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SLANT AND PSEUDO-SLANT SUBMANIFOLDS IN LCS-MANIFOLDS

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Abstract. We show new results on when a pseudo-slant submanifold is a LCS-manifold. Necessary and sufficient conditions for a submanifold to be pseudo-slant are given. We obtain necessary and sufficient conditions for the integrability of distributions which are involved in the definition of the pseudo-slant submanifold. We characterize the pseudo-slant product and give necessary and sufficient conditions for a pseudo-slant submanifold to be the pseudo-slant product. Also we give an example of a slant submanifold in an LCS-manifold to illustrate the subject.

Keywords: slant submanifold, pseudo-slant submanifold, LCS-manifold

MSC 2010: 53C15, 53C25

1. Introduction

The differential geometry of slant submanifolds has shown an increasing development since B. Y. Chen defined slant submanifolds in complex manifolds as a natural generalization of both the holomorphic and totally real submanifolds [6]. Many authors have studied such slant submanifolds in almost Hermitian manifolds. In [8], Lotto introduced the concept of slant submanifolds of a Riemannian manifold into an almost contact metric manifold. In [1], we defined and studied slant submanifolds of a Riemannian product manifold.

In [11], N. Papaghiuc has introduced a class of submanifolds in an almost Hermitian manifolds, called the semi-slant submanifolds, such that the class of proper CR-submanifolds and the class of slant submanifolds appear as particular cases in the class of semi-slant submanifolds.

Slant submanifolds of K-contact and Sasakian manifolds have been characterized by Cabrerizo et. al. in [4].

Carriazo defined and studied bi-slant submanifolds in almost Hermitian manifolds and simultaneously introduced the notion of pseudo-slant submanifolds in S-manifolds in [5]. The contact version of pseudo-slant submanifolds has been defined and studied by V. A. Khan and M. A. Khan in [7].

Recently Shaikh [12] introduced the notion of Lorentzian concircular structure manifolds (briefly, LCS-manifolds), giving an example which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [9] and also by Mihai and Rosca [10]. Then Shaikh and Baishya ([13]) investigated the applications of LCS-manifolds to the general theory of relativity and cosmology. The LCS-manifolds are also studied by Shaikh, Kim and Hui [14].

Motivated by the studies of the above authors, in the present paper we consider the pseudo-slant submanifolds of a LCS-manifold. The paper is organized as follows. Section 2 is concerned with preliminaries. Section 3 is devoted to the study of slant and pseudo-slant submanifolds of LCS-manifolds with the existence of slant submanifolds in LCS-manifold. We present an interesting example to illustrate the subject.

2. Preliminaries

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type (0,2) such that for each point $p \in M$, the tensor $g_p\colon T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-,+,\dots,+)$, where T_pM denotes the tangent vector space of M at p and $\mathbb{R}$ is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (non-spacelike, null, spacelike) if it satisfies $g_p(v,v) < 0$ ($\leqslant 0$, $= 0$, > 0, respectively) [2].

Definition 2.1. In a Lorentzian manifold (M,g), a vector field P is said to be concircular [15], if the (1,1)-tensor field A defined by

$$g(X,P) = A(X)$$

for all $X \in \Gamma(TM)$ satisfies

$$(\bar{\nabla}_X A)(Y) = \alpha\{g(X,Y) + \omega(X)A(Y)\},$$

where α is a non-zero scalar and ω is a closed 1-form and $\bar{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Let $\overline{M}$ be an n-dimensional Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1. \tag{2.1}$$

Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$g(X, \xi) = \eta(X) \tag{2.2}$$

the equation of the form

$$(\bar{\nabla}_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\} \quad (\alpha \neq 0) \tag{2.3}$$

holds for all vector fields $X, Y \in \Gamma(T\overline{M})$, where $\bar{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfying

$$\bar{\nabla}_X \alpha = (X\alpha) = d\alpha(X) = \varrho\eta(X), \tag{2.4}$$

ϱ being the scalar function given by $\varrho = -(\xi\alpha)$. If we put

$$\varphi X = \frac{1}{\alpha}\bar{\nabla}_X \xi, \tag{2.5}$$

then from (2.3) and (2.5) we have

$$\varphi X = X + \eta(X)\xi, \tag{2.6}$$

from which it follows that φ is a symmetric (1,1) tensor; it is called the structure tensor of the manifold.

Definition 2.2. The Lorentzian manifold $(\overline{M}, g)$ together with the unit timelike concircular vector field ξ, its associated 1-form η and a (1,1) tensor field φ is called a Lorentzian concircular structure manifold (briefly, LCS-manifold), [12].

For the sake of brevity, we denote the Lorentzian concircular structure manifold by the LCS-manifold in the rest of this paper.

LCS-manifolds, as a special case, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [9].

In a LCS-manifold ($n > 2$), the following relations hold;

$$\eta(\xi) = -1,\ \varphi\xi = 0,\ \eta(\varphi X) = 0,\ g(\varphi X, \varphi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.7}$$

$$\varphi^2 X = X + \eta(X)\xi, \tag{2.8}$$

$$S(X,\xi) = (n-1)(\alpha^2 - \varrho)\eta(X), \tag{2.9}$$

$$R(X,Y)\xi = (\alpha^2 - \varrho)[\eta(Y)X - \eta(X)Y], \tag{2.10}$$

$$R(\xi,Y)Z = (\alpha^2 - \varrho)[g(Y,Z)\xi - \eta(Z)Y], \tag{2.11}$$

$$(\bar{\nabla}_X\varphi)Y = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{2.12}$$

$$(X\varrho) = d\varrho(X) = \beta\eta(X), \tag{2.13}$$

$$R(X,Y)Z = \varphi R(X,Y)Z + (\alpha^2 - \varrho)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \tag{2.14}$$

for all $X, Y, Z \in \Gamma(TM)$ [12].

Let M be a submanifold of a LCS-manifold $\overline{M}$ with the induced metric g. Also, let ∇ and $\nabla^\perp$ be the induced connections on the tangent bundle TM and the normal bundle $T^\perp M$ of M, respectively. Then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.15}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^\perp_X V \tag{2.16}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, where h and A_V are the second fundamental form and the shape operator (corresponding to the normal vector field V), respectively, for the immersion of M into $\overline{M}$. The second fundamental form h and the shape operator A_V are related by

$$g(h(X,Y),V) = g(A_V X, Y) \tag{2.17}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$.

3. Pseudo-slant submanifolds of LCS-manifolds

Let M be a submanifold of a LCS-manifold $\overline{M}$. Then for any $X \in \Gamma(TM)$ we can write

$$\varphi X = \tau X + \nu X, \tag{3.1}$$

where τX is the tangential component and νX is the normal component of φX.

Also, for any $V \in \Gamma(T^{\perp}M)$, φV can be written in the following way:

$$\varphi V = tV + nV, \tag{3.2}$$

where tV and nV are also the tangential and normal components of φV, respectively. From (3.1) and (3.2) we can derive that the tensor fields τ, ν, t and n are also symmetric because φ is symmetric.

Throughout the paper, we consider ξ to be tangent to M. The submanifold M is said to be invariant if ν is identically zero, i.e., $\varphi X \in \Gamma(TM)$ for any $X \in \Gamma(TM)$. Also, M is said to anti-invariant if τ is identically zero, that is $\varphi X \in \Gamma(T^{\perp}M)$ for any $X \in \Gamma(TM)$.

Furthermore, for submanifolds tangent to the structure vector field ξ, there is another class of submanifolds which are called slant submanifolds.

Definition 3.1. Let M be a submanifold of a LCS-manifold $\overline{M}$. For each non-zero vector X tangent to M at x, the angle $\theta(x)$, $0 \leqslant \theta(x) \leqslant \frac{1}{2}\pi$ between φX and τX is called the slant angle or the Wirtinger angle. If the slant angle is constant, then the submanifold is also called the slant submanifold. Invariant and anti-invariant submanifolds are particular slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{1}{2}\pi$, respectively. A slant submanifold is said to be proper if the slant angle θ lies strictly between 0 and $\frac{1}{2}\pi$, i.e., $0 < \theta < \frac{1}{2}\pi$ [3].

Now, we will give the definition of the pseudo-slant submanifolds which are a generalization of the slant submanifolds.

Definition 3.2. Let $\overline{M}$ be a LCS-manifold and M an immersed submanifold in $\overline{M}$. We say that M is a pseudo-slant submanifold of a LCS-manifold $\overline{M}$ if there exist two orthogonal distributions D and $D^{\perp}$ such that

(i) TM admits the orthogonal direct decomposition

$$TM = D \oplus D^{\perp}, \quad \xi \in \Gamma(D),$$

(ii) the distribution D is slant with slant angle $\theta \neq 0$, that is, the angle between $\varphi(D)$ and D is constant,

(iii) the distribution $D^{\perp}$ is anti-invariant, that is, $\varphi(D^{\perp}) \subseteq (T^{\perp}M)$.

From the above definition, it is obvious that if $\theta = 0$ or $\theta = \frac{1}{2}\pi$, then the pseudo-slant submanifold becomes a semi-invariant submanifold or an anti-invariant submanifold, respectively. On the other hand, if we denote the dimensions of D and $D^{\perp}$ by d_1 and d_2, respectively, then we have the following cases:

(i) if $d_1 = 0$, then M is an anti-invariant submanifold,

(ii) if d_2 and $\theta = 0$, then M is an invariant submanifold,

(iii) if $d_2 = 0$ and $\theta \neq 0$, then M is a proper slant submanifold. A pseudo-slant submanifold is called proper if $d_1.d_2 \neq 0$, $\theta \neq 0$ and $\theta \neq \frac{1}{2}\pi$.

Now, let M be a pseudo-slant submanifold of a LCS-manifold $\overline{M}$. The orthogonal complement of $\varphi D^{\perp}$ in the normal bundle $T^{\perp}M$ is an invariant subbundle of $T^{\perp}M$ and is denoted by μ. We have the direct decomposition

$$T^{\perp}M = \varphi D^{\perp} \oplus \mu. \tag{3.3}$$

The covariant derivatives $\nabla\tau$ and $\nabla\nu$ are defined by

$$(\nabla_X \tau)Y = \nabla_X \tau Y - \tau(\nabla_X Y) \tag{3.4}$$

and

$$(\bar{\nabla}_X \nu)Y = \nabla^{\perp}_X \nu Y - \nu(\nabla_X Y) \tag{3.5}$$

for all X, $Y \in \Gamma(TM)$. The canonical structures τ and ν on a submanifold M are said to be parallel if $\nabla\tau = 0$ and $\bar{\nabla}\nu = 0$, respectively.

Now, we put $Q = \tau^2$; then ∇Q can be defined by

$$(\nabla_X Q)Y = \nabla_X QY - Q\nabla_X Y \tag{3.6}$$

for any $X, Y \in \Gamma(TM)$.

By using (3.4) and (3.6) it can be easily shown that for a submanifold M of a LCS-manifold $\overline{M}$, if there is a function λ on M such that

$$(\nabla_X \tau)Y = \lambda\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\} \tag{3.7}$$

for any $X, Y \in \Gamma(TM)$, then we have

$$(\nabla_X Q)Y = \lambda\{g(X, \tau Y)\xi + \eta(Y)\tau X\}. \tag{3.8}$$

Furthermore, taking into account (2.12), (3.1), (3.2), (3.4) and (3.5), we can find

$$(\nabla_X \tau)Y = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y) + \eta(Y)X\} + A_{\nu Y}X + th(X,Y) \tag{3.9}$$

and

$$(\nabla_X \nu)Y = nh(X,Y) - h(X,\tau Y) \tag{3.10}$$

for any $X, Y \in \Gamma(TM)$.

Also, by using (2.10), (3.7) and (3.9), it can be proved by direct calculation that

$$(\nabla_X \tau)Y = \alpha \cdot \lambda\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\} \tag{3.11}$$

if and only if

$$A_{\nu Y}X - A_{\nu X}Y = \frac{\alpha(\lambda-1)}{\alpha^2-\varrho}R(X,Y)\xi, \tag{3.12}$$

where α and ϱ can be given by (2.4).

Similarly, from (3.5) and (3.8), we can derive that

$$(\nabla_X \nu)Y = \eta(X)\nu\tau Y + \eta(Y)\nu\tau X \tag{3.13}$$

if and only if

$$A_{nV}Y - A_V\tau Y = g(Y,\tau tV)\xi + \eta(Y)\tau tV \tag{3.14}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$.

Here we note that invariant and anti-invariant submanifolds are special cases of pseudo-slant submanifolds. We know that the case $\nu = 0$ implies that $\varphi = \tau$ and so $\tau^2 = I + \eta \otimes \xi$. For an anti-invariant submanifold of a LCS-manifold $\overline{M}$ we have $\tau = 0$. If M is a proper slant submanifold in a LCS-manifold $\overline{M}$, we will prove that $\tau^2 X = \cos^2\theta(X + \eta(X)\xi)$ for any $X \in \Gamma(TM)$. This relation includes the invariant and anti-invariant case for $\theta = 0$ and $\theta = \frac{1}{2}\pi$, respectively.

Theorem 3.1. *Let M be a submanifold of a* LCS*-manifold $\overline{M}$ such that ξ is tangent to M. Then M is a slant submanifold if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$\tau^2 = \lambda(I + \eta \otimes \xi). \tag{3.15}$$

Moreover, if θ is the slant angle of M, then it satisfies $\lambda = \cos^2\theta$.

Proof. If M is a slant submanifold with slant angle θ, then we have

$$\cos\theta = \frac{g(\varphi X, \tau X)}{\|\varphi X\|\|\tau X\|} = \frac{\|\tau X\|}{\|\varphi X\|} \tag{3.16}$$

for any $X \in \Gamma(TM)$. On the other hand, for any $X \in \Gamma(TM)$, taking account of τ being symmetric and (3.16), we have

$$g(\tau^2 X, X) = g(\tau X, \tau X) = \cos^2\theta g(\varphi X, \varphi X) = \cos^2\theta g(X, \varphi^2 X)$$
$$= \cos^2\theta g(X, X + \eta(X)\xi).$$

Since g is a Riemannian metric, this implies that $\tau^2 = \cos^2\theta(I + \eta \otimes \xi)$. If we put $\lambda = \cos^2\theta$, we get our result that λ is also constant because θ is constant.

Conversely, we now assume the relation (3.15) holds. Then from (2.7) and (3.1), we obtain

$$\cos\theta(x) = \frac{g(\varphi X, \tau X)}{\|\varphi X\|\|\tau X\|} = \frac{g(\tau X, \tau X)}{\|\varphi X\|\|\tau X\|} = \frac{g(\tau^2 X, X)}{\|\varphi X\|\|\tau X\|}$$
$$= \lambda\frac{g(X, X + \eta(X)\xi)}{\|\varphi X\|\|\tau X\|} = \lambda\frac{g(X, X) + \eta^2(X)}{\|\varphi X\|\|\tau X\|}$$
$$= \frac{g(\varphi X, \varphi X)}{\|\varphi X\|\|\tau X\|} = \lambda\frac{\|\varphi X\|}{\|\tau X\|}.$$

Also, by using (3.16), we conclude that $\cos^2\theta(x) = \lambda$, where $\theta(x)$ is constant because λ is a constant, and so M is slant. $\square$

Corollary 3.1. *Let M be a slant submanifold of a* LCS*-manifold $\overline{M}$ with slant angle θ. Then for any $X, Y \in \Gamma(TM)$ we have*

$$g(\tau X, \tau Y) = \cos^2\theta[g(X, Y) + \eta(X)\eta(Y)], \tag{3.17}$$
$$g(\nu X, \nu Y) = \sin^2\theta[g(X, Y) + \eta(X)\eta(Y)]. \tag{3.18}$$

Proof. Taking account of τ being symmetric and Theorem 3.1, direct calculation gives (3.17). To prove (3.18), it is enough to take into account (2.7) and (3.1). $\square$

Proposition 3.1. *Let M be a slant submanifold of a* LCS*-manifold $\overline{M}$. Then $\nabla Q = 0$ if and only if M is an anti-invariant submanifold of $\overline{M}$.*

Proof. We denote the slant angle of M by θ. For any $X, Y \in \Gamma(TM)$, since $Q = \tau^2$ and M is a slant submanifold, we have

$$Q(\nabla_X Y) = \cos^2\theta\{\nabla_X Y + \eta(\nabla_X Y)\xi\}. \tag{3.19}$$

On the other hand, differentiating covariant derivative of $QY = \cos^2\theta[Y + \eta(Y)\xi]$ in the direction of X and using (2.3) and (2.5), we obtain

$$\begin{aligned}\nabla_X QY &= \cos^2\theta\{\nabla_X Y + X\eta(Y)\xi + \eta(Y)\nabla_X\xi\} \\ &= \cos^2\theta\{\nabla_X Y + \alpha g(\varphi X, Y)\xi + \eta(\nabla_X Y)\xi + \eta(Y)\alpha\varphi X\}.\end{aligned} \tag{3.20}$$

On the other hand, from (3.6), (3.19) and (3.20) we have

$$(\nabla_X Q)Y = \nabla_X QY - Q\nabla_X Y = \alpha\cos^2\theta\{g(\tau X, Y)\xi + \eta(Y)\tau X\}, \tag{3.21}$$

which implies that $\nabla Q = 0$ if and only if $\tau = 0$ or $\theta = \frac{1}{2}\pi$. Both the cases verify that M is an anti-invariant submanifold. □

Lemma 3.1. *Let M be a pseudo-slant submanifold of a* LCS*-manifold $\overline{M}$. Then at each point p of M, Q_p has only one eigenvalue $\lambda = \cos^2\theta$.*

Proof. The proof is similar to that in [8], so we omit it. □

Theorem 3.2. *Let M be a submanifold of a* LCS*-manifold $\overline{M}$ such that $\xi \in \Gamma(TM)$. Then M is a slant submanifold if and only if*

(1) *The endomorphism $Q|_D$ has only one eigenvalue at each point of M.*
(2) *There exists a function $\lambda\colon M \to (0, 1]$ such that*

$$(\nabla_X Q)Y = \alpha\cdot\lambda\{g(X, \tau Y)\xi + \eta(Y)\tau X\} \tag{3.22}$$

for any $X, Y \in \Gamma(TM)$. Furthermore, if θ is the slant angle of M, then it satisfies $\lambda = \cos^2\theta$.

Proof. If M is a slant submanifold of a LCS-manifold $\overline{M}$ with slant angle θ, then Lemma 3.1 and (3.21) imply that the relations (1) and (2) are satisfied.

Conversely, let $\lambda(p)$ be the only eigenvalue of $Q|_D$ at each point $p \in M$. Moreover, let $Y \in \Gamma(D)$ be a unit vector associated with λ, that is, $QY = \lambda Y$. Then by virtue of (2) and differentiating the covariant derivative of $QY = \lambda Y$ in the direction of X we have

$$\nabla_X QY = \nabla_X(\lambda Y),$$
$$(\nabla_X Q)Y + Q(\nabla_X Y) = X(\lambda)Y + \lambda\nabla_X Y,$$
$$\lambda\{g(\tau X, Y)\xi + \eta(Y)\tau X\} + Q(\nabla_X Y) = X(\lambda)Y + \lambda\nabla_X Y.$$

So we arrive at

$$\begin{aligned} X(\lambda)g(Y, Y) &= -g(\lambda\nabla_X Y, Y) + g(Q\nabla_X Y, Y) \\ &= g(\nabla_X Y, \lambda Y) - g(\nabla_X Y, QY) = 0, \end{aligned}$$

that is, λ is a constant function. In order to prove that M is a slant submanifold, it is enough to show that there is a constant μ such that $Q = \mu(I + \eta\otimes\xi)$. For $X \in \Gamma(TM)$ we can write $X = \overline{X} + \eta(X)\xi$, where $\overline{X} = X - \eta(X)\xi \in \Gamma(D)$. So we have $QX = Q\overline{X}$ and $Q\overline{X} = \lambda\overline{X}$ because $Q|_D = \lambda I$, that is, $QX = \lambda\overline{X} = \lambda(X - \eta(X)\xi)$. Taking $\lambda = \mu$, we get the desired assertion. □

Theorem 3.3. *Let M be a pseudo-slant submanifold of a* LCS*-manifold $\overline{M}$. Then the anti-invariant distribution $D^{\perp}$ is integrable if and only if*

$$A_{\nu U}V = A_{\nu V}U \tag{3.23}$$

for any $U, V \in \Gamma(D^{\perp})$.

P r o o f. By using (2.12), (2.16) and (3.1), we have

$$(\bar{\nabla}_U\varphi)V = \bar{\nabla}_U\varphi V - \varphi\bar{\nabla}_U V$$
$$\alpha g(U,V)\xi = -A_{\nu V}U + \nabla^{\perp}_U \nu V - \nabla^{\perp}_U \nu V - \varphi\nabla_U V - \varphi h(U,V)$$

for any $U, V \in \Gamma(D^{\perp})$. From the tangent components of this last equation we obtain

$$-\alpha g(U,V)\xi = A_{\nu V}U + \tau\nabla_U V + th(U,V),$$

which is equivalent to

$$f([U,V]) = A_{\nu U}V - A_{\nu V}U.$$

This proves our assertion. □

Theorem 3.4. *Let M be a pseudo-slant submanifold of a* LCS*-manifold $\overline{M}$. Then the slant distribution D is integrable if and only if*

$$\tau A_{\nu U}X = A_{\nu U}\tau X \tag{3.24}$$

for any $U \in \Gamma(D^{\perp})$ and $X \in \Gamma(D)$.

P r o o f. For any $X, Y \in \Gamma(D)$ and $U \in \Gamma(D^{\perp})$, by direct calculation we have

$$\begin{aligned} g([X,Y],U) &= g(\bar{\nabla}_X Y, U) - g(\bar{\nabla}_Y X, U) = g(\bar{\nabla}_Y U, X) - g(\bar{\nabla}_X U, Y) \\ &= g(\varphi\bar{\nabla}_Y U, \varphi X) - g(\varphi\bar{\nabla}_X U, \varphi Y) = g(\bar{\nabla}_Y \varphi U, \varphi X) - g(\bar{\nabla}_X \varphi U, \varphi Y) \\ &= g(\bar{\nabla}_Y \nu U, \nu X) + g(\bar{\nabla}_Y \nu U, \tau X) - g(\bar{\nabla}_X \nu U, \nu Y) - g(\bar{\nabla}_X \nu U, \tau Y). \end{aligned}$$

On the other hand, from (2.12), (2.15) and (2.16) we have

$$(\bar{\nabla}_X\varphi)U = \bar{\nabla}_X\varphi U - \varphi\bar{\nabla}_X U$$
$$-A_{\nu U}X + \nabla^{\perp}_X \nu U = \tau\nabla_X U + \nu\nabla_X U + th(X,U) + nh(X,U),$$

that is,

$$-A_{\nu U}X = \tau\nabla_X U + th(X,U) \tag{3.25}$$

and

$$(\nabla_X \nu)U = nh(X, U). \tag{3.26}$$

Also, by using (3.5) and (3.26), we conclude that

$$\begin{aligned}
g([X,Y],U) &= g(A_{\nu U}X, \tau Y) - g(A_{\nu U}Y, \tau X) + g(\nabla^{\perp}_{Y}\nu U, \nu X) - g(\nabla^{\perp}_{X}\nu U, \nu Y) \\
&= g(\tau A_{\nu U}X, Y) - g(A_{\nu U}\tau X, Y) + g((\nabla_Y \nu)U + \nu(\nabla_Y U), \nu X) \\
&\quad - g((\nabla_X \nu)U + \nu(\nabla_X U), \nu Y) \\
&= g(\tau A_{\nu U}X - A_{\nu U}\tau X, Y) + g(\nu(\nabla_Y U), \nu X) - g(\nu(\nabla_X U), \nu Y) \\
&= g(\tau A_{\nu U}X - A_{\nu U}\tau X, Y) + \sin^2\theta\{g(\nabla_Y U, X) - g(\nabla_X U, Y)\} \\
&= g(\tau A_{\nu U}X - A_{\nu U}\tau X, Y) + \sin^2\theta\{g(\nabla_X Y, U) - g(\nabla_Y X, U)\} \\
&= g(\tau A_{\nu U}X - A_{\nu U}\tau X, Y) + \sin^2\theta\{g([X,Y],U)\}.
\end{aligned}$$

So we conclude

$$\cos^2\theta g([X,Y],U) = g(\tau A_{\nu U}X - A_{\nu U}\tau X, Y),$$

which verifies our assertion. □

Next we will give an example of a slant submanifold in a LCS-manifold M to illustrate our results.

Example 3.1. Let $\mathbb{R}^7$ be the semi-Euclidean space endowed with the usual semi-Euclidean metric tensor $g = -\,\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}y_1^2 + \mathrm{d}y_2^2 + \mathrm{d}y_3^2$ and with coordinates $(t, x_1, x_2, x_3, y_1, y_2, y_3)$. We define the Lorentzian concircular structure on $\mathbb{R}^7$ by

$$\varphi\Big(\frac{\partial}{\partial t}\Big) = 0, \quad \varphi\Big(\frac{\partial}{\partial x_i}\Big) = \frac{\partial}{\partial x_i}, \quad \varphi\Big(\frac{\partial}{\partial y_i}\Big) = -\frac{\partial}{\partial y_i}, \quad 1 \leqslant i \leqslant 3$$

and

$$\xi = \frac{\partial}{\partial t}, \quad \eta = \mathrm{d}t.$$

Then for any vector field $Z = \lambda\partial/\partial t + \mu_i\partial/\partial x_i + \nu_i\partial/\partial y_i \in T(\mathbb{R}^7)$ we have

$$g(\varphi Z, \varphi Z) = \mu_i^2 + \nu_i^2, \quad g(Z,Z) = -\lambda^2 + \mu_i^2 + \nu_i^2, \quad \eta(\xi) = -1$$

and

$$\varphi^2 Z = \mu_i\frac{\partial}{\partial x_i} + \nu_i\frac{\partial}{\partial y_i} = Z + \eta(Z)\xi,$$

which implies that $g(\varphi Z, \varphi Z) = g(Z,Z) + \eta^2(Z)$.

Now, we consider the subspace M of $\mathbb{R}^7$ given by

$$\chi(s,u,v) = (s,u,v,k\sin u, k\sin v, -k\cos u, -k\cos v),$$

where k is a non-zero constant and s,u and v denote arbitrary parameters. By a direct calculation, we infer that the tangent space of M is spanned by

$$\xi = \frac{\partial}{\partial t}, \quad V_1 = \frac{\partial}{\partial x_2} + k\cos u\frac{\partial}{\partial x_3} + k\sin u\frac{\partial}{\partial y_2}, \quad V_2 = \frac{\partial}{\partial x_2} + k\cos v\frac{\partial}{\partial y_1} + k\sin v\frac{\partial}{\partial y_3}.$$

Furthermore, we obtain

$$\varphi V_1 = \frac{\partial}{\partial x_2} - k\cos u\frac{\partial}{\partial x_3} - k\sin u\frac{\partial}{\partial y_2}, \quad \varphi V_2 = \frac{\partial}{\partial x_2} - k\cos v\frac{\partial}{\partial y_1} - k\sin v\frac{\partial}{\partial y_3}.$$

So we conclude that

$$\cos\theta = \frac{g(V_1,\varphi V_1)}{\|\varphi V_1\|.\|V_1\|} = \frac{g(V_2,\varphi V_2)}{\|\varphi V_2\|.\|V_2\|} = \frac{1-k^2}{1+k^2},$$

that is, M is a slant submanifold of $\mathbb{R}^7$ with slant angle $\theta = \cos^{-1}((1-k^2)/(1+k^2))$.

For a pseudo-slant submanifold M of a LCS-manifold $\overline{M}$, if the distributions D and $D^{\perp}$ are totally geodesic in M, then M is called the pseudo-slant product of D and $D^{\perp}$.

The following theorem characterizes the pseudo-slant product.

Theorem 3.5. *Let M be a pseudo-slant submanifold of a* LCS*-manifold $\overline{M}$. Then M is a pseudo-slant product if and only if the second fundamental form h of M satisfies*

$$th(X,Z) = 0 \tag{3.27}$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(TM)$.

Proof. For any $X, Y \in \Gamma(D)$ and $U, V \in \Gamma(D^{\perp})$ we have

$$\begin{aligned} g(\nabla_X Y, U) &= -g(\nabla_X U, V) = -g(\bar{\nabla}_X U, V) = -g(\varphi\bar{\nabla}_X U, \varphi V) \\ &= -g(\bar{\nabla}_X \varphi U, \varphi Y) = -g(\bar{\nabla}_X \nu U, \tau Y) - g(\bar{\nabla}_X \nu U, \nu Y) \\ &= g(A_{\nu U}\tau Y, X) - g(\nu(\nabla_X U), \nu Y), \end{aligned}$$

that is,

$$\cos^2\theta g(\nabla_X U, Y) = -g(h(X,\tau Y), \nu U). \tag{3.28}$$

In the same way, we obtain

$$\begin{aligned}
g(\nabla_V U, X) &= g(\bar{\nabla}_V U, X) = -g(\bar{\nabla}_V X, U) = -g(\varphi\bar{\nabla}_V X, \varphi U) \\
&= -g(\bar{\nabla}_V \varphi X, \varphi U) = -g(\nabla^{\perp}_V \nu X, \nu U) - g(\bar{\nabla}_V \tau X, \nu U) \\
&= -g(h(\tau X, V), \nu U) - g((\nabla_V \nu)X + \nu(\nabla_V X), \nu U) \\
&= -g(h(\tau X, V), \nu U) - \sin^2\theta g(\nabla_V X, U),
\end{aligned}$$

that is,

$$\cos^2\theta g(\nabla_V X, U) = g(h(\tau X, V), \nu U), \tag{3.29}$$

which proves our assertion. □

Theorem 3.6. *Let M be a pseudo-slant submanifold of a* LCS*-manifold $\overline{M}$. If ν is parallel on D, then either M is a D-geodesic submanifold or $h(X,Y)$ is an eigenvector of n^2 with eigenvalue* $\cos^2\theta$.

Proof. Since $(\nabla_X \nu)Y = 0$ for any $X, Y \in \Gamma(D)$, from (3.10) we have

$$nh(X,Y) = h(X, \tau Y).$$

On the other hand, since D is a slant distribution and $\tau\xi = 0$, we obtain

$$n^2 h(X, Y + \eta(Y)\xi) = nh(X, \tau Y) = h(X, \tau^2 Y) = \cos^2\theta h(X, Y + \eta(Y)\xi).$$

This implies that either h vanishes on D or h is an eigenvector of n^2 with eigenvalue $\cos^2\theta$. □

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