Skew lattice structures on the financial events plane

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Abstract. In this paper we show that the plane of financial events (introduced recently by one of the authors) can be endowed, in a natural way, with skew lattice structures. These structures, far from being merely pure mathematical ones, have a precise financial dynamical meaning, indeed the real essence of the structures introduced in the paper is a dynamical one. Moreover this dynamical structures fulfill several meaningful properties. In the paper several theorems are proved about these structures and some applications are given.

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1 Preliminaries on skew lattices

Skew lattices represent the most studied class of non-commutative lattices. The study of non-commutative variations of lattices originates in Jordan's 1949 paper [15]. The current study of skew lattices began with the 1989 paper of Leech [13], where the fundamental structural theorems were proved. The importance of skew lattices lies in the structural role they play in the study of discriminator varieties, see Bignall and Leech [2]. A recent result of Cvetko-Vah and Leech states that if the set of idempotents $E(R)$ in a ring R is closed under multiplication then the join operation can be defined so that $E(R)$ forms a skew lattice, see [12] for the details.

1.1 Basic definitions

An algebraic structure $(S, \wedge, \vee)$ is said a *skew lattice* if

- both operations $\wedge$ and $\vee$ are associative;
- the two operations satisfy the absorption laws

$$x \wedge (x \vee y) = x \ , \quad (y \vee x) \wedge x = x$$

  and their corresponding dual relations.
- If one of the two operations $\wedge$, $\vee$ is commutative, then so is the other one, and we have a (commutative) lattice.

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This implies that all the elements of a skew lattice are idempotent for both operations, in other words the two equalities $x \wedge x = x$ and $x \vee x = x$ hold for all elements $x \in S$.

Definition 1.1. A skew lattice is said to be *cancellative* if

- the equality $x \wedge y = x \wedge z$ together with the dual relation $x \vee y = x \vee z$ imply the equality $y = z$;
- $x \wedge y = z \wedge y$ together with $x \vee y = z \vee y$ imply $x = z$.

Cancellation is equivalent to distributivity in the commutative case.

1.2 Green's equivalence relations

On a skew lattice $(S, \wedge, \vee)$ the three canonical *Green's equivalence relations* $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{D}$ on S are defined by the equivalences

$$\begin{aligned} a\mathcal{R}b &\Leftrightarrow (a \wedge b = b \textit{ and } b \wedge a = a) \Leftrightarrow (a \vee b = a \textit{ and } b \vee a = b) \\ a\mathcal{L}b &\Leftrightarrow (a \wedge b = a \textit{ and } b \wedge a = b) \Leftrightarrow (a \vee b = b \textit{ and } b \vee a = a) \end{aligned}$$

and by the equivalences

$$\begin{aligned} a\mathcal{D}b &\Leftrightarrow (a \wedge b \wedge a = a \textit{ and } b \wedge a \wedge b = b) \\ &\Leftrightarrow (a \vee b \vee a = a \textit{ and } b \vee a \vee b = b), \end{aligned}$$

for any points a, b in S.

The *Leechs First Decomposition Theorem* for skew lattices states that on any skew lattice $(S, \wedge, \vee)$ the Green's relation $\mathcal{D}$ is a congruence with respect to both the operations $\wedge$, $\vee$; each $\mathcal{D}$-class is a rectangular band and the quotient space $S/\mathcal{D}$ is a lattice, also referred to as the *maximal lattice image* of S. (See [13] for details).

1.3 Preorders induced by a skew lattice structure

On the underlying set S the *preorder induced by the skew lattice structure* $(\wedge, \vee)$ is the relation $\preceq$ on S defined by the equivalence

$$a \preceq b \Leftrightarrow a \wedge b \wedge a = a \Leftrightarrow b \vee a \vee b = b.$$

The preorder $\preceq$ determines (in the standard way) an equivalence relation, its *indifference relation*, which is nothing but the Green's equivalence $\mathcal{D}$. Consequently, the preorder on S induces a (partial) order on the lattice $S/\mathcal{D}$. When the quotient $S/\mathcal{D}$ is a chain with respect to that order, the skew lattice S itself is called a *skew chain.*

The *natural (partial) order* $\leq$ can be defined on S by the lattice structure, defining the majoration $x \leq y$ if and only if

$$x \wedge y = y \wedge x = x.$$

1.4 Right-handed and left-handed skew lattices

A skew lattice is *right-handed* if it satisfies the identities,

$$x \wedge y \wedge x = y \wedge x \ , \ \ x \vee y \vee x = x \vee y.$$

Hence the identities $x \wedge y = y$ and $x \vee y = x$ hold on each $\mathcal{D}$-class. *Left-handed* skew lattices are defined by the dual identities.

The *Leechs Second Decomposition Theorem* for skew lattices [13] states that " *On every skew lattice* $(S, \wedge, \vee)$ *the Green's relations* $\mathcal{R}$ *and* $\mathcal{L}$ *are congruencies with respect to both the operations* $\wedge$, $\vee$, *and* S *is isomorphic to the fiber product of a left-handed and a right-handed skew lattice over a common maximal lattice image, specifically to the fiber product* $S/\mathcal{R} \times_{S/\mathcal{D}} S/\mathcal{L}$.

1.5 Cosets

A skew lattice consisting of only two $\mathcal{D}$-classes is called *primitive*. The structure of primitive skew lattices was thoroughly studied in [14]. Let P be a primitive skew lattice with $\mathcal{D}$-classes A and B and assume $A > B$ on the quotient $P/\mathcal{D}$. For any point $b \in B$, the set

$$A \wedge b \wedge A = \{a \wedge b \wedge a' : a, a' \in A\}$$

is said to be a *coset* of A in B. Dually, a *coset* of B in A is any subset of the form $B \vee a \vee B$, for some $a \in A$.

All cosets of A in B and all cosets of B in A have equal power. It follows that, in the finite case, the power of each coset divides powers $|A|$ and $|B|$. The class B is partitioned by the cosets of A. Given $a \in A$, in each coset B_j of A in B there is exactly one element $b \in B$ such that $b < a$. Dually, given $b \in B$, in each coset A_i of B in A there is exactly one element $a \in A$ such that $b < a$. Given cosets A_i in A and B_j in B there is a natural bijection of cosets $\phi_{ji} : A_i \to B_j$, where $\phi_{ji}(x) = y$ iff $x > y$, i.e. iff $x \wedge y = y \wedge x = y$. Moreover, both operations $\wedge$ and $\vee$ on P are determined by the coset bijections. In the right handed case, the description of cosets can be simplified as it follows

$$A \wedge b \wedge A = b \wedge A \text{ and } B \vee a \vee B = B \vee a.$$

Indeed, for instance, $a \wedge b \wedge a' = (a \wedge b) \wedge (b \wedge a') = b \wedge a'$.

2 The space of financial events

In [7] the *space of financial events* is defined as the usual Cartesian plane $\mathbb{R}^2$. It is interpreted as the Cartesian product of a time-axis and a capital-axis. Every pair $e = (t, c)$ belonging to this plane is called a *financial event with time* t *and capital* c. If $c > 0$ $[c \geq 0$ $]$ then e is called a *strict credit* [*weak credit*], and if $c < 0$ $[c \leq 0$ $]$ then e is called a *strict debt* [*weak debt*]. If $c = 0$ then e is said a *null event*.

Let $i > -1$ and let

$$f_i(t, c) = (1 + i)^{-t} c.$$

The function f_i induces a preorder $\preceq_i$ on the space of financial events, defined by $(t_0, c_0) \preceq_i (t, c)$ if and only if $f_i(t_0, c_0) \leq f_i(t, c)$, which is further equivalent to

$$c_0(1+i)^{t-t_0} \leq c.$$

Following [7], the preorder $\preceq_i$ is called *the preorder induced by a separable capitalization factor* of rate i, since it corresponds to the *separable capitalization factor* of rate i, that is the function

$$f_i : h \mapsto (1+i)^h.$$

The preorder $\preceq_i$ induces an equivalence relation $\sim_i$ on $\mathbb{R}^2$, defined by $(t_0, c_0) \sim_i (t, c)$ if and only if $(t_0, c_0) \preceq_i (t, c)$ and $(t, c) \preceq_i (t_0, c_0)$, or equivalently,

$$(t_0, c_0) \sim_i (t, c) \Leftrightarrow f_i(t_0, c_0) = f_i(t, c).$$

The equivalence class containing an event (t_0, c_0) is given by

$$[(t_0, c_0)]_i = \{(t, (1+i)^{t-t_0} c_0) \mid t \in \mathbb{R}\}$$

and represents a smooth curve in the plane $\mathbb{R}^2$.

3 The space of financial events as a skew lattice

Definition 3.1. Given a fixed real $i > -1$, we define non-commutative meet $(\wedge_i)$ and non-commutative join $(\vee_i)$ of the space of financial events as follows:

$$(t_0, c_0) \wedge_i (t, c) = \begin{cases} (t, (1+i)^{t-t_0} c_0) & if \;\; (t_0, c_0) \preceq_i (t, c) \\ (t, c) & if \;\; (t, c) \preceq_i (t_0, c_0) \end{cases}$$

and

$$(t_0, c_0) \vee_i (t, c) = \begin{cases} (t_0, (1+i)^{t_0-t} c) & if \;\; (t_0, c_0) \preceq_i (t, c) \\ (t_0, c_0) & if \;\; (t, c) \preceq_i (t_0, c_0). \end{cases}$$

Remark 3.1. (Well posedness of the definitions). If $(t_0, c_0) \sim_i (t, c)$, then

$$(t, (1+i)^{t-t_0} c_0) = (t, c),$$

and the operations $\wedge_i$ is well defined. A similar observation shows that operation $\vee_i$ is well defined.

Remark 3.2. Note that the event $e_0 \wedge_i e$ has the time of the *second financial event* e and the event $e_0 \vee_i e$ has the time of *first financial event* e_0. It is evident that two events commute (with respect to the defined operations) if and only if they have the same time.

Theorem 3.3. *Let operations $\wedge_i$ and $\vee_i$ on the space of financial events be defined as above. Then $S_i = (\mathbb{R}^2, \wedge_i, \vee_i)$ is a skew lattice.*

Proof. We prove idempotency and associativity for operation $\wedge_i$. A dual proof can then be derived for operation $\vee_i$. Idempotency is immediate:

$$(t_0, c_0) \wedge (t_0, c_0) = (t_0, (1+i)^{t_0-t_0} c_0).$$

To see that $\wedge_i$ is associative, consider financial events $e_0 = (t_0, c_0)$, $e = (t, c)$ and $e' = (t', c')$. Consider $(e_0 \wedge_i e) \wedge_i e'$ and $e_0 \wedge_i (e \wedge_i e')$. One must check several cases for the order of events e_0, e and e' in respect to $\preceq_i$. We prove one of the non-trivial cases, the others are similar and shall be omitted. Assume that $e \preceq_i e_0 \preceq_i e'$. Then

$$(e_0 \wedge_i e) \wedge_i e' = e \wedge e' \quad \text{and} \quad e_0 \wedge_i (e \wedge_i e') = e \wedge_i e',$$

because $f_i(e \wedge_i e') = f_i(e) \leq f_i(e_0)$. The absorption follows from

$$e_0 \wedge_i (e_0 \vee_i e) = (t_0, c_0) \wedge_i (t_0, (1+i)^{t_0 - t} c) = (t_0, c_0)$$

if $e_0 \preceq_i e$, and

$$e_0 \wedge_i (e_0 \vee_i e) = e_0 \wedge_i e_0 = e_0,$$

if $e \preceq_i e_0$, and similar calculations. Therefore S_i is a skew lattice. □

4 Dynamical interpretation of the skew lattice operations

The definitions of the two operations can be restated in the following *dynamical way*.

Proposition 4.1. (Dynamical meaning of the operations). *Let*

$$\mu : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$$

be the action of the additive group of the real numbers $(\mathbb{R}, +)$ *upon the financial events plane defined by*

$$\mu(h, (t, c)) = (t + h, (1+i)^h c),$$

for every real h *and for every financial event* $e = (t, c)$. *Let us denote the financial event* $\mu(h, e)$ *simply by* $h.e$. *Then we have*

$$e_0 \wedge_i e = \begin{cases} (t - t_0).e_0 & if \;\; e_0 \preceq_i e \\ e & if \;\; e \preceq_i e_0 \end{cases}$$

and

$$e_0 \vee_i e = \begin{cases} (t_0 - t).e & if \;\; e_0 \preceq_i e \\ e_0 & if \;\; e \preceq_i e_0, \end{cases}$$

for every couple of financial event $e_0 = (t_0, c_0)$ *and* $e = (t, c)$.

Proof. It is simply a rewriting of the definitions by means of the action μ. □

Hence, the nature of the two definitions is dynamic.

Remark 4.2. For the use, in the context of financial events plane, of the dynamical systems, see [4], [6], [9] and [10]. Further research can be conducted by following [1] and [16].

Let us observe that the non commutativity of the lattice operations is a consequence of their dynamical nature. Let $e_0 = (t_0, c_0)$ and $e = (t, c)$ be two financial events, the difference $h = t - t_0$ is called *the time vector sending* e_0 *into* e.

Theorem 4.3. (Dynamical meaning of the non-commutativity). *Let $e_0 = (t_0, c_0)$ and $e = (t, c)$ be two financial events and let $h = t - t_0$ be the time vector sending e_0 into e. Then, the two commutation relations hold true:*

$$e_0 \wedge_i e = h.(e \wedge_i e_0) \ , \ \ e_0 \vee_i e = (-h).(e \vee_i e_0).$$

Proof. We have, for what concerns the meet,

$$e_0 \wedge_i e = \begin{cases} (t - t_0).e_0 & if \ \ e_0 \preceq_i e \\ (t - t).e & if \ \ e \preceq_i e_0 \end{cases}$$

and

$$e \wedge_i e_0 = \begin{cases} (t_0 - t).e & if \ \ e \preceq_i e_0 \\ (t_0 - t_0).e_0 & if \ \ e_0 \preceq_i e, \end{cases}$$

or, in equivalent form,

$$e_0 \wedge_i e = \begin{cases} (h).e_0 & if \ \ e_0 \preceq_i e \\ (0).e & if \ \ e \preceq_i e_0 \end{cases}$$

and

$$e \wedge_i e_0 = \begin{cases} (-h).e & if \ \ e \preceq_i e_0 \\ (-0).e_0 & if \ \ e_0 \preceq_i e. \end{cases}$$

It is clear, in each case, that $e_0 \wedge_i e = h.(e \wedge_i e_0)$. In a symmetric fashion we obtain the second result. □

Remark 4.4. The relations of commutation of the preceding theorem mean that the nature of non-commutativity is *dynamical* at all.

5 Financial interpretation of the skew lattice operations

Remark 5.1. *(Financial meaning of the operations).* Let e_0 and e be two financial events, we say that e_0 precedes e if the time (first projection) of e_0 is less than the time of e. From the financial point of view, the two operations, *when applied to a pair (e_0, e) of financial events such that e_0 precedes e*, describe the risk-aversion principle with respect to time. Indeed, let $e_0 = (t_0, c_0)$ and $e = (t, c)$ be two financial events in the chronological order (e_0, e), the meet of two events is always an event with time t and the join is an event at time t_0, in other words the decision-maker prefers (as shadow maximum) the events closest in the time (indeed he prefers the state at t_0 of the i-best event), also in the case the two events are equivalent at the rate i, and symmetrically, the decision-maker finds worst the events which are far in the future, even in the case of equivalence. Further, the decision-maker valuates as shadow minimum the events furthermost in the time (indeed he valuates infimum the state at t of the i-worst event), also in the case the two events are equivalent at the rate i, in other terms, the decision-maker finds worst the events which are far in the future, even in the case of equivalence.

Proposition 5.2. (Choice meaning of the operations). *The meet operation*

$$\wedge_i : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 : (e_0, e) \mapsto e_0 \wedge_i e,$$

is a choice function of the family of sets $(\inf(e_0, e))_{(e_0,e)\in\mathbb{R}^2\times\mathbb{R}^2}$*, that is a function*

$$c : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 : (e_0, e) \mapsto c(e_0, e),$$

such that $c(e_0, e) \in \inf(e_0, e)$*, for every pair* (e_0, e) *in* $\mathbb{R}^2 \times \mathbb{R}^2$*. Analogously, the join operation*

$$\vee_i : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 : (e_0, e) \mapsto e_0 \vee_i e,$$

is a choice function of the family of sets $(\sup(e_0, e))_{(e_0,e)\in\mathbb{R}^2\times\mathbb{R}^2}$*, that is, a function*

$$c : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 : (e_0, e) \mapsto c(e_0, e),$$

such that $c(e_0, e) \in \sup(e_0, e)$*, for every pair* (e_0, e) *in* $\mathbb{R}^2 \times \mathbb{R}^2$*.*

Proof. Let $e_0 = (t_0, c_0)$ and $e = (t, c)$ be two financial events, let us determine the set of the infima of the couple $\{e_0, e\}$, with respect to the preorder $\preceq_i$. We have

$$\inf(e_0, e) = \begin{cases} [e_0]_i & if \ \ e_0 \preceq_i e \\ [e]_i & if \ \ e \preceq_i e_0 \end{cases}.$$

Observing that the meet $e_0 \wedge_i e$ belongs to the set $\inf(e_0, e)$, the proof is complete. □

We can say more than the result of preceding proposition. Recall that, if $e_0 = (t_0, c_0)$ is a financial event, the evolution curve of e_0 is, by definition, the curve

$$\varepsilon(e_0) : \mathbb{R} \to \mathbb{R}^2 : t \mapsto (t - t_0).e_0.$$

Let

$$\varepsilon : \mathbb{R}^2 \to \mathcal{F}(\mathbb{R}, \mathbb{R}^2) : e_0 \mapsto \varepsilon(e_0)$$

be the application sending each financial event into the corresponding evolution curve and let $\varepsilon(\mathbb{R}^2)$ be the part of the function space $\mathcal{F}(\mathbb{R}, \mathbb{R}^2)$ image of the financial events plane by means of the application ε, i.e. the set of all the evolution curves in the financial events plane. The set $\varepsilon(\mathbb{R}^2)$ can be endowed with the total (linear) order defined by

$$\varepsilon(e_0) \leq_i \varepsilon(e) \ \ if \ and \ only \ if \ \ e_0 \preceq_i e,$$

for any financial events e_0 and e. Note that, for any two events e_0 and e, the infimum $\inf(\varepsilon(e_0), \varepsilon(e))$ of the two corresponding evolution curves (which is also a minimum) is a curve of evolution (either $\varepsilon(e_0)$ or $\varepsilon(e)$), and then a function of the time-axis $\mathbb{R}$ into the plane of financial events $\mathbb{R}^2$.

Theorem 5.3. *Let*

$$\varepsilon : \mathbb{R}^2 \to \mathcal{F}(\mathbb{R}, \mathbb{R}^2) : e_0 \mapsto \varepsilon(e_0)$$

be the application sending each financial event into the corresponding curve of evolution*. Then we have*

$$e_0 \wedge_i e = \inf(\varepsilon(e_0), \varepsilon(e))(\mathrm{pr}_1(e)) \ , \ \ e_0 \vee_i e = \sup(\varepsilon(e_0), \varepsilon(e))(\mathrm{pr}_1(e_0)),$$

or in other terms

$$e_0 \wedge_i e = \varepsilon(e_0) \wedge_i \varepsilon(e) \ (t) \ , \ \ e_0 \vee_i e = \varepsilon(e_0) \vee_i \varepsilon(e) \ (t_0),$$

for any event e_0*,* e *with times* t_0 *and* t *respectively.*

6 Basic properties of S_i

On a skew lattice $(S;\wedge,\vee)$ we introduce a right preorder $\geq_{\mathcal{R}}$ defined by $a \geq_{\mathcal{R}} b$ if and only if

$$(a \wedge b = b \textit{ and } a \vee b = a)$$

and a left preorder $\leq_{\mathcal{L}}$ defined by $a \leq_{\mathcal{L}} b$ if and only if

$$(a \wedge b = a \ \textit{ and } a \vee b = b),$$

for each a, b in S. Clearly, the Green's *equivalence relations* $\mathcal{R}$, $\mathcal{L}$ on S are induced by those preorder respectively, as indifference relations.

Theorem 6.1. *Let $\wedge_i$ and $\vee_i$ be the skew lattice operations on the space of financial events defined as above. Then:*

1) given financial events e_0 and e, the relation $e_0 \preceq_i e$ is equivalent to the equality $e_0 \wedge_i e \wedge_i e_0 = e_0$, which is further equivalent to $e \vee_i e_0 \vee_i e = e$. In other terms the preorder induced by the skew lattice structure coincides with the preorder $\preceq_i$;

2) the Green's relation $\mathcal{D}$ on S_i coincides with the indifference relation $\sim_i$;

3) the right preorder induced by the skew lattice structure is $\preceq_i$;

4) the Green's equivalence $\mathcal{R}$ coincides with the relation $\sim_i$;

5) the left preorder induced by the skew lattice structure is the natural order on each fiber $\{t\} \times \mathbb{R}$;

6) the maximal lattice image $S_i/\mathcal{D}$ is isomorphic to the chain $(\mathbb{R}, \min, \max)$, and the space of financial events is a skew chain.

Proof. 1) To see that the inequality $e_0 \preceq_i e$ is equivalent to the equality $e_0 \wedge_i e \wedge_i e_0 = e_0$, first assume that $e_0 \preceq_i e$. Direct calculation yields

$$e_0 \wedge_i e \wedge_i e_0 = e_0.$$

To prove the converse implication, let e_0 and e be such that $e_0 \wedge_i e \wedge_i e_0 = e_0$. Assume that $e \preceq_i e_0$, i.e. $f_i(t, c) \leq f_i(t_0, c_0)$. Then

$$e_0 = e_0 \wedge_i e \wedge_i e_0 = (t_0, (1 + i)^{t_0 - t} c),$$

which can only appear if $e \sim_i e_0$. So, if $e_0 \wedge_i e \wedge_i e_0 = e_0$, the only possibility is $e_0 \preceq_i e$. That $e_0 \wedge_i e \wedge_i e_0 = e_0$ is equivalent to $e \vee_i e_0 \vee_i e = e$ is a known fact in any skew lattice. 2) An immediate consequence is that relation $\mathcal{D}$ coincides with $\sim_i$. 3) Indeed, the right preorder is defined by $e_0 \geq_{\mathcal{R}} e$ if and only if

$$(e_0 \wedge_i e = e \;\&\; e_0 \vee_i e = e_0),$$

which means

$$e = e_0 \wedge_i e = \begin{cases} (t - t_0).e_0 & if \ \ e_0 \preceq_i e \\ e & if \ \ e \preceq_i e_0, \end{cases}$$

i.e. $e \preceq_i e_0$ and

$$e_0 = e_0 \vee_i e = \begin{cases} (t_0 - t).e & if \ \ e_0 \preceq_i e \\ e_0 & if \ \ e \preceq_i e_0, \end{cases}$$

i.e. $e \preceq_i e_0$.

4) It immediately follows from the preceding property, taking into account that $\sim_i$ is the indifference relation induced by the preorder $\preceq_i$ and the equivalence $\mathcal{R}$ is the indifference of the preorder $\geq_{\mathcal{R}}$.
5) Indeed, by definition of the left preorder, we have $e_0 \leq_{\mathcal{L}} e$, if and only if

$$(e_0 \wedge e = e_0 \text{ and } e_0 \vee e = e),$$

which means

$$e_0 = e_0 \wedge_i e = \begin{cases} (t - t_0).e_0 & if \;\; e_0 \preceq_i e \\ e & if \;\; e \preceq_i e_0, \end{cases}$$

i.e., $e_0 \preceq_i e$, that is $t = t_0$ and $(e_0)_2 \preceq_i (e)_2$; and

$$e = e_0 \vee_i e = \begin{cases} (t_0 - t).e & if \;\; e_0 \preceq_i e \\ e_0 & if \;\; e \preceq_i e_0, \end{cases}$$

i.e., $e = (t_0 - t).e$ and $e_0 \preceq_i e$, that is $t = t_0$ and $(e_0)_2 \preceq_i (e)_2$.
6) The $\mathcal{D}$-classes are given by f_i-images. It is clear that any functional f_i is surjective, therefore we deduce the claimed isomorphism $S_i/\mathcal{D} \cong (\mathbb{R}, \min, \max)$. □

Corollary 6.2. *Given $i > -1$, S_i is a cancellative skew lattice.*

Proof. It was proved in [11] that all skew chains are cancellative. □

Proposition 6.3. *Given $i > -1$, the skew chain S_i is right handed.*

Proof. Consider events $e_0 = (t_0, c_0)$ and $e = (t, c)$ and assume $e_0 \preceq_i e$, then

$$\begin{aligned}(e_0 \wedge_i e) \wedge_i e_0 &= (t, (1+i)^{t-t_0} c_0) \wedge_i (t_0, c_0) \\ &= (t_0, (1+i)^{t_0 - t}(1+i)^{t-t_0} c_0) = e_0 = e \wedge_i e_0\end{aligned}$$

and $(e \wedge_i e_0) \wedge_i e = e_0 \wedge_i e$, as we claimed. □

7 Binormality of S_i

Each equivalence class $[(t_0, c_0)]$ is determined by the value $f(t_0, c_0)$. The set

$$\{(0, f(t, c)) \mid (t, c) \in \mathbb{R}^2\}$$

is a sub-lattice of the skew lattice $\mathbb{R}^2$, and is isomorphic to the maximal lattice image $(\mathbb{R}, \min, \max)$; such a lattice is called a *lattice section.*
A skew lattice $(S, \wedge, \vee)$ is called *binormal* if it satisfies the identities

$$a \wedge b \wedge c \wedge a = a \wedge c \wedge b \wedge a \quad \text{and} \quad a \vee b \vee c \vee a = a \vee c \vee b \vee a.$$

A right-handed skew lattice is binormal if and only if it satisfies

$$b \wedge c \wedge a = c \wedge b \wedge a \quad \text{and} \quad a \vee b \vee c = a \vee c \vee b.$$

It follows from [14] that any skew lattice in which any maximal primitive sub-algebra $A \cup B$ has the property that A is a single coset of B in A and B is a single coset of A in B, is binormal.

Theorem 7.1. *Given any $i > -1$, the space of financial events S_i is a binormal skew lattice.*

Proof. Consider equivalence classes $A = [(t_A, c_A)]$ and $B = [(t_B, c_B)]$ with

$$f(t_B, c_B) < f(t_A, c_A).$$

Then $A \cup B$ is a primitive skew lattice, and $b \preceq_i a$ for any $b \in B$ and any $a \in A$. When is $b \leq a$ in respect to the natural *partial* order? In this case we obtain

$$(t_B, c_B) = (t_B, c_B) \wedge (t_A, c_A) = (t_A, (1+i)^{t_B - t_A} c_A),$$

which holds precisely when $t_A = t_B$. Therefore A is the single coset of B in A and B is the single coset of A in B. □

If $(S, \wedge_S, \vee_S)$ and $(T, \wedge_T, \vee_T)$ are skew lattice, then a *homomorphism of skew lattices* is any map $h : S \to T$ satisfying

$$h(x \wedge_S y) = h(x) \wedge_T h(y)$$

and the dual relation

$$h(x \vee_S y) = h(x) \vee_T h(y),$$

for all $x, y \in S$. A bijective homomorphism of skew lattices is called an *isomorphism of skew lattices*.

Corollary 7.2. *Algebraically, each skew lattice S_i is isomorphic to the direct product $\mathbb{R} \times C$ occurring when $i = 0$. Here $C = \{(0, c) \mid c \in \mathbb{R}\}$ is a right-rectangular skew lattice with the operations given by*

$$(0, c) \wedge_0 (0, d) = (0, d) \quad \text{and} \quad (0, c) \vee_0 (0, d) = (0, c).$$

In particular these various S_i are all isomorphic skew lattices.

8 A financial application

In this section we clarify the financial meaning of the skew lattice operations by means of the *order of compound interest with total time-risk aversion*, just introduced in the following subsection.

8.1 The order of compound capitalization with total time-risk aversion

Let $i > 0$ be a positive rate of interest and let $\leq'_i$ be the binary relation defined on the open half-plane of strict credits by $e_0 \leq'_i e$ if and only if

$$e_0 \leq_i e \quad \text{and} \quad t_0 \geq t,$$

for any two strict credits e_0 and e of time t_0 and t respectively. The relation $\leq'_i$ is an order, in fact it is a preorder since it is the conjunction of two preorders; moreover, it is an order since e_0 is indifferent, with respect to the preorder $\leq'_i$, to an event e if and only if e belongs to the set-curve of evolution generated by e_0 and $t_0 = t$, considered that for each time there is only one event on a curve of evolution with that time.

From a financial point of view this new order represents the rationality of a decision-maker that takes into account not only the compound capitalization at rate i of the market but that is completely risk-averse in time, indeed if $e_0 <_i e$ but $t_0 < t$, one does not consider e preferable to e_0 but incomparable with e_0, just for the inequality $t_0 < t$.

8.2 The application

The following theorem shows the relation between the preorder $\leq_i'$ and the skew lattice operations.

Theorem 8.1. *We have* $e_0 \leq_i' e$ *if and only if*

$$e \wedge_i e_0 = e_0 \text{ and } t_0 \geq t,$$

or, equivalently,

$$e \vee_i e_0 = e \text{ and } t_0 \geq t,$$

for any two strict credits e_0 *and* e *of time* t_0 *and* t *respectively.*

We present further a possible practical application. In decision problems one of the basic points of investigation is to find suprema and infima of the constraint with respect to a given preorder.

Proposition 8.2. *a) Let* K *be a compact subset of the financial events plane contained in the open half-plane of the strict credits. Then the supremum of* K *with respect to the order* $\leq_i'$ *is the non-commutative join of any event* e *with maximum* f_i*-value (at least one there exists by the Weierstrass theorem) with any event* e_0 *of* K *with minimum time (at least one exists by Weierstrass theorem) in the order* (e, e_0)*:*

$$\sup_{\leq_i'} K = e \vee_i e_0.$$

If e_0' *and* e' *are any two events such that* $f_i(e') = \min_K f_i$ *and* $\mathrm{pr}_1(e_0') = \max_K \mathrm{pr}_1$*, then* $\inf_{\leq_i'} K = e' \vee_i e_0'$.

b) Let K *be a compact subset of the financial events plane contained in the open half-plane of the strict debts. Then the supremum of* K *with respect to the order* $\leq_i'$ *is the non-commutative join of any event* e *with maximum* f_i*-value (at least one there exists by the Weierstrass theorem) with any event* e_0 *of* K *with maximum time (at least one exists by Weierstrass theorem) in the order* (e, e_0)*:* $\sup_{\leq_i'} K = e \vee_i e_0$.

If e_0' *and* e' *are any two events such that* $f_i(e') = \min_K f_i$ *and* $\mathrm{pr}_1(e_0') = \min_K \mathrm{pr}_1$*, then* $\inf_{\leq_i'} K = e' \vee_i e_0'$.

Remark 8.3. For the use and determination of extrema and Pareto boundaries, in the context of Decision Theory, see [3], [5] and [8].

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