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Size of a hydrogen atom in the expanding universe

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Abstract. I take a simple model of the hydrogen atom in a universe without spatial curvature. The Maxwell equations are formulated on the background cosmic spacetime. For a class of cosmic metrics, which includes the de Sitter universe, these equations admit solutions corresponding to an atom whose radius remains strictly constant during the expansion. In the Einstein–de Sitter universe approximate calculations show that the atom expands, but at a rate which is negligible compared with the general cosmic expansion.

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1. Introduction

Which entities participate in the expansion of the universe? Granted that clusters of galaxies expand, one can ask whether each galaxy expands, whether the expansion applies to the solar system, and even to atoms. The fact is, as has been emphasized recently by Anderson [1], that cosmological theory does not give a clear statement of the scale at which the expansion begins to apply. There have, however, been several attempted and partial answers, as the following brief history shows (for further historical material see [2]).

The question of whether the solar system participates in the expansion seems first to have been considered by McVittie [3], who discovered a model of the universe containing a mass point which could represent the Sun. He studied planetary orbits in the model but came to no definite conclusion. The question was taken up by Järnefelt [4, 5] who used McVittie's model and concluded that planetary orbits do not participate in the expansion. Dicke and Peebles [6], using general arguments, decided that the binary orbits of charged or uncharged particles are unaffected by the expansion. Noerdlinger and Petrosian [7] considered clusters or superclusters of galaxies and found that they would expand, but at a rate depending on the ratio of their density to the cosmic density.

A different approach was initiated by Einstein and Straus (E–S) [8, 9]. They showed that a Schwarzschild solution could be embedded in a pressure-free expanding universe. In the Schwarzschild vacuum region ('vacuole'), test particles, representing planets, would be totally unaffected by the expansion, so it was concluded that the solar system does not expand.

It has since been realized that the E–S model does not settle the question. In the first place it requires a certain relation between the radius of the vacuole, the mass of the Schwarzschild particle and the cosmic density. This relation is not fulfilled in the case of the solar system [10]. Secondly, the E–S model is spherically symmetric, whereas the astronomical systems to which one wishes to apply it are usually not. Moreover, subsequent theoretical work has made it doubtful whether the E–S model can be extended to non-spherical systems [11, 14].

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Gautreau [12] put forward a model of a particle embedded in an inhomogeneous, pressure-free expanding universe. He examined geodesics to study the influence of the expansion on planetary orbits. He concluded that in this model the orbits do expand. His work will be referred to again in section 5.

Interest in this topic has been renewed recently. Anderson [1] has used the method of Einstein, Infeld and Hoffmann to study a pair of gravitating particles in an Einstein–de Sitter (E–deS) universe. He concludes that the expansion does have an effect on the motion of the particles, though if they are in circular orbit it does not expand. Cooperstock *et al* [13] have studied the effect of the cosmic expansion on local dynamics, using Fermi normal coordinates, and conclude that 'the expansion affects all scales, but the magnitude of the effect is essentially negligible for local systems, even at the scale of galactic clusters'.

In an investigation of a different kind [10] I showed by an exact solution of the Einstein–Maxwell equations that local distributions of a certain sort of matter definitely participate in the expansion. The matter concerned was electrically counterpoised dust: this is dust carrying an electric charge density such that the gravitational attraction and the electric repulsion balance. A body composed of such matter would double its size in a few billion years. It is therefore interesting to study whether an atom, which at a simple classical level can be regarded as charged matter held stationary by inertial (centrifugal) force, also participates in the expansion.

In this paper I consider the effect on a hydrogen atom of the expansion of a Friedmann–Robertson–Walker (FRW) universe with no spatial curvature. I form the Maxwell equations with this as background, and set up the Bohr model in its ground state. In a class of cases—which includes the de Sitter universe—an exact analysis of the equations shows that the electron's orbit is unchanged during the expansion. In general, approximation is necessary and I limit myself to the E–deS model. I calculate approximately the change in radius of the electron's orbit in one revolution, and compare it with the displacement, in the same cosmic time, of a comoving uncharged test particle. The ratio of the two quantities, though positive, is negligibly small. The conclusion is that the hydrogen atom effectively does not participate in the cosmic expansion.

The plan of the paper is as follows. In section 2 I set up the equations of motion of the electron in the FRW universe with no spatial curvature. Section 3 gives the special exact solution of the equations, and in section 4 I use an approximation method to solve the equations in the E–deS case. The physical significance of the results is discussed in section 5 and there is a brief concluding section 6. There are two appendices.

2. The equations

I take a FRW universe with no spatial curvature

$$\mathrm{d}s^2 = -[R(t)]^2\left(\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin\theta^2\,\mathrm{d}\phi^2\right)\right) + c^2\,\mathrm{d}t^2, \tag{1}$$

where R is a dimensionless function and c is the speed of light, and consider the Maxwell equations on this background metric:

$$F_{ik} = \kappa_{i;k} - \kappa_{k;i}, \tag{2}$$

$$(F^{ik})_{;k} = 4\pi J^i. \tag{3}$$

F^{ik}, κ_i and J^i are the electromagnetic field tensor, the potential and the current, respectively. Here and throughout the paper indices are raised and lowered by the metric (1), and covariant differentiation is with respect to (1). It will be assumed that J^i vanishes, charges being represented by point singularities.

The electron has mass m and charge q, and so, as we are neglecting radiative effects, its 4-velocity u^i is determined by the Lorentz equation

$$mu^i_{;k}u^k = qu^k F_k{}^i. \tag{4}$$

The field F^{ik} is provided by the positive charge of the proton situated permanently at the origin of the coordinates of (1). Assuming spherical symmetry we take

$$\kappa_i = P(t)\, r^{-1}\delta^4_i;$$

then (2) gives as the only components of F_{ik}:

$$F_{14} = -F_{41} = P(t)\, r^{-2}.$$

Raising indices by the metric (1) and using (3) with $i = 1$ and 4 we find

$$F^{41} = -F^{14} = Qc^{-3}r^{-2}R^{-3}, \tag{5}$$

where Q is a constant with the dimensions of charge, which we take to be that of the proton.

To find how the size of the atom changes during the cosmic expansion we need to solve (4) for the motion of the electron in the field (5)†. Taking the orbit to lie in the plane $\theta = \pi/2$, and numbering the coordinates

$$r = x^1, \qquad \theta = x^2, \qquad \phi = x^3, \qquad t = x^4,$$

we obtain from (4) three equations:

$$\frac{\mathrm{d}u^1}{\mathrm{d}s} - r(u^3)^2 + \frac{2\dot{R}}{R}u^1u^4 = \frac{qQ}{mcr^2R^3}u^4, \tag{6}$$

$$\frac{\mathrm{d}u^3}{\mathrm{d}s} + \frac{2}{r}u^1u^3 + \frac{2\dot{R}}{R}u^3u^4 = 0, \tag{7}$$

$$\frac{\mathrm{d}u^4}{\mathrm{d}s} + \frac{R\dot{R}}{c^2}((u^1)^2 + r^2(u^3)^2) = \frac{qQ}{mc^3r^2R}u^1. \tag{8}$$

The metric (1) gives a fourth equation

$$c^2(u^4)^2 - R^2((u^1)^2 + r^2(u^3)^2) = 1, \tag{9}$$

but only three of the four are independent. Equation (7) can be integrated to give the conservation of angular momentum:

$$u^3r^2R^2 = h, \tag{10}$$

where h is a constant.

The rest of the paper deals with the solution of these equations. In the general case approximations are necessary, but for certain forms of $R(t)$ there is an exact solution as I show in the next section.

3. An exact solution of the equations of motion

The product rR represents proper radial distance from the origin in the spacetime (1), and also areal radius, i.e. $4\pi(rR)^2$ is the area of a sphere r = constant, t = constant. Hence for a circular orbit not expanding at all with the universe it is reasonable to suppose

$$rR = a \tag{11}$$

† By using the metric (1) we are neglecting the gravitational fields of the proton and electron. This is justifiable because the ratio of the gravitational force between the particles to the electric one is about 10^{-40}.

where a is a constant. We shall find an exact solution of the equations of motion (6) and (8)–(10) by imposing the condition (11).

Differentiating (11) along the path of the electron we have

$$Ru^1 + r\dot{R}u^4 = 0, \tag{12}$$

and differentiating again

$$R\frac{\mathrm{d}u^1}{\mathrm{d}s} + 2\dot{R}u^1u^4 + r\ddot{R}(u^4)^2 + r\dot{R}\frac{\mathrm{d}u^4}{\mathrm{d}s} = 0, \tag{13}$$

where an overdot means $\mathrm{d}/\mathrm{d}t$. We now substitute (6), (8)–(10) and (12) into (13) and obtain, after a calculation,

$$\frac{h^2}{a^4} + \frac{qQ}{mca^3}\left(1 - \frac{a^2\dot{R}^2}{R^2c^2}\right)u^4 + \left(\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2}\right)(u^4)^2 + \frac{\dot{R}^2}{c^2R^2} = 0. \tag{14}$$

Substitution of (12) into (9) gives

$$(u^4)^2 = \left(1 + \frac{h^2}{a^2}\right)\left(c^2 - \frac{a^2\dot{R}^2}{R^2}\right)^{-1}, \tag{15}$$

and this can be inserted into (14) to give a differential equation in $R(t)$ for the existence of non-expanding circular orbits. *u^1 given by (12), u^3 by (10) and u^4 by (15) solve equations (6)–(9) provided $R(t)$ satisfies (14).*

There is one function $R(t)$ which by inspection satisfies (14). This is

$$R = \exp \alpha t, \tag{16}$$

where α is a constant. With this form of R, equations (14) with (15) reduces to an equation between the constants $a, h, qQ/m, c$ and α. R given by (16) refers to the de Sitter universe so we have shown that *in the de Sitter universe there are circular electron orbits which remain unchanged during the expansion.* This, of course, is no surprise if one recalls that the metric for the de Sitter universe can be written in static form by the transformation

$$\rho = r \exp \alpha t, \qquad \tau = t - (2\alpha)^{-1}\log(1 - \alpha^2r^2\exp 2\alpha t). \tag{17}$$

The unchanging orbits are then given by $\rho =$ constant.

The equation between constants to which (14) and (15) reduce in the de Sitter model is

$$\frac{h^2}{a^2} + \frac{a^2\alpha^2}{c^2} = -\frac{qQ}{mc^2a}\left[\left(1 + \frac{h^2}{a^2}\right)\left(1 - \frac{a^2\alpha^2}{c^2}\right)\right]^{1/2}; \tag{18}$$

qQ is, of course, negative. *Here, and throughout the paper, positive square roots are to be taken.* Equation (18) will be referred to again in section 4.

The de Sitter model is not the only member of (1) which has strictly constant circular orbits. Others are given by the differential equation obtained by eliminating u^4 between (14) and (15). It would be interesting to know their expansion function $R(t)$ and the equation of state of the matter they contain. Unfortunately the differential equation concerned is extremely complicated, so I take a different approach, namely, I consider nearly circular orbits in the Einstein–de Sitter universe.

4. Nearly circular orbits in the Einstein–de Sitter universe

Let us imagine an electron projected at an event $E \equiv [a/R(t_0), \pi/2, 0, t_0]$ with velocity such that initially $\mathrm{d}(rR)/\mathrm{d}s = 0$, where s is the parameter along the trajectory. As before we suppose there is a proton at the origin. If the path were strictly circular $\mathrm{d}^2(rR)/\mathrm{d}s^2$ would vanish initially, and, of course, both derivatives would remain permanently zero. To find a nearly circular path I shall assume that $\mathrm{d}^2(rR)/\mathrm{d}s^2$ is zero initially, and try to find how far the path diverges from a circle of radius a as time passes. This will be a measure of how the size of a hydrogen atom changes as the universe expands. Thus our initial conditions are

$$s = 0, \qquad rR = a, \qquad \mathrm{d}(rR)/\mathrm{d}s = 0, \qquad \mathrm{d}^2(rR)/\mathrm{d}s^2 = 0. \tag{19}$$

It is necessary to use approximations. Some details of the calculations are given in appendix A. The motion of the electron takes place in the plane $\theta = \pi/2$. Assume that r and t along the trajectory can be expanded as follows:

$$r = r_0\left(1 + \sum_{n=1}^{\infty} b_n s^n\right), \tag{20}$$

$$t = t_0\left(1 + \sum_{n=1}^{\infty} c_n s^n\right), \tag{21}$$

where b_n and c_n are constants. For the Einstein–de Sitter universe†

$$R(t) = t^{2/3}. \tag{22}$$

Using (21) we can expand $R(t)$ along the trajectory in terms of s, and then form rR and expand that too:

$$rR = r_0 t_0^{2/3}\left(1 + \sum_{n=1}^{\infty} k_n s^n\right) \tag{23}$$

where k_n are constants determined in terms of b_n and c_n; the first two are zero because of (19) and the third is given in appendix A.

The object of this section is to express k_3 in terms of known quantities a, m, c, t_0 and qQ; this will tell us how the electron's orbit begins to diverge from a circle. We find that to satisfy (19)

$$b_1 + \tfrac{2}{3}c_1 = 0 \tag{24}$$

$$b_2 + \tfrac{2}{3}c_2 = \tfrac{5}{4}b_1^2. \tag{25}$$

At this stage it is useful to introduce some further notation. Let

$$x := 2a/3ct_0, \tag{26}$$

$$p := h^2/a^2, \tag{27}$$

$$y := (1+p)^{1/2}(1-x^2)^{-1/2}, \tag{28}$$

$$A := |qQ|(mc^2 a)^{-1}. \tag{29}$$

All quantities on the left are dimensionless. Further relations between the coefficients come by forming $u^i = \mathrm{d}x^i/\mathrm{d}s$ and substituting into (9); note that because of (10) and (19),

$$u^3 = h/a^2 + \mathrm{O}(s^3).$$

We find, making use of (24) and (25),

$$b_1 = -xy/a, \tag{30}$$

$$c_1 = 3xy/2a, \tag{31}$$

$$b_2 = \frac{y^2(5-2x^2)}{9(ct_0)^2(1-x^2)}, \tag{32}$$

† As stated previously, $R(t)$ is a dimensionless function, so strictly we should write $R(t) = (t/\beta)^{2/3}$ where β is a constant with the dimensions of time; however, β would have no observational significance so I omit it here.

$$c_2 = -\frac{x^2y^2}{2(ct_0)^2(1-x^2)}, \tag{33}$$

$$c_3 = -\frac{x^2y^3}{54(ct_0)^3(1-x^2)^2}(13-17x^2+4x^4) - \tfrac{3}{2}x^2b_3. \tag{34}$$

To proceed further we must use one or other of the equations of motion (6) and (8). Let us substitute (10), (20) and (21) into (6). We find up to order s, dividing through by r_0:

$$(2b_2+6b_3s) - pa^{-2}(1+b_1s) + \tfrac{4}{3}(b_1+2b_2s)(c_1+2c_2s)(1+c_1s)^{-1} = -Aa^{-2}(1+b_1s)(c_1+2c_2s)ct_0, \tag{35}$$

where we have used the fact that qQ is negative. To zeroth order this equation gives

$$2b_2 - pa^{-2} + \tfrac{4}{3}b_1c_1 = -Aa^{-2}c_1ct_0, \tag{36}$$

whence, using (28) and (30)–(32), we find after a short calculation

$$\frac{(2-5x^2)}{2(1-x^2)}y^2 - Ay - 1 = 0. \tag{37}$$

This equation determines h in terms of known quantities.

At this stage it is useful to consider orders of magnitude. Let us use (in cgs units)

$$Q = 4.8\times10^{-10}, \qquad q/m = -5.3\times10^{17}, \qquad c = 3\times10^{10}, \qquad a = 5.3\times10^{-9}, \qquad t_0 = 3.15\times10^{17}, \tag{38}$$

so that

$$x \sim 10^{-36}, \qquad A \sim 5\times10^{-5}. \tag{39}$$

Then we can write for the solution of (37)

$$y = 1 + \tfrac{1}{2}A + \tfrac{1}{8}A^2 + \mathrm{O}(A^4), \tag{40}$$

terms containing x^2 being negligible compared with A^4. Equations (28) and (40) give

$$p = A + \tfrac{1}{2}A^2 + \mathrm{O}(A^3). \tag{41}$$

At this point it is convenient to return to (18) which applies to the de Sitter model. Since $\alpha = (\Lambda/3)^{1/2}$, the term $(a\alpha/c)^2$ is about 10^{-72} and may be neglected. Adopting the notation (27) and (29) we find that (18) is the same as (41).

The lowest approximation to (41), namely $p = A$, is equivalent to the classical formula for the angular velocity ω of a negative charge q in circular motion about a positive one Q:

$$\omega^2 = |Qq|/ma^3.$$

(To see this put $h = \omega a^2c^{-1}$, and use (27) and (29).)

The final step is to find b_3 from (35). This will determine c_3 by means of (34) and thence the coefficient k_3 in (23) which gives the initial change in radius of the orbit. From (35), equating coefficients of s we have

$$6b_3 - pa^{-2}b_1 + \tfrac{4}{3}\left[2(b_1c_2+b_2c_1) - b_1c_1^2\right] = -Aa^{-2}ct_0(2c_2+b_1c_1). \tag{42}$$

Using (30)–(33) we find from this, after a short calculation

$$b_3 = -\frac{py}{9a^2ct_0} + \frac{Ay^2(2+x^2)}{18a^2(1-x^2)ct_0} - \frac{8y^3(4-x^2)}{81(1-x^2)(ct_0)^3}. \tag{43}$$

Inserting the value of p in terms of y given by (28) and simplifying, using (26), (37) and (40), we find after a calculation

$$b_3 = -(ct_0)^{-3}\left[\tfrac{34}{81} + \tfrac{5}{9}A + \mathrm{O}(A^2)\right], \tag{44}$$

neglecting terms of order $a^2(ct_0)^{-5}$. Substituting this into (34) we find

$$c_3 = \mathrm{O}(a^2(ct_0)^{-5}), \tag{45}$$

which may be neglected in comparison with b_3.

As shown in appendix A, k_3, the coefficient of s^3 in the expansion of rR, is

$$k_3 = b_3 + \tfrac{2}{3}c_3 + \tfrac{2}{9}c_2(3b_1 - c_1) + \tfrac{1}{81}c_1\big(54b_2 - 9b_1c_1 + 4c_1^2\big). \tag{46}$$

We now have all the ingredients of the right-hand side, given by (30)–(34), (43) and (45). The leading term in the sum on the right-hand side of (46) is found to be

$$k_3 = \tfrac{2}{27}(ct_0)^{-3}, \tag{47}$$

the greatest error term being of order $A(ct_0)^{-3}$.

Let me summarize what has been done in this section. I assumed the initial condition (19) for a nearly circular orbit, and expanded r and t as in (20) and (21). I used the equations of motion (6)–(9) to find the coefficients b_i, c_i ($i = 1, 2, 3$). This enabled me to calculate k_3, the coefficient in (23) which determines, up to order s^3, how the trajectory diverges from a circle.

5. Physical interpretation

We wish to find how the size of our model of the hydrogen atom changes during the expansion. We therefore study the change in radius of the electron's orbit. Suppose, as in section 4, that the electron is projected from event $E \equiv [a/R(t_0), \pi/2, 0, t_0]$ so that its motion is nearly circular. The interval of a revolution is obtained from (10), the cosmic time of a revolution, t_{rev}, from (21), and the change in radius from (23).

Now suppose that a particle comoving with the cosmic fluid is released at the same event E; we can calculate the proper distance that it moves in time t_{rev}, and compare that with the change in radius of the orbit.

To carry out this programme we first integrate (10), using (23):

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = \frac{h}{(rR)^2} = \frac{h}{a^2}(1 - 2k_3s^3) + \mathrm{O}(s^4). \tag{48}$$

The term in s^3, which contains the extremely small k_3 (see equation (47)), may be neglected, and we have

$$s_{rev} = 2\pi a^2/h = 2\pi a/p^{1/2}. \tag{49}$$

The cosmic time corresponding to this is obtained from (21), the necessary c_n being obtained from (31), (33), (34) and (41):

$$t_{rev} = 2\pi a/(cA^{1/2}) + \mathrm{O}(aA^{1/2}/c). \tag{50}$$

Finally, the change in radius during this time is calculated from (23) and (47): it is, up to order s^3,

$$\overset{1}{\Delta}(rR) = ak_3s_{rev}^3 = \frac{16\pi^3a^4}{27A^{3/2}(ct_0)^3}. \tag{51}$$

On the other hand, the change in radial distance of a particle comoving with the cosmic matter during time t_{rev} is

$$\overset{2}{\Delta}(rR) = r_0\dot{R}\,t_{rev} + \mathrm{O}(t_{rev}^2),$$

where r_0, the comoving radial coordinate, is that of E, namely $a/R(t_0)$; the leading term of this is found to be

$$\overset{2}{\Delta}(rR) = \frac{4\pi a^2}{3A^{1/2}(ct_0)}. \tag{52}$$

We can now compare these two changes:

$$\frac{\overset{1}{\Delta}}{\overset{2}{\Delta}} = \frac{4\pi^2 a^2}{9A(ct_0)^2} \sim 10^{-67}. \tag{53}$$

This means that for practical purposes the hydrogen atom does not expand with the universe.

The change in radius, though minute, is positive, and it is interesting to consider this further. One possible reason for it follows from work by Gautreau [12]. Across any sphere of fixed radius, such as the one $rR = a$ we have been considering, there is, in the cosmological model, a flux of matter due to the expansion. Thus the mass inside the sphere, and therefore its gravitational attraction, is diminishing, which will cause orbits to spiral outwards. In our application a flux of cosmic matter through the atom is obviously absurd and can be dismissed as a defect of the model. However, a calculation, given in appendix B, shows that not all of (51) can be accounted for in this way. The interesting possibility remains that there is some increase in the atom's size due to genuine relativistic effects.

6. Conclusion

The conclusion of the paper is that the cosmic expansion has a negligible effect on the size of a hydrogen atom. By this is meant that the distance between the electron and proton increases very much less rapidly than the distance between two uncharged particles, with the same initial separation, and moving with the background cosmic fluid. This result is in qualitative agreement with that of Anderson [1]. It is in contrast with a previous investigation [10], which showed that some electrically charged matter does participate in the expansion.

It is an interesting question as to how matter with structure is affected by the expansion of the universe. It is possible that bodies with different structures, such as the Sun and the Earth, may be affected differently. The work in this paper suggests that atoms may be able to resist the expansion.

It should be noted that the expansion of the Einstein–de Sitter universe does have a minute effect on the atomic radius, as was remarked in section 5. On the other hand, in the de Sitter universe, the electron's orbit remains strictly constant during the expansion, as appeared in section 3. A possible explanation is that in the E–deS case the cosmic matter has a frame-dragging effect, whereas in the empty de Sitter universe this effect is absent.

Appendix A. Expansion of rR

Using (20)–(22), we wish to write rR in terms of s, so that we can impose conditions (19) and calculate k_3 in (23). We have, up to order s^3,

$$R(t) = t_0^{2/3}\left[1 + \tfrac{2}{3}c_1 s + \left(\tfrac{2}{3}c_2 - \tfrac{1}{9}c_1^2\right)s^2 + \left(\tfrac{2}{3}c_3 - \tfrac{2}{9}c_1c_2 + \tfrac{4}{81}c_1^3\right)s^3 + \cdots\right]$$

so

$$rR = r_0 t_0^{2/3}\Big\{1 + \left(b_1 + \tfrac{2}{3}c_1\right)s + \tfrac{1}{9}\left(9b_2 + 6c_2 + 6b_1c_1 - c_1^2\right)s^2 \\ + \left[b_3 + \tfrac{2}{3}c_3 + \tfrac{2}{9}c_2(3b_1 - c_1) + \tfrac{1}{81}c_1\left(54b_2 - 9b_1c_1 + 4c_1^2\right)\right]s^3 + \cdots\Big\}.$$

Imposing (19) we obtain (24) and (25), and the coefficient k_3 of s^3 is seen to be given by (46).

Appendix B. Flux of matter through a fixed sphere

Here I consider the change in the electron's orbit due to flux of dust through the sphere $rR = a$. The calculation is Newtonian, and I will denote the radius of the orbit by ρ. Supposing that the orbit is circular and unchanging we have

$$\frac{h^2}{\rho^3} = \frac{|qQ|}{m\rho^2} + \frac{GM}{\rho^2} + \frac{4}{3}\pi G\mu\rho, \tag{B1}$$

where G is the constant of gravitation, μ the density of dust within the sphere and M the mass of the proton. Now suppose that μ and ρ depend on the time; multiply (54) through by ρ^3 and differentiate with respect to t. (Strictly, a term $\mathrm{d}^2\rho/\mathrm{d}t^2$ enters the equation of motion but this is negligible compared with the leading terms in (54).) The result is

$$\left[\frac{|qQ|}{m} + GM + \frac{16}{3}\pi G\mu\rho^3\right]\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{4}{3}\pi G\rho^4\frac{\mathrm{d}\mu}{\mathrm{d}t}.$$

Let us now insert the value of the density in the E–deS model, namely $\mu = (6\pi Gt^2)^{-1}$, and neglect the second and third terms in the square brackets on the left which are minute compared with the first; then

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{4m\rho^4}{9|qQ|t^3}. \tag{B2}$$

Suppose this change takes place at time t_0 and that ρ then has the value a. We can calculate the change in radius in a revolution by multiplying (55) by the time t_{rev} given in (50). We find, using (29)

$$\overset{3}{\Delta}\rho = \frac{8\pi a^4}{9A^{3/2}(ct_0)^3}. \tag{B3}$$

It will be seen that the ratio of this to the relativistic value (51) is $3/(2\pi^2)$, i.e. about 15%.

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