

SKEW CONNECTIONS IN VECTOR BUNDLES AND THEIR PROLONGATIONS

JURAJ VIRSIK

(Received 5 January 1970)

Communicated by E. Strzelecki

The paper is closely related to [1] and [2]. A skew connection in a vector bundle E as defined here is a pseudo-connection (in the sense of [1]) which can be changed into a connection by transforming separately the bundle E itself and the bundle of its differentials, i.e. one-forms on the base with values in E . The properties of skew connections are thus expected to be only "algebraically" more complicated than those of connections; especially one can follow the pattern of [1], and prolong them to obtain higher order semi-holonomic and non-holonomic pseudo-connections. It is shown in this paper that under some circumstances the main theorem of [1] or [2] applies also to skew connections.

Let M be a fixed (finite-dimensional, C^∞ -differentiable) manifold, E a (finite-dimensional over the reals, C^∞ -differentiable) vector bundle with base M and fibre type R^n . Let the dimension of M be m . We shall always suppose that the structure group of a vector bundle is the maximal linear group (i.e. $GL(n, R)$ in the case of E), and neglect the question of its possible reducibility. Let F be another vector bundle over M , $p : E \rightarrow M$ and $p' : F \rightarrow M$ the corresponding projections. A C^∞ -map $\Phi : E \rightarrow F$ (a diffeomorphism), such that $p'\Phi = p$ and Φ is linear on each fibre, is called a *bundle morphism* (isomorphism). If $T(M)$ and $T(M)^*$ are the tangent and cotangent bundles respectively to M , denote $T^1(E) = E \oplus E \otimes T(M)^*$, and by $S^1(E)$ the vector bundle over M of all one-jets of local sections of E . Denoting by R the trivial bundle $M \times R$, we have clearly $S^1(R) = T^1(R)$. Note that the fibres of both $S^1(E)$ and $T^1(E)$ have the same dimension, and that $E \otimes T(M)^*$ can be regarded as a subbundle of both $T^1(E)$ and $S^1(E)$, identifying it with $\text{Ker } \pi_T$ and $\text{Ker } \pi_S$ respectively, where $\pi_T : T^1(E) \rightarrow E$ and $\pi_S : S^1(E) \rightarrow E$ are the natural bundle projections. In [1] we have defined a *pseudo-connection* in E as a bundle isomorphism $H : S^1(E) \rightarrow T^1(E)$, and we have seen that it corresponds to a usual connection iff $\pi_T H = \pi_S$ and H is the identity on $E \otimes T(M)^*$.

Let $H = H_1 + H_2$ be a pseudo-connection, where $H_1 : S^1(E) \rightarrow E$ and $H_2 : S^1(E) \rightarrow E \otimes T(M)^*$ are its natural components. It is called a *skew connection* iff it preserves the subbundle $E \otimes T(M)^*$, i.e. iff $\pi_S(X) = 0 \Rightarrow H_1(X) = 0$. We have the evident

LEMMA 1. *A pseudo-connection H is a skew connection iff any of the two conditions is satisfied:*

- (A) *There is a bundle morphism $A : E \rightarrow E$ such that $H_1 = A\pi_S$;*
- (B) *There is a bundle morphism $Q : E \otimes T(M)^* \rightarrow E \otimes T(M)^*$ such that $\pi_S(X) = 0 \Rightarrow H(X) = H_2(X) = Q(X)$.*

Note that if such A or Q exists for a pseudo-connection H , then both they exist, are uniquely determined and invertible (i.e. bundle isomorphisms). Call A the *first* and Q the *second tensor* of the skew connection H . A pseudo-connection is thus a connection iff it is a skew connection with trivial (i.e. identity) first and second tensors. A skew connection is called a *relative connection with respect to a bundle isomorphism $A : E \rightarrow E$* (or briefly an *A-connection*) if its first and second tensors are A and $A \otimes id_{(T)M^*}$ respectively.

REMARK. A pseudo-connection is a skew connection, iff its components $\Gamma_{k\beta}^{h\alpha}(h, k = 1, \dots, n; \alpha, \beta = 0, 1, \dots, m)$ in coordinate neighbourhoods (c.f. [1]) satisfy $\Gamma_{k0}^{hi} = 0$ ($h, k = 1, \dots, n; i = 1, \dots, m$). In this case Γ_{k0}^{h0} are the components of the first, and Γ_{kj}^{hi} the components of the second tensors.

Both the groups $\text{Aut } S^1(E)$ or $\text{Aut } T^1(E)$, of all bundle automorphisms of $S^1(E)$ or $T^1(E)$ respectively, act freely and transitively (to the right or left respectively) on the set $PC(E)$ of all pseudo-connections in E . Each element $B \in \text{Aut } T^1(E)$ is uniquely determined by a ‘matrix of tensors’ $(B_{ik})_{i,k=1,2}$, where $B_{11} : E \rightarrow E, B_{12} : E \rightarrow E \otimes T(M)^*, B_{21} : E \otimes T(M)^* \rightarrow E, B_{22} : E \otimes T(M)^* \rightarrow E \otimes T(M)^*$ are bundle morphisms subject only to the condition that the morphism $(X+Y) \mapsto (B_{11}(X)+B_{21}(Y))+(B_{12}(X)+B_{22}(Y))$ of $T^1(E)$ onto itself be invertible.

THEOREM 1. *The subset $SC(E) \subset PC(E)$ of skew connections in E is one of the orbits in $PC(E)$ with respect to the action of the subgroup $\mathcal{B} \subset \text{Aut } T^1(E)$ characterized by the condition $B_{21} = 0$.*

The proof is evident. Note that $B_{21} = 0$ implies the invertibility of both B_{11} and B_{22} .

THEOREM 2. *If H is a skew connection in E , its first and second tensors being A and Q respectively, and $B \in \mathcal{B}$, then the first and second tensors of the skew connection BH are $B_{11}A$ and $B_{22}Q$ respectively.*

The proof is again evident as well as that of the

COROLLARY. *Given any pair $A : E \rightarrow E, Q : E \otimes T(M)^* \rightarrow E \otimes T(M)^*$ of bundle isomorphisms, there is a unique orbit $C_{AQ}(E) \subset PC(E)$, with respect to the action of the subgroup $\mathcal{B}_0 \subset \mathcal{B} \subset \text{Aut } T^1(E)$, consisting of all the skew connections in E admitting A and Q as their first and second tensors. The subgroup \mathcal{B}_0 is characterized by the condition $B_{21} = 0, B_{11} = id_E, B_{22} = id_{E \otimes TT(M)^*}$.*

If H is a skew connection, A and Q its tensors as above, let B be defined by the quadrupole $B_{11} = A^{-1}$, $B_{21} = B_{12} = 0$, $B_{22} = Q^{-1}$. Then $H^0 = BH$ is a connection in E called the *associated with H connection*. Conversely, if H^0 is a connection in E , A, Q arbitrary bundle automorphisms as above, then $H = B^{-1}H^0$, where B^{-1} is the inverse of B as above, is a skew connection admitting A and Q as its first and second tensors respectively. Explicitly

$$H = j_T^1 A \pi_T H^0 + j_T^{1*} Q \pi_T^* H^0,$$

where $T^1(E)$ is represented by the direct sum diagram

$$\begin{array}{ccc} E & \xrightarrow{j_T} & T^1(E) & \xleftarrow{j_T^{1*}} & E \otimes T(M)^* \\ & \xleftarrow{\pi_T} & & \xrightarrow{\pi_T^*} & \end{array}$$

(c.f. [1]). There is hence a natural one-to-one-correspondence between $SC(E)$ and all the triples consisting of connections in E and bundle automorphisms $A : E \rightarrow E, Q : E \otimes T(M)^* \rightarrow E \otimes T(M)^*$.

REMARK. If $B \in \text{Aut } T^1(E)$, $B' \in \mathcal{B}_0 B$ then $B'_{21} = B_{21}, B'_{22} = B_{22}$; if moreover $B_{21} = 0$, then also $B'_{11} = B_{11}$. Thus the tensors B_{21}, B_{22} are invariants of the right cosets with respect to \mathcal{B}_0 ; i.e. given $H \in PC(E)$, the tensors $B_{21} = B_{21}(H)$ and $B_{22} = B_{22}(H)$ corresponding to any automorphism of $T^1(E)$ taking H into a connection are ‘invariants of the pseudo-connection H ’. It is a skew connection iff $B_{21}(H) = 0$; in that case also $B_{11} = B_{11}(H)$ is an ‘invariant’ and evidently $B_{11}(H)^{-1}$ and $B_{22}(H)^{-1}$ coincide with the first and second tensors of the skew connection H .

Let $\Phi : E \rightarrow E$ be a bundle morphism. We have then also bundle morphisms $S^1(\Phi) : S^1(E) \rightarrow S^1(E)$ and $T^1(\Phi) : T^1(E) \rightarrow T^1(E)$ (c.f. [1]); S^1 and T^1 are functors from the category of vector bundles over M into itself. A skew connection H in E is called Φ -invariant if $T^1(\Phi)H = HS^1(\Phi)$. We have again an evident

LEMMA 2. *If $H \in SC(E)$ is Φ -invariant, then so is any skew connection BH , where $B \in \mathcal{B}_0$ and B_{11} commutes with Φ , B_{22} with $\Phi \otimes id_{T(M)^*}$.*

COROLLARY. *A skew connection is Φ -invariant if the associated connection is Φ -invariant and Φ commutes with the first tensor, $\Phi \otimes id_{T(M)^*}$ with the second tensor.*

A skew connection is called *regular*, if it is A -invariant, where A is its first tensor. Thus such $H \in SC(E)$ is regular if its associated connection is A -invariant and $A \otimes id_{T(M)^*}$ commutes with the second tensor; especially an A -connection is regular if its associated connection is A -invariant.

REMARK. The Φ -invariancy of a connection H , i.e. the condition $HS^1(\Phi) = T^1(\Phi)H$, is equivalent with the condition $\nabla_X(\Phi f) = \Phi \nabla_X f$ for any local section X of $T(M)$ and any local section f of E , where $\nabla_X f = \langle X, H_2(j^1 f) \rangle$ is the co-

variant derivative induced by the connection H (c.f. [1], p. 144). In other words H is Φ -invariant iff the absolute differential of Φ is zero. This gives also the local conditions for the regularity of a skew connection in terms of its components Γ_k^h and Γ_{ki}^h as

$$\partial_i \Gamma_k^s + \sum_{h=1}^n (\Gamma_{hi}^s \Gamma_k^h - \Gamma_{ki}^h \Gamma_h^s) = 0$$

for each $i = 1, \dots, m; s, k = 1, \dots, n$.

If E and F are two vector bundles over M , H_E and H_F connections in E and F respectively, then they induce natural connections $H_E(\oplus)H_F$ in $E \oplus F$ and $H_E(\otimes)H_F$ in $E \otimes F$; there is also a connection H_{E^*} in the dual bundle E^* induced by the connection H_E (see e.g. again [1] including the notations). Trying to generalize this to arbitrary skew connections H_E and H_F with the first tensors A_E and A_F , the second tensors Q_E and Q_F respectively, we first pass to the associated connections H_E^0, H_F^0 , form $H_E^0(\oplus)H_F^0$ or $H_E^0(\otimes)H_F^0$ or $H_{E^*}^0$ as above, and introduce $H_E(\oplus)H_F$ or $H_E(\otimes)H_F$ or H_{E^*} as the skew connections with these associated connections and the tensors ‘naturally’ connected with those of H_E and H_F . In the case of the direct sum this means that we put $A_{E \oplus F} = A_E \oplus A_F$, $Q_{E \oplus F} = Q_E \oplus Q_F$ for the tensors of $H_E(\oplus)H_F$, but in the case of the tensor product, to obtain the second tensor reasonably linked with Q_E and Q_F , one has to suppose that $Q_E = P_E \otimes R$, $Q_F = P_F \otimes R$, where $P_E : E \rightarrow E$, $P_F : F \rightarrow F$, $R : T(M)^* \rightarrow T(M)^*$ are some bundle automorphisms. We shall refer to this situation by saying that H_E and H_F are R -linked. Now if the skew connections H_E and H_F are R -linked, we define the tensors of $H_E(\otimes)H_F$ and H_{E^*} by $A_{E \otimes F} = A_E \otimes A_F$, $Q_{E \otimes F} = P_E \otimes P_F \otimes R$ and $A_{E^*} = (A_E)^*$, $Q_{E^*} = (P_E)^* \otimes R$. Note that if H_E is an A_E -connection, H_F an A_F -connection, then they are linked by the identity and $H_E(\otimes)H_F$ is an $(A_E \otimes A_F)$ -connection.

An easy consequence of Lemma 3.1 and 3.2 in [1] is

LEMMA 3. If $\Phi : E \rightarrow E$, $\Psi : F \rightarrow F$ are bundle morphisms, H_E and H_F connections in E and F respectively, then

$$H_E S^1(\Phi)(\otimes)H_F S^1(\Psi) = (H_E(\otimes)H_F)S^1(\Phi \otimes \Psi)$$

and

$$T^1(\Phi)H_E(\otimes)T^1(\Psi)H_F = T^1(\Phi \otimes \Psi)(H_E(\otimes)H_F).$$

LEMMA 4. Let $\Phi : E \rightarrow E$, $\Psi : F \rightarrow F$ be bundle morphisms. Let H_E be a Φ -invariant connection in E , H_F a Ψ -invariant connection in F . Then

- (a) $H_E(\oplus)H_F$ is $(\Phi \oplus \Psi)$ -invariant,
- (b) $H_E(\otimes)H_F$ is $(\Phi \otimes \Psi)$ -invariant,
- (c) H_{E^*} is Φ^* -invariant.

PROOF. (a) If $E \oplus F$ is represented by

$$\begin{array}{ccccc}
 & \xleftarrow{\pi_E} & & \xrightarrow{\pi_F} & \\
 E & & E \oplus F & & F \\
 & \xrightarrow{j_E} & & \xleftarrow{j_F} &
 \end{array}$$

then $H_E(\oplus)H_F = T^1(j_E)H_E S^1(\pi_E) + T^1(j_F)H_F S^1(\pi_F)$ (c.f. (3.23) in [1]) and hence $H_E S^1(\Phi) = T^1(\Phi)H_E, H_F S^1(\Psi) = T^1(\Psi)H_F$ implies

$$\begin{aligned}
 (H_E(\oplus)H_F)S^1(\Phi \oplus \Psi) &= T^1(j_E)H_E S^1(\Phi\pi_E) + T^1(j_F)H_F S^1(\Psi\pi_F) \\
 &= T^1(j_E\Phi)H_E S^1(\pi_E) + T^1(j_F\Psi)H_F S^1(\pi_F) = T^1(\Phi \oplus \Psi)T^1(j_E)H_E S^1(\pi_E) \\
 &\quad + T^1(\Phi \oplus \Psi)T^1(j_F)H_F S^1(\pi_F).
 \end{aligned}$$

(b) follows directly from Lemma 3.3 in [1].

(c) Denoting by $c : E \otimes E^* \rightarrow R$ the natural contraction, we have $c(id_E \otimes \Phi^*) = c(\Phi \otimes id_{E^*})$ and thus applying this, Lemma 3 and (b) to the relation $T^1(c)(H_E(\otimes)H_{E^*}) = S^1(c)$, (c.f. (3.49) in [1]), we get

$$\begin{aligned}
 T^1(c)(H_E(\otimes)[H_{E^*}S^1(\Phi^*)]) &= T^1(c)(H_E(\otimes)H_{E^*})(S^1(id_E) \otimes S^1(\Phi^*)) \\
 &= S^1(c)(S^1(\Phi) \otimes S^1(id_{E^*})) = T^1(c)([H_E S^1(\Phi)](\otimes)H_{E^*}) \\
 &= T^1(c)(T^1(\Phi)H_E](\otimes)H_{E^*}) = T^1(c)T^1(id_E \otimes \Phi^*)(H_E(\otimes)H_{E^*}) \\
 &= T^1(c)(H_E(\otimes)[T^1(\Phi^*)H_{E^*}]).
 \end{aligned}$$

Now according to the uniqueness property in Lemma 3.5 in [1], the proof is completed.

COROLLARY. *Let H_E and H_F be regular skew connections in E and F respectively, their tensors being A_E or $Q_E = P_E \otimes R$, and A_F or $Q_F = P_F \otimes R$. Let A_E commute with P_E and A_F with P_F . Then the skew connections $H_E(\oplus)H_F, H_E(\otimes)H_F$ and H_{E^*} are regular.*

PROOF. It is sufficient to show that $(A_E \oplus A_F) \otimes id_{T(M)^*}$ commutes with $Q_E \oplus Q_F$, and $A_E \otimes A_F \otimes id_{T(M)^*}$ with $P_E \otimes P_F \otimes R$, as well as $(A_E)^* \otimes id_{T(M)^*}$ with $(P_E)^* \otimes R$; but this is obvious from the assumptions.

This corollary is useful for the prolongation procedure of skew connections. First let us recall briefly some basic notions and notations from [1], (c.f. also [2]).

For each integer $q \geq 1$ denote by S^q, \bar{S}^q and \tilde{S}^q the covariant functors from the category of vector bundles over M into itself which are defined by means of the holonomic, semi-holonomic and non-holonomic jet prolongations respectively in the sense of Ch. Ehresmann. We put $E = S^0(E) = \bar{S}^0(E) = \tilde{S}^0(E)$ as well as $E = T^0(E) = \bar{T}^0(E) = \tilde{T}^0(E)$ and define for each $q \geq 1$ recurrently

$$\begin{aligned}
 T^q(E) &= T^{q-1}(E) \oplus E \otimes (\bigcirc^q T(M)^*) \\
 \bar{T}^q(E) &= \bar{T}^{q-1}(E) \oplus E \otimes (\otimes^q T(M)^*) \\
 \tilde{T}^q(E) &= \tilde{T}^{q-1}(E) \oplus \tilde{T}^{q-1}(E) \otimes T(M)^*,
 \end{aligned}
 \tag{1}$$

giving rise to the functors $T^q, \bar{T}^q, \hat{T}^q$ from the category of vector bundles into itself. Let $\pi_S^q : S^q(E) \rightarrow S^{q-1}(E), \pi_S^q : \bar{S}^q(E) \rightarrow \bar{S}^{q-1}(E)$ and $\pi_S = \bar{\pi}_S^q : \bar{S}^q(E) = S^1(\bar{S}^{q-1}(E)) \rightarrow \bar{S}^{q-1}(E)$, or correspondingly $\pi_T^q, \bar{\pi}_T^q$ and $\hat{\pi}_T^q$ (c.f. (1)) be the natural surjections. Let further $i_S^q : S^q(E) \rightarrow \bar{S}^q(E), i_S^q : \bar{S}^q(E) \rightarrow \hat{S}^q(E)$ denote the natural injections as well as i_T^q and \hat{i}_T^q in the other case. It is known (c.f. [1]) that i_S^q can be splitted into injections

$$i_S^q : \bar{S}^q(E) \xrightarrow{i_S^{q'}} S^1(\bar{S}^{q-1}(E)) \xrightarrow{S^1(i_S^{q-1})} S^1(\hat{S}^{q-1}(E)) = \hat{S}^q(E),$$

and analogously

$$i_T^q : \bar{T}^q(E) \xrightarrow{i_T^{q'}} T^1(T^{q-1}(E)) \xrightarrow{T^1(i_T^{q-1})} T^1(\hat{T}^{q-1}(E)) = \hat{T}^q(E).$$

Here the morphism $i_T^{q'}$ is determined by

$$i_T^{q'} : e \otimes \sum_{k=0}^q \omega_1^k \otimes \dots \otimes \omega_k^k \mapsto e \otimes \sum_{k=0}^{q-1} \omega_1^k \otimes \dots \otimes \omega_k^k + e \otimes \sum_{k=0}^{q-1} [\omega_1^{k+1} \otimes \dots \otimes \omega_k^{k+1}] \otimes \omega_{k+1}^{k+1},$$

where $e \in E, \omega_i^k \in T(M)^*$ for $i = 1, \dots, k; k = 0, \dots, q; \omega_0^0 = (1, x) \in \mathbf{R}$ and $x \in M$ is the point ‘over which’ these elements are taken.

One also identifies $E \otimes (\bigcirc^q T(M)^*)$ with both the subbundles $\text{Ker } \pi_S^q \subset S^q(E)$ as well as $\text{Ker } \pi_T^q \subset T^q(E)$, and $E \otimes^q T(M)^*$ with both the subbundles $\text{Ker } \bar{\pi}_S^q \subset \bar{S}^q(E)$ as well as $\text{Ker } \bar{\pi}_T^q \subset \bar{T}^q(E)$.

A holonomic or semi-holonomic or non-holonomic *pseudo-connection of order* $q \geq 1$ in E is a bundle isomorphism $HH^q : S^q(E) \rightarrow T^q(E)$ or $SH^q : \bar{S}^q(E) \rightarrow \bar{T}^q(E)$ or $NH^q : \hat{S}^q(E) \rightarrow \hat{T}^q(E)$ respectively. Given a sequence $\{HH^q\}_{q=1}^\infty$ or $\{SH^q\}_{q=1}^\infty$ or $\{NH^q\}_{q=1}^\infty$ of pseudo-connections in E , then it is called a sequence of holonomic or semi-holonomic or non-holonomic *connections* if for each $q \geq 1, \pi_T^q HH^q = HH^{q-1} \pi_S^q; HH^q|_{E \otimes (\bigcirc^q T(M)^*)} = id$, with $HH^0 = id_E$, or $\bar{\pi}_T^q SH^q = SH^{q-1} \bar{\pi}_S^q; SH^q|_{E \otimes (\bigotimes^q T(M)^*)} = id$, with $SH^0 = id_E$, or $\pi_T NH^q = NH^{q-1} \pi_S; NH^q|_{S^{q-1}(E) \otimes T(M)^*} = NH^{q-1} \otimes id_{T(M)^*}$, with $NH^0 = id_E$.

REMARK. These definitions are in accordance with the definitions of higher order connections in vector bundles in [3], [4] or [5]. On the other hand a higher order connection as introduced by C. Ehresmann corresponds in the case of vector bundles to a surconnection (and not connection) of P. Libermann (c.f. [3]). See also [6] for the relation of these two definitions.

As in [1], we restrict our interest to the semi-holonomic and non-holonomic cases. The following sequences of first order pseudo-connections have been also introduced in [1]:

$$\{\hat{H}_S^q\}, \text{ with } \hat{H}_S^q : S^1(\hat{S}^{q-1}(E)) \rightarrow T^1(\hat{S}^{q-1}(E));$$

$$\{\hat{H}_T^q\}, \text{ with } \hat{H}_T^q : S^1(\hat{T}^{q-1}(E)) \rightarrow T^1(\hat{T}^{q-1}(E));$$

$$\begin{aligned} &\{\tilde{H}_S^q\}, \text{ with } \tilde{H}_S^q : S^1(\tilde{S}^{q-1}(E)) \rightarrow T^1(\tilde{S}^{q-1}(E)); \\ &\{\tilde{H}_T^q\}, \text{ with } \tilde{H}_T^q : S^1(\tilde{T}^{q-1}(E)) \rightarrow T^1(\tilde{T}^{q-1}(E)). \end{aligned}$$

Such a sequence $\{\tilde{H}_S^q\}$ (or $\{\tilde{H}_T^q\}$) is called *reducible* to a sequence $\{\bar{H}_S^q\}$ (or $\{\bar{H}_T^q\}$) if for each $q \geq 1$ the relation $\tilde{H}_S^q S^1(i_S^{q-1}) = T^1(i_S^{q-1})\bar{H}_S^q$ (or $\tilde{H}_T^q S^1(i_T^{q-1}) = T^1(i_T^{q-1})\bar{H}_T^q$) holds. A sequence

$$(a) \{SH^q\}; \quad (b) \{\bar{H}_S^q\}; \quad (c) \{\bar{H}_T^q\}$$

of pseudo-connections is called *regular* if for each $q \geq 1$ the following condition is satisfied:

(a) $\pi_T^q SH^q = SH^{q-1}A^{q-1}\pi_S^q$ for some sequence $\{A^{q-1}\}$ of automorphisms $A^{q-1} : \bar{S}^{q-1}(E) \rightarrow \bar{S}^{q-1}(E)$ or, equivalently, $\pi_S^q (SH^q)^{-1} = (SH^{q-1})^{-1}(B^{q-1})^{-1}\pi_T^q$ for some sequence $\{B^{q-1}\}$ of automorphisms $B^{q-1} : \bar{T}^{q-1}(E) \rightarrow \bar{T}^{q-1}(E)$;

(b) $\pi_T \bar{H}_S^q i_S^{q'} = A^{q-1}\pi_S^q$ and $T^1(A^{q-1}\pi_S^q)\bar{H}_S^{q+1}i_S^{q+1'} = \bar{H}_S^q i_S^{q'} A^q \pi_S^{q+1}$ for some sequence $\{A^{q-1}\}$ of automorphisms as sub (a);

(c) $\pi_S(\bar{H}_T^q)^{-1}i_T^{q'} = (B^{q-1})^{-1}\pi_T^q$ and $S^1((B^{q-1})^{-1}\pi_T^q)(\bar{H}_T^{q+1})^{-1}i_T^{q+1'} = (\bar{H}_T^q)^{-1}i_T^{q'}(B^q)^{-1}\pi_T^{q+1}$ for some sequence $\{B^{q-1}\}$ of automorphisms as sub (a).

The relations

$$(2) \quad NH^q = T^1(NH^{q-1})\tilde{H}_S^q \langle \Rightarrow \rangle \tilde{H}_S^q = T^1(NH^{q-1})^{-1}NH^q$$

and

$$(3) \quad NH^q = \tilde{H}_T^q S^1(NH^{q-1}) \langle \Rightarrow \rangle \tilde{H}_T^q = NH^q S^1(NH^{q-1})^{-1}$$

define a ‘one-to-one-to-one’ correspondence $\{\tilde{H}_S^q\} \sim \{NH^q\} \sim \{\tilde{H}_T^q\}$ between the three sequences dealt with in the non-holonomic case. The main theorem in [1] states that if there is a triple of sequences in such a correspondence, then the following conditions are equivalent:

(I) $\{NH^q\}$ is reducible to a regular sequence $\{SH^q\}$ with the automorphisms $\{A^{q-1}\}$ (or $\{B^{q-1} = SH^{q-1}A^{q-1}(SH^{q-1})^{-1}\}$);

(II) $\{\tilde{H}_S^q\}$ is reducible to a regular sequence $\{\bar{H}_S^q\}$ with the automorphisms $\{A^{q-1}\}$;

(III) $\{\tilde{H}_T^q\}$ is reducible to a regular sequence $\{\bar{H}_T^q\}$ with the automorphisms $\{B^{q-1}\}$.

In particular it has been shown there that if H is a (first order) connection in E , h a (first order) connection in the tangent bundle $T(M)$, then one can get ‘by prolongation’ sequences which satisfy (III) and hence all the above conditions. This can be generalized with some restrictions to the case where H is a skew connection in E , h a skew connection in $T(M)$.

Thus suppose $H \in SC(E)$ with the tensors A and Q , $h \in SC(T(M))$ with the tensors a and q are R -linked skew connections, i.e. $Q = P \otimes R$, $q = p \otimes R$ for

some fixed bundle automorphism $R : T(M)^* \rightarrow T(M)^*$. We have already seen that one can construct then two canonical sequences $\{\bar{H}_T^q\}$ and $\{\tilde{H}_T^q\}$, where each \bar{H}_T^q ($q \geq 1$) is a skew connection in $\bar{T}^{q-1}(E)$ with the first tensor $\bar{A}_T^q = A \otimes \sum_{k=0}^q \otimes^k a^*$, the second tensor $\bar{Q}_T^q = \bar{P}_T^q \otimes R$, with $\bar{P}_T^q = P \otimes \sum_{k=0}^q \otimes^k p^*$, and each \tilde{H}_T^q ($q \geq 1$) is a skew connection in $\tilde{T}^{q-1}(E)$ with the first tensor $\tilde{A}_T^q = A \otimes (\otimes^{q-1}(id_R \otimes a^*))$, the second tensor $\tilde{Q}_T^q = \tilde{P}_T^q \otimes R$, with $\tilde{P}_T^q = P \otimes (\otimes^{q-1}(id_R \otimes p^*))$. Denote by $\{(\bar{H}_T^q)^0\}$ and $\{(\tilde{H}_T^q)^0\}$ the sequences of the corresponding associated connections – they are constructed from the associated to H and h connections H^0 and h^0 respectively as in [1].

LEMMA 5. *The sequence $\{\tilde{H}_T^q\}$ is reducible to the sequence $\{\bar{H}_T^q\}$, i.e. for each $q \geq 1$,*

$$H_T^q S^1(i_T^{q-1}) = T^1(i_T^{q-1})\bar{H}_T^q.$$

PROOF. Such a relation certainly holds for the sequences $\{(\tilde{H}_T^q)^0\}$ and $\{(\bar{H}_T^q)^0\}$ (c.f. [1] or [2]). On the other hand the relation between skew connections and associated connections gives in this case

$$(4) \quad \begin{aligned} \tilde{H}_T^q &= j_T^1 \tilde{A}_T^q \pi_T(\tilde{H}_T^q)^0 + j_T^{1*}(\tilde{P}_T^q \otimes R)\pi_T^*(\tilde{H}_T^q)^0, \\ \bar{H}_T^q &= j_T^1 \bar{A}_T^q \pi_T(\bar{H}_T^q)^0 + j_T^{1*}(\bar{P}_T^q \otimes R)\pi_T^*(\bar{H}_T^q)^0, \end{aligned}$$

and thus by (2.14-15) and (2.67-68) of [1] we get subsequently

$$\begin{aligned} \tilde{H}_T^q S^1(i_T^{q-1}) &= j_T^1 \tilde{A}_T^q \pi_T T^1(i_T^{q-1})(\bar{H}_T^q)^0 + j_T^{1*}(\tilde{P}_T^q \otimes R)\pi_T^*(T^1(i_T^{q-1})(\bar{H}_T^q)^0) \\ &= j_T^1 \tilde{A}_T^q i_T^{q-1} \pi_T(\bar{H}_T^q)^0 + j_T^{1*}(\tilde{P}_T^q \otimes R)(i_T^{q-1} \otimes id_{T(M)^*})\pi_T^*(\bar{H}_T^q)^0 \\ &= j_T^1 i_T^{q-1} \bar{A}_T^q \pi_T(\bar{H}_T^q)^0 + j_T^{1*}(i_T^{q-1} \otimes id_{T(M)^*})(\bar{P}_T^q \otimes R)\pi_T^*(\bar{H}_T^q)^0 \\ &= j_T^1 i_T^{q-1} \pi_T j_T^1 \bar{A}_T^q \pi_T(\bar{H}_T^q)^0 + j_T^{1*}(i_T^{q-1} \otimes id_{T(M)^*})\pi_T^* j_T^{1*}(\bar{P}_T^q \otimes R)\pi_T^*(\bar{H}_T^q)^0 \\ &= T^1(i_T^{q-1})\bar{H}_T^q. \end{aligned}$$

Here we have used the obvious relations

$$A_T^q i_T^{q-1} = i_T^{q-1} \bar{A}_T^q \text{ and } \tilde{P}_T^q i_T^{q-1} = i_T^{q-1} \bar{P}_T^q.$$

THEOREM 3. *Let H be a skew connection in E with the tensors A and $Q = P \otimes R$ which is regular and such that A commutes with P . Let h be a skew connection in $T(M)$ with a trivial first tensor (i.e. $a = id_{T(M)}$) and the second tensor $q = p \otimes R$ (especially let h be a connection in $T(M)$). Then the canonical sequence $\{\tilde{H}_T^q\}$ of skew connections is reducible to the canonical sequence $\{\bar{H}_T^q\}$, which is regular.*

PROOF. According to Lemma 5, all we have to prove is that $\{\bar{H}_T^q\}$ is regular. By the corollary of Lemma 4 we easily conclude, that each skew connection \bar{H}_T^q is regular, i.e. \bar{A}_T^q -invariant, i.e. $T^1(\bar{A}_T^{q+1})\bar{H}_T^{q+1} = \bar{H}_T^{q+1}S^1(\bar{A}_T^{q+1}) \Rightarrow S^1(\bar{\pi}_T^q)S^1(\bar{A}_T^{q+1})^{-1}(\bar{H}_T^{q+1})^{-1}i_T^{q+1} = S^1(\bar{\pi}_T^q)(\bar{H}_T^{q+1})^{-1}T^1(\bar{A}_T^{q+1})^{-1}i_T^{q+1}$. Now we have evidently $S^1(\bar{\pi}_T^q)S^1(\bar{A}_T^{q+1})^{-1} = S^1(\bar{A}_T^q)^{-1}S^1(\bar{\pi}_T^q)$, and from (4) we also derive $T^1(\bar{\pi}_T^q)\bar{H}_T^{q+1} = \bar{H}_T^q S^1(\bar{\pi}_T^q)$, i.e. $S^1(\bar{\pi}_T^q)(\bar{H}_T^{q+1})^{-1} = (\bar{H}_T^q)^{-1}T^1(\bar{\pi}_T^q)$. Finally by

(2.64) of [1] we get $T^1(\bar{\pi}_T^q)T^1(\bar{A}_T^{q+1})^{-1}i_T^{q+1'} = T^1(\bar{A}_T^q)^{-1}i_T^{q'}\bar{\pi}_T^{q+1}$ and this completes the proof, since we have

$$(5) \quad T^1(\bar{A}_T^q)^{-1}i_T^{q'} = i_T^{q'}(\bar{A}_T^{q+1})^{-1}$$

because of $a = id_{T(M)}$.

The results just obtained can be summarized in the following way: If the assumptions of Theorem 3 are satisfied – especially if H is a regular relative connection in E and h a connection in $T(M)$ – then the prolongation procedure described in [1] and [2] ‘works’ in essentially the same manner as for connections. That is, we get a canonical sequence $\{NH^q\}$ of non-holonomic pseudo-connections in E reducible to a regular sequence $\{SH^q\}$ of semi-holonomic pseudo-connections in E , and they are uniquely connected also with a sequence $\{\tilde{H}_S^q\}$ of first order pseudo-connections in the nonholonomic jet prolongations of E reducible to a regular sequence $\{\bar{H}_S^q\}$ of pseudo-connections in the semi-holonomic jet prolongations of E . Since $\{SH^q\}$ is regular, $\bar{\pi}_S^q(X) = 0 \Rightarrow \bar{\pi}_T^q SH^q(X) = 0$, and we have also

THEOREM 4. *Under the assumptions of Theorem 3, all the \tilde{H}_S^q and \bar{H}_S^q are skew connections.*

PROOF. By (4.8-9) of [1], $\tilde{H}_S^q = T^1(NH^{q-1})^{-1}NH^q = T_1(NH^{q-1})^{-1}\tilde{H}_T^q S^1(NH^{q-1})$, i.e. $\pi_T \tilde{H}_S^q = (NH^{q-1})^{-1}\tilde{A}_T^q NH^{q-1} \pi_S$, which proves that \tilde{H}_S^q is a skew connection. Similarly $\pi_S(X) = 0 \Rightarrow \pi_S S^1(SH^{q-1})(X) = SH^{q-1} \pi_S(X) = 0$ and thus also $\pi_T \bar{H}_T^q S^1(SH^{q-1}) = 0$ which means by (4.44) of [1] that $SH^{q-1} \pi_T \bar{H}_S^q = 0$, i.e. \bar{H}_S^q is a skew connection.

One can define, in an evident manner, the functors $\bar{T}^q, \tilde{T}^q, \bar{S}^q, \tilde{S}^q$ from the category of vector bundles over M into itself. Note that for $A : E \rightarrow E$ we have by our notations now $\bar{T}^q(A) = \bar{A}_T^{q+1}, \tilde{T}^q(A) = \tilde{A}_T^{q+1}$, and $\tilde{S}^q(A) = S^1(\tilde{S}^{q-1}(A))$ recurrently also satisfies

$$(6) \quad i_S^q \tilde{S}^q(A) = \tilde{S}^q(A) i_S^q.$$

THEOREM 5. *If H is a regular A -connection in E and h a connection in $T(M)$ then the canonical prolongations are such that each \tilde{H}_T^q is a $\tilde{T}^{q-1}(A)$ -connection, each \bar{H}_T^q is a $\bar{T}^{q-1}(A)$ -connection, each \tilde{H}_S^q is a $\tilde{S}^{q-1}(A)$ -connection, and each \bar{H}_S^q is a $\bar{S}^{q-1}(A)$ -connection.*

PROOF. The statement is evident for the \tilde{H}_T^q and \bar{H}_T^q from their construction. We shall first show that for $q \geq 1$

$$(7) \quad NH^q \tilde{S}^q(A) = \tilde{T}^q(A) NH^q.$$

This being evident for $q = 1$, we proceed by induction using (2) and get $NH^q \tilde{S}^q(A) = \tilde{H}_T^q S^1(NH^{q-1} \tilde{S}^{q-1}(A)) = \tilde{H}_T^q S^1(\tilde{T}^{q-1}(A)) S^1(NH^{q-1}) = \tilde{T}^q(A) \tilde{H}_T^q S^1(NH^{q-1}) = \tilde{T}^q(A) NH^q$, because by the Corollary of Lemma 4 the skew connection \tilde{H}_T^q is regular. Using this relation we have as in the proof of the preceding theorem

$\pi_T \tilde{H}_S^q = (NH^{q-1})^{-1} \tilde{T}^{q-1}(A) NH^{q-1} \pi_S = \tilde{S}^{q-1}(A) \pi_S$. Also if $X \in \text{Ker } \pi_S = \tilde{T}^{q-1}(E) \otimes T(M)^* \subset S^1(\tilde{T}^{q-1}(E))$ then $\tilde{H}_T^q(X) = (T^{q-1}(A) \otimes id_{T(M)^*})(X)$ and thus by (2), $\tilde{T}^{q-1}(A) \otimes id_{T(M)^*} = NH^q|_{\text{Ker } \pi_S \subset S^1(\tilde{S}^{q-1}(E))} [(NH^{q-1})^{-1} \otimes id_{T(M)^*}]$, i.e. by (7), $NH^q|_{\text{Ker } \pi_S} = NH^{q-1} \tilde{S}^{q-1}(A) \otimes id_{T(M)^*}$. But then for $X \in \text{Ker } \pi_S \subset S^1(\tilde{S}^{q-1}(E))$ we have again by (2), $\tilde{H}_S^q(X) = T^1(NH^{q-1})^{-1} NH^q(X) = [(NH^{q-1})^{-1} \otimes id_{T(M)^*}] [NH^{q-1} \tilde{S}^{q-1}(A) \otimes id_{T(M)^*}](X)$, from where we conclude that \tilde{H}_S^q is a $\tilde{S}^{q-1}(A)$ -connection. As for the \bar{H}_S^q , consider (6) together with the reducibility condition of $\{\tilde{H}_S^q\}$ to $\{\bar{H}_S^q\}$. From the just proved result about \tilde{H}_S^q we get $i_S^{q-1} \pi_T \bar{H}_S^q = \pi_T T^1(i_S^{q-1}) \bar{H}_S^q = \tilde{S}^{q-1}(A) \pi_S S^1(i_S^{q-1}) = \tilde{S}^{q-1}(A) i_S^{q-1} \pi_S = i_S^{q-1} \tilde{S}^{q-1}(A) \pi_S$, and hence $\pi_T \bar{H}_S^q = \tilde{S}^{q-1}(A) \pi_S$, because i_S^{q-1} is injective. Also if $X \in \text{Ker } \pi_S \subset S^1(\tilde{S}^{q-1}(E))$, then $S^1(i_S^{q-1})(X) \in \text{Ker } \pi_S \subset S^1(\tilde{S}^{q-1}(E))$ and thus by the already proved result about \tilde{H}_S^q we have $T^1(i_S^{q-1}) \bar{H}_S^q(X) = \tilde{H}_S^q S^1(i_S^{q-1})(X) = (\tilde{S}^{q-1}(A) \otimes id_{T(M)^*}) S^1(i_S^{q-1})(X) = (\tilde{S}^{q-1}(A) i_S^{q-1} \otimes id_{T(M)^*})(X) = (i_S^{q-1} \otimes id_{T(M)^*})(\tilde{S}^{q-1}(A) \otimes id_{T(M)^*})(X) = T^1(i_S^{q-1})(\tilde{S}^{q-1}(A) \otimes id_{T(M)^*})(X)$, and this proves the last relation because of the injectivity of $T^1(i_S^{q-1})$.

THEOREM 6. *Under the assumptions of Theorem 5, all the relative connections $\tilde{H}_T^q, \bar{H}_T^q, \tilde{H}_S^q, \bar{H}_S^q$ are regular.*

PROOF. It is again evident from the Corollary of Lemma 4 that this is true for \tilde{H}_T^q and \bar{H}_T^q . Thus we have only to prove $T^1(\tilde{S}^{q-1}(A)) \tilde{H}_S^q = \tilde{H}_S^q \tilde{S}^q(A)$, and $T^1(\bar{S}^{q-1}(A)) \bar{H}_S^q = \bar{H}_S^q S^1(\bar{S}^{q-1}(A))$. The first relation follows by (2) and (3) from (7) as $\tilde{H}_S^q \tilde{S}^q(A) = T^1(NH^{q-1})^{-1} NH^q \tilde{S}^q(A) = T^1((NH^{q-1})^{-1} \tilde{T}^{q-1}(A)) NH^q = T^1(\tilde{S}^{q-1}(A)) T^1(NH^{q-1})^{-1} NH^q = T^1(\tilde{S}^{q-1}(A)) \tilde{H}_S^q$. The second relation is obtained from this, the reducibility of $\{\tilde{H}_S^q\}$ to $\{\bar{H}_S^q\}$ and (6) as $T^1(i_S^{q-1}) \bar{H}_S^q S^1(\bar{S}^{q-1}(A)) = \tilde{H}_S^q S^1(i_S^{q-1} \bar{S}^{q-1}(A)) = \tilde{H}_S^q S^1(\tilde{S}^{q-1}(A)) S^1(i_S^{q-1}) = T^1(\tilde{S}^{q-1}(A)) \tilde{H}_S^q S^1(i_S^{q-1}) = T^1(\tilde{S}^{q-1}(A)) T^1(i_S^{q-1}) \bar{H}_S^q = T^1(i_S^{q-1}) T^1(\bar{S}^{q-1}(A)) \bar{H}_S^q$, Q.E.D., since $T^1(i_S^{q-1})$ is injective.

REMARK. Restricting ourselves to the most important case of a skew connection, namely to that of a relative connection, we have seen here that ‘the prolongation procedure works’ only if the initial (regular) relative connection in E is ‘pushed’ by a (strict) connection in $T(M)$. This is due to our definition of the regularity of the sequence $\{\tilde{H}_T^q\}$. If H and h were both arbitrary regular relative connections, we would still get the prolonged sequence $\{\tilde{H}_T^q\}$ reducible to $\{\bar{H}_T^q\}$, however not necessarily regular, the obstacle being essentially only with the relation (5). It seems likely that one could generalize the notion of a relative connection (most probably by developing the formalism in the category of ‘all’ vector bundles rather than only of those over a fixed M), and get a deeper condition for the ‘initial’ correlation (of the relative connection in $T(M)$ to the relative connection in E) in order to ‘let the prolongation procedure work’.

References

- [1] J. Virsik, Non-holonomic connections on vector bundles, *Czech. Math. J.* 17 (1967), I: 108—147; II: 200—224.
- [2] J. Virsik, Über Zusammenhänge höherer Ordnungen in Vektorraumbündeln, *Math. Nachrichten* 34 (1967), 219—227.
- [3] E. Feldmann, The geometry of immersions, *Bull. Amer. Math. Soc.* 69 (1963), 693—698.
- [4] P. Libermann, Sur la géométrie des prolongements des espaces fibrés vectoriels, *Ann. Inst. Fourier, Grenoble* 14 (1964), 145—172.
- [5] W. Pohl, Connections in differential geometry of higher order, *Transactions Amer. Math. Soc.* 125 (1966), 310—325.
- [6] N. Que, Du prolongement des espaces fibrés et des structures infinitésimales, *Ann. Inst. Fourier, Grenoble* 17 (1967), 157—223.

Monash University