d^{(i)})$ is the pole structure of the i-th puncture. The use of this formula will be illustrated in Sec. 2.3.

In Table 1, the tableau with one column of height 4 is also listed. In general, a tableau with one column of height N does not apparently have a corresponding superconformal tail, because the rule explained above would associate a tail of the form

$$\cdots \times SU(N) \times SU(N - 1) \times \cdots \times SU(2) \times \text{``SU(1)''}. \tag{2.8}$$

One also finds that none of the ϕ_d are allowed to have poles at the 'puncture' corresponding to this tableau.

This sounds problematic, but by using a string-theoretic construction one can make sense of it. Consider a brane configuration in type IIA string theory shown in Fig 1. There, vertical lines represent NS5-branes and horizontal lines D4-branes suspended

between them, as discussed in [2]. Thus this configuration shows the tail of a quiver with the gauge groups (2.8) where the "SU(1) part" just decouples in the infrared limit. Still, this configuration can be lifted to a configuration of a connected M5-brane in M-theory. Its rewriting produces a Riemann surface with simple punctures and one extra 'puncture', at which none of ϕ_d has poles. This extra puncture is not totally devoid of physical meaning, as its position encodes the separation of the last two NS5-branes, which roughly corresponds to the 'gauge coupling' of the 'SU(1) gauge group'. Therefore we find that it is natural to associate the Young tableau with one column of height N to this type of puncture.

2.2 Punctures and associated flavor symmetries

Let us now consider the flavor symmetry associated to a puncture labeled by a given Young tableau. As we saw, a tail (2.4) of the SU(N) quiver gives a number of simple punctures and a puncture associated to the Young tableau with rows of width $d_r - d_{r+1}$, $d_{r+1} - d_{r+2}$, ..., $d_{n-1} - d_n$. One finds that the $U(1)$ symmetry acting on each of the bifundamental hypermultiplets is carried by the simple punctures, and the flavor symmetries of k_a fundamental hypermultiplets of SU(d_a) gauge groups are associated to the punctures labeled by the tableau. We can easily read off the flavor symmetry from a given tableau: Let l_h be the number of columns of height h. Then, for each $l_h \neq 0$ there are l_h fundamental hypermultiplets coupled to one of the gauge groups in the tail, which gives U(l_h) symmetry. The overall U(1) is carried by the simple puncture closest to the puncture labeled by the tableau. Therefore the flavor symmetry is given by

$$S \left[\prod_{l_h > 0} U(l_h) \right]. \tag{2.9}$$

For the tails of SU(4) quivers, these flavor symmetries are listed in Table 1.

Let us note one curious mathematical fact: for a given tableau, we may associate an embedding of SU(2) into SU(N) described by the decomposition of the fundamental representation of SU(N) into the irreducible representations of SU(2), given by

$$N \rightarrow \underbrace{1 + 1 + \cdots + 1}_{l_1} + \underbrace{2 + \cdots + 2}_{l_2} + \cdots . \tag{2.10}$$

Then, its commutant inside SU(N) is easily seen to agree with (2.9).

2.3 SCFT with SU(N)^3 flavor symmetry

With the interpretation of the G-curve as the compactification of the A_{N-1} theory, one can easily derive various new types of S-duality, generalizing the ones found in [5]. As an
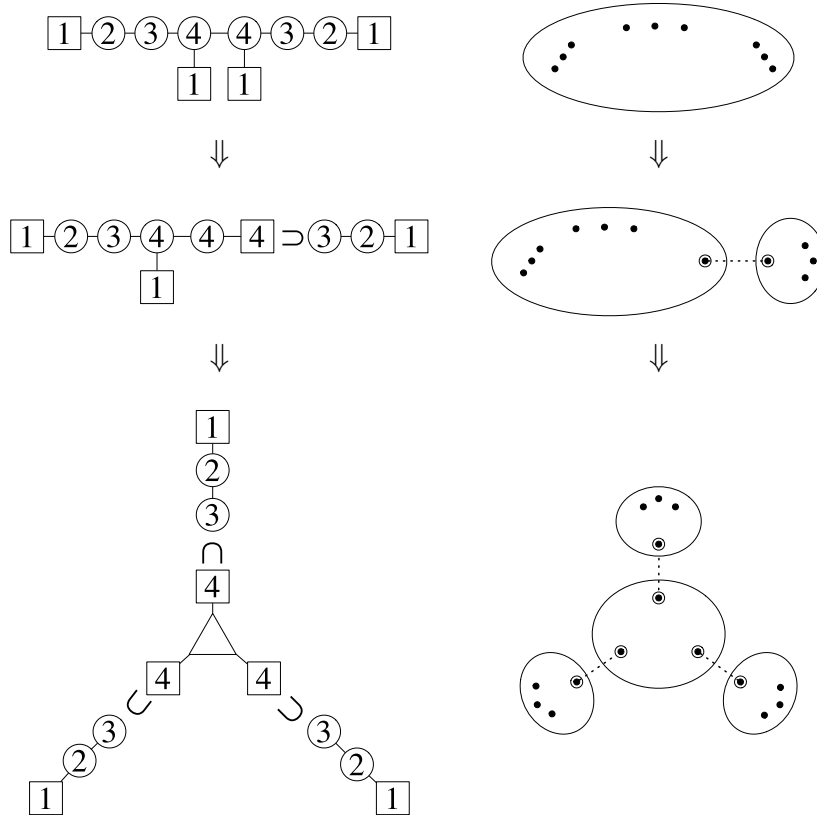
Figure 2: Construction of $T_{\mathrm{SU}(4)}$. A circle or a box with n inside is an $\mathrm{SU}(n)$ gauge group or flavor symmetry, respectively. A line connecting two objects is a bifundamental hypermultiplet. The symbol $\subset$ means that the subgroup of the flavor symmetry indicated couples to the corresponding gauge field. The triangle with three $\mathrm{SU}(4)$ flavor symmetry attached stands for the $T_{\mathrm{SU}(4)}$ theory.

example, let us recall the construction of a class of SCFTs with $\mathrm{SU}(N)^3$ flavor symmetry in [1], whose gravity dual was found in [6]. The general method was detailed in these papers, so we use a specific example of an $\mathrm{SU}(4)$ quiver to illustrate the procedure.

We start from the linear quiver shown in the first row of Fig. 2, and go to the region of the moduli space where three necks develop in the G-curve. Originally one has a $\mathbb{CP}^1$ with nine punctures of type $\bullet$; we split off three spheres, each with three simple punctures. Each endpoint of the necks becomes a full puncture $\odot$. Let us first split one sphere with three simple punctures, see the second row of Fig. 2. The $\mathrm{SU}(3)$ group in the tail

$$\mathrm{SU}(3) \times \mathrm{SU}(2) \tag{2.11}$$

becomes weakly coupled, and gauges the subgroup of the $\mathrm{SU}(4)$ flavor symmetry asso-

ciated to the puncture. We repeat the process three times, and arrive at the situation shown in the third row of Fig. 2. The resulting theory was called $T[A_3]$ in [1] and T_4 in [6]. We call it $T_{\mathrm{SU}(4)}$ to reduce possible later confusion.

$T_{\mathrm{SU}(4)}$ does not have a marginal coupling constant, because a configuration consisting of three points on a sphere has no modulus. The pole structure at each of the punctures is that $\phi_{2,3,4}$ has poles of order $1, 2, 3$. Applying the formula (2.7), one concludes that $T_{\mathrm{SU}(4)}$ has one operator of dimension 3 and two operators of dimension 4. Then one can check that the quivers shown in Fig. 2 have the same number of Coulomb branch operators for each scaling dimension.

The central charges a and c of this theory can also be easily calculated because they are independent of exactly marginal deformations. It is more intuitive to parametrize the central charges a and c using the effective number of hyper- and vector multiplets n_v and n_h, as defined by the relation:

$$a = \frac{5n_v + n_h}{24}, \qquad c = \frac{2n_v + n_h}{12}. \tag{2.12}$$

n_v and n_h of the total theory are obtained by counting the number of multiplets at the original weakly-coupled limit:

$$n_v(\text{total}) = 52, \qquad n_h(\text{total}) = 64. \tag{2.13}$$

In the regime where the three necks develop, one has three tails, each with

$$n_v(\text{tail}) = 11, \qquad n_h(\text{tail}) = 8, \tag{2.14}$$

Therefore we have

$$n_v(T_{\mathrm{SU}(4)}) = 52 - 3 \times 11 = 19, \qquad n_h(T_{\mathrm{SU}(4)}) = 64 - 3 \times 8 = 40. \tag{2.15}$$

3. 6d D_N theory and SO–USp quivers

3.1 Preliminary comments on SO and USp gauge groups

Having reviewed the construction the SU quivers, here we start the analysis of the SO–USp quivers. First we need to recall rudiments of these gauge groups, and also a few properties of hypermultiplets.

What is usually called a hypermultiplet in the representation R of a group G consists of an $\mathcal{N} = 1$ chiral multiplet in the representation R and another in the conjugate representation R^*. When we have N copies of them the flavor symmetry is at least U(N). When R is strictly real, it enhances to USp($2N$), as can be understood

from the form of the $\mathcal{N} = 1$ superpotential. When R is pseudo-real, one $\mathcal{N} = 1$ chiral multiplet in R forms an $\mathcal{N} = 2$ hypermultiplet, which is called a *half-hypermultiplet* in R. When we have N copies of them, the flavor symmetry is $SO(N)$.

Now let us consider an $SO(n)$ gauge theory with N_f massless hypermultiplets in the vector representation of dimension n. It has $USp(2N_f)$ flavor symmetry because the vector representation is strictly real. Here and in the following we use the convention that the fundamental representation of $USp(2n)$ is of dimension $2n$; thus in our notation $USp(2) \simeq SU(2)$ and $USp(4) \simeq SO(5)$ at the level of Lie algebra. The gauge coupling constant is marginal when $N_f = n - 2$.

Next consider a $USp(2n)$ gauge theory with N_f hypermultiplets in the vector representation. The flavor symmetry is then $SO(2N_f)$, and the theory becomes superconformal when $N_f = 2n + 2$. It will be important that the vector representation, which is $2n$-dimensional, is pseudo-real. This implies that one can form half-hypermultiplets, although one cannot have an odd number of half-hypermultiplets because of Witten's global anomaly. One can still gauge the subgroup $SO(d) \times SO(2N_f - d) \subset SO(2N_f)$ for odd d, preserving $\mathcal{N} = 2$ supersymmetry.

Therefore one can naturally consider a quiver theory with alternating gauge groups

$$\cdots \times SO(d_a) \times USp(d_{a+1} - 2) \times SO(d_{a+2}) \times USp(d_{a+3} - 2) \times \cdots \qquad (3.1)$$

with bifundamental half-hypermultiplets between consecutive gauge groups, and possibly with extra hypermultiplets in the fundamental representation for the a-th gauge group. Here the bifundamental representation is the tensor product of the vector representation of SO group and the fundamental representation of USp group. We let k_a be twice the number of hypermultiplets in the vector representation if the a-th gauge group is SO, while we let it be the number of half-hypermultiplets in the fundamental representation if the a-th gauge group is USp. Then the flavor symmetry is $USp(k_a)$ and $SO(k_a)$, respectively. For convenience we define $\delta_a = 0$ when the a-th gauge group is SO, and $\delta_a = 2$ when it is USp.

The requirement of marginality of each of the gauge coupling constants can be written succinctly as

$$k_a = d_{a-1} + d_{a+1} - 2d_a, \qquad (3.2)$$

exactly as in the case of the quiver of SU gauge groups (2.2). One such superconformal quiver is shown in Fig 3. There, a box stands for a flavor symmetry, and a circle a gauge symmetry; a gray one with n inside is an $SO(n)$ group, and a black one is an $USp(n)$ group; a line stands for a half-hypermultiplet in the bifundamental representation. The theory shown thus has the gauge group
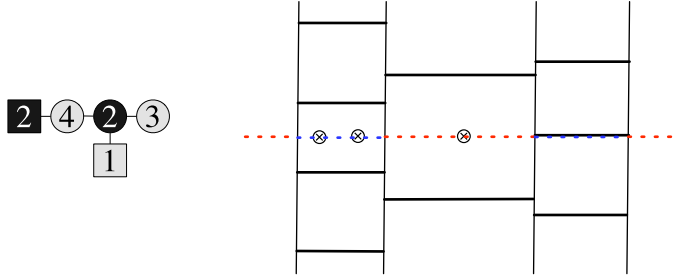
$$SO(4) \times USp(2) \times SO(3) \qquad (3.3)$$

Figure 3: On the left: an example of SO–USp quiver gauge theory. A circle or a box stands for a gauge group or a flavor symmetry, respectively. A gray object with n inside is an $SO(n)$ group, a black object with n inside is a $USp(n)$ group. On the right: the brane configuration realizing the quiver. The vertical lines stand for NS5-branes, the horizontal lines D4-branes suspended between them, and $\otimes$ D6 branes. The dotted line represents the O4-plane. The color distinguishes two types of O4-planes.

with bifundamentals, one extra hypermultiplet in the vector representation of $SO(4)$, and one extra half-hypermultiplet in the fundamental representation of $USp(2)$.

3.2 From type IIA brane configuration to the G-curve

Let us realize the quiver theory introduced in the previous subsection via a system of NS5, D4 and D6 branes with orientifolds in type IIA string theory, which is schematically shown in Fig. 3. These systems and the corresponding Seiberg-Witten curves were first analyzed by [12–14]; the subtler aspects of the orientifolding procedure was later clarified in [7, 8, 15].

We start from the flat ten-dimensional spacetime, and put $n + 1$ NS5 branes extending along directions $x^{0,1,2,3,4,5}$. We perform the orientifolding, flipping directions $x^{4,5,7,8,9}$, which introduces an O4-plane in the system. One important aspect is that an O4$^-$-plane becomes an O4$^+$-plane when it crosses an NS5-brane, and vice versa. We define δ_a to be 0 or 2 depending on the type of the O4-plane between a-th and $(a+1)$-st NS5 brane, so that we have an SO gauge group when $\delta_a = 0$ and a USp gauge group when $\delta_a = 2$. We analogously define δ_0 and δ_{n+1}. δ_a accounts the difference of D4-charge carried by an O4$^-$-plane and an O4$^+$ plane.

We then suspend $d_a - \delta_a$ D4-branes between the a-th and $(a+1)$-th NS5-branes. The a-th gauge group is $G_a = SO(d_a)$ if $\delta_a = 0$ and $= USp(d_a - \delta_a)$ if $\delta_a = 2$. We denote by k_a the number of D6-branes, extending along $x^{0,1,2,3,7,8,9}$, between the a-th and $(a+1)$-th NS5-branes; for simplicity we put all D6-branes on top of the O4-plane. This configuration realizes in the low-energy limit the quiver gauge theory specified by the sequences of numbers (d_a), (δ_a) and (k_a), as discussed in the previous subsection.

One example is depicted in Fig. 3: There, a vertical line stands for a NS5-brane, a horizontal solid line a D4-brane, a horizontal dotted line an O4-plane and an $\otimes$ a D6-brane. An O4$^-$-plane is in blue and an O4$^+$-plane is in red. We have the gauge group $SO(4) \times USp(2) \times SO(3)$ with bifundamentals between consecutive gauge factors, and extra hypermultiplets in the fundamental representations for $SO(4)$ and $USp(2)$. The properties of the O4$^+$ plane to the left and to the right of the D6-brane on top of it is known to be slightly different, and the plane to the right is, properly speaking, an $\widetilde{O4}^+$-plane, which is important to guarantee that there is no Witten's global anomaly in the low energy gauge theory, see [8] for details.

Let us now consider a quiver with the gauge groups

$$USp(2N-2) \times SO(2N) \times USp(2N-2) \times \cdots \times SO(2N) \times USp(2N-2), \qquad (3.4)$$

with a total of $n = 2s+1$ gauge factors, and with $2N$ massless half-hypermultiplets in the fundamental representation for each of the two $USp(2N)$ gauge groups at the ends to make them superconformal. The Seiberg-Witten curve is given by [13]

$$F(v,t) = v^{2N}t^{n+1} + P_1(v^2)t^n + P_2(v^2)t^{n-1} + \cdots + P_n(v^2)t + v^{2N} = 0, \qquad (3.5)$$

where

$$P_i(v^2) = c_i v^{2N} + u_i^{(2)}v^{2N-2} + u_i^{(4)}v^{2N-4} + \cdots + u_i^{(2N)}. \qquad (3.6)$$

Here $u_i^{(2k)}$ is the Casimir of degree $2k$ of the i-th gauge group, except for $u_i^{(2N)}$ when the i-th gauge group is $USp(2N-2)$, for which no such Casimir exists. In fact, the zeros of $F(v,t)$ at $v = 0$ all need to be double zeros:

$$F(0,t) = u_1^{(2N)}t^n + u_2^{(2N)}t^{n-1} + \cdots + u_n^{(2N)}t = \alpha t^2 \prod_{i=1}^{s-1}(t-q_i)^2 \qquad (3.7)$$

for some choice of α and q_i. In particular this forces $u_1^{(2N)} = u_n^{(2N)} = 0$. This condition leaves s independent parameters α and q_i, which encode the Casimir operators of s $SO(2N)$ gauge factors.

This condition is necessary to prevent a so-called "t-configuration," i.e. a transversal intersection of a single M5-brane with the M-theory orientifold plane [7, 8].

The Seiberg-Witten curve can be rewritten in Gaiotto's form

$$0 = x^{2N} + \varphi_2 x^{2N-2} + \varphi_4 x^{2N-4} + \cdots + \varphi_{2N}, \qquad (3.8)$$

where $x = v\,dt/t$ is a holomorphic differential on the G-curve $\Sigma = \mathbb{CP}^1$ parametrized by t. φ_{2k} encodes the vevs of Coulomb branch operators:

$$\varphi_{2k} = \frac{u_1^{(2k)}t^n + u_2^{(2k)}t^{n-1} + \cdots + u_n^{(2k)}t}{\prod(t-t_a)}\left(\frac{dt}{t}\right)^{2k}. \qquad (3.9)$$
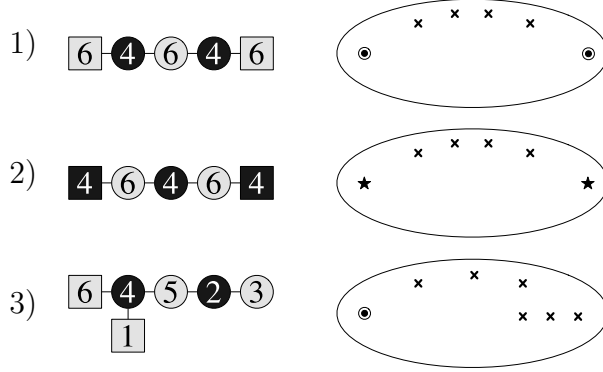
Figure 4: Examples of SO–USp quivers and their G-curves. Simple punctures are marked by ×. There are two types of full punctures. Each of the punctures labeled by ⊙ or ⋆ has one SO(2N) or USp(2N − 2) flavor symmetry, respectively.

Here the t_a are defined by

$$\prod(t - t_a) = t^{n+1} + c_1 t^n + \cdots + c_n t + 1 \tag{3.10}$$

which encodes the gauge coupling constants. We see that φ_{2k} is allowed to have a simple pole at $n + 1$ points where $t = t_i$, whereas it is allowed to have poles of order $2k - 1$ at two points $t = 0$ and $t = \infty$.

As for φ_{2N}, the condition (3.7) means that it can be written as

$$\varphi_{2N} = \varphi_{\tilde{N}}^2 \quad \text{where} \quad \varphi_{\tilde{N}} = \frac{t \prod(t - q_i)}{\prod(t - t_a)^{1/2}} \left(\frac{dt}{t}\right)^N. \tag{3.11}$$

$\varphi_{\tilde{N}}$ has $\mathbb{Z}_2$ monodromy around $t = t_a$, with a pole of order $1/2$; whereas it has no monodromy around $t = 0, \infty$ and has poles of order $N - 1$.

The G-curve is shown as the first example in Fig. 4; there, we have taken $2N = 6$ and $n = 3$. φ_{2k} has simple poles and $\varphi_{\tilde{N}}$ behaves as $\sim 1/t^{1/2}$ at the punctures denoted by × with the local coordinate t chosen such that the puncture is at $t = 0$. φ_{2k} has poles of order $2k - 1$ and $\varphi_{\tilde{N}}$ has poles of order $N - 1$ at the punctures denoted by ⊙.

Again, this system can be thought of as the compactification of the six-dimensional D_N theory on Σ, with prescribed sets of singularities for the worldvolume fields. Recall that the six-dimensional D_N theory arises as the low-energy theory on a stack of $2N$ M5-branes on top of the $\mathbb{R}^5/\mathbb{Z}^2$ M-theory orientifold, or equivalently of the compactification of type IIB string theory on an ALE space of type D_N. This theory has operators of dimension $2, 4, \ldots, 2N - 2$ and one extra operator of dimension N, which become the differentials φ_2, φ_4, $\ldots$, φ_{2N-2} and $\varphi_{\tilde{N}}$, respectively. The Lie algebra of type D_N has one outer automorphism, under which operators of dimension $2, 4, \ldots, 2N - 2$ are even

but the extra one of dimension N is odd; it is the Pfaffian. The analysis above shows that the simple puncture $\times$ has a $\mathbb{Z}_2$ monodromy of this outer automorphism associated to it. This agrees with the known fact that the transversal intersection of an M5-brane with the M-theory orientifold $\mathbb{R}^5/\mathbb{Z}_2$ screened by an even number of M5-branes has an associated $\mathbb{Z}_2$ charge [8].

Let us next consider the quiver with the gauge groups

$$\mathrm{SO}(2N) \times \mathrm{USp}(2N-2) \times \cdots \times \mathrm{USp}(2N-2) \times \mathrm{SO}(2N), \qquad (3.12)$$

with a total of $2s+1$ gauge groups. There are bifundamental hypermultiplets as always, and $N-1$ hypermultiplets in the fundamental representation for each of the $\mathrm{SO}(N)$ groups at the ends. The Seiberg-Witten curve is given again by (3.5), but the condition on the double zeros is now

$$F(0,t) = u_1^{(2N)}t^n + u_2^{(2N)}t^{n-1} + \cdots + u_n^{(2N)}t = \alpha t \prod_{i=1}^{s}(t-q_i)^2. \qquad (3.13)$$

There are s simple punctures at t_i as before, but the pole structure at $t=0,\infty$ is now different: φ_{2k} still has poles of order $2k-1$, but $\varphi_{\tilde{N}}$ has a pole of order $N-1/2$. In particular there is a $\mathbb{Z}_2$ monodromy around $t=0,\infty$. We label this type of punctures by $\star$. The case $2N=6$ is shown in the second line of Fig. 4.

The same exercise can be repeated with the quiver of the form

$$\mathrm{USp}(2N-2) \times \mathrm{SO}(2N-1) \times \mathrm{USp}(2N-4) \times \mathrm{SO}(2N-3) \times \cdots \times \mathrm{USp}(2) \times \mathrm{SO}(3), \quad (3.14)$$

with bifundamentals between each pair of two consecutive gauge groups as always, and $2N+1$ extra fundamental half-hypermultiplets on the leftmost $\mathrm{USp}(2N-2)$ gauge group. This quiver theory has a total of $2N-2$ gauge groups, and we find that the resulting G-curve has $2N$ punctures of type $\times$ and one puncture of type $\odot$; see Appendix B for details. The case $2N=6$ is shown as the third example in Fig. 4.

3.3 SCFT with $\mathrm{SO}(2N)^3$ flavor symmetry

By making use of the interpretation of SO–USp quivers as compactifications of the six-dimensional D_N theory, one can easily find their various infinitely strongly-coupled limits. As an exercise let us construct a theory with no marginal coupling constant and with $\mathrm{SO}(2N)^3$ flavor symmetry, which we denote as $T_{\mathrm{SO}(2N)}$. The construction for the A_N theory was reviewed in Sec. 2.3, which we closely follow.

For concreteness, let us first consider the case $2N=6$. Take two copies of the quiver theory of the third example of Fig. 4, and introduce an $\mathrm{SO}(6)$ gauge group which gauges the $\mathrm{SO}(6)$ flavor symmetry of the original one. One then has the linear
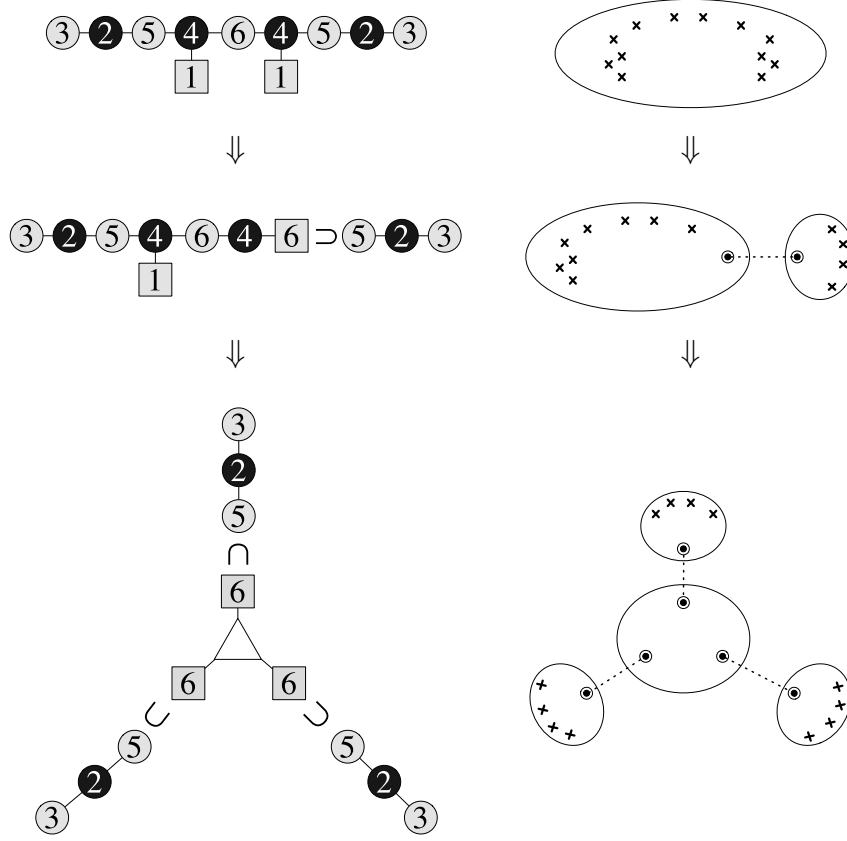
Figure 5: Construction of $T_{\mathrm{SO}(6)}$. The triangle with three SO(6) flavor symmetries attached stands for the $T_{\mathrm{SO}(6)}$. The symbol $\subset$ between the SO(6) flavor symmetry and the SO(5) gauge symmetry signifies that SO(5) $\subset$ SO(6) is gauged.

quiver shown in the first row of Fig. 5, whose associated G-curve is a $\mathbb{CP}^1$ with twelve punctures of type $\times$. Let us go to the region of the moduli space where three necks develop. We split off three spheres, each with four punctures of type $\times$. Each of the endpoints of the necks becomes a full puncture marked by $\odot$.

Let us first split off one sphere with three simple punctures, see the second row of Fig. 5. The group SO(5) in the tail

$$\mathrm{SO}(5) \times \mathrm{USp}(2) \times \mathrm{SO}(3) \tag{3.15}$$

becomes weakly coupled, and gauges the subgroup of the SO(6) flavor symmetry associated to the puncture. We repeat the process three times, and arrive at the situation shown in the third row of Fig. 5. Again, n_v and n_h of the total theory are obtained by counting the number of multiplets: first, the total theory has

$$n_v(\text{total}) = 67, \qquad n_h(\text{total}) = 64. \tag{3.16}$$

In the regime where the three necks develop, one has three tails, each with

$$n_v(\text{tail}) = 16, \qquad n_h(\text{tail}) = 8. \tag{3.17}$$

Therefore we have

$$n_v(T_{\text{SO}(6)}) = 67 - 3 \times 16 = 19, \qquad n_h(T_{\text{SO}(6)}) = 64 - 3 \times 8 = 40. \tag{3.18}$$

These numbers are exactly the same as those for $T_{\text{SU}(4)}$, (2.15). Recall that the SU(4) quivers arose from the compactification of the worldvolume theory on four coincident M5 branes, whereas the SO(6)–USp(4) quivers arose from six coincident M5 branes on top of the $\mathbb{R}^5/\mathbb{Z}_2$ orientifold singularity. These two systems give the same low-energy six-dimensional $A_3 \simeq D_3$ (2,0) theory. Therefore, they should result in the same SCFT in four dimensions, because we compactified the same theory on the same surface, with the same number of the same type of defects. The agreement of n_v and n_h of $T_{\text{SU}(4)}$ and $T_{\text{SO}(6)}$ is expected from the six-dimensional viewpoint, but is quite nontrivial from the perspective of four-dimensional gauge theory.

The construction of $T_{\text{SO}(2N)}$ can be done analogously. We start from a linear quiver with $6N - 9$ gauge groups

$$\text{SO}(3) \times \text{USp}(2) \times \cdots \text{USp}(2N - 4) \times \text{SO}(2N - 1) \times$$
$$\text{USp}(2N - 2) \times \text{SO}(2N) \times \cdots \times \text{SO}(2N) \times \text{USp}(2N - 2) \times$$
$$\text{SO}(2n - 1) \times \text{USp}(2N - 4) \times \cdots \times \text{USp}(2) \times \text{SO}(3), \quad (3.19)$$

with a bifundamental half-hypermultiplet between each pair of two consecutive groups, and one half-hypermultiplet in the fundamental for the first and the last USp$(2N - 2)$ gauge groups. The G-curve then is a sphere with $3(2N - 2)$ punctures of type $\times$. One can split off three spheres with $2N - 2$ punctures each, thus decoupling three tails of the form

$$\text{SO}(3) \times \text{USp}(2) \times \cdots \text{USp}(2N - 4) \times \text{SO}(2N - 1). \tag{3.20}$$

This results in a theory described by a G-curve with three punctures of type $\odot$. We then have

$$n_v(T_{\text{SO}(2N)}) = \frac{8N^3}{3} - 7N^2 + \frac{10N}{3}, \qquad n_h(T_{\text{SO}(2N)}) = \frac{8N^3}{3} - 4N^2 + \frac{4N}{3}. \tag{3.21}$$

Now we can paste multiple copies of $T_{\text{SO}(2N)}$ by gauging SO$(2N)$ groups to find a four-dimensional realization of the compactification of the six-dimensional D_N theory. It would be interesting to extend the holographic analysis of [6] to this case and reproduce n_h and n_v from the gravity solution.

It is natural ask if there is a theory whose G-curve is a sphere with three punctures of type $\star$ and with $\mathrm{USp}(2N - 2)^3$ flavor symmetry. This is impossible because one cannot have $\mathbb{Z}_2$ monodromy at three points on the sphere. Instead it is possible to construct a theory whose G-curve is a sphere with two punctures of type $\star$ and one punctures of type $\odot$, by performing a similar procedure to the one presented above. The flavor symmetry is then $\mathrm{SO}(2N) \times \mathrm{USp}(2N - 2)^2$. The important point is that two punctures of type $\star$ appear when a G-curve degenerates and develops a neck with $\mathbb{Z}_2$ monodromy around it.

3.4 Tails, tableaux and flavor symmetries

Let us now classify possible types of superconformal tails of the SO–USp quivers. We found in Sec. 3.1 that the requirement of the marginality of coupling constants implies

$$d_1 < d_2 < \cdots < d_l = d_{l+1} = \cdots = d_r > d_{r+1} > \cdots d_n. \tag{3.22}$$

We let $2N = d_l = \cdots = d_r$. Then we can associate a Young tableau with rows of widths $d_r - d_{r+1}$, $d_{r+1} - d_{r+2}, \ldots$, as was the case for the tails of SU quivers. There are two crucial differences, however. One is that we need to distinguish the cases for which the last gauge group is SO or USp; the other is that not all of the tableaux are allowed because d_a for a USp gauge group needs to be even.

For a given tail, let us then associate a tableau with the following rule:

- If it ends with a USp group, associate a tableau, with gray boxes, whose rows are of width $d_r - d_{r+1}$, $d_{r+1} - d_{r+2}$, ... One has $2N$ boxes in total.

- If it ends with an SO group, associate a tableau, with black boxes, whose rows are again of width $d_r - d_{r+1}$, $d_{r+1} - d_{r+2}$, ..., *except* the last row, for which we let the width be $d_n - d_{n+1} - 2 = d_n - 2$. This procedure is consistent because the smallest SO group one can have is SO(3). One has $2N - 2$ boxes in total.

To help grasp the procedure, we list all the tails of SO(6)–USp(4) quivers in Table 2 and in Table 3. Note that a tableau with one row of $2N$ gray boxes corresponds to the puncture of type $\odot$, the tableau with one column of $2N - 2$ black boxes to the puncture of type $\times$, and the tableau with one row of $2N - 2$ black boxes to the puncture of type $\star$ that we used in the previous subsection; we use these notations interchangeably.

Let l_h be the number of columns of height h in a given tableau. One finds in general that

- l_h for even h is even for a gray tableau; it just guarantees that d_a be even for USp gauge groups. Then one can associate an embedding of SU(2) into SO($2N$),

Tableau	Alias	Flavor	Quiver	G-curve
	$\odot$	SO(6)	6–**4**–6	
		USp(2) × SO(2)	6–**4**–6–**2**–2 / **2**	
		SO(3)	6–**4**–5–**2**–3 / 1	
		SO(2)	6–**4**–4–**0** / 2	
		none	6–**4**–5–**2**–3–**0**–1 / 1	

Table 2: One class of punctures of the six-dimensional D_N theories are marked by SO tableaux, which encode embeddings of SU(2) into SO(2N). On the right of the each tableau are the associated flavor symmetry and a quiver theory whose G-curve has a corresponding puncture. A puncture whose tableau consists of one row of width $2N$ is a full puncture $\odot$. Quivers with USp(0) 'gauge group' need to be understood as a shorthand for the brane configurations, as explained in the text.

$\rho : \text{SU}(2) \to \text{SO}(2N)$, given by the decomposition of the vector representation $2N$ of SO(2N) under SU(2) via

$$2N \to \underbrace{1 + 1 + \cdots + 1}_{l_1} + \underbrace{2 + \cdots + 2}_{l_2} + \cdots . \qquad (3.23)$$

Recall that the irreducible representation of SU(2) of dimension h for even h is pseudo-real. The embedding above is possible because l_h copies of this irreducible representation can be strictly real, l_h being even. Thus we call such a tableau an SO(2N) tableau.

- Similarly, l_h for odd h is even for a black tableau. Again, this just guarantees that d_a is even for USp gauge groups. Let us then associate an embedding ρ of SU(2) into USp(2N − 2), given by the decomposition of the fundamental representation

Tableau	Alias	Flavor	Quiver	G-curve
	$\star$	USp(4)	6–4–6–4	
		USp(2)	6–4–5–2 / 1	
		SO(2)	6–4–4 / 2	
	$\times$	none	6–4–5–2–3 / 1	

Table 3: Another class of punctures of the six-dimensional D_N theories are marked by USp tableaux, which encode SU(2) embeddings into USp($2N-2$). On the right of the each tableau are the associated flavor symmetry and a quiver theory whose G-curve has a corresponding puncture. A puncture whose tableau consists of one column of height $2N-2$ is a simple puncture $\times$, and a puncture whose tableau consists of one row of width $2N-2$ is a full puncture $\star$.

$2N-2$ of USp($2N-2$) under SU(2) via

$$2N - 2 \to \underbrace{1+1+\cdots+1}_{l_1}+\underbrace{2+\cdots+2}_{l_2}+\cdots. \tag{3.24}$$

Recall that the irreducible representation of SU(2) of dimension h for odd h is strictly real. The embedding above is possible because l_h copies of this irreducible representation can be pseudo-real, l_h being even. Thus we call such a tableau a USp($2N-2$) tableau.

In this way, we associate a tableau for each superconformal tail, which encodes its information concisely. One can observe the following facts concerning superconformal tails and the flavor symmetries associated to them:

- For a tail ending in a USp group, the flavor symmetry associated to it is

$$\prod_{h:\mathrm{odd},l_h\geq 2} \mathrm{SO}(l_h) \times \prod_{h:\mathrm{even},l_h\geq 2} \mathrm{USp}(l_h), \tag{3.25}$$

which coincides exactly with the commutant inside SO($2N$) of the SU(2) embedding associated to the tableau labeling the tail.
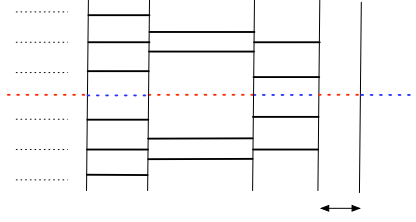
Figure 6: Brane configuration involving "USp(0)" part. It would correspond to a quiver with gauge groups $\cdots \times SO(6) \times USp(4) \times SO(4) \times$ "USp(0)".

- Similarly, for a tail ending in an SO group, the flavor symmetry is

$$\prod_{h:\text{odd},l_h \geq 2} USp(l_h) \times \prod_{h:\text{even},l_h \geq 2} SO(l_h), \tag{3.26}$$

which agrees with the commutant inside $USp(2N-2)$ of the SU(2) embedding associated to the tableau.

Conversely, given a $USp(2N-2)$ tableau one can always construct a superconformal tail ending in an SO gauge group, and given an $SO(2N)$ tableau one can write down a tail ending in a USp gauge group. However, there is one class of exceptions which are $SO(2N)$ tableaux consisting of just two columns, associated to the decomposition

$$2N \rightarrow (2N-k) + k. \tag{3.27}$$

Here k is odd unless N is even, in which case $k = N$ is also allowed. Naive application of the algorithm above associates a superconformal tail of the form

$$\cdots \times SO(6) \times USp(4) \times SO(4) \times \text{``}USp(0)\text{''} \tag{3.28}$$

which does not make sense in a purely gauge-theoretic setting. However, as was the case for SU quivers, one can still write down a brane configuration corresponding to this situation (Fig 6) and consistently lift it to M-theory.[1]

Finally let us discuss the behavior of φ_{2k} and $\varphi_{\tilde{N}}$ at the punctures, which can be found by a careful analysis of the Seiberg-Witten curves. One finds that it is not sufficient to specify the degrees of the poles for each φ_{2k} or $\varphi_{\tilde{N}}$, contrary to the case of the A_N theory. For example, at the puncture associated to the SO(6) tableau corresponding to $6 \rightarrow 3+3$, we find the following two conditions, whose derivation can be found in Appendix B:

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[1] This situation might be related to the appearance of the gluino condensate for USp(0) in the framework of Dijkgraaf-Vafa [16]. It would be interesting to clarify the relation; the author thanks Masaki Shigemori for discussion.

- φ_2, $\varphi_{\bar{3}}$ and φ_4 have poles of degree $1, 1, 2$ respectively; and

- $(\varphi_2)^2 - 4\varphi_4$ has only a simple pole.

The second condition guarantees that the M5-brane wrapping the Seiberg-Witten curve does not have a single, transversal intersection with the M-theory orientifold.

This sounds slightly puzzling, considering the fact that the D_3 theory in six dimensions is equivalent to the A_3 theory, for which the defects were classified and such a polynomial constraint was not found. Indeed, the decomposition $6 \to 3 + 3$ of the **6** of SO(6) corresponds to $4 \to 3 + 1$ of the **4** of SU(4), for which the pole structure was just that all $\phi_{2,3,4}$ have simple poles at the puncture, see Table 1. But upon further reflection this is exactly what is expected. Say that an element of the Cartan subalgebra of SO(6) acts on the **6** as

$$\mathrm{diag}(a, -a, b, -b, c, -c); \tag{3.29}$$

then it acts on the **4** as

$$\mathrm{diag}(a + b + c, a - b - c, -a + b - c, -a - b + c)/2. \tag{3.30}$$

Therefore ϕ_4 and φ_4 cannot be just equated; instead they satisfy

$$2\phi_2 = \varphi_2, \qquad 16\phi_4 = \varphi_2^2 - 4\varphi_4. \tag{3.31}$$

Thus we find that the condition on the worldvolume fields as found from the D_3 theory is the same as the one found from the point of view of the A_3 theory.

3.5 Duality with SCFT with E_7 flavor symmetry

Having analyzed general punctures of the D_N theory, we can now have some more fun. Let us start from the quiver with gauge groups $\mathrm{USp}(4) \times \mathrm{SO}(5) \times \mathrm{USp}(2)$, shown in the first row of Fig. 7. As before, we can go to a region where four punctures of type $\times$ collide, decoupling a tail with gauge groups $\mathrm{SO}(5) \times \mathrm{USp}(2) \times \mathrm{SO}(3)$. The resulting strongly-coupled theory has no marginal coupling because there are only three punctures on the sphere. There is only one Coulomb branch operator and its dimension is four, because the original theory contained three operators of dimension two and two of dimension four, whereas the decoupled tail has three dimension-two and one dimension-four operators.

This suggests that this theory is the E_7 SCFT of Minahan-Nemeschansky [9]. One can perform many tests of the proposal: one can easily check that the central charges a and c agree with what were found in [5]; and the flavor symmetry manifest in this description is naturally a subgroup of E_7,

$$\mathrm{SO}(6)^2 \times \mathrm{SO}(3) \simeq \mathrm{SU}(4)^2 \times \mathrm{SU}(2) \subset E_7. \tag{3.32}$$
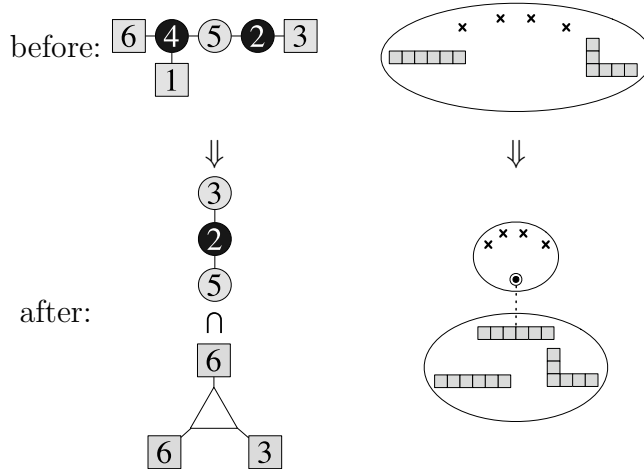
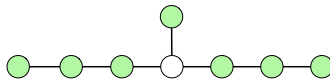**Figure 7:** S-duality involving the SCFT with E_7 flavor symmetry

Figure 8: Extended Dynkin diagram of E_7 showing subgroup $SU(4)^2 \times SU(2)$

This subgroup comes from the decomposition of the extended Dynkin diagram, see Fig. 8. The flavor symmetry central charges of this subgroup agree with those of E_7, which were found in [5].

The S-duality found in [5] involving the E_7 SCFT started from $USp(4)$ gauge theory with twelve half-hypermultiplets, which is exactly the first example in Table 2. The infinitely strongly-coupled limit corresponds to collapsing two singularities of type $\times$. The analysis above indicates that this procedure results in one Young tableau with columns of height $3, 1, 1, 1$. It would be fruitful to analyze which defects can arise when two defects of general type collide. Such collisions should provide a wealth of new S-dualities.

4. Discussion

In this paper, we generalized the construction of [1], which realized many four-dimensional SCFTs as compactifications of the six-dimensional A_{N-1} theory, to the D_N theory. We utilized this construction to find a new class of isolated SCFTs with $SO(2N)^3$ flavor symmetry, which arise in strongly-coupled limits of linear quivers of SO and USp gauge groups. We also saw how the E_7 SCFT of Minahan and Nemeschansky arises from this construction.

In [1], it was noted that the types of tails of SU superconformal quivers for six-dimensional A_{N-1} theory can be naturally associated to Young tableaux; we can naturally associate an embedding of SU(2) into SU(N) to such Young tableau, whose commutant inside SU(N) gave the flavor symmetry of that tail. In this paper, the analysis was extended to alternating SO–USp quivers for the six-dimensional D_N theory and it was found that the tails of such quivers can naturally be associated to an embedding of SU(2) into either SO($2N$) or USp($2N-2$); again, the flavor symmetry associated to the tail is given by the commutant of that embedding of SU(2).

We also saw that the simplest kinds of defects of the D_N theory have $\mathbb{Z}_2$ monodromy for the Pfaffian operator. This is suggestive in that the Pfaffian is odd under the outer automorphism of the D_N Lie algebra, whose quotient is exactly USp($2N-2$), which was used in the labeling of the tails. It would be interesting to consider defects associated to other outer automorphisms of A_{N-1} or D_4, and identify their realizations using quiver gauge theory.

The most pressing issue is to find out how the association to the defects of an embedding of SU(2) into SU, SO or USp groups can be intrinsically understood from a six-dimensional point of view, and how these embeddings control the behavior of the scalar fields around them. These defects are of codimension two. Therefore, if we compactify the six-dimensional $(2,0)$ theory on a torus parallel to the worldvolume of the defects, we obtain surface operators of the $\mathcal{N} = 4$ super Yang-Mills in four dimensions. The study of such surface operators was initiated in [17]. There, it was found that embeddings of SU(2) naturally appear which specify the singular part of the field configuration around the defect. Therefore, the problem seems to be in identifying which of the possible defects of four-dimensional $\mathcal{N} = 4$ super Yang-Mills descend from those of six-dimensional theories. It would be interesting to pursue this relation further.

Acknowledgments

A. SO(4)–USp(2) quivers

It is instructive to analyze the simplest case of the SO–USp quiver, namely the case

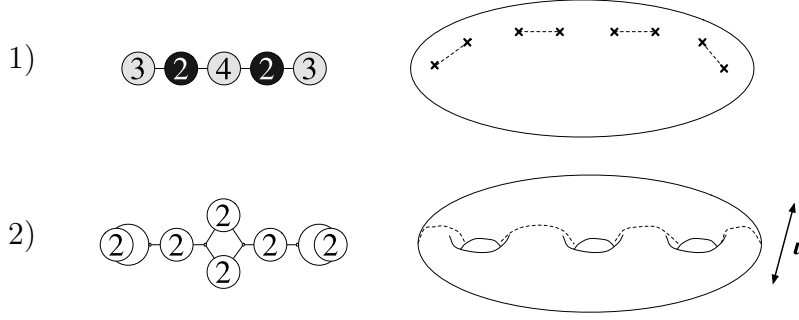**Figure 9:** An SO(4)–USp(2) quiver and the same quiver as an SU(2) generalized quiver. Corresponding G-curves are also shown. In the first description the curve is a sphere with eight punctures; in the second it is a hyperelliptic genus-three curve.

with SO(4)–USp(2) using our formalism. Consider a linear quiver theory with the gauge group

$$SO(3) \times USp(2) \times SO(4) \times \cdots USp(2) \times SO(3), \tag{A.1}$$

with s SO(4), $s + 1$ USp(2) and two SO(3) factors. The case $s = 1$ is shown in the diagram 1) of Fig. 9. Following the procedure explained in the main part, we find that the Seiberg-Witten curve is specified by the G-curve $\Sigma = \mathbb{CP}^1$ with $2s + 6$ punctures of the same type; we have two quadratic differentials on Σ, φ_2 and $\varphi_{\tilde{2}}$ corresponding to two Casimirs of SO(4), namely the trace of the square and the Pfaffian. At each puncture, $\varphi_2 \sim (dt)^2/t$ and $\varphi_{\tilde{2}} \sim (dt)^2/t^{1/2}$ where t is a local coordinate for which the puncture is at $t = 0$.

Now let us recall that $SO(4) \simeq SU(2) \times SU(2)$, and the vector representation of SO(4) is the tensor product of the doublets of each of the SU(2) factors; also that $SO(3) \simeq SU(2)$ and the vector representation of SO(3) is in the tensor product of two doublets. We neglect possible issues coming from the global structure of the groups. This should not cause any problems as long as we consider theories on the flat $\mathbb{R}^4$.

Then the quiver can also be presented as in the diagram 2) of Fig. 9 in the notation of [1]. In this case the G-curve is a genus-$(s + 2)$ Riemann surface Ξ and there is a quadratic differential ϕ_2 on it. An important constraint is that in the description as an SO(4)–USp(2) quiver, we cannot independently vary the coupling constants of two SU(2) factors of SO(4). It is natural to guess that this requirement translates to the fact that the curve Ξ is hyperelliptic. Indeed, it gives the correct number of marginal coupling constants because the number of the moduli of hyperelliptic curves of genus $s + 2$ is $2s + 3$, which agrees with the number of gauge factors in the quiver (A.1).

Now Ξ is equipped with the hyperelliptic involution ι which flips the two sheets; the fixed points are exactly the branch points on Σ. We can split ϕ_2 on Ξ into even

and odd parts under ι:

$$\phi_2 = \phi_2^+ + \phi_2^-. \tag{A.2}$$

We then regard $\phi_2^\pm$ as differentials on Σ.

Pick a puncture on Σ and take the local coordinate t so that the puncture is at $t = 0$. The local coordinates on Ξ is then $s = t^{1/2}$. ϕ_2^+ is holomorphic and even in s, which translates to the condition

$$\phi_2^+ \sim (ds^2) \sim (dt^2)/t, \tag{A.3}$$

implying that ϕ_2^+ has a simple pole at the branch points. Similarly, ϕ_2^- behaves as

$$\phi_2^- \sim s(ds^2) \sim (dt^2)/t^{1/2}. \tag{A.4}$$

Therefore we can identify φ_2 with ϕ_2^+ and $\varphi_{\bar{2}}$ with ϕ_2^-.

B. Curves for general SO–USp quivers

Here we provide some details of the derivation of the Seiberg-Witten curves for general linear quiver gauge theories with alternating SO and USp gauge groups. The brane construction was reviewed in Sec. 3.2, see Fig. 3 for a drawing of the system.

Let us first recall how D6 branes lift to a Taub-NUT space in M-theory. Let N_f be the total number of D6 branes. When all of them are at $x^4 = x^5 = 0$, the resulting Taub-NUT space is given as a complex manifold by the equation

$$yz = v^{N_f}, \tag{B.1}$$

and the orientifolding in M-theory acts by sending $v \to -v$. The action of orientifolding on y and z depends on the quiver; for simplicity we assume that y is fixed for now.

The origin $y = z = v = 0$ is blown up as long as the positions of D6-branes along x^6 directions are distinct. The blown-up, smooth manifold is given by introducing extra pairs of local coordinates (y_i, z_i) at the i-th nut $i = 1, 2, \ldots, N_f$, such that

$$y_1 z_1 = y_2 z_2 = \cdots = v, \tag{B.2}$$

and

$$y = y_1, \quad z_1 = 1/y_2, \quad z_2 = 1/y_3, \quad \ldots, \quad z_{N_f} = z. \tag{B.3}$$

There are $N_f - 1$ $\mathbb{CP}^1$'s parametrized by $y_{2,3,\ldots,N_f}$ which we call C_i, see Fig. 10. We analogously define the loci $z = 0$ and $y = 0$ to be C_0, C_{N_f}. The relations (B.2), (B.3) imply that the orientifolding fixes C_{even} but that it acts by multiplication by -1 on C_{odd}. Therefore the M-theory orientifold exists at C_{even}, but not at C_{even}.
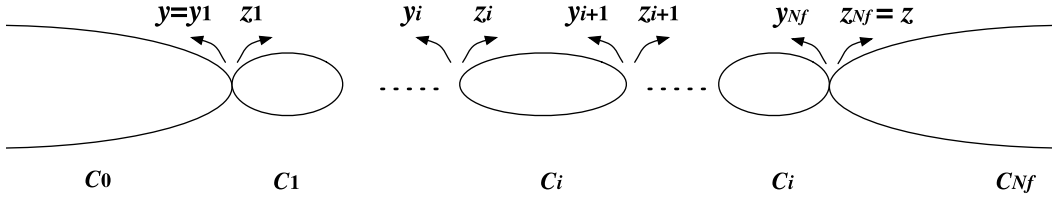
Figure 10: Schematic description of the Taub-NUT space, the lift of D6-branes. C_1 to C_{N_f} are blown-up two-cycles.

Now let us consider an SO–USp superconformal quiver gauge theory for the D_N theory, specified by integers (d_a), (δ_a) and (k_a), see Sec. 3.2 for notations. The Seiberg-Witten curve is a curve in the Taub-NUT space discussed above [13, 14], given by

$$F(v,t) = v^{2N}t^{n+1} + v^{2N-d_1}P_1(v)t^n + v^{2N-d_2}P_2(v)t^{n-2} + \cdots$$
$$+ v^{2N-d_n}P_n(v)t + v^{2N} = 0. \quad \text{(B.4)}$$

where $t = y_k$ and k is the number of D6-branes of the left hand tail of the superconformal quiver. $P_a(v)$ is a polynomial of degree d_a, even or odd in v according to the parity of d_a:

$$P_a(v) = c_a v^{d_a} + u_a^{(2)}v^{d_a-2} + u_a^{(4)}v^{d_a-4} + \cdots . \quad \text{(B.5)}$$

In the semi-classical regime, c_a encodes the gauge coupling constant, whereas $u_a^{(2k)}$ encodes the vacuum expectation values of the adjoint scalar field of the a-th gauge groups, *except* the constant term of $P_a(v)$ for which the gauge group is USp. These constant terms are determined by the requirement that the resulting M5-brane intersects the cycles C_i in a manner consistent with orientifolding in M-theory. The main condition is that an M5 brane cannot intersect with the M-theory orientifold five-plane transversally; an even number of M5-branes need to intersect at a point on an orientifold five-plane. This condition was first formulated in [7]. Refer to [8] for more details.

Let us define $x = v \, dt/t$ and

$$\varphi_{2k} = \frac{u_1^{(2k)}t^n + u_2^{(2k)}t^{n-1} + \cdots + u_n^{(2k)}t}{\prod(t - t_a)}\left(\frac{dt}{t}\right)^{2k}, \quad \text{(B.6)}$$

where we defined $u_a^{(2k)} = 0$ when $d_a < 2k$, and $\prod(t - t_a) = t^{n+1} + c_1 t^n + \cdots + 1$. Then one can rewrite the curve above into Gaiotto's form:

$$0 = x^{2N} + \varphi_2 x^{2N-2} + \varphi_4 x^{2N-4} + \cdots + \varphi_{2N}. \quad \text{(B.7)}$$
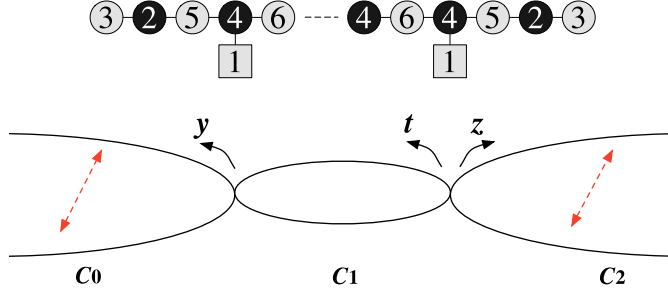
Figure 11: A quiver, and the corresponding Taub-NUT space in which its Seiberg-Witten curve is embedded. Orientifolding fixes C_1, but acts as multiplication by -1 on C_0 and C_2.

The structure of the poles at the punctures at $t = 0, \infty$ can be readily found from the form (B.6), but the conditions imposed on the constant terms of $P_a(v)$ by the consistency of the M-theory orientifold are rather intricate, and the author has not found a concise way to express them for a general sequence of gauge groups. Instead they are illustrated through two examples, which were used in the main part of the paper.

The first example is the quiver drawn in Fig. 11, which was the main topic of Sec. 3.3. The Taub-NUT space $yz = v^2$ is given in the right hand side of the same figure. The red broken arrow on C_0 and C_2 indicates that the orientifolding sends $y \to -y$ and $z \to -z$. The Seiberg-Witten curve was given in (B.4). t is the local coordinate of C_1. Let $n = 2b + 7$ be the total number of gauge groups; b is the number of SO(6) gauge groups. The consistent way to eliminate extra parameters in P_{even} is then to set the constant parts of P_2, P_4, P_{2b+4} and P_{2b+6} to zero, and to require

$$F(0,t) = \alpha t^5 \prod_{i=1}^{b-1} (t - q_i)^2 \tag{B.8}$$

for some complex numbers α, q_i. Indeed, the intersection of C_1 with the Seiberg-Witten curve is given by the double zeros q_i, as required by the consistency of the M-theory orientifold [8]. Furthermore, the intersection with C_0 is given by

$$u_1^{(2)} y^2 - u_3^{(4)} = 0 \tag{B.9}$$

which is compatible with the orientifolding action. The same can be said for C_2. Under these constraints, one finds that φ_2, φ_4, ..., φ_{2N} all have simple poles at $t = 0$. This translates to the behavior $\sim 1/t^{1/2}$ for $\varphi_{\tilde{N}}$.

The second example is the quiver drawn in Fig. 12; USp(0) needs to be taken as a shorthand for the corresponding brane configuration. This time C_0 and C_2 are both fixed by the orientifolding. The Seiberg-Witten curve is
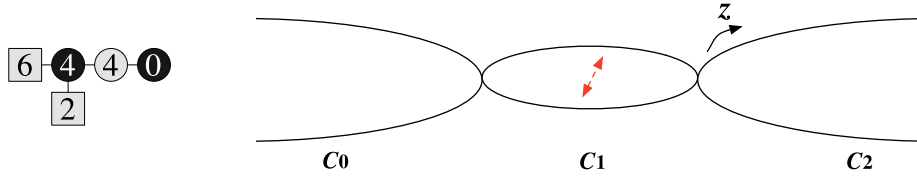
Figure 12: Another quiver, and the corresponding Taub-NUT in which its Seiberg-Witten curve is embedded. Orientifolding fixes C_0 and C_1, but acts as multiplication by -1 on C_1.

$$0 = v^6 y^4 + (c_1 v^6 + u_1^{(2)} v^4 + u_1^{(4)} v^2 + u_1^{(6)}) y^3$$
$$+ v^2 (c_2 v^4 + u_2^{(2)} v^2 + u_2^{(4)}) y^2 + v^4 (c_3 v^2 + u_3^{(2)}) y + v^6. \quad \text{(B.10)}$$

The intersection of this curve with C_2 parameterized by z is determined by

$$u_1^{(6)} + u_2^{(4)} z + u_3^{(2)} z^2 + z^3 = 0. \quad \text{(B.11)}$$

Now, C_2 is a fixed locus of the M-theory orientifold, and no M5-brane is wrapped on it. Therefore, the intersection needs to be a double zero when an M5-brane intersects on it. This requires $u_1^{(6)} = 0$ and $4u_2^{(4)} = (u_3^{(2)})^2$, which leads to a simple pole in φ_2 and a double pole in φ_4 such that $4\varphi_4 - (\varphi_2)^2$ only has a simple pole. This property was crucial when we matched this defect of the D_3 theory with the simple puncture of the A_3 theory. In a similar manner, one finds that a quiver ending with groups $SO(2k)$–$USp(k-2)$ will allow a simple pole in φ_k and a double pole in φ_{2k}, such that $4\varphi_{2k} - (\varphi_k)^2$ only has a simple pole.

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