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SKEW DERIVATIONS AND $U_q(\mathrm{sl}(2))$†

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ABSTRACT
This note first describes the basic properties of the skew derivations on the polynomial ring $k[X]$. As a consequence of these properties it is proved that the q-analogue of the enveloping algebra of sl(2), $U_q(\mathrm{sl}(2))$, has a unique action on $\mathbf{C}[X]$, where "action" means that $\mathbf{C}[X]$ is a module algebra in the Hopf algebra sense. This depends on the fact that the generators of a subalgebra of $U_q(\mathrm{sl}(2))$ described by Woronowicz must act via an automorphism, and the skew derivations associated to it.

1. Skew derivations

Let A be an algebra over a field k, and fix $\sigma \in \mathrm{Aut}_k A$. A *skew derivation* [O] of A is a k-linear map $\delta : A \to A$ such that

$$\delta(ab) = \delta(a)b + \sigma(a)\delta(b) \qquad \text{for all } a, b \in A.$$

Since the definition depends on σ, we call δ a *σ-derivation*. The set of all σ-derivations is denoted by $\mathrm{Der}_\sigma A$. Note that $\sigma - 1 \in \mathrm{Der}_\sigma A$, and if $\delta \in \mathrm{Der}_\sigma A$, then $\sigma\delta\sigma^{-1} \in \mathrm{Der}_\sigma A$.

Suppose that A is commutative. Then $\mathrm{Der}_\sigma A$ is a left A-module, where A acts by left multiplication. The power rule for σ-derivations becomes

$$\delta(a^n) = (a^{n-1} + a^{n-2}\sigma(a) + \cdots + \sigma(a)^{n-1})\delta(a)$$

$$= \frac{(a^n - \sigma(a^n))}{a - \sigma(a)}\,\delta(a) \qquad \text{if } a \neq \sigma(a).$$

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More generally, if f is a function of a, the rule for differentiating a composition of functions becomes (when $a \neq \sigma(a)$)

$$\delta(f) = (f - \sigma(f))(a - \sigma(a))^{-1}\delta(a).$$

If $A = k[a_1, \ldots, a_n]$ is any finitely generated k-algebra, then a σ-derivation is completely determined by $\delta(a_1), \ldots, \delta(a_n)$. For the free algebra $F = k\langle X_1, \ldots, X_n\rangle$, the $\delta(X_j)$ may be chosen arbitrarily. Let I be an ideal of F. If $\sigma(I) \subset I$, and $\delta(I) \subset I$, then σ induces an automorphism of F/I, and δ induces a σ-derivation on F/I. To check that $\sigma(I) \subset I$, and $\delta(I) \subset I$, it is enough to check that $\sigma(r_\lambda) \in I$, and $\delta(r_\lambda) \in I$ for generators r_λ of the ideal I.

We now examine the polynomial ring $k[X]$. Let ∂ be the σ-derivation given by $\partial(X) = 1$. Since $\delta \in \mathrm{Der}_\sigma k[X]$ is determined by $\delta(X)$, if $\delta(X) = p$, then $\delta = p\partial$. Thus $\mathrm{Der}_\sigma k[X]$ is the free $k[X]$-module with basis ∂. What are the eigenvalues of the action of σ on $\mathrm{Der}_\sigma k[X]$ given by $\delta \mapsto \sigma\delta\sigma^{-1}$?

Lemma 1.1. *Let $\sigma \in \mathrm{Aut}_k k[X]$ with $\sigma(X) = \alpha X + \beta$ where $\sigma, \beta \in k$. Consider the eigenvalues for the action of σ on $\mathrm{Der}_\sigma k[X]$.*

(a) *The only possible eigenvalues are α^{n-1}, $n = 0, 1, 2, \ldots$.*

(b) *Suppose that α is not a root of 1. Then the eigenvectors are $(X + \beta(\alpha - 1)^{-1})^n\partial$ with corresponding eigenvalues α^{n-1}, $n = 0, 1, 2, \ldots$.*

Proof. Let $\delta = p\partial$. We view $\sigma, p, \delta \in \mathrm{End}_k k[X]$ with p acting on $k[X]$ by left multiplication. Thus $\sigma\delta\sigma^{-1} = \sigma p\sigma^{-1}\sigma\partial\sigma^{-1}$. Now $\sigma\partial\sigma^{-1}(X) = \alpha^{-1}$, so $\sigma\partial\sigma^{-1} = \alpha^{-1}\partial$; and $\sigma p\sigma^{-1} = \sigma(p)$ because $\sigma p\sigma^{-1}(f) = \sigma(p \cdot \sigma^{-1}(f)) = \sigma(p)\cdot f$.

Thus $\sigma\delta\sigma^{-1} = \alpha^{-1}\sigma(p)\partial$, and δ is an eigenvector, with $\sigma\delta\sigma^{-1} = \mu\delta$ if and only if $\sigma(p) = p(\alpha X + \beta) = \alpha\mu p(X)$. Thus we must find eigenvectors for the action of σ on $k[X]$.

If $\alpha = 1$ the result is trivial. If $\alpha \neq 1$, we may set $Y = X + \beta(\alpha - 1)^{-1}$. Since $\sigma(Y^n) = \alpha^n Y^n$, the eigenvectors for the σ action on $\mathrm{Der}_\sigma k[X]$ are the $Y^n\partial$ having eigenvalue α^{n-1} ∎

Corollary 1.2. *Let $\sigma(X) = \alpha X + \beta$ where α is not a root of 1. If $\delta_1, \delta_2 \in \mathrm{Der}_\sigma k[X]$ satisfy $\sigma\delta_1\sigma^{-1} = \mu\delta_1$ and $\sigma\delta_2\sigma^{-1} = \mu^{-1}\delta_2$ for some $1 \neq \mu \in k$, then (up to scalar multiples) the only possibilities are*

$$\delta_1 = \partial, \quad \delta_2 = (X + \beta(\alpha - 1)^{-1})^2\partial \quad \textit{and} \quad \mu = \alpha^{-1}$$

or

$$\delta_1 = (X + \beta(\alpha - 1)^{-1})^2\partial, \quad \delta_2 = \partial \quad \textit{and} \quad \mu = \alpha.$$

2. Some Hopf algebras involving skew derivations

This section gives two examples of non-commutative, and non-cocommutative Hopf algebras, both involving skew derivations. Recall that, if H is a Hopf algebra with co-multiplication $\Delta : H \to H \otimes H$, given by $\Delta(h) = \Sigma_{(h)} h_{(1)} \otimes h_{(2)}$, then an algebra A is an *H-module algebra* if A is an H-module, and H "measures" A; that is, $h \cdot 1 = \varepsilon(h)1$ and

$$h \cdot (ab) = \sum_{(h)} (h_{(1)} \cdot a)(h_{(2)} \cdot b) \qquad \text{for all } a, b \in A.$$

An element $\delta \in H$ is called *σ-primitive* if $\Delta(\delta) = \delta \otimes 1 + \sigma \otimes \delta$ for some $0 \neq \sigma \in H$. The coassociativity of H forces $\Delta(\sigma) = \sigma \otimes \sigma$; that is, σ is group-like. The properties of the antipode s of H imply that $s(\sigma) = \sigma^{-1} \in H$ and that $s(\delta) = -\sigma^{-1}\delta$. Hence if A is a H-module algebra, then σ acts on A as an automorphism, and δ acts as a σ-derivation.

Example 2.1. Fix $0 \neq \alpha \in k$, and define $H = k[Y] * \langle \sigma \rangle$ to be the skew group ring of the group $\langle \sigma \rangle \cong \mathbf{Z}$, over the polynomial ring $k[Y]$ where the action is $\sigma(Y) = \alpha Y$. Thus in H, $\sigma Y = \alpha Y \sigma$. Make H into a Hopf algebra by defining

$$\Delta(\sigma) = \sigma \otimes \sigma, \quad \Delta(Y) = Y \otimes 1 + \sigma \otimes Y, \quad \varepsilon(\sigma) = 1,$$

$$\varepsilon(Y) = 0, \quad s(\sigma) = \sigma^{-1}, \quad s(Y) = -\sigma^{-1}Y.$$

Thus H is neither commutative, nor co-commutative.

The commutative polynomial ring $A = k[X]$ may be made into an H-module algebra with σ acting as the automorphism $\sigma(X) = \alpha^{-1}X$ and Y acting as the σ-derivation $\partial \in \mathrm{Der}_\sigma k[X]$, i.e. $Y(X) = 1$. Thus H is isomorphic to the subalgebra $k[\sigma, \sigma^{-1}, \partial]$ of $\mathrm{End}_k k[X]$.

As a Hopf algebra, H is similar in spirit to [S, p. 89] and to [T]. However, those examples were not represented as skew derivations. The connection between skew derivations and Hopf algebras was pointed out to one of us by Kharchenko [K]; he used the tensor algebra on the vector space generated by *all* skew derivations of an arbitrary algebra A to construct a Hopf algebra. Our H is the "smallest" non-cocommutative subalgebra of his construction. We thank M. Artin for suggesting we look at the special case $A = k[X]$.

Example 2.2. This example reappears in Section 4 in connection with $U_q(\mathrm{sl}(2))$. Fix $0 \neq \gamma \in k$, and define $A = k\langle x, y \rangle / \langle xy - \gamma yx \rangle$. Define $\sigma \in$

$\mathrm{Aut}\,k\langle x, y\rangle$ by $\sigma(x) = \gamma x$ and $\sigma(y) = \gamma^{-1}y$. Consider the σ-derivations ∂_1, ∂_2 on $k\langle x, y\rangle$ defined by

$$\partial_1(x) = 0, \quad \partial_1(y) = x \quad \text{and} \quad \partial_2(x) = y, \quad \partial_2(y) = 0.$$

Since $\langle xy - \gamma yx\rangle$ is stable under σ, and $\partial_i(xy - \gamma yx) = 0$, we may view $\sigma \in \mathrm{Aut}_k A$ and $\partial_1, \partial_2 \in \mathrm{Der}_\sigma A$. Let $H = k[\partial_1, \partial_2, \sigma, \sigma^{-1}]$ be the subalgebra of $\mathrm{End}_k A$ generated by these elements.

In H the following relations hold:

$$\sigma\partial_1 = \gamma^2\partial_1\sigma, \qquad \sigma\partial_2 = \gamma^{-2}\partial_2\sigma \tag{2.3}$$

$$\partial_1\partial_2 - \gamma^{-2}\partial_2\partial_1 = (\gamma^2 - 1)^{-1}(\sigma^2 - 1). \tag{2.4}$$

Notice that (2.4) says $\partial_1\partial_2 - \gamma^{-2}\partial_2\partial_1$ is a σ^2-derivation of A. It is not difficult to show that H is defined by precisely these relations: first use the Diamond Lemma [B] to show that the algebra defined by the relations (2.3) and (2.4) has a basis $\sigma^k\partial_1^i\partial_2^j$, then check that these elements acting on A are linearly independent.

Make H into a Hopf algebra by defining

$$\Delta(\sigma) = \sigma \otimes \sigma, \quad \Delta(\partial_i) = \partial_i \otimes 1 + \sigma \otimes \partial_i, \quad \varepsilon(\sigma) = 1, \quad \varepsilon(\partial_i) = 0,$$

$$s(\sigma) = \sigma^{-1}, \quad s(\partial_1) = -\sigma^{-1}\partial_1, \quad s(\partial_2) = -\sigma^{-1}\partial_2.$$

This algebra H, which is neither commutative nor co-commutative, first appeared in [W], and is isomorphic to a subalgebra of $U_q(\mathrm{sl}(2))$; see (3.1) and (3.4).

3. $U_q(\mathrm{sl}(2))$ and its action on $\mathbf{C}[X]$

This section concerns the action of $U_q(\mathrm{sl}(2))$, the q-analogue of the enveloping algebra of sl(2), on $\mathbf{C}[X]$. *Fix $0 \neq q \in \mathbf{C}$, not a root of unity*. As defined by Jimbo [J] and Drinfeld [D], $U_q(\mathrm{sl}(2)) = \mathbf{C}[E, F, K, K^{-1}]$ is defined by relations

$$KE = q^2EK, \quad KF = q^{-2}FK, \quad EF - FE = (K^2 - K^{-2})/(q^2 - q^{-2}).$$

Make $U_q(\mathrm{sl}(2))$ a Hopf algebra by defining

$$\begin{array}{lll} \Delta(E) = E \otimes K^{-1} + K \otimes E, & \Delta(F) = F \otimes K^{-1} + K \otimes F, & \Delta(K) = K \otimes K, \\ s(E) = -q^{-2}E, & s(F) = -q^2F, & s(K) = K^{-1}, \\ \varepsilon(E) = 0, & \varepsilon(F) = 0, & \varepsilon(K) = 1. \end{array}$$

Independently of Drinfeld and Jimbo, Woronowicz [W, Table 7, p. 150] defined an algebra which, in retrospect, is isomorphic to a subalgebra of $U_q(\mathrm{sl}(2))$. We will denote this algebra (which depends on a parameter $0 \neq \nu \in \mathbf{C}$) by $W_\nu = \mathbf{C}[\nabla_0, \nabla_1, \nabla_2]$, with the relations

$$\nu \nabla_2 \nabla_0 - \nu^{-1} \nabla_0 \nabla_2 = \nabla_1,$$

$$\nu^2 \nabla_1 \nabla_0 - \nu^{-2} \nabla_0 \nabla_1 = (1 + \nu^2) \nabla_0,$$

$$\nu^2 \nabla_2 \nabla_1 - \nu^{-2} \nabla_1 \nabla_2 = (1 + \nu^2) \nabla_2.$$

Lemma 3.1. *Suppose that $\nu = q^2$. Then there is an injective algebra homomorphism $W_\nu \to U_q(\mathrm{sl}(2))$ defined by*

$$\nabla_0 \mapsto -qFK,$$

$$\nabla_1 \mapsto qEK,$$

$$\nabla_2 \mapsto (K^4 - 1)/(q^{-4} - 1).$$

Proof. First consider the subalgebra $\mathbf{C}[EK, FK, K^4, K^{-4}]$ of $U_q(\mathrm{sl}(2))$. The defining relations are

$$K^4(EK) = q^8(EK)K^4, \qquad K^4(FK) = q^{-8}(FK)K^4, \tag{3.2}$$

$$(EK)(FK) - q^{-4}(FK)(EK) = (K^4 - 1)/(q^4 - 1). \tag{3.3}$$

Consequently, the proposed images of $\nabla_0, \nabla_1, \nabla_2$ satisfy the defining relations of W_ν. Hence the proposed algebra homomorphism exists. It follows from the Diamond Lemma [B] that W_ν has a basis $\nabla_2^i \nabla_0^j \nabla_1^k$ with $i, j, k \in \mathbf{N}$, and that $U_q(\mathrm{sl}(2))$ has a basis $E^i F^j K^k$ $(i, j \in \mathbf{N}, k \in \mathbf{Z})$. The injectivity of the given map follows. ∎

Thus we may identify W_ν with its image in $U_q(\mathrm{sl}(2))$. We shall consider the slightly larger algebra $W_q := \mathbf{C}[EK, FK, K^2, K^{-2}]$. Notice that W_q is a sub-Hopf algebra of $U_q(\mathrm{sl}(2))$; the K^2 term is required by consideration of $\Delta(EK)$, and the K^{-2} term is required by consideration of $s(K^2)$. Although $W_\nu \subset W_q$, W_ν is not a Hopf subalgebra; this is the reason we prefer to focus on W_q.

Theorem 3.4. *Suppose that $\mathbf{C}[X]$ is a $W_q(\mathrm{sl}(2))$-module algebra with K^2 acting as the automorphism $\sigma(X) = \alpha X + \beta$. Set $Y = X + \beta(\alpha - 1)^{-1}$. Then (up to an automorphism of W_q) there are two possibilities:*

(1) *$\alpha = q^{-4}$ and $EK \mapsto \partial$, $FK \mapsto -q^{-4} Y^2 \partial$,*

(2) *$\alpha = q^4$ and $EK \mapsto -q^4 Y^2 \partial$, $FK \mapsto \partial$,*

where ∂ is the σ-derivation $\partial(Y) = 1$. Furthermore, there is no loss of generality in assuming that $\beta = 0$.

PROOF. Notice that K^2 is group-like, and EK and FK are K^2-primitive. Therefore K^2 must act as an automorphism, and EK and FK act as skew derivations with respect to this automorphism. Write $\sigma, \delta_1, \delta_2$ for the images of K^2, EK, FK in $\mathrm{End}_{\mathbf{C}}\mathbf{C}[X]$. After (3.2) and (3.3) the following relations hold:

$$\sigma\delta_1\sigma^{-1} = q^4\delta_1, \qquad \sigma\delta_2\sigma^{-1} = q^{-4}\delta_2, \tag{3.5}$$

$$\delta_1\delta_2 - q^{-4}\delta_2\delta_1 = (\sigma^2 - 1)/(q^4 - 1). \tag{3.6}$$

Since q^4 is an eigenvalue for the σ action on $\mathrm{Der}_\sigma\mathbf{C}[X]$, it follows from (1.1) that q^4 is a power of α, and hence α is not a root of unity. Set $Y = X + \beta(\alpha - 1)^{-1}$. As in (1.1), Y is a σ-eigenvector, and replacing X by Y, we may take $\beta = 0$.

By (1.2) either

(1) $\delta_1 = \gamma_1\partial$, $\delta_2 = \gamma_2 Y^2\partial$ and $\alpha = q^{-4}$

or

(2) $\delta_1 = \gamma_1 Y^2\partial$, $\delta_2 = \gamma_2\partial$ and $\alpha = q^4$,

where γ_1 and γ_2 are scalars to be determined by the requirement that (3.6) holds. To determine $\gamma := \gamma_1\gamma_2$ we compute the action of the expressions in (3.6) on Y^n. In case (1)

$$\gamma(\partial Y^2\partial - \alpha Y^2\partial^2) : Y^n \mapsto \gamma(1 - \alpha^n)(1 + \alpha^n)(1 - \alpha)^{-1}Y^n,$$

$$(\sigma^2 - 1)/(q^4 - 1) : Y^n \mapsto \alpha(\alpha^{2n} - 1)(1 - \alpha)^{-1}Y^n.$$

Therefore $\gamma = -\alpha$. In case (2)

$$\gamma(Y^2\partial^2 - \alpha^{-1}\partial Y^2\partial) : Y^n \mapsto \gamma(1 - \alpha^n)(1 + \alpha^n)\alpha^{-1}(\alpha - 1)^{-1}Y^n,$$

$$(\sigma^2 - 1)/(q^4 - 1) : Y^n \mapsto (\alpha^{2n} - 1)(\alpha - 1)^{-1}Y^n.$$

Therefore $\gamma = -\alpha$.

The map $EK \mapsto \gamma_1 EK$, $FK \mapsto \gamma_1^{-1}FK$, $K^2 \mapsto K^2$ is an automorphism of W_q. Thus, up to an automorphism of W_q, we may assume that $\gamma_1 = 1$, and so $\gamma_2 = \gamma = -\alpha$, or $\gamma_2 = 1$ and $\gamma_1 = -\alpha$. ∎

COROLLARY 3.7. *Suppose that* $\mathbf{C}[X]$ *is a* $U_q((2))$*-module algebra. Then (up to isomorphism of module algebras, and automorphisms of* $U_q(\mathrm{sl}(2))$ *there are two possibilities*:

(1) $K = \sigma : X \mapsto q^{-2}X$, $\quad E = \partial\sigma^{-1}$, $\quad F = -q^{-4}X^2\partial\sigma^{-1}$,

(2) $K = \sigma : X \mapsto q^2X$, $E = -q^4X^2\partial\sigma^{-1}$, $F = \partial\sigma^{-1}$,
where ∂ is the σ-derivation $\partial(X) = 1$.

PROOF. Instead of proving (3.6) up to an automorphism of W_q, set $X = \gamma_1^{-1}Y$ and $X = \gamma_2^{-1}Y$ in cases (1) and (2). Thus X is a K^2-eigenvector of eigenvalue α. Write $K(X) = \xi X$; thus $\xi^2 = \alpha$ in the notation of (3.6). In case (1), $KE(X) = \sigma\partial\sigma^{-1}(X) = \sigma\partial(\xi^{-1}X) = \sigma(\xi^{-1}) = \xi^{-1}$, and $EK(X) = \partial(X) = 1$. However, $KE = q^2EK$ so $\xi^{-1} = q^2$ and $\xi = q^{-2}$ in case (1). The second possibility is obtained in a similar manner. ■

The statement of (3.7) may be slightly changed to avoid mention of automorphisms of $U_q(\mathrm{sl}(2))$.

COROLLARY 3.8. *Suppose that* $\mathbf{C}[X]$ *is a* $U_q(\mathrm{sl}(2))$*-module algebra. There exists* $Y \in \mathbf{C}[X]$ *such that* $\mathbf{C}[Y] = \mathbf{C}[X]$, *and one of the following two possibilities occurs*:
(1) $K = \sigma : Y \mapsto q^{-2}Y$, $E = \partial\sigma^{-1}$, $F = -q^{-4}Y^2\partial\sigma^{-1}$,
(2) $K = \sigma : Y \mapsto q^2Y$, $E = -q^4Y^2\partial\sigma^{-1}$, $F = \partial\sigma^{-1}$,
where ∂ is the σ-derivation $\partial(Y) = 1$.

This section was motivated by analogy with the action of $U(\mathrm{sl}(2))$ on $\mathbf{C}[X]$ as differential operators. That action is given by

$$E = \partial, \quad H = -2X\partial, \quad F = -X^2\partial,$$

where $\partial = d/dX$. Observe that $\mathbf{C}[X]$ is the dual of a Verma module, and contains the trivial module.

4. A "base affine space" for $U_q(\mathrm{sl}(2))$

Recall the natural action of $U(\mathrm{sl}(2))$ acting as differential operators on the commutative ring $\mathbf{C}[X, Y]$. The action is obtained as follows. Let sl(2) act on $\mathbf{C}^2$ in the obvious way. There is a unique extension of this to an action on $S(\mathbf{C}^2)$, the symmetric algebra, such that sl(2) acts as derivations. Explicitly the action is given by

$$E = X\partial/\partial Y, \quad H = X\partial/\partial X - Y\partial/\partial Y, \quad F = Y\partial/\partial X.$$

The decomposition of $S(\mathbf{C}^2) = \bigoplus_n S^n(\mathbf{C}^2)$ into its homogeneous components is an sl(2)-module decomposition, and $S^n(\mathbf{C}^2)$ is the unique $(n+1)$-dimensional sl(2)-module.

What is the analogue of this for $U_q(\mathrm{sl}(2))$?

THEOREM 4.1 ([L], [R3]). *Suppose that q is not a root of unity. Then, for each $n > 0$ there are precisely 4 simple $U_q(\mathrm{sl}(2))$-modules (up to isomorphism) of dimension n.*

THEOREM 4.2. *If q is not a root of unity, then for each $n \geqq 1$, W_v $(v = q^2)$ has exactly one simple module of dimension n.*

PROOF. This follows from [W, Theorem 5.4], with the proviso that inverting the element K^4 has eliminated all except one of the 1-dimensional modules for W_v. See also [BS]. ■

THEOREM 4.3. *Let $A = \mathbf{C}[x, y]$ where $xy = q^2yx$. There is an action of $W_q(\mathrm{sl}(2)) = C[EK, FK, K^{\pm 2}]$ on A such that*

(a) *A is a $W_q(\mathrm{sl}(2))$-module algebra;*

(b) *each homogeneous component $A_n = \bigoplus_{1 \leqq i \leqq n} \mathbf{C}x^iy^{n-i}$ is a simple $W_q(\mathrm{sl}(2))$-module of dimension $n + 1$;*

(c) *each A_n remains simple over the subalgebra $\mathbf{C}[EK, FK, K^{\pm 4}] \cong W_v$ $(v = q^2)$;*

(d) *A is a $U_q(\mathrm{sl}(2))$-module algebra, and the action of E, F, K on A_1 is $E(x) = 0$, $E(y) = qx$, $F(x) = q^{-1}y$, $F(y) = 0$, $K(x) = qx$, $K(y) = q^{-1}y$.*

PROOF. This follows at once from Example 2.2 and Lemma 3.1. Define K^2 to act via the automorphism $x \mapsto q^2x$, $y \mapsto q^{-2}y$ and EK, FK to act as the σ^2-derivations

$$EK : x \mapsto 0, y \mapsto x, \qquad FK : x \mapsto y, y \mapsto 0.$$

It is routine to check (b) and (c). ■

Clearly the same question can be asked for $U_q(\mathfrak{g})$. Let G be the simply connected, connected algebraic group with $\mathrm{Lie}\, G = \mathfrak{g}$, let B be a Borel subgroup, with unipotent radical N. The action of $U(\mathfrak{g})$ as differential operators on $\mathcal{O}(G/N)$ is such that each finite-dimensional simple $\mathfrak{g}$-module appears in $\mathcal{O}(G/N)$ with multiplicity 1. What is the analogue of $\mathcal{O}(G/N)$ for $U_q(\mathfrak{g})$? In effect we are asking for a quantum version of Borel–Weil–Bott. The action of $\mathrm{sl}(2)$ on $S^n(\mathbf{C}^2)$ may be interpreted as an action on the global sections of the line bundle $\mathcal{O}_{\mathbf{P}^1}(n)$. Pursuing this analogy, one may interpret A as the homogeneous coordinate ring of the "quantum projective line", and the homogeneous components A_n as the sections of line bundles.

FINAL REMARKS. Consider the relationship between the three different algebras $U_q(\mathrm{sl}(2))$, $W_q(\mathrm{sl}(2))$ and W_ν with $\nu = q^2$. There are inclusions as follows:

$$\begin{aligned} U_q(\mathrm{sl}(2)) &= \mathbf{C}[E, F, K^{\pm 1}] \\ &\supset W_q(\mathrm{sl}(2)) = \mathbf{C}[EK, FK, K^{\pm 2}] \supset W_\nu = \mathbf{C}[EK, FK, K^{\pm 4}]. \end{aligned}$$

The first two are Hopf algebras, but the last one is not. If $n > 0$, then $U_q(\mathrm{sl}(2))$ has 4 distinct n-dimensional simple modules, $W_q(\mathrm{sl}(2))$ has 2 distinct n-dimensional simple modules and W_ν has a unique n-dimensional simple module. In terms of irreducible representations, W_ν is the most like $U(\mathrm{sl}(2))$.

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