Functional Analysis,
Approximation and
Computation
9 (2) (2017), 1–5

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Skew exactness and range-kernel orthogonality III

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Abstract. We record some "range-kernel orthogonality" for the elementary operators defining Hankel and Toeplitz operators.

Introduction

This note is a sort of follow-up visit to our earlier paper [H3], in which we feel we may have left gaps in a couple of the arguments, specifically Theorem 5 and Theorem 6; we have also belatedly noticed the connection between the "derivations" involved and the Brown-Halmos definitions [BH] of Hankel and Toeplitz operators.

1. Hermitian elements of a Banach algebra

Recall [BD] that a Banach algebra element $a \in A$ is said to be *hermitian*, written $a \in \mathrm{Re}(A)$, provided its *numerical range* is real:

1.1 $$V_A(a) \equiv \{\varphi(a) : \varphi \in \mathrm{State}(A)\} \subseteq \mathbf{R}\ ,$$

where the "states" of A

1.2 $$\mathrm{State}(A) = \{\varphi \in A^\dagger : \|\varphi\| = 1 = \varphi(1)\}$$

are the norm one linear functionals which achieve their norm at the identity $1 \in A$. The real-linear subspace $\mathrm{Re}(A) = H \subseteq A$ is an example of a "hermitian subspace" [H4] of A: we have

1.3 $$1 \in H\ ;\ H \cap iH = O \equiv \{0\}\ .$$

We introduce the complex-linear *Palmer subspace*

1.4 $$\mathrm{Reim}(A) = \mathrm{Re}(A) + i\mathrm{Re}(A)$$

and observe that the "real and imaginary parts" of $a = h + ik \in \mathrm{Reim}(A)$ are well-defined, with

$$x : \mathrm{Reim}(A) \to \mathrm{Re}(A)\ ,\ y : \mathrm{Reim}(A) \to \mathrm{Re}(A)$$

2010 *Mathematics Subject Classification*. 47A05.
Keywords. Hermitian subspace; Palmer subspace; Orthogonality.
Received: 20 Dececmber 2016; Accepted: 18 May 2017
Communicated by Dragan S. Djordjević
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given by setting

1.5 $$a = x(a) + iy(a) \ (a \in \mathrm{Reim}(A)) \ ,$$

and then there is an *involution* $* : \mathrm{Reim}(A) \to \mathrm{Reim}(A)$ given by

1.6 $$(h + ik)^* = h - ik \ (\{h, k\} \subseteq \mathrm{Re}(A)) \ .$$

If we further observe [BD],[Sp] that $H = \mathrm{Re}(A)$ satisfies

1.7 $$H = \mathrm{cl}(H) \perp iH \ ,$$

the subspaces of "real" and "imaginary" elements are closed and mutually "orthogonal", then it follows that $\mathrm{Reim}(A)$ is also a closed subspace of A, on which the involution $a \to a^*$ is a bounded (real-)linear operator. Here, for $E + F \subseteq X$, "orthogonality" $E \perp F$ is according to James and Birkhoff:

1.8 $$E \perp F \iff \mathrm{AND}_{x \in E}\Big(\|x\| = \mathrm{dist}(x, F)\Big) \ .$$

The Palmer subspace is not in general closed under multiplication; it is however [BD],[Sp] closed under the Lie bracket: if $\{a, b\} \subseteq \mathrm{Reim}(A)$ then

1.9 $$[a, b] \equiv ab - ba \in \mathrm{Reim}(A) \ ,$$

with

1.10 $$[a, b]^* = [b^*, a^*] \ .$$

2. Self-commutant approximants

Specializing to the algebra $A = B(X)$ of bounded operators on a Banach space X, we claim that

Lemma 1 *If X is a Banach space then*

2.1 $$T \in \mathrm{Reim}\, B(X) \Longrightarrow T^{-1}(0) \cap T^{*-1}(0) \subseteq \mathrm{Re}(T)^{-1}(0) \perp \mathrm{Re}(T)(X) \ .$$

Proof. If $\{H, K\} \subseteq \mathrm{Re}\, B(X)$ then

2.2 $$T = H + iK \Longrightarrow T^{-1}(0) \cap T^{*-1}(0) = H^{-1}(0) \cap K^{-1}(0) \ ,$$

and now by Sinclair's theorem ([Si] Proposition 1; [F] Corollary 7)

2.3 $$H^{-1}(0) \perp H(X) \bullet$$

If in particular $T \in B(X)$ satisfies the *Fuglede* condition,

2.4 $$T^{-1}(0) \subseteq T^{*-1}(0) \ ,$$

then Lemma 1 says that

$$T^{-1}(0) \perp \mathrm{Re}(T)(X) \ .$$

If in particular A and B are Banach algebras and M a *Banach (A,B)-bimodule* then [H4] tuples $a \in A^n$ and $b \in B^n$ combine to give "elementary" operators

$$L_a \circ R_b : m \mapsto \sum_j a_j m b_j \ (M \to M) \ ;$$

we focus particularly on the "generalized inner derivation"

$$L_a - R_b : m \mapsto am - mb$$

and its multiplicative analogue

$$L_a R_b - I : m \mapsto amb - m$$

associated with $(a, b) \in A \times B$. For example if $(a, b) = (v, u)$ are the *backward and forward shifts* in $A = B(X)$ with $X = \ell_2$ then [BH] $(L_a R_b - I)^{-1}(0)$ and $(L_a - R_b)^{-1}(0)$ are respectively the *Toeplitz* and the *Hankel* operators.

Generally when $A = M = B$,

Theorem 2 *If A is a Banach algebra and* $\{a, b\} \subseteq \mathrm{Reim}(A)$ *then*

2.5 $$b \in (L_a - R_a)^{-1}(0) \cap (L_{a^*} - R_{a^*})^{-1}(0) \Longrightarrow \|a\| \le \|a - [b, b^*]\| \text{ and } \|b\| \le \|b - [a, a^*]\| .$$

Proof. This is Lemma 1 with $X = A$ and $T = L_b - R_b$, together with the observation that

2.6 $$(L_a - R_a)(b) = 0 \Longleftrightarrow (L_b - R_b)(a) = 0 ,$$

and also (1.10)

2.7 $$(L_{a^*} - R_{a^*})(b) = (L_a - R_a)(b)^* = 0 .$$

Indeed if $(L_b - R_b)(a) = 0 = (L_b - R_b)^*(a)$ then, with $b = h + ik$ and arbitrary $c \in A$,

$$\|a\| \le \|a - \mathrm{Re}(T)(c)\| \equiv \|a - (L_h - R_h)(c)\| \ :$$

but now

$$c = 2ik \Longrightarrow (L_h - R_h)(c) = (L_b - R_b)(b^*) = [b, b^*] ;$$

for the second part interchange a and b •

Theorem 2 is a cosmetic improvement of Theorem 5 of [H3], in that the Fuglede condition (2.4) is withheld from $T = L_a - R_a$, and also plugs a small gap (2.7) in the argument there.

3. Multiplicative commutants

Theorem 6 of [H3] needs more than cosmetic adjustment. If we write, for arbitrary $c \in A$,

3.1 $$D_c = L_c - R_c \ ; \ \triangle_c = L_c R_c - I ,$$

then we recall [H1] that if $\{a, b\} \subseteq A$ then

3.2 $$\begin{pmatrix} D_{a+b} \\ D_{ab} \end{pmatrix} = \begin{pmatrix} I & I \\ R_b & L_a \end{pmatrix} \begin{pmatrix} D_a \\ D_b \end{pmatrix}$$

and hence always

3.3 $$D_a^{-1}(0) \cap D_b^{-1}(0) \subseteq D_{a+b}^{-1}(0) \cap D_{ab}^{-1}(0) .$$

Conversely (taking [H2],[HH] the "adjugate" !)

3.4 $$\begin{pmatrix} L_a & -I \\ -R_b & I \end{pmatrix} \begin{pmatrix} D_{a+b} \\ D_{ab} \end{pmatrix} = \begin{pmatrix} L_a - R_b & 0 \\ 0 & L_a - R_b \end{pmatrix} \begin{pmatrix} D_a \\ D_b \end{pmatrix} ;$$

and hence if $L_a - R_b$ is one one there is equality in (3.3). We also have $ST - I = S(T - I) + (S - I)$, so that

3.5 $$(S - I)^{-1}(0) \cap (T - I)^{-1}(0) \subseteq (ST - I)^{-1}(0) ;$$

in particular

3.6 $$\triangle_a\triangle_b = L_bR_b\triangle_a + \triangle_b = L_{ab}R_{ba} - I\ (\neq \triangle_{ab})\ ,$$

and hence

3.7 $$\triangle_a^{-1}(0) \cap \triangle_b^{-1}(0) \subseteq (L_{ab}R_{ba} - I)^{-1}(0)\ .$$

As a supplement to the first row of (3.2) we have also

3.8 $$D_{a+b} = (L_a - R_b) + (L_b - R_a)\ ,$$

so that

3.9 $$(L_a - R_b)^{-1}(0) \cap (L_b - R_a)^{-1}(0) \subseteq D_{a+b}^{-1}(0)\ ,$$

giving cosmetic improvement of the sort of two-variable extension of Theorem 2 noticed by Mansour and Bouzenada ([MB] Theorem 3.1).

If we make the assumption

3.10 $$\mathrm{Reim}(A)^2 \subseteq \mathrm{Reim}(A)\ ,$$

that the Palmer subspace is a subalgebra (so that [BD],[Sp] it is in fact a C* algebra) then, in place of Theorem 6 of [H3],

Theorem 3 *If* $\{a, b\} \subseteq \mathrm{Reim}(A)$ *then*

3.11 $$b \in (L_aR_a - I)^{-1}(0) \cap (L_{a^*}R_{a^*} - I)^{-1}(0) \Longrightarrow \mathrm{AND}_{c\in A}\Big(\|b\| \leq \|b + c - a^*acaa^*\|\Big)\ .$$

Proof. This is (3.7) with $b = a^*$:

3.12 $$(L_aR_a - I)^{-1}(0) \cap (L_{a^*}R_{a^*} - I)^{-1}(0) \subseteq (L_{a^*a}R_{aa^*} - I)^{-1}(0) \perp (L_{a^*a}R_{aa^*} - I)(A)\ .$$

The multiplicative assumption of course guarantees that the product of commuting hermitian operators in the middle of (3.12) is again hermitian •

We at the same time have cosmetic improvement of Duggal's ([Du] Theorem 2.6) version:

3.13 $$b \in (L_aR_a - I)^{-1}(0) \cap (L_{a^*}R_{a^*} - I)^{-1}(0) \Longrightarrow b^*b \in (L_a - R_a)^{-1}(0)\ .$$

An old example of Anderson and Foias says that the multiplicative assumption (3.10) cannot be omitted from Theorem 3; if

3.14 $$0 \neq a^* = a = a^2 \neq 1 \in A = B(X)\ ,$$

then ([AF] Example 5.8) $L_aR_a \in B(A)$ is not hermitian, and the orthogonality at the end of (3.12) is liable to fail.

If $a = u \in A = B(X)$ with $X = \ell_2$ is the forward shift then $(L_a - R_a)^{-1}(0)$ consists of the *analytic Toeplitz operators* while $a^* = v$ is the backward shift, and $(L_{a^*} - R_{a^*})^{-1}(0)$ the *co-analytic* Toeplitz operators; in this case the intersection on the left hand side of (2.5) reduces to the scalar multiples of the identity. When $a = u$ then $(L_{a^*}R_a - I)^{-1}(0)$ consists of the Toeplitz operators; $(L_aR_{a^*} - I)^{-1}(0)$ however, and therefore the intersection, reduces to the co-analytic Toeplitz operators.

Putting $a = 1$ in (2.5) shows ([MB] Corollary 3.2) that

3.12 $$b \in \mathrm{Reim}(A) \Longrightarrow \|1 - [b, b^*]\| \geq 1\ .$$

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