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SLANT AND LEGENDRE CURVES IN BIANCHI-CARTAN-VRANCEANU GEOMETRY

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Abstract. We study Legendre and slant curves for Bianchi-Cartan-Vranceanu metrics. These curves are characterized through the scalar product between the normal at the curve and the vertical vector field and in the helix case they have a proper (non-harmonic) mean curvature vector field. The general expression of the curvature and torsion of these curves and the associated Lancret invariant (for the slant case) are computed as well as the corresponding variant for some particular cases. The slant (particularly Legendre) curves which are helices are completely determined.

Keywords: Bianchi-Cartan-Vranceanu metric; slant curve; Legendre curve; Lancret invariant; helix

MSC 2010: 53D15, 53B25, 53A55, 53C25

1. Preliminaries

Among the Riemannian manifolds of non-constant sectional curvature a special rôle is played by the homogeneous spaces with a large isometry group. Due to the recent approach of Hamilton-Perelman to the Poincaré conjecture (by means of Ricci flow), great interest is paid to dimension three.

It is well-known that the maximum dimension of the isotropy group of a 3-dimensional manifold is 6 and that there is no metric with 5-dimensional group. The Bianchi-Cartan-Vranceanu spaces are certain 3-dimensional homogeneous Riemannian manifolds with 6- and 4-dimensional isometry group. They form a two parameters family (with parameters denoted as l and m) containing, among others, some remarkable 3-manifolds: $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$ and the 3-dimensional Heisenberg group Nil_3. Recently, several studies have been devoted to special submanifolds in these spaces: parallel surfaces [2], biharmonic curves [6], [12] and [13], constant angle surfaces [15], graphs of constant mean curvature [19], biharmonic surfaces [21], higher order parallel and totally umbilical surfaces [23].

The aim of this paper is to continue the study of submanifolds, more precisely 1-dimensional submanifolds. In fact, two types of curves are discussed in these spaces: θ-slant curves (with θ an arbitrary angle considered between the tangent vector field and the vertical vector field $E_3 = \partial/\partial z$) and particularly at $\theta = \pi/2$, Legendre curves. We choose this subject for two reasons:

1) the rôle of Legendre curves in almost contact geometry is remarkable and well-known; in [4] the reader finds an excellent survey on these curves,

2) although the literature on Legendre curves is rich ([3], [5], [7], [16], [20], [24], [25]), slant curves have been studied until now only for the Sasakian geometry in [12], for the contact pseudo-Hermitian geometry in [14], for the f-Kenmotsu geometry in [10], in normal almost contact geometry in [8] and for warped products in [9]. Although some Bianchi-Cartan-Vranceanu metrics are almost contact metrics we prefer in the present work a unified treatment in order to emphasize the common properties of these metrics. Another feature of the present paper, as compared with [8], is that here we do not use the structural tensor field φ of $(1,1)$-type, which is a main tool in almost contact geometry.

In Section 3 we obtain a characterization for all these curves in terms of orthogonality between the E_3 and the normal vector field. Based on this result we derive the expression of the Frenet frame as well as the curvature and torsion. By defining the Lancret invariant of a θ-slant curve γ (with $\theta \neq 0, \pi$) as

$$\text{Lancret}(\gamma) = \frac{\cos\theta}{|\sin\theta|}, \tag{1.1}$$

we obtain the expression of this invariant in the Bianchi-Cartan-Vranceanu setting.

Also, in Section 3 we prove an important property of slant non-Legendre curves: in the non-geodesic case and for $l \neq 0$ these are Bertrand curves, i.e., there exists an affine dependence between curvature and torsion. In the case $m = 0$ we provide the general expression of slant (particularly Legendre) curves and their curvature, torsion and Lancret invariant, as well as of slant helices.

The last section is devoted to examples and we completely discuss three manifolds: the Euclidean space, the Heisenberg group Nil_3 and the 3-dimensional sphere $\mathbb{S}^3$. Some results already known from [8] and [12] are reobtained with particular choices of the parameters; but to the best of our knowledge, until now there have been no general expressions for Legendre curves in Nil_3 nor for their curvature and torsion in $\mathbb{S}^3$.

2. Bianchi-Cartan-Vranceanu 3 geometries and curves in Riemannian setting

Fix two real numbers l and m and denote by M^3_m the manifold $\{(x,y,z)\in\mathbb{R}^3;\ F(x,y,z)=1+m(x^2+y^2)>0\}$. We shall consider on M^3_m *the Bianchi-Cartan-Vranceanu metric* [23], page 343,

$$g_{l,m}=\frac{1}{F^2}\,\mathrm{d}x^2+\frac{1}{F^2}\,\mathrm{d}y^2+\Big(\mathrm{d}z+\frac{ly}{2F}\,\mathrm{d}x-\frac{lx}{2F}\,\mathrm{d}y\Big)^2. \tag{2.1}$$

An important feature of these metrics is their S^1-invariance, i.e., the invariance with respect to transformations

$$\begin{pmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{pmatrix}=\begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}. \tag{2.2}$$

Other remarks concerning these metrics are in [23] and the particular cases are discussed here in the last section.

An orthonormal basis is

$$E_1=F\frac{\partial}{\partial x}-\frac{ly}{2}\frac{\partial}{\partial z},\quad E_2=F\frac{\partial}{\partial y}+\frac{lx}{2}\frac{\partial}{\partial z},\quad E_3=\frac{\partial}{\partial z} \tag{2.3}$$

with the dual basis

$$\omega_1=\frac{\mathrm{d}x}{F},\quad \omega_2=\frac{\mathrm{d}y}{F},\quad \omega_3=\mathrm{d}z+\frac{ly}{2F}\,\mathrm{d}x-\frac{lx}{2F}\,\mathrm{d}y. \tag{2.4}$$

We need the covariant derivatives of E_i; if ∇ is the Levi-Civita connection of $g_{l,m}$ we have

$$\nabla_{E_1}E_1=2myE_2,\quad \nabla_{E_2}E_2=2mxE_1,\quad \nabla_{E_3}E_3=0 \tag{2.5}$$

$$\begin{cases}\nabla_{E_1}E_2=-2myE_1+\frac{1}{2}lE_3, & \nabla_{E_2}E_1=-2mxE_2-\frac{1}{2}lE_3\\ \nabla_{E_1}E_3=\nabla_{E_3}E_1=-\frac{1}{2}lE_2, & \nabla_{E_2}E_3=\nabla_{E_3}E_2=\frac{1}{2}lE_1.\end{cases} \tag{2.6}$$

From (2.5_3) it results that E_3 is a *geodesic vector field,* i.e., its integral curves are geodesics of $g_{l,m}$. In the following we call this vector field *vertical.* A straightforward computation yields that E_3 is also a Killing vector field; for the set of all Killing vector fields see [22].

Next, following [4], page 164, we recall the notion of a Frenet curve in a $(2n+1)$-dimensional manifold: Let m be an integer with $1\leqslant m\leqslant 2n+1$. The curve $\gamma\colon I\subseteq\mathbb{R}\to M$ parametrized by the arc length s is called an *m-Frenet curve* on M

if there exist m orthonormal vector fields $U_1 = \gamma', U_2, \dots, U_m$ along γ and $(m-1)$ positive smooth functions $k_1, \dots, k_{m-1}$ of s such that:

$$\nabla_{\gamma'} U_1 = k_1 U_2, \quad \nabla_{\gamma'} U_2 = -k_1 U_1 + k_2 U_3, \quad \dots, \quad \nabla_{\gamma'} U_m = k_{m-1} U_{m-1}. \tag{2.7}$$

The function k_j is called the j-th curvature of γ, and γ is

a) a geodesic if $m = 1$; then we get the well-known equation $\nabla_{\gamma'}\gamma' = 0$;

b) a *circle* if $m = 2$ and k_1 is a constant: then we have $\nabla_{\gamma'} E_1 = k_1 E_2$, $\nabla_{\gamma'} E_2 = -k_1 E_1$;

c) a *helix* of order m if $k_1, \dots, k_{m-1}$ are constants. The Frenet curve γ is called *non-geodesic* if $k_1 > 0$ everywhere on I and in dimension 3 it is called a *generalized helix* if $k_2/k_1 = \text{const}$.

3. Slant and Legendre curves in Bianchi-Cartan-Vranceanu geometry

Let γ be a 3-Frenet curve in $(M^3_m, g_{l,m})$ for which we denote the Frenet frame as usual $(T = \gamma', N, B)$ and let us consider the Frenet equations

$$\text{a) } \nabla_T T = kN, \quad \text{b) } \nabla_T N = -kT + \tau B, \quad \text{c) } \nabla_T B = -\tau N, \tag{3.1}$$

where k denotes the curvature of the curve γ and τ is its torsion.

The main notion of this paper is introduced as follows:

Definition. The vertical angle of γ is the function $\theta\colon I \to [0, \pi)$ given by:

$$\cos\theta(s) = g_{l,m}\Big(T(s), \frac{\partial}{\partial z}\Big) = g_{l,m}(T(s), E_3). \tag{3.2}$$

The curve γ is called a *slant curve* (or more precisely θ-*slant curve*) if θ is a constant function. In the particular case of $\theta = \pi/2$ the curve γ is called a *Legendre curve*.

In the following we suppose that γ is non-geodesic, i.e., $k > 0$ and then γ cannot be an integral curve of E_3 which means $\theta \neq 0, \pi$. Let $\gamma'(s) = (\gamma_1'(s), \gamma_2'(s), \gamma_3'(s))$ be the general form of $T(s)$ whose expression in the given frame

$$T = \frac{\gamma_1'}{F} E_1 + \frac{\gamma_2'}{F} E_2 + \Big[\frac{l(\gamma_2\gamma_1' - \gamma_1\gamma_2')}{2F} + \gamma_3'\Big] E_3. \tag{3.3}$$

Now we are able to prove the first important result:

Theorem 3.1. *A non-geodesic curve γ in $(M^3, g_{l,m})$ is a θ-slant curve if and only if its normal vector field N is $g_{l,m}$-orthogonal to E_3:*

$$\omega_3(N) = 0. \tag{3.4}$$

This yields the decomposition of E_3 in the Frenet frame:

$$E_3 = \cos\theta T + |\sin\theta| B. \tag{3.5}$$

Thus, a Legendre curve has $B = E_3$.

P r o o f. From (3.3) and (2.6) we have

$$\nabla_{\gamma'} E_3 = \frac{l\gamma_2'}{2F} E_1 - \frac{l\gamma_1'}{2F} E_2. \tag{3.6}$$

We derive covariantly the relation (3.2):

$$0 = -\theta' \sin\theta = g_{l,m}(kN, E_3) + g_{l,m}\Big(T, \frac{l\gamma_2'}{2} E_1 - \frac{l\gamma_1'}{2} E_2\Big) \tag{3.7}$$

and then $k\omega_3(N) = 0$. □

An important consequence of (3.5) is the expression of the Frenet frame with respect to E_i:

$$T \text{ as in (3.3)}, \quad N = \frac{\pm 1}{F|\sin\theta|}(-\gamma_2' E_1 + \gamma_1' E_2), \tag{3.8}$$
$$B = -\frac{\cos\theta}{F|\sin\theta|}(\gamma_1' E_1 + \gamma_2' E_2) + |\sin\theta| E_3$$

where for obtaining the above formulae we use two other equations:

1) the θ-slant condition reads

$$\frac{l(\gamma_2\gamma_1' - \gamma_1\gamma_2')}{2F} + \gamma_3' = \cos\theta \tag{3.9}$$

2) the unit length of γ' yields via (3.3)

$$(\gamma_1')^2 + (\gamma_2')^2 = F^2 \sin^2\theta. \tag{3.10}$$

The sign of N is fixed by the positivity of k and τ. In [18], page 155, the following notion is introduced: a non-geodesic curve is called a *slant helix* if the principal normal lines of γ make a constant angle with a fixed direction. Therefore, a slant curve is a slant helix with E_3 as the fixed direction. The second main result gives the curvature and torsion of γ:

Theorem 3.2. *If $\gamma\colon I\to (M^3, g_{l,m})$ is a θ-slant non-geodesic curve then its curvature and torsion are*

$$k=\frac{1}{F|\sin\theta|}\Big|\frac{\gamma_1'\gamma_2''-\gamma_2'\gamma_1''}{F}-l\gamma_3'\sin^2\theta+\frac{(\gamma_2\gamma_1'-\gamma_1\gamma_2')(4m-l^2)\sin^2\theta}{2}\Big|, \tag{3.11}$$

$$\tau=\Big|\frac{l}{2}+\Big|\frac{\gamma_1'\gamma_2''-\gamma_2'\gamma_1''}{F^2\sin^2\theta}-\frac{l\gamma_3'}{F}+\frac{(\gamma_2\gamma_1'-\gamma_1\gamma_2')(4m-l^2)}{2F}\Big|\cos\theta\Big|. \tag{3.12}$$

It follows that the Lancret invariant of slant curves in Bianchi-Cartan-Vranceanu geometry is

$$\mathrm{Lancret}_{\pm}(\gamma)=\frac{2\tau\pm l}{2k}. \tag{3.13}$$

A Legendre curve has

$$k=\frac{1}{F}\Big|\frac{\gamma_1'\gamma_2''-\gamma_2'\gamma_1''}{F}-l\gamma_3'+\frac{(\gamma_2\gamma_1'-\gamma_1\gamma_2')(4m-l^2)}{2}\Big|,\quad \tau=\frac{|l|}{2}. \tag{3.14}$$

P r o o f. We prove first that

$$k=|\delta\sin\theta|,\quad \tau=\Big|\frac{l}{2}+|\delta|\cos\theta\Big|, \tag{3.15}$$

where

$$\delta=\frac{1}{F\sin^2\theta}g_{l,m}(\nabla_{\gamma'}\gamma',-\gamma_2'E_1+\gamma_1'E_2). \tag{3.16}$$

From the first Frenet equation we have

$$k=g_{l,m}(\nabla_{\gamma'}\gamma',N) \tag{3.17}$$

and the expression (3.8_2) of N yields (3.15_1). By (3.5) we have

$$B=\frac{1}{|\sin\theta|}E_3-\frac{\cos\theta}{|\sin\theta|}T \tag{3.18}$$

and then

$$\nabla_{\gamma'}B=\frac{1}{|\sin\theta|}\nabla_{\gamma'}E_3-\frac{\cos\theta}{|\sin\theta|}\nabla_{\gamma'}T=\frac{l}{2F|\sin\theta|}[\gamma_2'E_1-\gamma_1'E_2]-\frac{\cos\theta}{|\sin\theta|}kN, \tag{3.19}$$

which gives (3.15_2) by choosing $+$ in (3.8_2). We deduce that $\cos\theta/|\sin\theta|=(\tau\pm l/2)/k$, which gives the Lancret invariant.

A long but straightforward computation yields

$$\delta = \frac{1}{F\sin^2\theta}\Big(\frac{\gamma_1'\gamma_2'' - \gamma_2'\gamma_1''}{F} - l\gamma_3'\sin^2\theta + \frac{(\gamma_2\gamma_1' - \gamma_1\gamma_2')(4m - l^2)\sin^2\theta}{2}\Big) \tag{3.20}$$

since

$$\begin{cases} g_{l,m}(\nabla_{\gamma'}\gamma', -\gamma_2' E_1) = -\dfrac{\mathrm{d}}{\mathrm{d}s}\Big(\dfrac{\gamma_1'\gamma_2'}{F}\Big) + \dfrac{\gamma_1'\gamma_2''}{F} + \gamma_2' g_{l,m}(\gamma', \nabla_{\gamma'}E_1), \\ g_{l,m}(\nabla_{\gamma'}\gamma', \gamma_1' E_2) = \dfrac{\mathrm{d}}{\mathrm{d}s}\Big(\dfrac{\gamma_1'\gamma_2'}{F}\Big) - \dfrac{\gamma_2'\gamma_1''}{F} - \gamma_1' g_{l,m}(\gamma', \nabla_{\gamma'}E_2). \end{cases} \tag{3.21}$$

We have

$$\begin{cases} \nabla_{\gamma'}E_1 = \Big[\dfrac{(8m - l^2)(\gamma_2\gamma_1' - \gamma_1\gamma_2')}{4F} - \dfrac{l\gamma_3'}{2}\Big]E_2 - \dfrac{l\gamma_2'}{2F}E_3, \\ \nabla_{\gamma'}E_2 = \Big[\dfrac{(l^2 - 8m)(\gamma_2\gamma_1' - \gamma_1\gamma_2')}{4F} + \dfrac{l\gamma_3'}{2}\Big]E_1 + \dfrac{l\gamma_1'}{2F}E_3. \end{cases} \tag{3.22}$$

and hence we get the conclusion. □

Corollary. *Suppose that $l \neq 0$. A θ-slant curve in $(M^3, g_{l,m})$ which is non-geodesic and non-Legendre is a Bertrand curve.*

P r o o f. Recall that γ is a Bertrand curve if there exists $a, b \in \mathbb{R}\setminus\{0\}$ such that

$$ak + b\tau = 1. \tag{3.23}$$

From (3.15) we get

$$|\delta| = \frac{k}{|\sin\theta|}, \quad \pm\tau = \frac{l}{2} + |\delta|\cos\theta.$$

Expressing $|\delta|$ from both the relations we have $-k\cos\theta/\sin\theta + (\pm\tau) = l/2$, which yields (3.23) with $a = -2\cos\theta/(l\sin\theta)$ and $b = \pm 2/l$. □

The case when the equations (3.9)–(3.10) are explicitly integrable is $m = 0$:

Theorem 3.3. *Let γ be a non-geodesic θ-slant curve in $(\mathbb{R}^3, g_{l,0})$. Then there exists a unit-length parametrization $\zeta(s) = (\cos u(s), \sin u(s))$ of the unit circle S^1 such that $\gamma = \gamma_u$ has the expression*

$$\gamma_u(s) = \Big(|\sin\theta|\int_0^s \zeta(t)\,\mathrm{d}t, s\cos\theta - \frac{l\sin^2\theta}{2}\Big[\int_0^s \Big(\cos u(t)\int_0^t \sin u(\varrho)\,\mathrm{d}\varrho - \sin u(t)\int_0^t \cos u(\varrho)\,\mathrm{d}\varrho\Big)\,\mathrm{d}t\Big]\Big). \tag{3.24}$$

Its curvature, torsion and Lancret invariant are

$$
(3.25)\qquad \begin{cases} k(s) = |u'(s) - l\cos\theta||\sin\theta| - \dfrac{l^2|\sin\theta|}{2}(\gamma_2(s)\gamma_1'(s) - \gamma_1(s)\gamma_2'(s)),\\ \tau(s) = \Big|\dfrac{l}{2} + \Big(|u'(s) - l\cos\theta| - \dfrac{l^2}{2}(\gamma_2(s)\gamma_1'(s) - \gamma_1(s)\gamma_2'(s)\Big)\cos\theta\Big|,\\ \mathrm{Lancret}(\gamma) = \dfrac{2\tau \pm l}{2k}. \end{cases}
$$

In particular, a Legendre curve in $(\mathbb{R}^3, g_{l,0})$ *has the expression*

$$
(3.26)\qquad \gamma_u^L(s) = \Big(\int_0^s \zeta(t)\,\mathrm{d}t, \\ -\frac{l}{2}\Big[\int_0^s\Big(\cos u(t)\int_0^t \sin u(\varrho)\,\mathrm{d}\varrho - \sin u(t)\int_0^t \cos u(\varrho)\,\mathrm{d}\varrho\Big)\,\mathrm{d}t\Big]\Big),
$$

and its curvature and torsion are

$$
(3.27)\qquad k(s) = |u'(s)| - \frac{l^2}{2}(\gamma_2(s)\gamma_1'(s) - \gamma_1(s)\gamma_2'(s)), \quad \tau(s) = \frac{|l|}{2}.
$$

P r o o f. Due to $F = 1$ the relation (3.10) implies the existence of $\zeta = \zeta(s)$ such that

$$
\gamma_1' = |\sin\theta|\cos u(s), \quad \gamma_2' = |\sin\theta|\sin u(s)
$$

and the integration gives (3.24). A straightforward computation yields the conclusion. □

The particular case $u(s) = ps$ with $p \neq 0$ gives the corresponding θ-slant curve

$$
(3.28)\qquad \gamma_p(s) = \Big(\frac{|\sin\theta|}{p}(\sin(ps), 1 - \cos(ps)), s\cos\theta - \frac{l\sin^2\theta}{2p}\Big(\frac{\sin(ps)}{p} - s\Big)\Big)
$$

with

$$
(3.29)\qquad \begin{cases} k(s) = |p - l\cos\theta||\sin\theta| - \dfrac{l^2|\sin\theta|^3}{2p}(\cos(ps) - 1),\\ \tau(s) = \Big|\dfrac{l}{2} + \Big[|p - l\cos\theta| - \dfrac{l^2\sin^2\theta}{2p}(\cos(ps) - 1)\Big]\cos\theta\Big| \end{cases}
$$

and the Legendre curve

$$
(3.30)\qquad \gamma_p^L(s) = \frac{1}{p}\Big(\sin(ps), 1 - \cos(ps), -\frac{l}{2}\Big(\frac{\sin(ps)}{p} - s\Big)\Big)
$$

with

$$k(s) = \left|p - \frac{l^2(\cos(ps)-1)}{2p}\right|, \quad \tau(s) = \frac{|l|}{2}. \tag{3.31}$$

Denote by h the second fundamental form of γ and by H its mean curvature field

$$H = \operatorname{trace}(h) = h(T,T) = \nabla_T T. \tag{3.32}$$

Then γ is called a *curve with proper mean curvature vector field* if there exists $\lambda \in C^\infty(\gamma)$ such that

$$\Delta H = \lambda H. \tag{3.33}$$

In particular, if $\lambda = 0$ then γ is known as a *curve with harmonic mean curvature vector field*. Here the Laplace operator Δ acts on the vector valued function H and is given by

$$\Delta H = -\nabla_T \nabla_T \nabla_T T. \tag{3.34}$$

Making use of Frenet equations, we can rewrite (3.33) as

$$-3k'kT + (k'' - k^3 - k\tau^2)N + (2k'\tau + k\tau')B = -\lambda k N. \tag{3.35}$$

It follows that both k and τ are constants, and the function λ becomes a constant too, namely

$$\lambda = \kappa^2 + \tau^2 \tag{3.36}$$

(see also Theorem 1.1 in [17]). In our framework we state the following result:

Theorem 3.4. *A non-geodesic θ-slant curve γ in $(M^3, g_{l,m})$ has a proper mean curvature vector field if and only if γ is a helix and then*

$$\lambda\ (:= \lambda_\theta) = \delta^2 + \frac{l^2}{4} + l|\delta|\cos\theta. \tag{3.37}$$

In particular, a helix Legendre curve satisfies

$$\lambda\ (:= \lambda_{\text{Legendre}}) = \delta^2 + \frac{l^2}{4} > 0. \tag{3.38}$$

Proof. We compute λ of (3.36) by using (3.15). □

Since δ appears as a main expression in all computations above we are interested in its other form. So, inspired by [6], we introduce the angle function $\beta = \beta(s)$ given by

$$\gamma_1' = F|\sin\theta|\cos\beta, \quad \gamma_2' = F|\sin\theta|\sin\beta. \tag{3.39}$$

Then one obtains

$$\delta = \beta' + 2m\sin\theta(\gamma_2\cos\beta - \gamma_1\sin\beta) - l\cos\theta. \tag{3.40}$$

Using this expression we can characterize the θ-slant helices in Bianchi-Cartan-Vranceanu manifolds. Namely, k and τ are constants if and only if δ is a constant and this condition $\delta' = 0$ reads

$$\frac{\beta''}{\beta'} = \frac{F'}{F} \tag{3.41}$$

and then $a\beta'(s) = F(\gamma(s))$ with a a positive number. Then we integrate (3.39):

Theorem 3.5. *The θ-slant helices in $(M^3_m, g_{l,m})$ are given by*

$$\begin{aligned}\gamma_a(s) = \Big(&a|\sin\theta|\sin\beta(s) + c_1, -a|\sin\theta|\cos\beta(s) + c_2, s\cos\theta \\ &+ \frac{l\sin\theta}{2}\Big[as\sin\theta + c_1\int_0^s \sin\beta(t)\,\mathrm{d}t - c_2\int_0^s \cos\beta(t)\,\mathrm{d}t\Big]\Big)\end{aligned} \tag{3.42}$$

where c_1, c_2 are real constants; here, $\beta = \beta(s)$ is a solution of the ordinary differential equation

$$a\beta' = 1 + m[(a\sin\theta\sin\beta + c_1)^2 + (-a\sin\theta\cos\beta + c_2)^2] \tag{3.43}$$

which determines also the possible F's where (3.42) holds. For $c_1 = c_2 = 0$ we get

$$\begin{aligned}\gamma_a(s) = \Big(&a|\sin\theta|\sin\Big(\Big(\frac{1}{a} + ma\sin^2\theta\Big)s\Big), \\ &- a|\sin\theta|\cos\Big(\Big(\frac{1}{a} + ma\sin^2\theta\Big)s\Big), \Big(\cos\theta + \frac{la\sin^2\theta}{2}\Big)s\Big)\end{aligned} \tag{3.44}$$

with

$$\begin{aligned}k &= \Big|\Big(\frac{1}{a} - ma\sin^2\theta - l\cos\theta\Big)\sin\theta\Big|, \\ \tau &= \Big|\frac{l}{2} + \Big|\frac{1}{a} - ma\sin^2\theta - l\cos\theta\Big|\cos\theta\Big|.\end{aligned} \tag{3.45}$$

The Legendre helices are

$$\gamma_a(s) = \Big(a\sin\Big(\Big(\frac{1}{a}+ma\Big)s\Big), -a\cos\Big(\Big(\frac{1}{a}+ma\Big)s\Big), \frac{la}{2}s\Big) \tag{3.46}$$

with

$$k = \Big|\frac{1}{a} - ma\Big|, \quad \tau = \frac{|l|}{2}, \quad \lambda_{\text{Legendre}} = \Big(\frac{1}{a}-ma\Big)^2 + \frac{l^2}{4}. \tag{3.47}$$

4. Examples

4.1. Euclidean geometry. $\mathbb{E}^3$: $m = l = 0$. (3.24)–(3.25) become

$$\gamma_u(s) = \Big(|\sin\theta|\int_0^s \zeta(t)\,\mathrm{d}t, s\cos\theta\Big) \tag{4.1}$$

with

$$k(s) = |u'(s)||\sin\theta| = \frac{\gamma_1'\gamma_2'' - \gamma_2'\gamma_1''}{|\sin\theta|}, \quad \tau(s) = |u'(s)|\cos\theta, \quad \text{Lancret}(\gamma) = \frac{\tau}{k}. \tag{4.2}$$

(3.26)–(3.27) become

$$\gamma_u^L(s) = \Big(\int_0^s \zeta(t)\,\mathrm{d}t, 0\Big) \tag{4.3}$$

with

$$k(s) = |u'(s)| = \gamma_1'\gamma_2'' - \gamma_2'\gamma_1'', \quad \tau(s) = 0. \tag{4.4}$$

It is a well known fact that the curvature of a unit-speed plane curve is $k(s) = \gamma_1'\gamma_2'' - \gamma_2'\gamma_1''$.

We get that γ_u from (4.1) is a helix if and only if u' is a constant, say p. Then γ_u is exactly γ_p given by (3.28) with $l = 0$ and then $\lambda_\theta = \lambda_{\text{Legendre}} = \lambda_u = (u')^2 = p^2$. Another (equivalent) expression is given by (3.44):

$$\gamma_a(s) = \Big(a|\sin\theta|\sin\frac{s}{a}, -a|\sin\theta|\cos\frac{s}{a}, s\cos\theta\Big) \tag{4.1 helices}$$

with $k = |\sin\theta|/a$ and $\tau = |\cos\theta|/a$.

4.2. The Heisenberg group. $\mathrm{Nil}_3\colon m = 0, l = -2$. (3.24)–(3.25) become

$$\gamma_u(s) = \biggl(|\sin\theta| \int_0^s \zeta(t)\,\mathrm{d}t, s\cos\theta + \sin^2\theta \biggl[\int_0^s \biggl(\cos u(t) \int_0^t \sin u(\varrho)\,\mathrm{d}\varrho - \sin u(t) \int_0^t \cos u(\varrho)\,\mathrm{d}\varrho\biggr)\,\mathrm{d}t\biggr]\biggr) \tag{4.5}$$

with

$$\begin{cases} k(s) = (|u'(s) + 2\cos\theta| - 2\gamma_2(s)\gamma_1'(s) + 2\gamma_1(s)\gamma_2'(s))|\sin\theta|, \\ \tau(s) = |(|u'(s) + 2\cos\theta| - 2\gamma_2(s)\gamma_1'(s) + 2\gamma_1(s)\gamma_2'(s))\cos\theta - 1|, \\ \mathrm{Lancret}(\gamma) = (\tau \pm 1)/k. \end{cases} \tag{4.6}$$

The above Lancret invariant was obtained for the general Sasakian 3-dimensional geometry in [12], page 362.

(3.26)–(3.27) become

$$\gamma_u^L(s) = \biggl(\int_0^s \zeta(t)\,\mathrm{d}t, \int_0^s \biggl(\cos u(t) \int_0^t \sin u(\varrho)\,\mathrm{d}\varrho - \sin u(t) \int_0^t \cos u(\varrho)\,\mathrm{d}\varrho\biggr)\,\mathrm{d}t\biggr) \tag{4.7}$$

with

$$k(s) = |u'(s)| - 2(\gamma_2(s)\gamma_1'(s) - \gamma_1(s)\gamma_2'(s)), \quad \tau(s) = 1. \tag{4.8}$$

(3.37)–(3.38) read

$$\lambda_\theta = \delta^2 + 1 - 2|\delta|\cos\theta, \quad \lambda_{\mathrm{Legendre}} = \delta^2 + 1 \tag{4.9}$$

where, from (3.40), $\delta(s) = u'(s) + 2\cos\theta$.

The θ-slant helices (3.44) are

$$\gamma_a(s) = \Bigl(a|\sin\theta|\sin\frac{s}{a}, -a|\sin\theta|\cos\frac{s}{a}, (\cos\theta - a\sin^2\theta)s\Bigr) \tag{4.9 helices}$$

with $k = |\sin(2\theta)|/a$ and $\tau = \bigl||1/a - 2\cos\theta|\cos\theta - 1\bigr|$ while the Legendre helices are

$$\gamma_a(s) = \Bigl(a\sin\frac{s}{a}, -a\cos\frac{s}{a}, -as\Bigr). \tag{4.10 helices}$$

The proper biharmonic curve of [12], page 364, is a γ_u of (4.5) with $u(s) = As + a$ while the non-helix slant curve of [12], page 365, corresponds to $u(s) = \ln s$.

4.3. The sphere. For $4m = l^2$ we have $\mathbb{S}^3(m)$. In particular, $m = 1,\ l = -2$ gives $\mathbb{S}^3$. (3.11)–(3.12) become for $\mathbb{S}^3$

$$k(s) = \frac{1}{|\sin\theta|(1+\gamma_1^2+\gamma_2^2)}\left|\frac{\gamma_1'\gamma_2'' - \gamma_2'\gamma_1''}{1+\gamma_1^2+\gamma_2^2} + 2\gamma_3'\sin^2\theta\right|, \tag{4.11}$$

$$\tau(s) = \left|\left|\frac{\gamma_1'\gamma_2'' - \gamma_2'\gamma_1''}{\sin^2\theta(1+\gamma_1^2+\gamma_2^2)} + \frac{\gamma_3'}{1+\gamma_1^2+\gamma_2^2}\right|\cos\theta - 1\right| \tag{4.12}$$

while (3.13) gives

$$\text{Lancret}(\gamma) = \frac{\tau \pm 1}{k}. \tag{4.13}$$

Then a Legendre curve on $\mathbb{S}^3$ satisfies

$$k(s) = \frac{1}{1+\gamma_1^2+\gamma_2^2}\left|\frac{\gamma_1'\gamma_2'' - \gamma_2'\gamma_1''}{1+\gamma_1^2+\gamma_2^2} + 2\gamma_3'\right|, \quad \tau(s) = 1. \tag{4.14}$$

Now, recall that E_3 is a Killing vector field; then our notion of a slant curve belongs to the class of *general helices* introduced in [1], page 1505. For the case of $\mathbb{S}^3$ the Lancret theorem from [1], page 1506, yields the same (Sasakian) Lancret invariant $(\tau \pm 1)/k$.

4.4. For $m > 0$ and $l = 0$ we have $M_m = S^2(4m) \times \mathbb{R}$.

4.5. If $m < 0$ and $l = 0$ then we have $M_m^3 = \mathbb{H}^2(4m) \times \mathbb{R}$ where $\mathbb{H}^2(k)$ is the hyperbolic plane of constant Gaussian curvature $k < 0$. For both the Cases 4.4 and 4.5 a slant curve has

$$k(s) = \frac{1}{F|\sin\theta|}\left|\frac{\gamma_1'\gamma_2'' - \gamma_2'\gamma_1''}{F} + 2m\sin^2(\gamma_1'\gamma_2 - \gamma_2'\gamma_1)\right|, \tag{4.15}$$

$$\tau(s) = \left|\frac{\gamma_1'\gamma_2'' - \gamma_2'\gamma_1''}{F^2\sin^2\theta} + \frac{2m(\gamma_2\gamma_1' - \gamma_1\gamma_2')}{F}\right||\cos\theta|, \tag{4.16}$$

which means that the Legendre curves in these ambient manifolds have vanishing torsion.

4.6. If $m > 0$ and $l \neq 0$ we get $SU(2)\setminus\{\infty\}$. The Legendre curves of this manifold have a constant non-null torsion. The general properties of the curves in $SU(2)$ are studied in [11].

4.7. If $m < 0$ and $l \neq 0$ we have $\widetilde{SL}(2,\mathbb{R})$. In the following, returning to the general case we use along γ the cylindrical coordinates $x(s) = r(s)\cos\beta(s),\ y(s) = r(s)\sin\beta(s),\ z(s) = z(s)$ and then (3.10) becomes

$$(r')^2 + r^2(\beta')^2 = [1 + mr^2]^2\sin^2\theta, \tag{4.17}$$

which yields

$$(\beta')^2 = \frac{[1+mr^2]^2 \sin^2\theta - (r')^2}{r^2}. \tag{4.18}$$

Now, we make the choice of a positive real constant $r = r_0$ and then

$$\beta(s) = \pm \frac{(1+mr_0^2)\sin\theta}{r_0} s. \tag{4.19}$$

We get the final expression of the curve for the positive sign in (4.19):

$$\gamma_{r_0} = \Big(r_0 \cos\Big(\frac{(1+mr_0^2)\sin\theta}{r_0} s\Big), r_0 \sin\Big(\frac{(1+mr_0^2)\sin\theta}{r_0} s\Big), \Big(\cos\theta + \frac{lr_0}{2}\Big)s\Big). \tag{4.20}$$

This curve is a helix with

$$\begin{aligned} k &= \Big|\frac{(1+mr_0^2)\sin^2\theta}{r_0} - l\sin\theta\cos\theta\Big|, \\ \tau &= \Big|\frac{l}{2} + \Big|\frac{(1+mr_0^2)\sin\theta}{r_0} - l\cos\theta\Big|\cos\theta\Big|. \end{aligned} \tag{4.21}$$

For Examples 4.4 and 4.5 this yields helices with

$$k = \frac{(1+mr_0^2)\sin^2\theta}{r_0}, \quad \tau = \frac{(1+mr_0^2)\sin\theta\cos\theta}{r_0}. \tag{4.22}$$

We can use the relation (4.22_1) in order to obtain slant curves with a prescribed curvature for the case 4.5.

Proposition 4.1. *Let $c > 0$ be given and $m = -\widetilde{m}^2 < 0$. Then on $\mathbb{H}(-4\widetilde{m}^2) \times \mathbb{R}$ the curve* (4.17) *with*

$$r_0 = \frac{\sqrt{c^2 + 4\widetilde{m}^2 \sin^4\theta} - c}{2\widetilde{m}^2 \sin^2\theta} \tag{4.23}$$

is a helix θ-slant curve with $k = c$ and $\tau = c\cos\theta/\sin\theta$.

P r o o f. The curve is

$$\gamma_{r_0} = \Big(r_0 \cos\Big(\frac{cs}{\sin\theta}\Big), r_0 \sin\Big(\frac{cs}{\sin\theta}\Big), s\cos\theta\Big)$$

and a simple computation yields the conclusion. □

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