

SKEW INVERSE POWER SERIES RINGS OVER A RING
WITH PROJECTIVE SOCLE

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Abstract. A ring R is called a right PS-ring if its socle, $\text{Soc}(R_R)$, is projective. Nicholson and Watters have shown that if R is a right PS-ring, then so are the polynomial ring $R[x]$ and power series ring $R[[x]]$. In this paper, it is proved that, under suitable conditions, if R has a (flat) projective socle, then so does the skew inverse power series ring $R[[x^{-1}; \alpha, \delta]]$ and the skew polynomial ring $R[x; \alpha, \delta]$, where R is an associative ring equipped with an automorphism α and an α -derivation δ . Our results extend and unify many existing results. Examples to illustrate and delimit the theory are provided.

Keywords: skew inverse power series ring; skew polynomial ring; annihilator; projective socle ring; flat socle ring

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1. INTRODUCTION

A right R -module M_R is called a PS-module if every simple submodule is projective and, equivalently, if its socle, $\text{Soc}(M_R)$, is projective. The study of PS-modules was initiated by Gordon in [3] and Nicholson and Watters in [13]. The class of modules with projective socles includes all nonsingular modules, regular modules, and modules with zero socle. The class of PS-modules is closed under direct sums and submodules. Xue in [19] proved that PS-modules are preserved by Morita equivalences and excellent extensions. A ring R is called a *left PS-ring* if ${}_R R$ is a PS-module. The class of rings with projective socles includes all semiprime rings, nonsingular rings, V -rings (i.e., rings all of whose simple right modules are injective) and PP rings (where a PP ring has every principal left ideal projective). In particular, every Baer ring is a PS-ring (where R is Baer if every left (or right) annihilator is generated by an idempotent). The notion of PS-rings is not left-right symmetric (see [13]). A ring

R is duo if each one-sided ideal of R is a two-sided ideal. In [19], Xue proved that a duo ring R is a PS-ring if and only if it is a right PS-ring.

Nicholson and Watters in [13] proved that R has a projective socle if and only if the full matrix ring $M_n(R)$ has a projective socle. Furthermore, it was proved in [13], Theorem 3.1, that if R is a left PS-ring, then so are the polynomial ring $R[x]$ and power series ring $R[[x]]$, but the converse is false. Also, in [12], Liu and Li showed that the commutative PS-ring condition is preserved by the generalized power series ring $R[[S; \leq]]$, where (S, \leq) is a positively strictly totally ordered monoid. Recently, Salem, Farahat and Abd-Elmalk in [16] investigated PS-modules over Ore extensions and skew generalized power series extensions.

Motivated by the results in [12], [13] and [16], we will prove that, if R is a right PS-ring, then the skew inverse power series ring $R[[x^{-1}; \alpha, \delta]]$ and skew polynomial ring $R[x; \alpha, \delta]$ are right PS-rings, where R is an α -compatible right quasi-duo ring with an automorphism α and an α -derivation δ . In particular, if R is a commutative ring which has a (flat) projective socle, then so is the differential inverse power series ring $R[[x^{-1}; \delta]]$ and the differential polynomial ring $R[x; \delta]$.

For a nonempty subset X of R , $r_R(X)$ (or $l_R(X)$) is used for the right (or left) annihilator of X over R . The right socle of R will be denoted by $\text{Soc}(R)$. Also, for a ring R , $M_n(R)$ denotes the ring of $n \times n$ matrices over R . Furthermore, we use \mathbb{Z} , \mathbb{N} , and \mathbb{Z}_n for the integers, positive integers, and integers modulo n , respectively.

2. PRELIMINARIES

Throughout this paper, R denotes an associative ring with identity, α an automorphism of R and δ an α -derivation of the ring R (i.e., δ is an additive operator on R with the property that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$). Then we denote by $R[[x^{-1}; \alpha, \delta]]$ the *skew inverse power series ring* over the coefficient ring R formed by the formal series $f(x) = \sum_{i=0}^{\infty} a_i x^{-i}$, where x is a variable and the coefficients a_i of the series $f(x)$ are elements of the ring R . In the ring $R[[x^{-1}; \alpha, \delta]]$, addition is defined as usual and multiplication is defined with respect to the relations

$$x^{-1}a = \sum_{i=1}^{\infty} \alpha^{-1}(-\delta\alpha^{-1})^{i-1}(a)x^{-i} \quad \text{for each } a \in R.$$

Skew inverse power series rings have wide applications. Not only do they provide interesting examples in noncommutative algebra, they have also been a valuable tool used first by Hilbert in the study of the independence of geometry axioms. The ring-theoretical properties of skew inverse power series rings have been investigated

by many authors (see [2], [10], [15], [14] and [17], for instance). Most of them have addressed either the case $\delta = 0$ and α an automorphism or the case where α is the identity. However, the recent surge of interest in quantum groups and quantized algebras has brought renewed interest in general skew inverse power series rings, due to the fact that many of these quantized algebras and their representations can be expressed in terms of iterated skew inverse power series rings. When we move from these “unmixed” skew inverse power series to the general case with an automorphism α and an α -derivation δ , we face a much greater challenge.

We denote by $R[x; \alpha, \delta]$ the Ore extension (*skew polynomial ring*) whose elements are the polynomials over R , addition is defined as usual and multiplication is subject to the relation $xa = \alpha(a)x + \delta(a)$ for any $a \in R$. Note that $R[x; \alpha, \delta]$ is written as $R[x; \alpha]$ and $R[x; \delta]$ when $\delta = 0$ and α is the identity map, respectively. Skew polynomial rings such as Weyl algebras and quantum groups have been a source of many interesting examples in noncommutative ring theory.

A ring R with unity is called *right (left) quasi-duo* if every maximal right (left) ideal of R is two-sided or, equivalently, every right (left) primitive homomorphic image of R is a division ring. Examples of right quasi-duo rings include, for instance, commutative rings, local rings, rings in which every nonunit has a (positive) power that is central, endomorphism rings of uniserial modules, and power series rings and rings of upper triangular matrices over any of the above-mentioned rings (see [20]). In particular, by [15], Theorem 2.7 (1), the skew inverse power series ring $R[[x^{-1}; \alpha, \delta]]$ over a right (left) quasi-duo ring R with an automorphism α and an α -derivation δ , is a right (left) quasi-duo ring. But the $n \times n$ full matrix rings over right quasi-duo rings are not right quasi-duo (for more details see [8], [9], and [20]).

According to Krempa in [7], an endomorphism α of a ring R is said to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$, for $a \in R$. A ring R is said to be α -*rigid* if there exists a rigid endomorphism α of R . In [4], Moussavi and Hashemi introduced α -compatible rings and studied their properties. A ring R is called α -*compatible* if for each $a, b \in R$, $ab = 0$ if and only if $a\alpha(b) = 0$.

Basic properties of rigid and compatible endomorphisms, proved by Hashemi and Moussavi in [4], Lemma 2.2 and 2.1, are summarized here:

Lemma 2.1. *Let α be an endomorphism of a ring R . Then:*

- (i) *if α is compatible, then α is injective;*
- (ii) *α is compatible if and only if for all $a, b \in R$, $\alpha(a)b = 0 \Leftrightarrow ab = 0$;*
- (iii) *the following conditions are equivalent:*
 - (1) *α is rigid;*
 - (2) *α is compatible and R is reduced;*
 - (3) *for every $a \in R$, $\alpha(a)a = 0$ implies that $a = 0$.*

3. MAIN RESULTS

We start with the following characterization of a left PS-ring involving maximal left ideals, which has been presented by Nicholson and Watters in [13], Theorem 2.4. This lemma plays a fundamental role in achieving our aim in this paper.

Lemma 3.1. *The following conditions are equivalent for a ring R :*

- (1) R is a left PS-ring.
- (2) If L is a maximal left ideal of R , then $r_R(L) = eR$, where $e^2 = e \in R$.

Theorem 3.2. *Let R be a right quasi-duo ring with an automorphism α and an α -derivation δ . If R is an α -compatible right PS-ring, then $R[[x^{-1}; \alpha, \delta]]$ and $R[x; \alpha, \delta]$ are right PS-rings.*

Proof. We prove the result for $R[[x^{-1}; \alpha, \delta]]$ only; the proof for $R[x; \alpha, \delta]$ is similar. Let L be a maximal right ideal of $S = R[[x^{-1}; \alpha, \delta]]$. By the right-sides version of Lemma 3.1, it is enough to show that $l_S(L) = Se$ for some idempotent $e^2 = e \in S$. Let I be the set of all trailing coefficients of the elements of L with 0 adjoined. If $I = R$, then $l_S(L) = 0$. So we assume $I \neq R$ and show that I is a maximal right ideal of R . Let $c \in R - I$. Then $c \notin L$, so $S = L + cS$. It follows that there exist $f(x) = \sum_{i=0}^{\infty} a_i x^{-i} \in L$ and $g(x) = \sum_{j=0}^{\infty} b_j x^{-j} \in S$ such that $1 = f(x) + cg(x)$. Thus $1 = a_0 + cb_0$. If $a_0 = 0$, then $1 \in cR$ and so $R = I + cR$. If $a_0 \neq 0$, then $a_0 \in I$ and again $R = I + cR$. Hence I is a maximal right ideal of R . Since R is a right PS-ring, there exists an idempotent $e^2 = e \in R$ such that $l_R(I) = Re$. We will show that $l_S(L) = Se$. If $eL \not\subseteq L$, then $S = L + eL$. So there exist $a, b \in I$ such that $1 = a + eb$. Since $eI = 0$, it follows that $1 \in I$, a contradiction. Therefore $eL \subseteq L$. Now, suppose that $f(x) = \sum_{i=0}^{\infty} a_i x^{-i} \in L$. We claim that $ea_i = 0$ for each $0 \leq i$. To the contrary, suppose that j is the minimum index such that $ea_j \neq 0$. Since ea_j is the trailing coefficient of $ef(x) \in L$, we have $ea_j \in I$. So $ea_j = e(ea_j) = 0$, a contradiction. Thus $Se \subseteq l_S(L)$. Conversely, we prove that for each $g(x) = \sum_{j=0}^{\infty} b_j x^{-j} \in l_S(L)$, we have $b_j x^{-j} = b_j x^{-j} e$ for each $0 \leq j$. To the contrary, suppose that k is the minimum index such that $b_k x^{-k} \neq b_k x^{-k} e$. Let $f(x) = \sum_{i=0}^{\infty} a_i x^{-i} \in L$. Since for each $j, j < k, b_j x^{-j} = b_j x^{-j} e$, we have $\left(\sum_{j=0}^{k-1} b_j x^{-j}\right) f(x) = \left(\sum_{j=0}^{k-1} b_j x^{-j} e\right) f(x) = 0$. On the other hand, $0 = g(x) f(x) = \left(\sum_{j=0}^{k-1} b_j x^{-j}\right) f(x) + \left(\sum_{j=k}^{\infty} b_j x^{-j}\right) f(x)$. Thus we have $\left(\sum_{j=k}^{\infty} b_j x^{-j}\right) \left(\sum_{i=0}^{\infty} a_i x^{-i}\right) = 0$ and so $b_k \alpha^{-k}(a_0) = 0$. Since α is surjective,

$b_k = \alpha^{-k}(c)$ for some $c \in R$. So $ca_0 = 0$. Hence $c \in l_R(I) = Re$. Thus $c = ce$. Hence $b_k = \alpha^{-k}(c) = \alpha^{-k}(ce) = b_k\alpha^{-k}(e)$. Since R is a right quasi-duo ring, it follows that $l_R(I)$ is an ideal. Therefore, $c\delta(e) \in l_R(I)$ and so $c\delta(e) = c\delta(e)e$. On the other hand, we have $c\delta(e) = c\delta(e)e + c\alpha(e)\delta(e)$. Hence $\alpha^k(b_k)\alpha(e)\delta(e) = 0$. Now, the α -compatibility of R implies that $b_k\alpha^{-k}(e)\delta(e) = b_k\delta(e) = 0$. By using again the α -compatibility of R , we get $b_k\alpha^p(\delta(e)) = 0$ for each p . Similarly, we have $b_k\alpha^p(\delta^l(e)) = 0$ for each p and $0 < l$. Thus $b_kx^{-k} = b_kx^{-k}e$, a contradiction. So we get $b_jx^{-j} = b_jx^{-j}e$ for each $n \leq j$ and consequently, $g(x) = g(x)e$. It follows that $l_S(I) \subseteq Se$, and the proof is complete. \square

Corollary 3.3. *Let R be a commutative ring with an automorphism α and an α -derivation δ . If R is an α -compatible PS-ring, then so are $R[[x^{-1}; \alpha, \delta]]$ and $R[x; \alpha, \delta]$.*

Corollary 3.4. *Let R be a right quasi-duo ring, $S = R[[x^{-1}; \delta_1]] \dots [[x^{-1}; \delta_n]]$ be an iterated differential inverse power series ring, where each δ_i is a derivation of $R[[x^{-1}; \delta_1]] \dots [[x^{-1}; \delta_{i-1}]]$. If R has right projective socle, then so does S .*

Proof. The result follows from [14], Theorem 2.4 (1), and Theorem 3.2. \square

Corollary 3.5. *Let R be a commutative ring with a derivation δ . If R has a projective socle, then so does $R[x; \delta]$.*

Recall from [18] that R is a *left FS-ring* if $\text{Soc}({}_R R)$ is flat. This class includes all SF-rings (i.e., rings whose simple modules are flat). Obviously PS-rings are FS-rings. Also, the class of right FS-rings is closed under the polynomial extensions, direct products, and excellent extensions (for more details see [11] and [18]).

By combining [18], Proposition 8, and Corollary 3.3, we obtain the following.

Corollary 3.6. *Let R be a commutative ring with an automorphism α and an α -derivation δ . If R is an α -compatible FS-ring, then so are $R[[x^{-1}; \alpha, \delta]]$ and $R[x; \alpha, \delta]$.*

Corollary 3.7. *Let R be a commutative ring with a derivation δ . If R has a flat socle, then so does $R[x; \delta]$.*

The following example shows that there is a large class of noncommutative rings which satisfy conditions of Theorem 3.2.

Example 3.8. (i) Let R be the ring of upper triangular $n \times n$ matrices over a division ring D . An easy computation shows that $\text{Soc}({}_R R)$ consists of all matrices of R with only nonzero entries in the first row. Hence R is a left PS-ring. On the

other hand, from [20], Proposition 2.1, it follows that R is a left quasi-duo ring. Hence $R[[x^{-1}; \delta]]$ has a projective socle by Theorem 3.2, for any derivation δ of R .

(ii) Let $S = \mathbb{Z}_2[x]/(x^2)$, and $R = \begin{pmatrix} \mathbb{Z}_2 & S \\ 0 & S \end{pmatrix}$, where \mathbb{Z}_2 is the ring of integers modulo 2. Then by [13], Example 2.7 (1), R is a left PS-ring. Also, R is a left quasi-duo ring, by [6], Proposition 10. Thus $R[[x^{-1}; \delta]]$ has a projective socle by Theorem 3.2, for any derivation δ on R .

(iii) Let R_1 be a right PS-ring, right quasi-duo ring and c a central invertible element of R_1 . Put $R := R_1[[y]]$. Let α be the automorphism on R determined by $\alpha(y) = cy$. For each $x \in R$, consider $\delta_x: R \rightarrow R$ given by $\delta_x(r) = \alpha(r)x - xr$ for each $r \in R$. Then δ_x is an α -derivation of R for each $x \in R$. It is easy to see that R is an α -compatible ring. On the other hand, by Theorem 3.2 and [15], Theorem 2.7 (1), R is a right PS-ring and right quasi-duo ring. Therefore, by Theorem 3.2, $R[[x^{-1}; \alpha, \delta]]$ and $R[x; \alpha, \delta]$ are right PS-rings.

The following example shows that there exists a commutative ring R with an α -derivation δ such that $R[[x^{-1}; \alpha, \delta]]$ and $R[x; \alpha, \delta]$ have a projective socle but R itself does not have a flat socle. Hence the converse of Corollaries 3.3 and 3.6 is not true in general.

Example 3.9. Let $R = \mathbb{Z}_2[x]/(x^2)$, where (x^2) is a principal ideal generated by x^2 of the polynomial ring $\mathbb{Z}_2[x]$ over the field \mathbb{Z}_2 of two elements. Now, let α be the identity map on R and we define an α -derivation δ on R by $\delta(\bar{x}) = 1$, where $\bar{x} = x + (x^2)$ in R . Then by [15], Example 4.8, the skew inverse power series ring $R[[x^{-1}; \alpha, \delta]]$ is a Baer ring and hence it has a projective socle. Also, by [1], Example 11,

$$R[x; \alpha, \delta] = R[x; \delta] \cong M_2(\mathbb{Z}_2[y^2]) \cong M_2(\mathbb{Z}_2[t]).$$

Since $\mathbb{Z}_2[t]$ is a principal integral domain, $\mathbb{Z}_2[t]$ is a Prüfer domain (i.e., all finitely generated ideals are invertible). So by [5], Exercise 3, page 17, $M_2(\mathbb{Z}_2[t])$ is Baer. Therefore $R[x; \alpha, \delta] = R[x; \delta]$ is Baer and hence it has a projective socle. But by [13], Example 2.7 (2), R has not a flat socle.

The following example shows that right PS-rings are independent of right quasi-duo rings and do not imply each other.

Example 3.10. (i) Suppose that R is a right PS-ring. Then by [13], Theorem 4.1, the $n \times n$ full matrix ring $M_n(R)$ is a right PS-ring for all positive integers $n \geq 2$. But $M_n(R)$ is not a right quasi-duo ring, by [8], Corollary 4.5.

(ii) For a field \mathbb{F} , let R be the ring of matrices $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ with $a, b \in \mathbb{F}$. It is easy to see that R has not a projective socle, but clearly R is a quasi-duo ring.

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