

SKEINS AND HANDLEBODIES

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The Temperley-Lieb algebra is used to find a base for the vector space that is associated to a closed surface by the Topological Quantum Field Theory corresponding to the original Jones polynomial invariant.

1. Introduction. The Kauffman linear skein \mathcal{SM} of an oriented 3-manifold M , that has a (possibly empty) finite collection of (framed) points specified in its boundary, is defined as follows. Throughout, A will be a fixed complex number later to be chosen to be a primitive $4r$ th root of unity (though it is equally possible to work with the ring of Laurent polynomials in A , quotiented by the ideal generated by a cyclotomic polynomial). \mathcal{SM} is the vector space of formal linear sums of isotopy classes of framed links in M of disjoint simple closed curves and arcs that agree with the specification in ∂M , quotiented by the following relations.

$$(i) \quad \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = A \begin{array}{c} \cup \\ \cup \end{array} + A^{-1} \begin{array}{c} \cap \\ \cap \end{array}$$

$$(ii) \quad L \cup U = (-A^{-2} - A^2)L.$$

In (i) a framing on a curve is depicted by a parallel to the curve, and in (ii) U is a zero-framed unknotted component in a ball disjoint from the link L . It is often convenient to project M to some surface F (for example, S^3 less two points projects to S^2 , a handlebody projects to a disc-with-holes) and then \mathcal{SM} is interpreted as the linear skein \mathcal{SF} of link diagrams in F as in [5], [7], [10], the framings being determined by parallel curves in F . In particular the n th Temperley-Lieb algebra is the Kauffman skein of the ball with two sets of n points specified on its boundary. It is convenient to consider that via link diagrams in a rectangle with n specified points on the left edge, n points on the right edge, the product in the algebra coming from juxtaposition of the rectangles. Now, it is clear that $\mathcal{SS}^3 = \mathbb{C}$; in fact a zero-framed link corresponds in \mathcal{SS}^3 to its Jones polynomial evaluated at $t = A^{-4}$. Suppose that M is embedded in S^3 , that M' is the closure of $S^3 - M$ and that M and M' have the same specified framed points in their

common boundary. There is then a natural bilinear map

$$SM \times SM' \rightarrow \mathbb{C}$$

induced by the operation of taking framed links L and L' in M and M' and regarding $L \cup L'$ as an element of SS^3 . Such bilinear forms will be used in the usual way to associate to any element X of SM an element X^* of the dual of SM' . This idea was considered, for the Temperley-Lieb algebra, in some detail in [6] where it was noted that the bilinear form could become degenerate when A was a root of unity.

The $Sl_q(2, \mathbb{C})$ invariant introduced by Witten [12] for a closed 3-manifold “at a certain level” (here to be interpreted as “at a 4th primitive root of unity A ”), was in [7] and [8] shown to be essentially an element of SS^3 (*indirectly*) associated to any framed link in S^3 that produces the manifold by means of surgery. Atiyah and Witten [1] laid down the axioms for a “Topological Quantum Field Theory” within the framework of which it might be desirable to view quantum invariants. That involves the association of a vector space with every closed oriented surface and an element of that space to every oriented 3-manifold bounded by the surface. Vogel [11], paying due regard to framings of manifolds, explained how the relevant Topological Quantum Field Theory for the $Sl_q(2, \mathbb{C})$ invariant could be viewed in the context of the combinatorial methods of [7] and [8]. If a surface is regarded as the common boundary of a 3-manifold M in S^3 , and the closure of its complement M' as already considered, then the vector space to be associated to that surface is the quotient of SM by the kernel of the map $X \mapsto X^*$. The purpose of this paper is briefly to give a base for that quotient space, thus in principle determining its dimension, and thus giving, from this viewpoint (as explained in [1]), an approach to the $Sl_q(2, \mathbb{C})$ -invariant of the product of a closed surface and a circle.

2. Using the Temperley-Lieb algebra. In what follows diagrams represent elements of the (Kauffman) linear skein of link diagrams in a rectangle with n specified points on the left edge and n' on the right edge. An integer i beside an arc will denote the intended presence of i parallel copies of that arc. When $n = n'$ this skein is the Temperley-Lieb algebra TL_n . Recall (for example from [7] or [8]) that this is generated, as an algebra, by elements $1, e_1, e_2, \dots, e_{n-1}$ where 1 is n arcs going straight from the left side of the rectangle to the right, and that e_i is the same except that the i th arc doubles back to the

$(i + 1)$ th point on the left edge, the i th and $(i + 1)$ th points on the right edge being connected similarly. For generic values of A , TL_n contains a central idempotent $f^{(n)}$ with the properties that $f^{(n)}e_i = 0$ for all i , $f^{(n)}f^{(n)} = f^{(n)}$ and $1 - f^{(n)}$ belongs to, and is indeed the identity of, the ideal generated by $\{e_1, e_2, \dots, e_{n-1}\}$. In the diagrams that follow a small square will denote the presence of an $f^{(n)}$ the relevant value of n being deducible from the labels on the strings entering the square; this follows the convention of [2]. The number Δ_n will be defined by

$$\Delta_n = (-1)^n(A^{2(n+1)} - A^{-2(n+1)})/(A^2 - A^{-2}),$$

this being characterised by the Chebyshev recurrence relation

$$\Delta_{n+1} = \Delta_1\Delta_n - \Delta_{n-1}$$

where $\Delta_1 = -A^{-2} - A^2$ and $\Delta_0 = 1$. The inductive defining formula for $f^{(n)}$ is shown in Figure 1, $f^{(1)}$ being the identity in TL_1 .

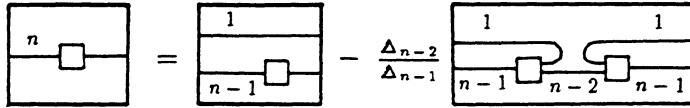


FIGURE 1

The formulae depicted in Figure 2 are immediately deducible (the third by induction on i).

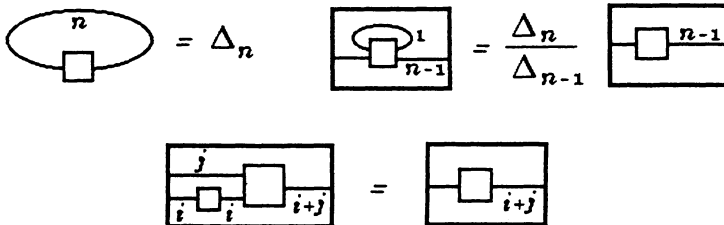


FIGURE 2

Note that when A is chosen to be a primitive $4r$ th root of unity, $\Delta_n \neq 0$ when $n \leq r - 2$, $\Delta_n = 0$ when $n = r - 1$ and $f^{(n)}$ is not defined in $n \geq r$.

Consider the diagram shown in Figure 3.

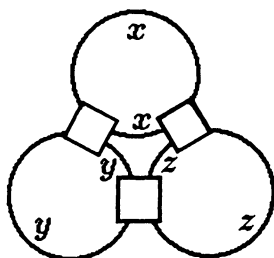


FIGURE 3

It consists of x parallel copies of a circle, y of another circle and z of a third with $f^{(x+y)}$, $f^{(y+z)}$ and $f^{(z+x)}$ inserted as shown. Let $\Gamma(x, y, z)$ be the element of SS^2 that this diagram represents. It will be important to know when $\Gamma(x, y, z)$ is and is not zero. This element is thus calculated in the following which serves also as an exercise in the use of the Temperley-Lieb algebra (the result of the lemma is inherent in [4] and implied in [2]; a simultaneously derived version of this proof appears in [9]). Here $\Delta_n!$ denotes $\Delta_n \Delta_{n-1} \Delta_{n-2} \cdots \Delta_1$, this being interpreted as 1 if n is -1 or zero.

LEMMA 1.

$$\Gamma(x, y, z) = (\Delta_{x+y+z}! \Delta_{x-1}! \Delta_{y-1}! \Delta_{z-1}!) / (\Delta_{y+z-1}! \Delta_{z+x-1}! \Delta_{x+y-1}!).$$

Proof. Consider the equations depicted in Figure 4. The first line follows from the defining relation in Figure 1 (together with $f^{(z)}e_{z-1} = 0$), the second line follows by iterating the first line (and using the third equality of Figure 2).

$$\begin{aligned} \left[\begin{array}{c} z \\ \text{---} \\ y \end{array} \right] \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} 1 \\ z-1 \\ y+z-2 \end{array} &= - \frac{\Delta_{y+z-3}}{\Delta_{y+z-2}} \left[\begin{array}{c} z \\ \text{---} \\ y \end{array} \right] \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} 1 \\ y+z-3 \end{array} \\ &= \frac{(-1)^{z-1} \Delta_{y-1}}{\Delta_{y+z-2}} \left[\begin{array}{c} z \\ \text{---} \\ y \end{array} \right] \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} z-1 \\ 1 \\ y+z-2 \end{array} \end{aligned}$$

FIGURE 4

Next, the defining relation in Figure 1 followed by a double application of Figure 4 produces the identity of Figure 5.

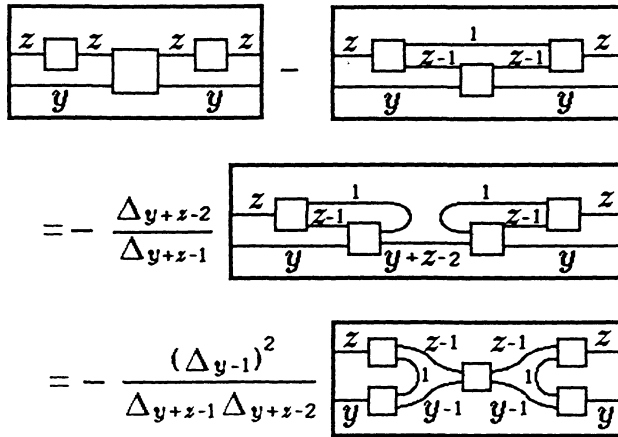


FIGURE 5

Now apply the identity of Figure 5 to the required object in Figure 3 using the second and third formula of Figure 2. The following recurrence relation results:

$$\Gamma(x, y, z) = \Gamma(x, y, z - 1)\Delta_{x+z}/\Delta_{x+z-1} - \Gamma(x + 1, y - 1, z - 1)(\Delta_{y-1})^2/(\Delta_{y+z-1}\Delta_{y+z-2}).$$

This is ready for a verification of the given formula by induction on z . That formula is clearly true when $z = 0$ and inserting it into this recurrence relation reduces the proof to a demonstration of the equality

$$\Delta_{x+y+z}\Delta_{z-1} = \Delta_{x+z}\Delta_{y+z-1} - \Delta_{y-1}\Delta_x.$$

The truth of this can however easily be checked either directly from the formula for Δ_n or using a double induction on

$$\Delta_{x+y} = \Delta_x\Delta_y - \Delta_{x-1}\Delta_{y-1}.$$

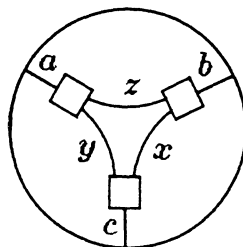


FIGURE 6

Consider the triad $T_{a,b,c}$ of Figure 6, introduced by Kauffman [2]. It is an element of \mathcal{SD} , the linear skein of the disc D , where now D

has an even number $a+b+c$ of specified boundary points partitioned into three subsets as shown. $T_{a,b,c}$ has the idempotents $f^{(a)}$, $f^{(b)}$, and $f^{(c)}$ placed just inside the disc adjacent to the three subsets. It is required that the integers x , y and z defined by $a = y+z$, $b = z+x$ and $c = x+y$ should all be non-negative. Suppose that D' is the disc complementary to D in S^2 with the same specified boundary points. By means of the bilinear form $\mathbb{S}D \times \mathbb{S}D' \rightarrow \mathbb{C}$, $T_{a,b,c}$ corresponds to the element $T_{a,b,c}^*$ of the dual of $\mathbb{S}D'$.

LEMMA 2. *Let A be a primitive $4r$ th root of unity. Then $T_{a,b,c}^*$ is non-zero if and only if $a + b + c \leq 2(r - 2)$.*

Proof. $\mathbb{S}D'$ has a base consisting of all diagrams in D' with no crossing. However, for all but one of these diagrams there is an arc from a point of one of the three specified subsets (for example that with a points) to another point of the same subset. The presence of the idempotent adjacent in $T_{a,b,c}$ to that subset ensures (using $f^{(a)}e_i = 0$) that $T_{a,b,c}^*$ annihilates such an element. There remains to consider the base element of $\mathbb{S}D'$ that consists of z arcs from the first boundary subset to the second such subset, x from the second to the third and y from the third to the first. Of course $T_{a,b,c}^*$ maps this element to $\Gamma(x, y, z)$. It follows from Lemma 1 that, as $x + y + z$ increases, this is non-zero until $\Delta_{x+y+z}! = 0$ and that occurs when $x + y + z = r - 1$.

From now onwards fix A as a primitive $4r$ th root of unity.

DEFINITION. A triple (a, b, c) of non-negative integers will be called admissible if $a+b+c$ is even, $a+b+c \leq 2(r-2)$, $a \leq b+c$, $b \leq c+a$ and $c \leq a+b$. This will be written $(a, b, c) \in \mathbb{A}$.

Note that the admissibility condition is just that a triple of non-negative integers (x, y, z) should exist as above and that $x+y+z \leq r-2$. In such circumstances define $\Theta(a, b, c) = \Gamma(x, y, z)$.

3. Independence in 3-manifolds. Consider now the following situation. Let M_a be a 3-manifold M in S^3 having, as specified framed points in ∂M , a points grouped in a small disc and N other points. These N points do not change in what follows. Let M'_a be the closure of the complement of M_a . Suppose that for each a , $0 \leq a \leq r-2$, $\{X_{i,a} : i \in \mathcal{I}(a)\}$ is a collection of elements of $\mathbb{S}M_a$, $\mathcal{I}(a)$ being some indexing set, and that each $X_{i,a}$ has the idempotent $f^{(a)}$ placed adjacent to the a points. Let $Y_{i,a,b,c}$ be obtained from $X_{i,a}$ by adding

the triad $T_{a,b,c}$ as in Figure 7 (the 2-dimensional projections will be considered only in a neighbourhood of the a points).

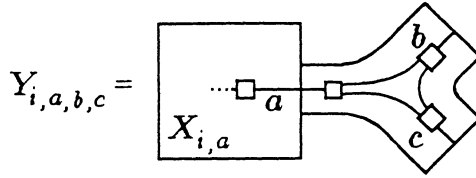


FIGURE 7

LEMMA 3. Suppose that the $\{X_{i,a}^* : i \in \mathcal{I}(a)\}$ are, for each a , independent elements of the dual of $\mathbb{S}M'_a$. Then, for any b and c , $\{Y_{i,a,b,c}^* : (a, b, c) \in \mathbb{A}, i \in \mathcal{I}(a)\}$ are independent.

Proof. Suppose $\sum_{i,a} \lambda_{i,a} Y_{i,a,b,c}^* = 0$. Then $\sum_{i,a} \lambda_{i,a} Z_{i,a,b,c,d}^* = 0$ where $Z_{i,a,b,c,d}$ is as shown in Figure 8. Note that in Figure 8, and in the remainder of the paper, a black dot is used, as in [2], to indicate the presence of a triad of the type of Figure 6.

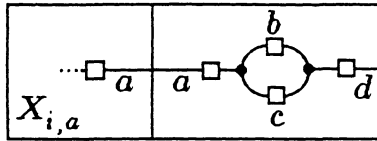


FIGURE 8

However it follows at once (see [2], for example) that

$$Z_{i,a,b,c,d}^* = \delta_{a,d} \Theta(b, c, d) \Delta_d^{-1} X_{i,d}^*.$$

So, letting $d = a$, for each a such that (a, b, c) is admissible,

$$\sum_{i \in \mathcal{I}(a)} \lambda_{i,a} \Theta(a, b, c) \Delta_a^{-1} X_{i,a}^* = 0.$$

But the admissibility means that $\Theta(a, b, c) \neq 0$ (and $\Delta_a \neq 0$) so $\lambda_{i,a} = 0$ for all a and i .

COROLLARY. If triads are added together to form a tree

$$X(a_1, a_2, \dots, a_s; i_1, i_2, \dots, i_{s-3})$$

as in Figure 9, then, fixing a_1, a_2, \dots, a_s and allowing i_1, i_2, \dots, i_{s-3} to vary in all ways so that admissibility occurs at each node of the tree, the elements $\{X(a_1, a_2, \dots, a_s; i_1, i_2, \dots, i_{s-3})^*\}$ are independent.

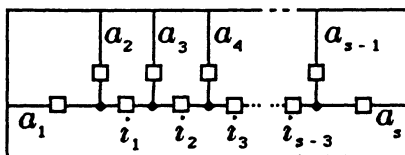


FIGURE 9

Suppose that an even number $a_1 + a_2 + \dots + a_s$ of points are specified on the boundary of a disc D the points grouped into s subsets each containing a_i points. Then, it is easy to see, in the following way, that the duals of *all* diagrams that have, for all i , the idempotent $f^{(a_i)}$ adjacent to the i th subset, span the same subspace as $\{X(a_1, a_2, \dots, a_s; i_1, i_2, \dots, i_{s-3})^*\}$. This is because in the Temperley-Lieb algebra TL_n , $1 = f^{(n)} + R$ where R is some sum of products of the e_i . As $f^{(r-1)^*} = 0$ the dual of any diagram cutting a chord of the disc in more than $r - 2$ points may be replaced by the sum of duals of diagrams cutting the chord fewer times. For $n \leq r - 2$ the dual of a diagram whose arcs cross a given chord n times may be replaced by a sum of duals of diagrams, in one of which the arcs cross the chord n times but are then decorated with an $f^{(n)}$ and in the others the arcs cross fewer than n times. At any stage an inadmissible triad can be neglected as its dual is the zero map. When $s = 4$ this means there are two obvious bases for the same space. The situation is depicted in Figure 10 in which the diagrams represent elements of the dual space to SD' when D has $a + b + c + d$ specified points in ∂D . This is familiar from [2]; the summation is over all i for which (i, b, c) and (i, a, d) are admissible. The elements $\left\{ \begin{smallmatrix} a & b & i \\ c & d & j \end{smallmatrix} \right\}$ of this change of base matrix are referred to as (quantised) $6j$ -symbols.

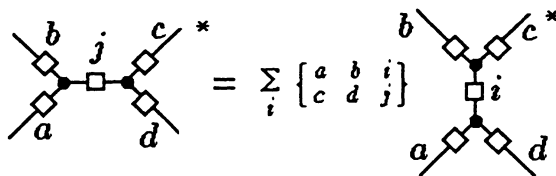


FIGURE 10

There now follows a somewhat technical little lemma that will prove useful.

LEMMA 4. *Suppose that (b, b, a) and (a, c, c) are admissible triples. Then there is some j , $0 \leq j \leq r - 2$, such that the element of SS^2 depicted by Figure 11 is non-zero.*

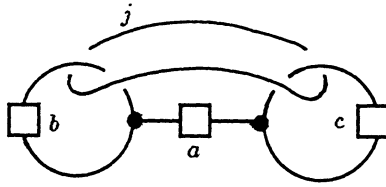


FIGURE 11

Proof. Note that the idempotent $f^{(j)}$ does *not* appear on the j strands. Suppose that, to the contrary, the diagram represents zero for every j . Then, using the above-mentioned base change, the summation shown in Figure 12 is zero.

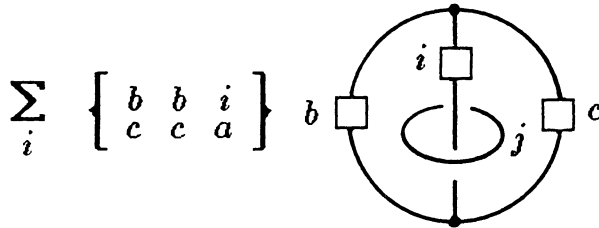


FIGURE 12

However, by Lemma 6 of [7], this means that

$$\sum_i \begin{Bmatrix} b & b & i \\ c & c & a \end{Bmatrix} (\alpha_i)^j \Theta(b, i, c) = 0,$$

where $\alpha_i = -A^{2(i+1)} - A^{-2(i+1)}$. Because A is a primitive $4r$ th root of unity $\alpha_0, \alpha_1, \dots, \alpha_{r-2}$ are all distinct. Thus the Vandermonde matrix $(\alpha_i)^j$ is non-singular, and so there can be no non-trivial linear relation between its rows (compare [7]). Thus

$$\begin{Bmatrix} b & b & i \\ c & c & a \end{Bmatrix} \Theta(b, i, c) = 0$$

for all i for which (b, i, c) is admissible. But then $\Theta(b, i, c) \neq 0$ so that $\begin{Bmatrix} b & b & i \\ c & c & a \end{Bmatrix}$ is zero for all such i . That however means that a whole row of a change of base matrix is zero, a contradiction.

Note that it follows that the lemma is equally true if the idempotent $f^{(j)}$ is now inserted on the j strands.

COROLLARY. Let $U_{a,b,j}$ be the element of SD in Figure 13. Then $U_{a,b,j}^* = \mu_{a,b,j} T_{a,j,j}^*$, and for each a and b with (a, b, b)

admissible, there exists a j , with $0 \leq j \leq r - 2$ and (a, j, j) admissible, such that $\mu_{a,b,j} \neq 0$.

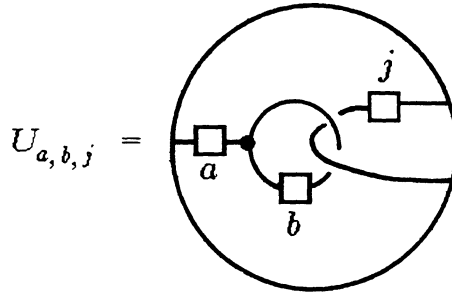


FIGURE 13

Proof. $U_{a,b,j}^*$ and $T_{a,j,j}^*$ both annihilate all but the same single base element of $\mathbb{S}D'$. Thus they differ by multiplication by a scalar $\mu_{a,b,j}$. However by Lemma 4, for each a and b , with (a, b, b) admissible, there is some j for which $U_{a,b,j}^*$ is not the zero map.

Return now to the notation at the beginning of this section in which $\{X_{i,a} : i \in \mathcal{I}(a)\}$ is a collection of elements of $\mathbb{S}M_a$ each having the idempotent $f^{(a)}$ placed adjacent to the set of a points in ∂M . Suppose that $W_{i,a,b}$ is as shown in Figure 14.

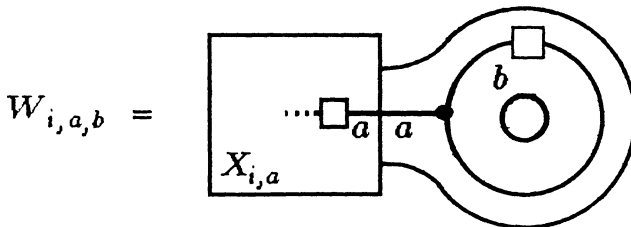


FIGURE 14

Note that $W_{i,a,b}$ is in the linear skein of a new 3-manifold in S^3 obtained by adding a 1-handle to M in a trivial manner.

LEMMA 5. *Suppose that the $\{X_{i,a}^* : i \in \mathcal{I}(a)\}$ are, for each a , independent elements of the dual of $\mathbb{S}M'_a$. Then $\{W_{i,a,b}^* : (a, b, b) \in \mathbb{A}, i \in \mathcal{I}(a)\}$ are independent.*

Proof. Suppose that $\sum_{i,a,b} \lambda_{i,a,b} W_{i,a,b}^* = 0$, the sum being over all (i, a, b) such that $0 \leq a \leq r - 2$, $i \in \mathcal{I}(a)$ and (a, b, b) is

admissible. Then, for all j and k ,

$$\sum_{i,a,b} \lambda_{i,a,b} W_{i,a,b,j,k}^{+*} = 0$$

where $W_{i,a,b,j,k}^+$ is shown in Figure 15.

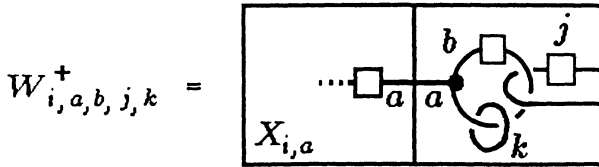


FIGURE 15

Then

$$\sum_{i,a,b} \lambda_{i,a,b} (\alpha_b)^k (X_{i,a} \cup U_{a,b,j})^* = 0$$

where, as before, the $(\alpha_b)^k$ are the elements of a non-singular Vandermonde matrix. Hence, for each b ,

$$\sum_{i,a} \lambda_{i,a,b} (X_{i,a} \cup U_{a,b,j})^* = 0.$$

By the corollary to Lemma 4, for each b

$$\sum_{i,a} \lambda_{i,a,b} (X_{i,a} \cup \mu_{a,b,j} T_{a,j,j})^* = 0.$$

Then, by Lemma 3, if (a, j, j) is admissible $\lambda_{i,a,b} \mu_{a,b,j} = 0$. But the corollary to Lemma 4 states that, for each a and b , such a j can be selected so that $\mu_{a,b,j} \neq 0$. Hence $\lambda_{i,a,b} = 0$ for all (a, b, i) for which $i \in \mathcal{J}(a)$ and (a, b, b) is admissible.

THEOREM. *Suppose that M is a handlebody in S^3 (with no specified points in its boundary), and that M' , the closure of its complement, is also a handlebody. Let A be a primitive 4th root of unity. A base for the quotient of $\mathcal{J}M$ by the kernel of the natural map $\mathcal{J}M \rightarrow \text{Hom}(\mathcal{J}M', \mathbb{C})$ is as follows. Project M to a disc with holes. The duals of all diagrams as shown in Figure 16 that have an admissible triad at each node form the required base.*

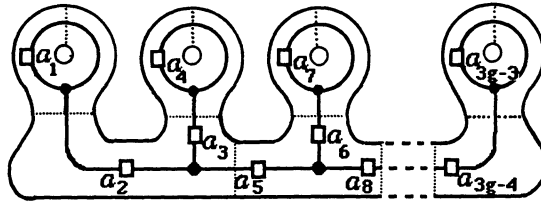


FIGURE 16

Proof. That the given set spans the quotient follows in a way similar to the remarks following Lemma 3: The infinite spanning set consisting of the duals of all diagrams with no crossing and no null-homotopic closed curve can, as there explained, be reduced to a finite set in which no diagram cuts any of the dotted chords of Figure 16 in more than $r - 2$ points, and the Temperley-Lieb idempotents can likewise be inserted where the diagrams cross those chords. The only remaining diagrams with non-zero duals are those consisting of admissible triads between the dotted chords. Independence is an immediate consequence of Lemmas 3 and 5.

Note 1. Other bases can be obtained by modifying the above base at two adjacent nodes of Figure 16, using the $6j$ -symbols as the change of base matrix as in Figure 10.

Note 2. The idempotents can be removed from all the triads of Figure 16 and the result is still a base (the result is clearly still a spanning set and it has the same number of elements as has the base).

Note 3. If M is a ball the dimension of the quotient space is 1. If M is a solid torus the dimension is $r - 1$ and if M has genus 2 the dimension is $(r^3 - r)/6$.

Note 4. If one begins with a closed connected orientable surface and specifies in it a collection of simple closed curves that separate the surface into copies of a disc with two holes ('pairs of pants'), spanning those curves with discs, then adding 3-balls, produces a handlebody bounded by the surface. The discs give a decomposition of the handlebody into triad-like pieces ready for the construction of a base of the relevant (quotient) vector space as above.

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