Skew information-based coherence generating power of quantum channels

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Abstract We study the ability of a quantum channel to generate quantum coherence when it applies to incoherent states. We define the measure of coherence generating power (CGP) for a generic quantum channel to be the average coherence generated by the quantum channel acting on a uniform ensemble of incoherent states based on the skew information-based coherence measure. We present explicitly the analytical formulae of the CGP for any arbitrary finite dimensional unitary channels. We derive the mean value of the CGP over the unitary groups and investigate the typicality of the normalized CGP. Furthermore, we give an upper bound of the CGP for the convex combinations of unitary channels. Detailed examples are provided to calculate exactly the values of the CGP for the unitary channels related to specific quantum gates and for some qubit channels.

1. Introduction

Quantum coherence is a distinctive feature of quantum systems associated with the superposition principle. It plays pivotal roles in quantum thermodynamics [1], quantum metrology [2] and quantum biology [3]. The quantification of quantum coherence from a mathematical perspective, however, has only been considered not long ago in [4], where a rigorous framework for coherence measures has been proposed. The past few years have witnessed a great interest in quantifying quantum coherence by utilizing various distance measures such as relative entropy, l_1-norm, intrinsic randomness, robustness of coher-

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ence, max-relative entropy, fidelity, affinity, skew information, generalized α-z-relative Rényi entropy, logarithmic coherence number, Schatten-p-norm and Fisher information, etc. [4–21]. On the other hand, the problem of coherence distillation and coherence dilution have also been discussed [22–28], and a complete theory of one-shot coherence distillation has been formulated [29]. The average quantum coherence over the pure state decompositions of a mixed quantum state has been discussed in [30]. Feasible methods have been introduced to detect and estimate the coherence by constructing coherence witnesses for any finite-dimensional states [31]. Quantum coherence from other resource-theoretical perspectives such as no-broadcasting of quantum coherence [32,33], interconversion between quantum coherence and quantum entanglement or quantum correlations [?,34–39] have also been studied extensively. The study on coherence of quantum channels has also attracted much attention [40–46].

The coherence measure defined in [4] is basis-dependent. To get rid of the influence of the basis, the average coherence with respect to mutually unbiased bases or all the basis sets have been discussed [47,48]. On the other hand, random pure quantum states provide new perspectives for various phenomena in quantum physics and quantum information processing [49]. Average coherence based on the relative entropy of coherence and its typicality for random pure states and random mixed states have been derived in [50,51], and the average subentropy, coherence and entanglement of random mixed quantum states have been discussed in [52].

The concepts of cohering power and de-cohering power of generic quantum channels have been initially introduced by Mani and Karimipour [53]. By optimization on the output coherence, the coherence generating power (CGP) of a quantum channel has been defined to quantify the power of a channel in generating quantum coherence. Many examples have been given to the qubit channels including those induced by quantum gates. Different kinds of operations which either preserve or generate coherence have also been studied [54, 55]. It was Zanardi *et al.* [56, 57] who first utilize probabilistic averages to study the CGP. By introducing a measure based on the average coherence generated by the channel acting on a uniform ensemble of incoherent states, a new method in quantifying the CGP of unitary channels has been formulated. The coherence measure based on the Hilbert-Schmidt norm has been exploited to derive explicitly the analytical formulae of the CGP.

However, the Hilbert-Schmidt norm measure is not a well-defined coherence measure since it does not possess the expected monotonicity property. In [58] by using the well-defined relative entropy of coherence measure, Zhang *et al.* has studied the quantification of CGP for a generic quantum channel via probabilistic averages and derived explicitly the analytical formulae of CGP for unitary channels and deduced an upper bound for the CGP for unital quantum channels. Since the skew information-based coherence is also a well-defined coherence measure that can be experimentally measured, it is of significance to calculate the CGP of a generic quantum channel under the skew information-based

coherence, instead of the relative entropy of coherence. In this paper, we will solve this problem.

The paper is arranged as follows. In Section 2, we first recall the concepts of skew information and skew information-based coherence. Then by adopting the probabilistic averages, we define the coherence generating power of a generic quantum channel with respect to skew information-based coherence. In Section 3, we present an explicit analytical formula of the CGP via skew information-based coherence for any unitary channels and calculate the CGP of unitary channels induced by some specific quantum gates. Based on the formula given in Section 3, we further compute the mean valued of the CGP and discuss the typicality for the normalized CGP in Section 4. In Section 5, we study the CGP for convex combinations of unitary channels and derive the CGP for some important qubit channels. Finally, we give some concluding remarks in Section 6.

2. CGP of quantum channels under skew information-based coherence

Let $\mathcal{H} = \mathbb{C}^N$ be a Hilbert space of dimension N, and B($\mathcal{H}$), S($\mathcal{H}$) and D($\mathcal{H}$) be the set of all bounded linear operators, Hermitian operators and density operators on $\mathcal{H}$, respectively. Denote by U(N) the group of all $N \times N$ unitary matrices.

Fix an orthonormal basis $\{|k\rangle\}_{k=1}^N$ of $\mathcal{H}$. The set of incoherent states, which are diagonal in this basis, can be written as $\mathcal{I} = \{\delta \in \mathrm{D}(\mathcal{H}) | \delta = \sum_{k=1}^N p_k |k\rangle\langle k|, \ p_k \geq 0, \ \sum_k^N p_k = 1\}$. Let Λ be a CPTP map $\Lambda(\rho) = \sum_n K_n \rho K_n^\dagger$, where K_n are Kraus operators satisfying $\sum_n K_n^\dagger K_n = I_N$ with I_N the identity operator on $\mathcal{H}$. K_n are called incoherent Kraus operators if $K_n^\dagger \mathcal{I} K_n \in \mathcal{I}$ for all n, and the corresponding Λ is called an incoherent operation.

A well-defined coherence measure $C(\cdot)$ of a quantum state ρ should satisfy the following conditions [4]:

- (C1) (Faithfulness) $C(\rho) \geq 0$ and $C(\rho) = 0$ iff ρ is incoherent;

- (C2) (Convexity) $C(\cdot)$ is convex in ρ;

- (C3) (Monotonicity) $C(\Lambda(\rho)) \leq C(\rho)$ for any incoherent operation Λ;

- (C4) (Strong monotonicity) $C(\cdot)$ does not increase on average under selective incoherent operations, i.e., $C(\rho) \geq \sum_n p_n C(\varrho_n)$, where $p_n = \mathrm{Tr}(K_n \rho K_n^\dagger)$ are probabilities and $\varrho_n = \frac{K_n \rho K_n^\dagger}{p_n}$ are the post-measurement states, K_n are incoherent Kraus operators.

The skew information-based coherence $C_S(\rho)$ of a quantum state ρ with respect to a fixed orthonormal basis $\{|k\rangle\}_{i=1}^N$ in an N-dimensional Hilbert space H is defined by [9],

$$C_S(\rho) = \sum_{k=1}^N I(\rho, |k\rangle\langle k|) = 1 - \sum_{k=1}^N \langle k|\sqrt{\rho}|k\rangle^2, \tag{1}$$

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where $I(\rho, |k\rangle\langle k|) = -\frac{1}{2}\text{Tr}\{[\sqrt{\rho}, |k\rangle\langle k|]\}^2$ is the skew information of the state ρ with respect to the projector $|k\rangle\langle k|$, $k = 1, 2, \cdots, N$. $C_S(\rho)$ is shown to be a well-defined coherence measure which satisfies the required properties of a coherence measure in the framework of [4]. It is of pivotal importance with meaningful physical interpretations and can be experimentally implemented. The advantage of this coherence measure is that it has an analytic expression. Also, an operational meaning in connection with quantum metrology has been revealed. The distribution of this coherence measure among the multipartite systems has been investigated and a corresponding polygamy relation has been proposed. It is also found that this coherence measure provides the natural upper bounds of quantum correlations prepared by incoherent operations. Moreover, it is shown that this coherence measure can be experimentally measured [9]. Since the skew information-based coherence measure (1) is well-defined and can be analytically expressed, it is of great significance both theoretically and practically, and worth evaluating the CGP of unitary channels based on this measure. Note that $C_S(\rho)$ attains the maximal value $1 - \frac{1}{N}$ at the maximal coherent sate $|\psi\rangle = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} e^{i\theta_j}|j\rangle$.

Quantum ensembles are formulated by specifying probability measures on $\text{D}(\mathbb{C}^N)$. The uniqueness for such measures cannot be guaranteed, while the Fubini-Study (FS) measure is the only natural measure in defining random pure states [59].

Conventionally, it is not easy to deal with the emerged monotone metrics when $N > 2$, we have to take great efforts when we consider a Riemannian geometry on $\text{D}(\mathbb{C}^N)$. However, for some special monotone metrics, the measures induced from them would be easier to tackle with. Recall that in flat space, the Euclidean measure is decomposed into a product. We can use the same technique here. The set of quantum mixed states in the form $\rho = U\Lambda U^\dagger$, with Λ a fixed diagonal matrix having strictly positive eigenvalues, is a flag manifold $\mathbf{F}^{(N)} = \text{U(N)}/[\text{U(1)}]^N$. If the chosen eigenvalues and eigenvectors are independent, and the eigenvectors are drawn according to the invariant Haar measure, $\text{d}\mu_{\text{Haar}}(W) = \text{d}\mu_{\text{Haar}}(UW)$, then we can assume that a probability distribution in $\text{D}(\mathbb{C}^N)$ possess the invariance with respect to unitary rotations, $P(\rho) = P(W\rho W^\dagger)$ [59].

Combining the two measures, a product measure on the Cartesian product of the flag manifold and the simplex $\mathbf{F}^{(N)} \times \Delta_{N-1}$ can be defined: $\text{d}\omega(\rho) = \text{d}\mu_{\text{Haar}}(U) \times \text{d}\mu(\Lambda)$, which induces the corresponding probability distribution, $P(\rho) = P_{\text{Haar}}(\mathbf{F}^{(N)}) \times P(\Lambda)$, where the first factor denotes the natural, unitarily invariant distribution on the flag manifold $\mathbf{F}^{(N)} = \text{U(N)}/[\text{U(1)}]^N$ induced by the Haar measure on U(N). Note that the Haar measure on U(N) is unique while there is no unique choice for μ [59].

The measures used frequently over $\text{D}(\mathbb{C}^N)$ can be obtained by taking partial trace over a M-dimensional environment of an ensemble of pure states distributed according to the unique, unitarily invariant FS measure on the space $\mathbb{CP}^{MN-1}$ of pure states of the composite system. There is a simple physical motivation for such measures: they can be used if anything is known about the density matrix, apart from the dimensionality M

of the environment. When $M = 1$, we get the FS measure on the space of pure states. Since the rank of ρ is limited by M, when $M \geq N$ the induced measure covers the full set of $\mathrm{D}(\mathbb{C}^N)$. Since the pure state $|\psi\rangle$ is drawn according to the FS measure, the induced measure is of the product form $P(\rho) = P_{\mathrm{Haar}}(\mathbf{F}^{(N)}) \times P(\Lambda)$. Hence the distribution of the eigenvectors of ρ is determined by the Haar measure on $\mathrm{U}(N)$ [59].

The general measure for the joint probability distribution of spectrum $\Lambda = \{\lambda_1, \ldots, \lambda_N\}$ of ρ is given by [60],

$$\mathrm{d}\omega_{N,M}(\Lambda) = C_{N,M} \delta \left(1 - \sum_{j=1}^{N} \lambda_j \right) \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 \prod_{j=1}^{N} \lambda_j^{M-N} \theta(\lambda_j) \mathrm{d}\lambda_j, \qquad (2)$$

where δ is the Dirac delta function, the theta function θ ensures that ρ is positive definite, and $C_{N,M}$ is the normalization constant,

$$C_{N,M} = \frac{\Gamma(NM)}{\prod_{j=0}^{N-1} \Gamma(N - j + 1)\Gamma(M - j)}.$$

In this paper we take $N = M$. In this scenario, we deal with non-Hermitian square random matrice characteristic of the *Ginibre ensemble* [61, 62] and obtain the Hilbert-Schmidt measure [59]. Denote $\mathrm{d}\omega_{N,N} = \mathrm{d}\omega_{\mathrm{HS}}$ and $C_{N,N} = C_N^{\mathrm{HS}}$. Thus we have [60, 63]

$$\mathrm{d}\omega_{\mathrm{HS}}(\rho) = \mathrm{d}\mu_{\mathrm{Haar}}(U) \times \mathrm{d}\mu(\Lambda)$$

for $\rho = U\Lambda U^\dagger$. Here $\mathrm{d}\mu(\Lambda)$ is given by [60, 63],

$$\mathrm{d}\mu(\Lambda) = C_N^{\mathrm{HS}} \delta \left(1 - \sum_{j=1}^{N} \lambda_j \right) |\Delta(\lambda)|^2 \prod_{j=1}^{N} \mathrm{d}\lambda_j, \qquad (3)$$

where $\Delta(\lambda) = \prod_{1 \leq k < l \leq N} (\lambda_l - \lambda_k)$ and

$$C_N^{\mathrm{HS}} = \frac{\Gamma(N^2)}{\Gamma(N + 1) \prod_{j=1}^{N} \Gamma(j)^2}. \qquad (4)$$

Let Φ be a quantum channel, i.e., a trace-preserving completely positive linear map, which maps an incoherent state Λ to $\Phi(\Lambda)$. By employing the technique of probabilistic averages [56–58], we define the measure of coherence generating power (CGP) $\mathbf{CGP_S}(\Phi)$ of Φ to be the average skew information-based coherence generated by the quantum channel acting on a uniform ensemble of incoherent states,

$$\mathbf{CGP_S}(\Phi) := \int_{\mathcal{I}} \mathrm{d}\mu(\Lambda) C_S(\Phi(\Lambda)), \qquad (5)$$

where $\mathcal{I}$ denotes the set of incoherent states, μ is the probability measure on a uniform ensemble of incoherent states and $\mathrm{d}\mu(\Lambda)$ is given in Eq. (3). Obviously, for incoherent quantum channels Φ_{IO}, one has $\mathbf{CGP_S}(\Phi_{IO}) = 0$ since $\Phi_{IO}(\Lambda)$ is always incoherent.

In the following we calculate $\mathbf{CGP_S}(\Phi)$ for unitary channels Φ_U such that $\Phi_U(\Lambda) = U\Lambda U^\dagger$, where U denotes unitary transformations and $\dagger$ the transpose and conjugation.

3. CGP of unitary channels under skew information-based coherence

We first calculate the CGP of unitary channels under skew information-based coherence, see proof in Appendix A.

Theorem 3.1 For any given $N \times N$ unitary matrix U, the CGP of the unitary channel Φ_U is given by

$$\mathbf{CGP_S}(U) := \mathbf{CGP_S}(\Phi_U)$$
$$= \left(1 - \frac{1}{N^2(N-1)}\left[\left(\sum_{k=1}^N I_{kk}^{(\frac{1}{2})}\right)^2 - \sum_{k,l=1}^N \left(I_{kl}^{(\frac{1}{2})}\right)^2\right]\right)\left(1 - \frac{1}{N}\sum_{k,i=1}^N |U_{ki}|^4\right), \quad (6)$$

where $I_{kl}^{(\frac{1}{2})} = \sum_{r=0}^{\min(k,l)}(-1)^{k+l}\binom{\frac{1}{2}}{k-r}\binom{\frac{1}{2}}{l-r}\frac{\Gamma(\frac{3}{2}+r)}{r!}$.

Below we present an estimation on the lower and upper bounds on $\mathbf{CGP_S}(U)$.

Proposition 3.1 For any unitary channel Φ_U,

$$0 \leq \mathbf{CGP_S}(U) \leq \mathrm{CGP}_N,$$

where

$$\mathrm{CGP}_N := \left(1 - \frac{1}{N}\right)\left(1 - \frac{1}{N^2(N-1)}\left[\left(\sum_{k=1}^N I_{kk}^{(\frac{1}{2})}\right)^2 - \sum_{k,l=1}^N \left(I_{kl}^{(\frac{1}{2})}\right)^2\right]\right). \quad (7)$$

The lower bound is saturated iff $|U_{ki}| \cdot |U_{kj}| = 0$ for all $k, i, j = 1, 2, \cdots, N$ with $i \neq j$, while the upper bound is saturated iff $|U_{ki}|^2 = 1/N$ for all $k, i = 1, 2, \cdots, N$.

Proof. Since U is unitary, we have $\sum_{i=1}^N |U_{ki}|^2 = 1$ for $k = 1, 2, \cdots, N$. Then

$$\sum_{k,i=1}^N |U_{ki}|^4 = \sum_{k=1}^N \left(\sum_{i=1}^N (|U_{ki}|^2)^2\right) \leq \sum_{k=1}^N \left(\sum_{i=1}^N |U_{ki}|^2\right)^2 = N.$$

By Eq. (6), we obtain that $\mathbf{CGP_S}(U) \geq 0$. It is easy to see that the lower bound is saturated, i.e., $\mathbf{CGP_S}(U) = 0$, iff $\sum_{k,i=1}^N |U_{ki}|^4 = N$ iff $\sum_{i=1}^N (|U_{ki}|^2)^2 = \left(\sum_{i=1}^N |U_{ki}|^2\right)^2$ iff $|U_{ki}| \cdot |U_{kj}| = 0$ for all $k, i, j = 1, 2, \cdots, N$ with $i \neq j$.

On the other hand, noting that $\sum_{k,i=1}^N |U_{ki}|^2 = N$ for $k, i = 1, 2, \cdots, N$, and utilizing the Lagrange multiplier method, one can check that the minimal value of $\sum_{k,i=1}^N |U_{ki}|^4$ is 1, which is attained iff $|U_{ki}|^2 = 1/N$ for all $k, i = 1, 2, \cdots, N$. This implies from (6) that $\mathbf{CGP_S}(U) \leq \mathrm{CGP}_N$, and the upper bound is saturated iff $|U_{ki}|^2 = 1/N$ for all $k, i = 1, 2, \cdots, N$. $\square$

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Note that $|U_{ki}| \cdot |U_{kj}| = 0$ for all $k, i, j = 1, 2, \cdots, N$ with $i \neq j$ implies that at least one of the elements in each row of the matrix is 0. For example, when $N = 2$, the unitary $U \in \mathrm{U}(2)$ is in the following form,

$$
\begin{pmatrix} u & -v \\ \bar{v} & \bar{u} \end{pmatrix}, \quad u, v \in \mathbb{C}, \quad |u|^2 + |v|^2 = 1.
$$

In order for the $\mathbf{CGP_S}(U)$ to reach the lower bound 0, the unitary U is in either of the following forms,

$$
\begin{pmatrix} 0 & -e^{\sqrt{-1}\phi} \\ e^{-\sqrt{-1}\phi} & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} e^{\sqrt{-1}\varphi} & 0 \\ 0 & e^{-\sqrt{-1}\varphi} \end{pmatrix}.
$$

A set of orthonormal bases $\{e_k\}$ with $e_k = \{|0\rangle_k, |1\rangle_k, \cdots, |N-1\rangle_k\}$ for a Hilbert space $H = \mathbb{C}^N$ is called mutually unbiased bases (MUBs) if [64, 65] $|_k\langle i|j\rangle_l| = 1/\sqrt{N}$ holds for all $i, j \in \{0, 1, \cdots, N-1\}$ and $k \neq l$. From Proposition 3.1, it can be seen that if the base $\{|i\rangle\}_{i=1}^N$ and the base $\{U|i\rangle\}_{i=1}^N$ are mutually unbiased, the unitary channel Φ_U reaches the maximal value of CGP. For example, the unitary U satisfying that $\langle s|U|t\rangle = 1/\sqrt{N} \exp(\sqrt{-1}\frac{2\pi}{N}st)(s, t = 1, 2, \cdots, N)$ has the maximal CGP.

From (7), for $N = 2$ and $N = 3$ we have

$$
\mathrm{CGP}_2 = \frac{1}{2}\left(1 - \frac{3\pi}{16}\right) \approx 0.205
$$

and

$$
\mathrm{CGP}_3 = \frac{2}{3}\left(1 - \frac{103\pi}{512}\right) \approx 0.245,
$$

respectively. In Fig. 1 we plot the maximal value CGP_N of $\mathbf{CGP_S}(U)$ as a function of $N = 2^m$ for $N = 2, ..., 10$. It shows that as N increases, CGP_N approaches to 0.28.
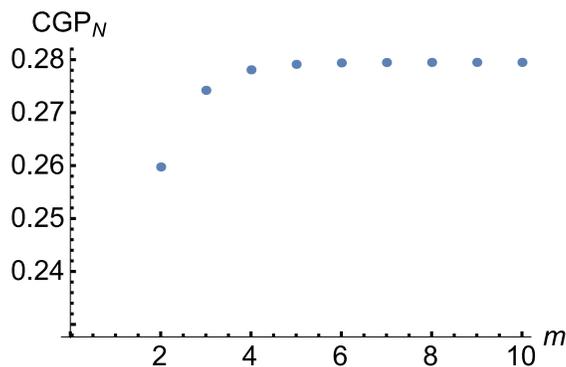
Figure 1: The upper bound CGP_N of $\mathbf{CGP_S}(U)$ as a function of $N = 2^m$.

Next, as examples we calculate the CGP for some specific unitary channels by using

Theorem 3.1. First, for $N = 2$ it follows from Eq. (6) that

$$\mathbf{CGP_S}(U) = \left(1 - \frac{3\pi}{16}\right)\left(1 - \frac{1}{2}\sum_{k,i=1}^{2}|U_{ki}|^4\right). \tag{8}$$

Example 3.1 Consider the Hadamard gate $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. From Eq. (8) we have $\mathbf{CGP_S}(H) = \frac{1}{2}(1 - \frac{3\pi}{16}) \approx 0.205$.

Example 3.2 Consider the unitary transformation $U_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. By (8), we have the CGP of the unitary channel related to U_θ,

$$\mathbf{CGP_S}(U_\theta) = \frac{1}{2}\left(1 - \frac{3\pi}{16}\right)\sin^2 2\theta. \tag{9}$$

Fig. 2 shows the coherence generating power $\mathbf{CGP_S}(U_\theta)$ of U_θ as the function of $\theta \in [0, \pi]$. It can be seen that the maximal value of $\mathbf{CGP_S}(U_\theta)$ is $\frac{1}{2}(1 - \frac{3\pi}{16}) \approx 0.205$, which is attained at $\theta = \pi/4$ and $\theta = 3\pi/4$.
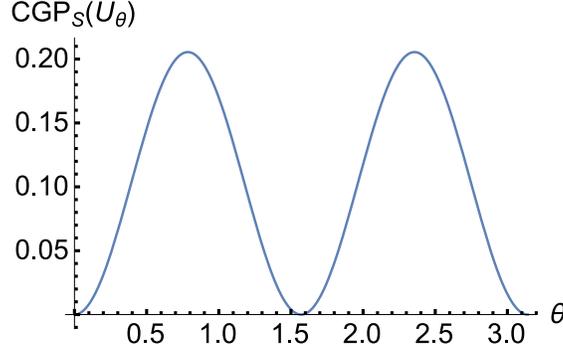
Figure 2: The skew information-based coherence generating power of U_θ with respect to the parameter θ.

When $N = 4$, it follows from Eq. (6) that

$$\mathbf{CGP_S}(U) = \left(1 - \frac{54545\pi}{262144}\right)\left(1 - \frac{1}{4}\sum_{k,i=1}^{4}|U_{ki}|^4\right). \tag{10}$$

Example 3.3 (Square root of swap gate) The $\sqrt{\text{swap}}$ gate is an important quantum gate since any quantum multi-qubit gates can be generated by combining $\sqrt{\text{swap}}$ and single qubit gates, which is given by

$$\sqrt{\text{swap}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(1+i) & \frac{1}{2}(1-i) & 0 \\ 0 & \frac{1}{2}(1-i) & \frac{1}{2}(1+i) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

From Eq. (10), we have

$$\mathbf{CGP_S}(\sqrt{\text{swap}}) = \frac{1}{4}\left(1 - \frac{54545\pi}{262144}\right) \approx 0.087.$$

Example 3.4 For a partial swap operator [66], one has $U_t \in \mathcal{U}(\mathbb{C}^d \otimes \mathbb{C}^d)$: $U_t = \sqrt{t}\mathbb{I}_d \otimes \mathbb{I}_d + \mathrm{i}\sqrt{1-t}\,S$, where $S = \sum_{i,j=1}^{d} |ij\rangle\langle ji|$ and $t \in [0,1]$. When $d = 2$, we have

$$U_t = \begin{pmatrix} \sqrt{t} + \sqrt{1-t}\mathrm{i} & 0 & 0 & 0 \\ 0 & \sqrt{t} & \sqrt{1-t}\mathrm{i} & 0 \\ 0 & \sqrt{1-t}\mathrm{i} & \sqrt{t} & 0 \\ 0 & 0 & 0 & \sqrt{t} + \sqrt{1-t}\mathrm{i} \end{pmatrix}.$$

Then it follows from Eq. (10) that

$$\mathbf{CGP_S}(U_t) = t(1-t)\left(1 - \frac{54545\pi}{262144}\right), \quad t \in [0,1]. \tag{11}$$

We plot $\mathbf{CGP_S}(U_t)$ as the function of $t \in [0,1]$ in Fig. 3. It is found that the maximal value of $\mathbf{CGP_S}(U_t)$ is $\mathbf{CGP_S}(U_{\frac{1}{2}}) = \frac{1}{4}\left(1 - \frac{54545\pi}{262144}\right) \approx 0.087$ attained at $t = \frac{1}{2}$.
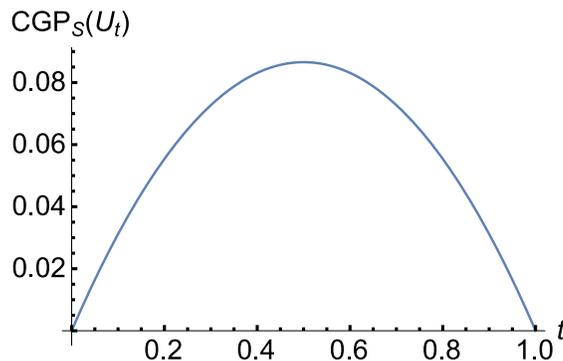
Figure 3: The skew information-based coherence generating power of U_t with respect to the parameter t.

It is pointed out that [58] the possible values of the relative entropy-based CGP $\mathbf{CGP_R}$ form the closed interval $[0, \ln N - H_N + 1]$, where $H_N = \sum_{n=1}^{N} 1/n$, and both $\mathbf{CGP_R}(H)$ and the maximal value of $\mathbf{CGP_R}(U_\theta)$ ($\theta \in [0,\pi]$) reach the maximal CGP of qubit unitary channels, $\ln 2 - 1/2 \approx 0.193$. In comparison, $\mathbf{CGP_R}(\sqrt{\text{swap}}) = \frac{1}{2}\ln 2 \approx 0.347$ is greater than the maximal CGP of unitary channels given by 4×4 unitary matrices, $\ln 4 - H_4 + 1 \approx 0.303$, while the maximal value of $\mathbf{CGP_R}(U_t)$, $t \in [0,1]$, $\frac{1}{4}(2\ln 2 - 1) \approx 0.097$, is less than it.

Note that similarly, for skew information-based CGP $\mathbf{CGP_S}$, both $\mathbf{CGP_S}(H)$ and the maximal value of $\mathbf{CGP_S}(U_\theta)$ ($\theta \in [0,\pi]$) reach the maximal CGP of unitary channels given by 2×2 unitary matrices, $\frac{1}{2}(1 - \frac{3\pi}{16}) \approx 0.205$. However, both $\mathbf{CGP_S}(\sqrt{\text{swap}})$

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and the maximal value of $\mathbf{CGP_S}(U_t)$ for $t \in [0,1]$, $\frac{1}{4}\left(1 - \frac{54545\pi}{262144}\right) \approx 0.087$, are less than the maximal CGP of unitary channels given by 4×4 unitary matrices, $\frac{3}{4}\left(1 - \frac{54545\pi}{262144}\right) \approx 0.26$. Moreover, for the same unitary channel given in the above examples, the skew information-based CGP $\mathbf{CGP_S}$ are less than the relative entropy-based CGP $\mathbf{CGP_R}$ calculated in [58].

4. CGP as a random variable over the unitary group under skew information-based coherence

We now regard $\mathbf{CGP_S}(U)$ as a random variable over the group of $N \times N$ unitary matrices $U(N)$ equipped with the Haar measure $\mathrm{d}\mu_{\text{Haar}}(U)$. We calculate the mean value of $\mathbf{CGP_S}(U)$ for random unitary channels.

If G is a locally compact group, there is, up to a constant multiple, a unique regular Borel measure μ_L that is invariant under left translation. Here left translation invariance of a measure μ means that $\mu(M) = \mu(gM)$ for all measurable sets M and $g \in G$. Regularity means that $\mu(M) = \inf\{\mu(\mathcal{O}) : M \subseteq \mathcal{O},\ \mathcal{O} \text{ open}\} = \sup\{\mu(\mathcal{C}) : M \supseteq \mathcal{C},\ \mathcal{C} \text{ compact}\}$. Such a measure is called a left-invariant Haar measure. It has the properties that any compact set has finite measure and any nonempty open set has positive measure. Left-invariance of the measure amounts to left-invariance of the corresponding integral,

$$\int_G f(g'g)\mathrm{d}\mu_L(g) = \int_G f(g)\mathrm{d}\mu_L(g)$$

for any Haar integral function f on G and any $g' \in G$ [67]. We denote this left-invariant Haar measure by μ_{Haar} here.

Theorem 4.1 The mean value of $\mathbf{CGP_S}(U)$ is given by

$$\mathbb{E}_U[\mathbf{CGP_S}(U)] = \frac{N-1}{N+1}\left(1 - \frac{1}{N^2(N-1)}\left[\left(\sum_{k=1}^N I_{kk}^{(\frac{1}{2})}\right)^2 - \sum_{k,l=1}^N \left(I_{kl}^{(\frac{1}{2})}\right)^2\right]\right). \quad (12)$$

where $I_{kl}^{(\frac{1}{2})} = \sum_{r=0}^{\min(k,l)}(-1)^{k+l}\binom{\frac{1}{2}}{k-r}\binom{\frac{1}{2}}{l-r}\frac{\Gamma(\frac{3}{2}+r)}{r!}$.

Proof. From Eq. (6), the mean value of $\mathbf{CGP_S}(U)$ for unitary channels is given by

$$\mathbb{E}_U[\mathbf{CGP_S}(U)]$$

$$= \int_{U(N)}\mathrm{d}\mu_{\text{Haar}}(U)\left(1 - \frac{1}{N^2(N-1)}\left[\left(\sum_{k=1}^N I_{kk}^{(\frac{1}{2})}\right)^2 - \sum_{k,l=1}^N \left(I_{kl}^{(\frac{1}{2})}\right)^2\right]\right)\left(1 - \frac{1}{N}\sum_{k,i=1}^N |U_{ki}|^4\right) \quad (13)$$

where μ_{Haar} is a unitarily invariant uniform Haar measure. Noting that the Haar measure

is left-invariant, we obtain

$$\mathbb{E}_U[\mathbf{CGP_S}(U)]$$

$$= \left(1 - \frac{1}{N^2(N-1)}\left[\left(\sum_{k=1}^{N} I_{kk}^{(\frac{1}{2})}\right)^2 - \sum_{k,l=1}^{N}\left(I_{kl}^{(\frac{1}{2})}\right)^2\right]\right)\left(1 - \frac{1}{N}\int_{U(N)} \mathrm{d}\mu_{\mathrm{Haar}}(U)\sum_{k,i=1}^{N} |U_{ki}|^4\right)$$

$$= \left(1 - \frac{1}{N^2(N-1)}\left[\left(\sum_{k=1}^{N} I_{kk}^{(\frac{1}{2})}\right)^2 - \sum_{k,l=1}^{N}\left(I_{kl}^{(\frac{1}{2})}\right)^2\right]\right)\left(1 - N\int_{U(N)} \mathrm{d}\mu_{\mathrm{Haar}}(U)|U_{11}|^4\right), \quad (14)$$

where $U_{11} = \langle 1|U|1\rangle$. From the proof of Theorem 1 in [68], we have

$$\int_{U(N)} \mathrm{d}\mu_{\mathrm{Haar}}(U)|U_{11}|^4 = (N-1)B(3, N-1) = (N-1)\frac{\Gamma(3)\Gamma(N-1)}{\Gamma(N+2)}$$

$$= (N-1)\frac{2}{(N+1)N(N-1)} = \frac{2}{N(N+1)}. \quad (15)$$

Substituting (15) into (14) one gets (12). $\square$

Comparing (12) in Theorem 4.1 with (19) in Theorem 4 of [68], we see that the mean value $\mathbb{E}_U[\mathbf{CGP_S}(U)]$ of skew information-based CGP for random unitary channels coincides with average skew information-based coherence $\mathbb{E}_\rho[C_I(\rho)]$ for random quantum mixed states. This fact can be explained as follows. On the one hand, any quantum mixed state can be diagonalized, i.e., $\rho = U\Lambda U^\dagger$, where Λ is a diagonal matrix and U is a unitary operator. Hence, for a random quantum mixed state, the randomness resides in both the diagonal matrix Λ and the unitary operator U. On the other hand, according to Eq. (2), the CGP of a unitary channel with respect to skew information-based coherence is defined via probabilistic averages, in which the probability measure is on a uniform ensemble of incoherent states (diagonal matrices). Therefore, it is not surprising that these two quantities are equal.

Define the normalized CGP $\widetilde{\mathbf{CGP_S}}(U) := \mathbf{CGP_S}(U)/\mathrm{CGP}_N \leq 1$. Then it follows that $\mathbb{E}_U[\widetilde{\mathbf{CGP_S}}(U)] = \frac{N}{N+1}$, which coincides with the normalized CGP based on the Hilbert-Schmidt norm of coherence presented in [56]. Let (X, d_1) and (Y, d_2) be two metric spaces and $T : X \to Y$ be a mapping. T is called a Lipschitz continuous mapping on X with the Lipschitz constant η, if there exists $\eta > 0$ such that $d_2(T(x), T(y)) \leq \eta d_1(x, y)$ holds for all $x, y \in X$ [69].

Let $X : \mathrm{U(N)} \to \mathbb{R}$ be a Lipschitz continuous function from the unitary group to the real line with a Lipschitz constant K, i.e., $|X(U) - X(V)| \leq K\|U - V\|_2$, with $\|\cdot\|_2$ denoting the Hilbert-Schmidt norm of a matrix A, i.e., $\|A\|_2 := \sqrt{\mathrm{Tr}A^\dagger A}$ [70]. Let $U \in \mathrm{U(N)}$ be chosen uniformly at random. Then for any $\epsilon > 0$, we have the Levy's Lemma for Haar-distributed $N \times N$ unitaries [71]: $\Pr\{|X(U) - \mathbb{E}[X(U)]| \geq \epsilon\} \leq \exp\left(\frac{-N\epsilon^2}{4K^2}\right)$, where Pr denotes the probability of a random event. Using this version of Levy's Lemma,

we can similarly obtain the typicality of CGP, similar to the one presented in [56],

$$\Pr\left\{\widetilde{\mathbf{CGP}_{\mathbf{S}}}(U) \geq 1 - \frac{2}{N^{1/3}}\right\} \geq 1 - \exp\left(\frac{-N^{1/3}}{256}\right).$$

5. CGP of mixed unitary channels under skew information-based coherence

In this section, we consider the convex combinations of unitary channels of the form $\Phi(\cdot) = \sum_m p_m U_m \cdot U_m^\dagger$. In this case, $\sqrt{\Phi(\Lambda)} = \sqrt{\sum_m p_m U_m \Lambda U_m^\dagger}$. In general it is difficult to compute $\mathbf{CGP}_{\mathbf{S}}(\Phi)$ since $\sqrt{\Phi(\Lambda)}$ is hard to tackle. We present an upper bound for this class of channels.

Theorem 5.1 For mixed unitary channels $\Phi(\cdot) = \sum_m p_m U_m \cdot U_m^\dagger$, we have

$$\mathbf{CGP}_{\mathbf{S}}(\Phi) \leq \sum_m p_m \mathbf{CGP}_{\mathbf{S}}(U_m). \tag{16}$$

Proof. Note that the skew information-based coherence is a well-defined coherence measure which satisfies the convexity under classical mixing [9],

$$C_S\left(\sum_n q_n \rho_n\right) \leq \sum_n q_n C_S(\rho_n),$$

where $q_n \geq 0$ and $\sum_n q_n = 1$. It follows that

$$C_S(\Phi(\Lambda)) = C_S\left(\sum_m p_m U_m \Lambda U_m^\dagger\right) \leq \sum_m p_m C_S(U_m \Lambda U_m^\dagger).$$

Therefore, by the definition (2) we get

$$\mathbf{CGP}_{\mathbf{S}}(\Phi) = \int_\Gamma d\mu(\Lambda) C_S(\Phi(\Lambda)) \leq \sum_m p_m \int_\Gamma d\mu(\Lambda) C_S(U_m \Lambda U_m^\dagger) = \sum_m p_m \mathbf{CGP}_{\mathbf{S}}(U_m).$$

This completes the proof. $\square$

As applications, we consider the Pauli channels defined by

$$\Phi(\rho) = \sum_{m=0}^3 p_m \sigma_m \rho \sigma_m, \quad p_m \geq 0, \quad \sum_{m=0}^3 p_m = 1, \tag{17}$$

where $\sigma_0 = I$, and σ_m, $m = 1, 2, 3$, are the standard Pauli matrices. When $p_1 = p_2 = p_3 = p$, one has the depolarizing channel. When $p_1 = p$, $p_2 = p_3 = 0$ and $p_1 = p_2 = 0$, $p_3 = p$ one gets the bit-flipping channel and the phase-flipping channel, respectively. The case $p_2 = p$ and $p_1 = p_3 = 0$ corresponds to the bit-phase-flipping channel. From Eq. (8), it is easy to check that $\mathbf{CGP}_{\mathbf{S}}(\sigma_m) = 0$ for $m = 0, 1, 2, 3$. Hence, for any Pauli channel Φ we have $\mathbf{CGP}_{\mathbf{S}}(\Phi) = 0$ from (16).

As another example, consider the (unital) amplitude damping channel $\Phi(\rho) = \sum_{n=1}^{2} E_n \rho E_n^\dagger$ with

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\gamma} \end{pmatrix}.$$

We have $\Phi(\Lambda) = \Lambda$. It follows from (6) that $\mathbf{CGP_S}(\Phi) = 0$.

For the (nonunital) amplitude damping channel $\Phi(\rho) = \sum_{n=1}^{2} E_n \rho E_n^\dagger$ with

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix},$$

we have

$$\Phi(\Lambda) = \begin{pmatrix} \lambda_1 + \gamma\lambda_2 & 0 \\ 0 & (1-\gamma)\lambda_2 \end{pmatrix},$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$. This implies that $C_S(\Phi(\Lambda)) = 0$, and thus $\mathbf{CGP_S}(\Phi) = 0$.

More generally, by the property (8) given in [11], it can be seen that if the Kraus operators E_n of the channel Φ and the reference basis $\{|k\rangle\}$ satisfy $E_n|k\rangle\langle k| = |k\rangle\langle k|E_n$ for all n and k, then we have $C_S(\Phi(\rho)) \leq C_S(\rho)$, and thus $C_S(\Phi(\Lambda)) \leq C_S(\Lambda)$. For such channels, it holds that $\mathbf{CGP_S}(\Phi) = 0$.

6. Conclusions and discussions

We have introduced the measure of the coherence generating power (CGP) of a quantum channel, which is the average coherence generated by the channel acting on a uniform ensemble of incoherent states. By adopting the technique of probabilistic averages, we have successfully derived the explicit analytical formulae of CGP for any arbitrary finite dimensional unitary channels. Furthermore, we have formulated the mean value of the CGP over the unitary group, and investigated the typicality of the normalized CGP. Moreover, we have also presented an upper bound on CGP for mixed unitary channels (the convex combination of unitary channels) by using the convexity of the skew information-based coherence. Detailed examples have been provided for quibt channels.

In [56, 57] Zanardi *et al.* have studied the coherence generating power of unitary channels based on the coherence measure of Hilbert-Schmidt norm, in which the measure is in fact not well-defined and the computation involves only integrals in uniform Haar measure over pure states. Instead, the authors in [58] derived the formulae of CGP for unitary channels based on the relative entropy of coherence. The skew information-based coherence measure we adopted in this paper is also well-defined and has many important operational interpretations.

The obtained results enrich and complement the ones given in [58]. It is also worth pointing out that as the dimension $N \to \infty$, the CGP of unitary channels in [56, 57] approaches to 0, while the CGP of unitary channels in [58] does not always approach to

0. The mean value of the CGP for random unitary channels also approaches to 0 when $N \to \infty$. In comparison, numerical results show that our CGP of unitary channels and the mean value of the CGP over the unitary group both approaches to a positive number close to 0.28. Our results may shed some new light on the studies of coherence generating power of quantum channels.

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Competing interests

The authors declare no competing interests.

Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Appendix A: Proof of Theorem 3.1

Proof of Theorem 3.1 Suppose that the spectral decomposition of Λ is $\Lambda = \sum_{j=1}^{N} \lambda_j |j\rangle\langle j|$. Then $\sqrt{\Lambda} = \sum_{j=1}^{N} \sqrt{\lambda_j}|j\rangle\langle j|$. Consider the unitary channel $\Phi_U(\Lambda) = U\Lambda U^\dagger$. We have

$$
\begin{aligned}
\int_{\mathcal{I}} \mathrm{d}\mu(\Lambda) C_S(\Phi_U(\Lambda)) &= \int_{\mathcal{I}} \mathrm{d}\mu(\Lambda)\left[1 - \sum_{k=1}^{N}\langle k|U\sqrt{\Lambda}U^\dagger|k\rangle^2\right] \\
&= \int_{\mathcal{I}} \mathrm{d}\mu(\Lambda) - \sum_{k=1}^{N}\int_{\mathcal{I}} \mathrm{d}\mu(\Lambda)\left\langle k^{\otimes 2}|U^{\otimes 2}\sqrt{\Lambda}^{\otimes 2}(U^{\otimes 2})^\dagger)|k^{\otimes 2}\right\rangle \\
&= 1 - \sum_{k=1}^{N}\left\langle k^{\otimes 2}|U^{\otimes 2}\int_{\mathcal{I}} \mathrm{d}\mu(\Lambda)\sqrt{\Lambda}^{\otimes 2}(U^{\otimes 2})^\dagger)|k^{\otimes 2}\right\rangle.
\end{aligned} \tag{18}
$$

Since

$$\int_{\mathcal{I}} \mathrm{d}\mu(\Lambda)\sqrt{\Lambda}^{\otimes 2}$$

$$= \int_{\mathcal{I}} \mathrm{d}\mu(\Lambda) \sum_{1\leq i=j\leq N} \sqrt{\lambda_i\lambda_j}|ij\rangle\langle ij| + \int_{\mathcal{I}} \mathrm{d}\mu(\Lambda) \sum_{1\leq i\neq j\leq N} \sqrt{\lambda_i\lambda_j}|ij\rangle\langle ij|$$

$$= \int_{\mathcal{I}} \mathrm{d}\mu(\Lambda) \sum_{i=1}^{N} \lambda_i|ii\rangle\langle ii| + \int_{\mathcal{I}} \mathrm{d}\mu(\Lambda) \sum_{1\leq i\neq j\leq N} \sqrt{\lambda_i\lambda_j}|ij\rangle\langle ij|$$

$$= \frac{1}{N}\sum_{i=1}^{N}|ii\rangle\langle ii| + C_N^{\mathrm{HS}}\int_{\mathbb{R}_+^N} \sum_{1\leq i\neq j\leq N}\sqrt{\lambda_i\lambda_j}\delta\left(1-\sum_{j=1}^{N}\lambda_j\right)|\Delta(\lambda)|^2\prod_{j=1}^{N}\mathrm{d}\lambda_j|ij\rangle\langle ij|, \quad (19)$$

where $\Delta(\lambda) = \prod_{1\leq k<l\leq N}(\lambda_l - \lambda_k)$, and $\int_{\mathbb{R}_+^N}\sqrt{\lambda_i\lambda_j}\delta\left(1-\sum_{j=1}^{N}\lambda_j\right)|\Delta(\lambda)|^2\prod_{j=1}^{N}\mathrm{d}\lambda_j$ are the same for $i\neq j$, we only need to calculate

$$\int_{\mathbb{R}_+^N}\sqrt{\lambda_1\lambda_2}\delta\left(1-\sum_{j=1}^{N}\lambda_j\right)|\Delta(\lambda)|^2\prod_{j=1}^{N}\mathrm{d}\lambda_j.$$

Denote

$$F(t) = \int_{\mathbb{R}_+^N}\sqrt{\lambda_1\lambda_2}\delta\left(t-\sum_{j=1}^{N}\lambda_j\right)|\Delta(\lambda)|^2\prod_{j=1}^{N}\mathrm{d}\lambda_j.$$

By performing Laplace transform $(t\to s)$ of $F(t)$, and letting $\mu_j = s\lambda_j, j=1,2$, we get

$$\tilde{F}(s) = \int_{\mathbb{R}_+^N}\sqrt{\lambda_1\lambda_2}\exp\left(-s\sum_{j=1}^{N}\lambda_j\right)|\Delta(\lambda)|^2\prod_{j=1}^{N}\mathrm{d}\lambda_j$$

$$= s^{-(N^2+1)}\int_{\mathbb{R}_+^N}\sqrt{\mu_1\mu_2}\exp\left(-\sum_{j=1}^{N}\mu_j\right)|\Delta(\mu)|^2\prod_{j=1}^{N}\mathrm{d}\mu_j, \quad (20)$$

where $\Delta(\mu) = \prod_{1\leq k<l\leq N}(\mu_l - \mu_k)$. Utilizing the inverse Laplace transform $(s\to t)$: $\mathscr{L}^{-1}(s^\alpha) = \frac{t^{-\alpha-1}}{\Gamma(-\alpha)}$, we obtain

$$F(t) = \frac{t^{N^2}}{\Gamma(N^2+1)}\int_{\mathbb{R}_+^N}\sqrt{\mu_1\mu_2}\exp\left(-\sum_{j=1}^{N}\mu_j\right)|\Delta(\mu)|^2\prod_{j=1}^{N}\mathrm{d}\mu_j. \quad (21)$$

Thus

$$\int_{\mathbb{R}_+^N}\sqrt{\lambda_1\lambda_2}\delta\left(1-\sum_{j=1}^{N}\lambda_j\right)|\Delta(\lambda)|^2\prod_{j=1}^{N}\mathrm{d}\lambda_j$$

$$= \frac{1}{\Gamma(N^2+1)}\int_{\mathbb{R}_+^N}\sqrt{\mu_1\mu_2}\exp\left(-\sum_{j=1}^{N}\mu_j\right)|\Delta(\mu)|^2\prod_{j=1}^{N}\mathrm{d}\mu_j$$

$$= \frac{(N-2)!\prod_{j=1}^{N}\Gamma(j)^2}{\Gamma(N^2+1)}\left[\left(\sum_{k=1}^{N}I_{kk}^{(\frac{1}{2})}\right)^2 - \sum_{k,l=1}^{N}\left(I_{kl}^{(\frac{1}{2})}\right)^2\right], \quad (22)$$

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where we have used the following result [68],

$$\int_{\mathbb{R}_+^N} \sqrt{\mu_1 \mu_2} \exp\left(-\sum_{j=1}^N \mu_j\right) |\Delta(\mu)|^2 \prod_{j=1}^N \mathrm{d}\mu_j$$

$$= (N-2)! \prod_{j=1}^N \Gamma(j)^2 \left[\left(\sum_{k=1}^N I_{kk}^{(\frac{1}{2})}\right)^2 - \sum_{k,l=1}^N \left(I_{kl}^{(\frac{1}{2})}\right)^2\right], \qquad (23)$$

with $I_{kl}^{(\frac{1}{2})} = \sum_{r=0}^{\min(k,l)} (-1)^{k+l} \binom{\frac{1}{2}}{k-r}\binom{\frac{1}{2}}{l-r} \frac{\Gamma(\frac{3}{2}+r)}{r!}$.

From (6), (21) and (22) we obtain

$$\int_{\mathcal{I}} \mathrm{d}\mu(\Lambda)\sqrt{\Lambda}^{\otimes 2}$$

$$= \frac{1}{N}\sum_{i=1}^N |ii\rangle\langle ii| + \frac{1}{N^3(N-1)}\left[\left(\sum_{k=1}^N I_{kk}^{(\frac{1}{2})}\right)^2 - \sum_{k,l=1}^N \left(I_{kl}^{(\frac{1}{2})}\right)^2\right] \sum_{1\le i\ne j\le N} |ij\rangle\langle ij| \quad (24)$$

Combining (24) and (23), we get

$$\int_{\mathcal{I}} \mathrm{d}\mu(\Lambda) C_S(\Phi_U(\Lambda))$$

$$= 1 - \left(\frac{1}{N}\sum_{k,i=1}^N |U_{ki}|^4 + \frac{1}{N^3(N-1)}\left[\left(\sum_{k=1}^N I_{kk}^{(\frac{1}{2})}\right)^2 - \sum_{k,l=1}^N \left(I_{kl}^{(\frac{1}{2})}\right)^2\right]\sum_{k=1}^N \sum_{1\le i\ne j\le N} |U_{ki}|^2|U_{kj}|^2\right)$$

$$= 1 - \frac{1}{N}\sum_{k,i=1}^N |U_{ki}|^4 - \frac{1}{N^3(N-1)}\left[\left(\sum_{k=1}^N I_{kk}^{(\frac{1}{2})}\right)^2 - \sum_{k,l=1}^N \left(I_{kl}^{(\frac{1}{2})}\right)^2\right]\left(N - \sum_{k,i=1}^N |U_{ki}|^4\right)$$

$$= \left(1 - \frac{1}{N^2(N-1)}\left[\left(\sum_{k=1}^N I_{kk}^{(\frac{1}{2})}\right)^2 - \sum_{k,l=1}^N \left(I_{kl}^{(\frac{1}{2})}\right)^2\right]\right)\left(1 - \frac{1}{N}\sum_{k,i=1}^N |U_{ki}|^4\right),$$

which completes the proof. $\square$

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