Slant lightlike submanifolds of indefinite Sasakian manifolds

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Abstract. In this paper, we define and study both slant lightlike submanifolds and screen slant lightlike submanifolds of an indefinite Sasakian manifold. We provide non-trivial examples and obtain necessary and sufficient conditions for the existence of a slant lightlike submanifold.

1. Introduction

In [8], Duggal and Bejancu introduced the geometry of arbitrary lightlike submanifolds of semi-Riemannian manifolds. Since then, many authors have studied the geometry of lightlike hypersurfaces and lightlike submanifolds. Lightlike geometry has its applications in general relativity, particularly in black hole theory. Indeed, it is known that lightlike hypersurfaces are examples of physical models of Killing horizons in general relativity[13]. A Killing horizon is a lightlike hypersurface which is a local isometry horizon with respect to 1− parameter group. Physically, a particle on local isometry horizon of a 4− dimensional spacetime manifold may immediately be travelling at the speed of light along the single null generator, but standing still to relative to its surroundings. Roughly speaking, a Killing horizon is a lightlike hypersurface whose generating null vector can be normalized so as to coincide with one of the Killing vector. The surface of a black hole is described in terms of Killing horizon. This relation has its roots in Hawking's area theorem which states that if matter satisfies the dominant energy condition, then the area of the black hole can not decrease[15].

On the other hand, the theory of contact manifolds has its roots in differential equations, optics and phase space of a dynamical system (for details see [1]. Recently, Fritelli at all [12] gave a self-contained presentation of the null surface formulation of the Einstein based on the contact geometry of differential equation. The essential idea of the null surface formulation is to start from family of co-dimension one foliations of the spacetime manifold by hypersurfaces, fix the conformal structure of spacetime by requiring these hypersurface to be null and formulate the Einstein equations in terms of these data.

Let $\bar{M}$ be a Sasakian manifold with almost contact structure (ϕ, η, V) and M a Riemannian manifold isometrically immersed in $\bar{M}$ such that the structure vector field V is tangent to M. Then M is called invariant if $\phi(T_pM) = T_pM$, for every $p \in M$, where T_pM denotes the tangent space to M at the point p. M is called anti-invariant if $\phi(T_pM) \subset T_pM^{\perp}$ for every $p \in M$, where $T_pM^{\perp}$ denotes the normal space to

M at the point *p*. As a generalization of invariant and totally real submanifolds of almost contact metric manifolds, following Chen's definition [6], A. Lotta [16] and Cabrerizo et all [5] studied the geometry of slant submanifolds of a Sasakian manifold $\bar{M}$ as a real submanifold such that the angle between ϕX and $T_x M$ is constant for every vector $X \in T_x M$ and $x \in M$. The first author of this paper introduced lightlike slant submanifolds of indefinite Kaehler manifolds in [18] and [19]. On the other hand, in [10], Duggal-Şahin studied various lightlike submanifolds of indefinite Sasakian manifolds. However, the concept of slant lightlike submanifolds of indefinite Sasakian manifolds has not been studied as yet.

The objective of this paper is to introduce the notion of slant submanifolds of an indefinite Sasakian manifolds. We study the existence problem and establish an interplay between slant lightlike submanifolds and contact Cauchy Riemann (CR)-lightlike submanifolds [10].

Section 2 includes basic information on the lightlike geometry as needed in this paper. In section 3, we introduce the concept of slant lightlike submanifolds and give a non-trivial example. We show that, contrary to the Riemannian case, the geometry of slant lightlike and screen slant lightlike submanifolds is very different from the Riemannian case. We prove a characterization theorem and show that co-isotropic contact CR-lightlike submanifolds are slant lightlike submanifolds. Because, a slant lightlike submanifold of an indefinite Sasakian manifold do not contain invariant and screen real submanifolds, finally, in section 4, we introduce screen slant lightlike submanifoldds of indefinite Sasakian manifold and give an example of such submanifolds.

2. Preliminaries

An odd dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called a contact metric manifold [4] if there exists a $(1, 1)$ tensor field ϕ, a vector field V, called the characteristic vector field, and its 1-form η satisfying

$$
\begin{aligned}
\bar{g}(\phi X, \phi Y) &= \bar{g}(X, Y) - \epsilon \, \eta(X)\eta(Y), \quad \bar{g}(V, V) = \epsilon, \\
\phi^2(X) &= -X + \eta(X) V, \qquad \bar{g}(X, V) = \eta(X), \\
d\,\eta(X, Y) &= \bar{g}(X, \phi Y), \qquad \forall X, Y \in \Gamma(T\bar{M}),
\end{aligned}
\tag{1.1}
$$

where $\epsilon = 1$ or -1. It follows that $\phi V = 0$, $\quad \eta \circ \phi = 0$, $\quad \eta(V) = \epsilon$.

Then $(\phi, V, \eta, \bar{g})$ is called contact metric structure of $\bar{M}$. We say that M has a normal contact structure if $N_\phi + d\eta \otimes \xi = 0$, where N_ϕ is the Nijenhuis tensor field of ϕ [4]. A normal contact metric manifold is called a Sasakian manifold [20] for which we have

$$
\begin{aligned}
\bar{\nabla}_X V &= \phi X, \\
(\bar{\nabla}_X \phi) Y &= -\bar{g}(X, Y)V + \epsilon \, \eta(Y)X.
\end{aligned}
\tag{1.2}
\tag{1.3}
$$

We follow [8] for the notation and formulas used in this paper. A submanifold (M^m, g) immersed in a semi-Riemannian manifold $(\bar{M}^{m+n}, \bar{g})$ is called a lightlike submanifold if the metric g induced from $\bar{g}$ is degenerate and the radical distribution $Rad(TM)$ is of rank r, where $1 \le r \le m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, i.e.,

$$
TM = Rad\,(TM) \perp S(TM).
$$

Consider a screen transversal vector bundle $S(TM^\perp)$, which is a semi-Riemannian complementary vector bundle of $Rad(TM)$ in $TM^\perp$. Since, for any local basis $\{\xi_i\}$ of $Rad(TM)$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^\perp)$ in $[S(TM)]^\perp$ such that $\bar{g}(\xi_i, N_j) = \delta_{ij}$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$ [8, page 144]. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\bar{M}|_M$. Then,

$$
\begin{aligned}
tr(TM) &= ltr(TM) \perp S(TM^\perp), \\
T\bar{M}|_M &= S(TM) \perp [Rad(TM) \oplus ltr(TM)] \perp S(TM^\perp).
\end{aligned}
$$

Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM/Rad\,TM$ [14]. Following result is important to this paper.

Proposition 2.1 [8]. *The lightlike second fundamental forms of a lightlike submanifold M do not depend on $S(TM), S(TM^{\perp})$ and $ltr(TM)$.*

Throughout this paper, we will discuss the dependence (or otherwise) of the results on induced object(s) and refer [8] for their transformation equations.

Followings are four subcases of a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$.

Case 1: r-lightlike if $r < min\{m, n\}$;
Case 2: Co-isotropic if $r = n < m$; $S(TM^{\perp}) = \{0\}$;
Case 3: Isotropic if $r = m < n$; $S(TM) = \{0\}$;
Case 4: Totally lightlike if $r = m = n$; $S(TM) = \{0\} = S(TM^{\perp})$.

The Gauss and Weingarten formulas are:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \forall X, Y \in \Gamma(TM), \tag{1.4}$$
$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V, \forall X \in \Gamma(TM), V \in \Gamma(tr(TM)), \tag{1.5}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X, Y), \nabla^t_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(ltr(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $ltr(TM)$, respectively. The second fundamental form h is a symmetric $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$. Then we have

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{1.6}$$
$$\bar{\nabla}_X N = -A_N X + \nabla^l_X(N) + D^s(X, N), \tag{1.7}$$
$$\bar{\nabla}_X W = -A_W X + \nabla^s_X(W) + D^l(X, W), \quad \forall X, Y \in \Gamma(TM), \tag{1.8}$$

$N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Denote the projection of TM on $S(TM)$ by $\bar{P}$. Then, by using (1.4), (1.6)-(1.8) and taking account that $\bar{\nabla}$ is a metric connection we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{1.9}$$
$$\bar{g}(D^s(X, N), W) = \bar{g}(N, A_W X). \tag{1.10}$$

We set

$$\nabla_X \bar{P} Y = \nabla^*_X \bar{P} Y + h^*(X, \bar{P} Y), \tag{1.11}$$
$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{1.12}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$. By using above equations we obtain

$$\bar{g}(h^l(X, \bar{P} Y), \xi) = g(A^*_\xi X, \bar{P} Y), \tag{1.13}$$
$$\bar{g}(h^*(X, \bar{P} Y), N) = g(A_N X, \bar{P} Y), \tag{1.14}$$
$$\bar{g}(h^l(X, \xi), \xi) = 0 \quad, \quad A^*_\xi \xi = 0. \tag{1.15}$$

In general, the induced connection ∇ on M is not metric connection. Since $\bar{\nabla}$ is a metric connection, by using (1.6) we get

$$(\nabla_X g)(Y, Z) = \bar{g}(h^l(X, Y), Z) + \bar{g}(h^l(X, Z), Y). \tag{1.16}$$

However, it is important to note that $\nabla^\star$ is a metric connection on $S(TM)$. From now on, we briefly denote $(M, g, S(TM), S(TM^{\perp}))$ by M in this paper.

Definition 2.1 [10] *Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold tangent to the structure vector field V immersed in an indefinite Sasakian manifold $(\bar{M}, \bar{g})$. We say that M is a contact **CR**-lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:*

(A) *Rad TM is a distribution on M such that Rad TM ∩ φ(RadTM) = {0},*

(B) *There exist vector bundles D_0 and D' over M such that*

$$S(TM) = \{\phi(RadTM) \oplus D'\} \perp D_o \perp V,$$
$$\phi D_o = D_o, \phi(D') = L_1 \perp ltr(TM),$$

where D_0 is a non-degenerate distribution on M and L_1 is a vector sub bundle of $S(TM^\perp)$.

Thus, we have the following decomposition

$$TM = D \oplus V \oplus D', \tag{1.17}$$

where

$$D = Rad\, TM \perp \phi(Rad\, TM) \perp D_o. \tag{1.18}$$

A contact CR-lightlike submanifold is proper if $D_o \neq \{0\}$ and $L_1 \neq \{0\}$.

3. Slant lightlike submanifolds

We start with the following lemmas which will be useful for later results.

Lemma 3.1. *Let M be an $r-$ lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$. Suppose that $\phi RadTM$ is a distribution on M such that $RadTM \cap \phi RadTM = \{0\}$. Then $\phi ltr(TM)$ is subbundle of the screen distribution $S(TM)$ and $\phi RadTM \cap \phi ltr(TM) = \{0\}$*

Proof. Since by hypothesis $\phi RadTM$ is a distribution on M such that $\phi RadTM \cap RadTM = \{0\}$, we have $\phi RadTM \subset S(TM)$. Now we claim that $ltr(TM)$ is not invariant with respect to ϕ. Let us suppose that $ltr(TM)$ is invariant with respect to ϕ. Choose $\xi \in \Gamma(RadTM)$ and $N \in \Gamma(ltr(TM))$ such that $\bar{g}(N, \xi) = 1$. Then from the decomposition of a lightlike submanifold, we have $1 = \bar{g}(\xi, N) = \bar{g}(\phi\xi, \phi N) = 0$ due to $\phi\xi \in \Gamma(S(TM))$ and $\phi N \in \Gamma(ltr(TM))$. This is a contradiction, so $ltr(TM)$ is not invariant with respect to ϕ. Also ϕN does not belong to $S(TM^\perp)$, since $S(TM^\perp)$ is orthogonal to $S(TM)$, $\bar{g}(\phi N, \phi\xi)$ must be zero, but we have $\bar{g}(\phi N, \phi\xi) = \bar{g}(N, \xi) \neq 0$ for some $\xi \in \Gamma(RadTM)$, this is again a contradiction. Thus we conclude $\phi ltr(TM)$ is a distribution on M. Moreover, ϕN does not belong to $RadTM$. Indeed, if $\phi N \in \Gamma(RadTM)$, we would have $\phi^2 N = -N \in \Gamma(\phi RadTM)$, but this is impossible. Similarly, ϕN does not belong to $\phi RadTM$. Thus we conclude that $\phi ltr(TM) \subset S(TM)$ and $\phi RadTM \cap \phi ltr(TM) = \{0\}$.

Lemma 3.2. *Under the hypothesis of Lemma 3.1 and the spacelike characteristic vector field, if $r = q$, then any complementary distribution to $\phi(RadTM) \oplus \phi ltr(TM)$ in $S(TM)$ is Riemannian.*

Proof. Let $dim(\bar{M}) = m + n$ and $dim(M) = m$. Lemma 3.1 implies that $\phi ltr(TM) \oplus \phi RadTM \subset S(TM)$. We denote the complementary distribution to $\phi ltr(TM) \oplus \phi RadTM$ in $S(TM)$ by D'. Then we have a local quasi orthonormal field of frames on $\bar{M}$ along M

$$\{\xi_i, N_i, \phi\xi_i, \phi N_i, X_\alpha, W_a\}, i \in \{1, ..., r\}, \alpha \in \{3r + 1, ..., m\}, a \in \{r + 1, ..., n\},$$

where $\{\xi_i\}$ and $\{N_i\}$ are lightlike basis of $RadTM$ and $ltr(TM)$, respectively and $\phi\xi_i$, ϕN_i, $\{X_\alpha\}$ and $\{W_a\}$ are orthonormal basis of $S(TM)$ and $S(TM^\perp)$, respectively. From the basis $\{\xi_1, ..., \xi_r, \phi\xi_1, ..., \phi\xi_r, \phi N_1, ..., \phi N_r, N_1, ..., N_r\}$ of $ltr(TM) \oplus RadTM \oplus \phi RadTM \oplus \phi ltr(TM)$, we can construct an orthonormal basis $\{U_1, ..., U_{2r}, V_1, ..., V_{2r}\}$ as

follows

$$U_1 = \frac{1}{\sqrt{2}}(\xi_1 + N_1) \qquad U_2 = \frac{1}{\sqrt{2}}(\xi_1 - N_1)$$
$$U_3 = \frac{1}{\sqrt{2}}(\xi_2 + N_2) \qquad U_4 = \frac{1}{\sqrt{2}}(\xi_2 - N_2)$$
$$\cdots \qquad\qquad \cdots$$
$$\cdots \qquad\qquad \cdots$$
$$U_{2r-1} = \frac{1}{\sqrt{2}}(\xi_r + N_r) \qquad U_{2r} = \frac{1}{\sqrt{2}}(\xi_r - N_r)$$
$$V_1 = \frac{1}{\sqrt{2}}(\phi\xi_1 + \phi N_1) \qquad V_2 = \frac{1}{\sqrt{2}}(\phi\xi_1 - \phi N_1)$$
$$V_3 = \frac{1}{\sqrt{2}}(\phi\xi_2 + \phi N_2) \qquad V_4 = \frac{1}{\sqrt{2}}(\phi\xi_2 - \phi N_2)$$
$$\cdots \qquad\qquad \cdots$$
$$\cdots \qquad\qquad \cdots$$
$$V_{2r-1} = \frac{1}{\sqrt{2}}(\phi\xi_r + \phi N_r) \qquad V_{2r} = \frac{1}{\sqrt{2}}(\phi\xi_r - \phi N_r).$$

Hence, $Span\{\xi_i, N_i, \phi\xi_i, \phi N_i\}$ is a non-degenerate space of constant index $2r$. Thus we conclude that $RadTM \oplus \phi RadTM \oplus ltr(TM) \oplus \phi ltr(TM)$ is non-degenerate and of constant index $2r$ on $\bar{M}$. Since

$$index(T\bar{M}) = index(RadTM \oplus ltr(TM)) + index(\phi RadTM \oplus \phi ltr(TM)$$
$$+ index(D' \perp S(TM^\perp)),$$

we have $2q = 2r + index(D' \perp S(TM^\perp))$. Thus, if $r = q$, then $D' \perp S(TM^\perp)$ is Riemannian, i.e., $index(D' \perp (S(TM)^\perp)) = 0$. Hence D' is Riemannian.

Remark 3.2. As mentioned in the introduction, the purpose of this paper is to introduce the notion of slant lightlike submanifolds. To define this notion, one needs to consider angle between two vector fields. As we can see from section 2, a lightlike submanifold has two (radical and screen) distributions: The radical distribution is totally lightlike and therefore it is not possible to define angle between two vector fields of radical distribution. On the other hand, the screen distribution is non-degenerate. Thus one way to define slant notion is to choose a Riemannian screen distribution on lightlike submanifold, for which we use Lemma 3.2.

Proposition 3.1. *There exist no lightlike submanifolds of an indefinite almost contact manifold $\bar{M}$ such that the structure vector field V is belong to $RadTM$ or $ltr(TM)$.*

Proof. Suppose that M is a lightlike submanifold and $V \in \Gamma(RadTM)$. Then there exist a vector field $W \in \Gamma(ltr(TM))$ $g(N, V) = 1 \neq 0$. On the other hand from (1) we have

$$g(\phi N, \phi V) = g(V, N) - \eta(V)\eta(N).$$

Since V is null and $\phi V = 0$, we obtain

$$g(V, N) = 0.$$

This is a contradiction which proves our assertion.

From now on, we suppose that the structure vector field V is tangent to M. Then proposition 3.1 implies that $V \in \Gamma(S(TM))$. In this paper we assume that V is spacelike.

Definition 3.1. Let M be a $q-$ lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$. Then we say that M is a slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(A) $RadTM$ is a distribution on M such that

$$\phi RadTM \cap RadTM = \{0\}. \tag{3.1}$$

(B) For each non-zero vector field tangent to D at $x \in \mathbf{U} \subset M$, the angle $\theta(X)$ between ϕX and the vector space D_x is constant, that is, it is independent of the choice of $x \in \mathbf{U} \subset M$ and $X \in D_x$, where D is complementary distribution to $\phi RadTM \oplus \phi ltr(TM)$ in the screen distribution $S(TM)$.

This constant angle $\theta(X)$ is called slant angle of the distribution D. A slant lightlike submanifold is said to be proper if $D \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$. From the definition 3.1, we have the following decomposition:

$$TM = RadTM \perp S(TM), \qquad (3.2)$$
$$= RadTM \perp (\phi RadTM \oplus \phi ltr(TM)) \perp D. \qquad (3.3)$$

Remark 3.3. As per Proposition 2.1, Definition 3.1 does not depend on $S(TM)$ and $S(TM^\perp)$, but, it depends on the transformation equations (2.60) in [8, page 165], with respect to the screen second fundamental forms h^s. However, our conclusions of this paper do not change with respect to a change of h^s.

Example 3.1. Let $\bar{M} = (R_2^9, \bar{g})$ be a semi-Riemann manifold, where $\bar{g}$ is of signature $(-, +, +, +, -, +, +, +, +)$ with respect to canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}.$$

Then for any $\theta \in (0, \frac{\Pi}{2})$

$$X(u_1, u_2, u_3, u_4, u_5, u_6) = (u_1, u_2, u_3, 0, u_4, u_1, u_5 \cos\theta, u_5 \sin\theta, u_6),$$

defines a six-dimensional slant lightlike submanifolds M with slant angle θ in R_2^9. It is easy to see that

$$
\begin{aligned}
e_1 &= 2(\partial x_1 + \partial y_2 + y_1 \partial z) \\
e_2 &= 2(\partial x_2 + y_2 \partial z) \\
e_3 &= 2(\partial x_3 + y_3 \partial z) \\
e_4 &= 2(\partial y_1) \\
e_5 &= 2(\partial y_3 + \partial y_4) \\
e_6 &= 2\partial z = V
\end{aligned}
$$

form a local frame of TM. It is also easy to see that $RadTM = span\{e_1\}$ and $\phi RadTM = \{e_2 - e_4\}$. Thus $\phi RadTM$ is a distribution on M. Furthermore $D = span\{e_3, e_5\}$ is a slant distribution with slant angle θ. On the other hand the screen transversal bundle is spanned by

$$
\begin{aligned}
W_1 &= \partial x_4 \\
W_2 &= -\partial y_3 + \partial y_4
\end{aligned}
$$

and the lightlike transversal bundle is spanned by

$$N = (-\partial x_1 + \partial y_2).$$

Hence we can see that $\phi N = \frac{1}{2}(e_2 + e_4)$ and $g(\phi N, \phi z_1) = 1$.

Proposition 3.2. *Slant lightlike submanifolds do not include invariant and screen real lightlike submanifolds of an indefinite Sasakian manifold.*

Proof. Let M be a invariant or screen real lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, since $\phi RadTM = RadTM$, the first condition of slant lightlike submanifold is not satisfied which proves our assertion.

Proposition 3.3. *Let M be a slant lightlike submanifold of indefinite Sasakian manifold $\bar{M}$. If D is integrable then M is a contact CR-lightlike submanifold with $D_0 = \{0\}$.*

Proof. Since $\bar{\nabla}$ is a metric connection, for $X, Y \in \Gamma(D)$ using (1.2), we get

$$
\begin{aligned}
g([X, Y], V) &= g(\bar{\nabla}_X Y, V) - g(\bar{\nabla}_Y X, V) \\
&= -g(Y, \bar{\nabla}_X V) + g(X, \bar{\nabla}_Y V) \\
&= -g(Y, \phi X) + g(X, \phi Y) \\
&= 2g(\phi Y, X).
\end{aligned}
$$

Since D is integrable, we get $g(\phi Y, X) = 0$. Because D is Riemannian, it shows that $\phi(D) \subseteq tr(TM)$. Thus M is a contact CR-lightlike submanifold.

It is known that Contact CR-lightlike submanifolds also do not include invariant and real lightlike submanifolds [10]. Thus we may expect some relations between contact CR-lightlike submanifold and slant lightlike submanifold. Indeed we have the following.

Proposition 3.4. *Let M be a $q-$ lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$. Then any coisotropic contact CR-lightlike submanifold is a slant lightlike submanifold with $\theta = 0$. In particular, a lightlike real hypersurface of an indefinite Sasakian manifold $\bar{M}$ of index 2 is a slant lightlike submanifold with $\theta = 0$. Moreover, any contact CR-lightlike submanifold of $\bar{M}$ with $D_o = \{0\}$ is a slant lightlike submanifold with $\theta = \frac{\pi}{2}$.*

Proof. Let M be a $q-$ lightlike contact CR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, by definition of contact CR-lightlike submanifold, $\phi RadTM$ is a distribution on M such that $RadTM \cap \phi RadTM = \{0\}$. If M is coisotropic, then $S(TM^\perp) = \{0\}$, thus $L_2 = 0$ Then the complementary distribution to $\phi RadTM \oplus \phi ltr(TM)$ is D_o. Lemma 3.2 implies that D_o is Riemannian. Since D_o is invariant with respect to ϕ, it follows that $\theta = 0$. Our second assertion is clear due to a lightlike real hypersurface of $\bar{M}$ is coisotropic. Now, if M is contact CR-lightlike submanifold with $D_o = \{0\}$, then the complementary distribution to $\phi RadTM \oplus \phi ltr(TM)$ is D'. Since D' is anti-invariant with respect to ϕ, it follows that $\theta = \frac{\pi}{2}$. Thus proof is complete.

From Proposition 3.4, coisotropic contact CR-lightlike submanifolds, lightlike real hypersurfaces and contact CR-lightlike submanifolds with $D_o = \{0\}$ are some of the many more examples of slant lightlike submanifolds.

Theorem 3.1. Let M be a q lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then M is a slant lightlike submanifold if and only if

1. $\phi RadTM$ is a distribution on M such that

$$
\phi RadTM \cap RadTM = \{0\},
$$

2. $D = \{X \in \Gamma(D) | T^2 X = -\lambda(X - \eta(X))\xi\}$ is a distribution such that it is complementary to $\phi RadTM \oplus \phi ltr(TM)$.

Proof. First of all, we get

$$
\cos \theta(X) = \frac{g(\phi X, TX)}{|\phi X||TX|} = \frac{-g(X, \phi TX)}{|X||TX|} = \frac{-g(X, T^2 X)}{|X||TX|}.
$$

On the other hand, we have

$$
\cos \theta(X) = \frac{|TX|}{|X|}.
$$

Hence

$$
\cos^2 \theta(X) = \frac{-g(X, T^2 X)}{|X|^2}.
$$

Thus we have

$$T^2 X = -X + \eta(X)\xi,$$

which proves theorem.

We now denote the projections on $RadTM$, $\phi RadTM$, $\phi ltr(TM)$ and D in TM by P_1, P_2, P_3, P_4. Similarly we denote the projections on $ltr(TM)$ and $S(TM^\perp)$ by Q_1 and Q_2. Then we get

$$\phi X = P_1 X + P_2 X + P_3 X + P_4 X + \eta(X)V \tag{3.4}$$

for $X \in \Gamma(TM)$. Similarly, we write

$$W = Q_1 W + Q_2 W \tag{3.5}$$

for $W \in \Gamma(tr(TM))$. Now applying ϕ to (3.4) we have

$$\phi X = \phi P_1 X + \phi P_2 X + \phi P_3 X + f P_4 X + F P_4 X \tag{3.6}$$

where $f P_4 X$ (resp., $F P_4 X$) denotes the tangential part(resp. screen transversal part) of $\phi P_4 X$. Thus we get

$$\phi P_1 X \in \phi RadTM, \phi P_2 X \in \Gamma(RadTM) \tag{3.7}$$

$$\phi P_3 X \in \Gamma(ltr(TM)), f P_4 X \in \Gamma(D), \tag{3.8}$$

$$F P_4 X \in \Gamma(S(TM^\perp)).$$

Applying ϕ to (3.5) we get

$$\phi W = \phi Q_1 W + B Q_2 W + C Q_2 W,$$

where $B Q_2 W$ (resp. $C Q_2 W$) denotes the tangential (resp. screen transversal part) of $\phi Q_2 W$.

Now, using (3.4), Gauss-Weingarten formulas (2.3) – (2.5)and considering the decomposition of a slant lightlike submanifold, we obtain

$$
\begin{aligned}
-g(X,Y)V + \eta(Y)X \quad &= \quad \nabla_X P_1 Y + \nabla_X \phi P_2 Y - A_{\phi P_3 Y}X \\
&\quad + \nabla_X f P_4 Y - A_{F P_4 Y}X - \phi P_1 \nabla_X Y - \phi P_2 \nabla_X Y \\
&\quad - f P_4 \nabla_X Y) - \phi h^l(X,Y) - B h^s(X,Y),
\end{aligned}
$$

$$h^l(X,\phi P_1 Y) + h^l(X,\phi P_2 Y) + h^l(X, f P_4 Y) = -\nabla^l_X \phi P_3 Y - D^l(X, F P_4 Y) + \phi P_3 \nabla_X Y$$

and

$$
\begin{aligned}
h^s(X,\phi P_1 Y) + h^s(X,\phi P_2 Y) + h^s(X, f P_4 Y) \quad &= \quad -D^s(X,\phi P_3 Y) - \nabla^s_X F P_4 Y \\
&\quad + F P_4 \nabla_X Y) - C h^s(X,Y).
\end{aligned}
$$

In particular, for $Y \in \Gamma(RadTM)$, we have

$$
\begin{aligned}
\phi h^l(X,Y) + B h^s(X,Y) \quad &= \quad \nabla_X P_1 Y + \nabla_X \phi P_2 Y - \phi P_1 \nabla_X Y \\
&\quad - \phi P_2 \nabla_X Y - f P_4 \nabla_X Y \tag{3.9}
\end{aligned}
$$

and for $Y \in \Gamma(S(TM))$, we get

$$
\begin{aligned}
\phi h^l(X,Y) + B h^s(X,Y) \quad &= \quad \nabla_X \phi P_2 Y - A_{\phi P_3 Y}X \\
&\quad + \nabla_X f P_4 Y - A_{F P_4 Y}X - \phi P_1 \nabla_X Y - \phi P_2 \nabla_X Y \\
&\quad - f P_4 \nabla_X Y + g(X,Y)V - \eta(Y)X. \tag{3.10}
\end{aligned}
$$

Theorem 3.2. *Let M be a proper slant lightlike submanifold of indefinite Sasakian manifold $\bar{M}$. Then the induced connection ∇ is never a metric connection.*

Proof. Suppose that the induced connection is a metric connection then $\nabla_X \phi P_2 Y \in \Gamma(RadTM)$ and $h^l(X, Y) = 0$, thus we have

$$\nabla_X \phi P_2 Y - \phi P_2 \nabla_X Y \in \Gamma(RadTM).$$

Hence, we obtain

$$\phi P_1 \nabla_X Y + f P_4 \nabla_X Y) + B h^s(X, Y) - g(X, Y)V = 0.$$

Since M has the following decomposition

$$TM = RadTM \oplus \phi RadTM \oplus \phi ltr(TM) \oplus D \oplus V.$$

We get

$$\phi P_1 \nabla_X Y = 0,$$

$$f P_4 \nabla_X Y) = -B h^s(X, Y),$$

$$g(X, Y)V = 0.$$

Now taking $X = \phi N$ and $Y = \phi \xi$ then we obtain $g(N, \xi) = 0$. Thus $V = 0$. This is a contradiction. Thus M does not have a metric connection.

4. Screen slant lightlike submanifolds

In the previous section, we have seen that slant lightlike submanifolds do not contain invariant and screen real submanifolds. In this section, we introduce a new class which includes invariant lightlike submanifolds as well as screen real submanifolds. We first give the following lemma which will be useful to define screen slant notion on the screen distribution.

Lemma 4.1. *Let M be a $2q-$ lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with constant index $2q$ such that $2q < \dim(M)$ and the structure vector field is a spacelike vector field on $S(TM)$. Then the screen distribution $S(TM)$ of lightlike submanifold M is Riemannian.*

Proof Let $\bar{M}$ be a real $2k = m + n-$ dimensional indefinite Sasakian manifold and $\bar{g}$ be a semi-Riemannian metric on $\bar{M}$ of index $2q$. Let us assume that M be an $m-$ dimensional and $2q(< m)-$ lightlike submanifold of $\bar{M}$. Then we have a local quasi orthonormal field of frames on $\bar{M}$ along M

$$\{\xi_i, N_i, X_\alpha, W_a\}, i \in \{1, ..., 2q\}, \alpha \in \{2q + 1, ..., m\}, a \in \{2q + 1, ..., n\},$$

where $\{\xi_i\}$ and $\{N_i\}$ are lightlike basis of $RadTM$ and $ltr(TM)$, respectively and $\{X_\alpha\}$ and $\{W_a\}$ are orthonormal basis of $S(TM)$ and $S(TM^\perp)$, respectively. From the null basis $\{\xi_1, ..., \xi_{2q}, N_1, ..., N_{2q}\}$ of $ltr(TM) \oplus RadTM$, we can construct an orthonormal basis $\{U_1, ..., U_{4q}\}$ as follows

$$
\begin{array}{ll}
U_1 = \frac{1}{\sqrt{2}}(\xi_1 + N_1) & U_2 = \frac{1}{\sqrt{2}}(\xi_1 - N_1) \\
U_3 = \frac{1}{\sqrt{2}}(\xi_2 + N_2) & U_4 = \frac{1}{\sqrt{2}}(\xi_2 - N_2) \\
\qquad ... & \qquad ... \\
\qquad ... & \qquad ... \\
U_{4q-1} = \frac{1}{\sqrt{2}}(\xi_{2q} + N_{2q}) & U_{4q} = \frac{1}{\sqrt{2}}(\xi_{2q} - N_{2q}).
\end{array}
$$

Hence, $Span\{\xi_i, N_i\}$ is a non-degenerate space of constant index $2q$. Thus we conclude that $RadTM \oplus ltr(TM)$ is non-degenerate and constant index $2q$ on $\bar{M}$. Since

$$index(T\bar{M}) = index(RadTM \oplus ltr(TM)) + index(S(TM^\perp) \perp S(TM)),$$

we obtain that $S(TM) \perp S(TM^\perp)$ is constant index zero, that is, $S(TM)$ and $S(TM^\perp)$ are Riemannian vector bundles. Thus proof is complete. ∎

Thus Lemma 4.1 enables us to give the following definition.

Definition 4.1. Let $(M, g, S(TM))$ be a $2q-$ lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with constant index $2q < dim(M)$. Then we say that M is a screen slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(i) $RadTM$ is invariant with respect to ϕ, i.e. $\phi(RadTM) = RadTM$,
(ii) For each non-zero vector field X tangent to $S(TM)$ at $x \in U \subset M$, the angle $\theta(X)$ between ϕX and $S(TM)$ is constant, i.e., it is independent of the choice of x and $X \in \Gamma(S(TM))$.

We note that $\theta(X)$ is called the slant angle. We point out the following features:

(a) $RadTM$ is even dimensional,
(b) Screen slant lightlike submanifolds do not include real hypersurface.

Proposition 4.1. *Any invariant an screen real lightlike submanifolds are screen slant lightlike submanifolds with $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively.*

Proof. Let M be a invariant lightlike submanifold. Then$\phi(TM) = TM$. Hence $\phi(RadTM) = RadTM$ and $\phi(S(TM)) = S(TM)$. Thus the condition (i) is satisfied and $\theta = 0$. The other assertion can be proved in a similar way. ∎

From Proposition 4.1, we conclude that there are many examples of screen slant lightlike submanifolds. Now, we are going to give an example of proper screen slant submanifolds.

Example 4.1. Let M be a submanifold of R_2^2 given by the following equations

$$x_1 = u_1 sin\theta + u_2 cos\theta \quad , \quad y_1 = u_1 cos\theta - u_2 sin\theta$$
$$x_2 = 2cosu_3 \quad , \quad y_2 = 2sinu_3$$
$$x_3 = 2u_3 sinu_4 \quad , \quad y_3 = 2u_3 cosu_4$$
$$x_4 = u_2 \quad , \quad y_4 = u_1$$
$$z = u_5.$$

Then the tangent bundle of TM is spaned by

$$z_1 = sin\theta\partial x_1 + cos\theta\partial y_1 + \partial y_4 + sin\theta y_1\partial z,$$
$$z_2 = cos\theta\partial x_1 - sin\theta\partial y_1 + (cos\theta y_1 + y_4)\partial z,$$
$$z_3 = 2(-sinu_3\partial x_2 + sinu_4\partial x_3 + cosu_3\partial y_2 + cosu_4\partial y_3) + (-sinu_3 y_2 + sinu_4 y_3)\partial z,$$
$$z_4 = 2(u_3 cosu_4\partial x_3 - u_3 sinu_4\partial y_3) + u_3 y_3 cosu_4\partial z,$$
$$Z = \partial z.$$

Then it follows that M is $2-$lightlike submanifold with $RadTM = span\{z_1, z_2\}$. It is easy to see that $\phi z_1 = z_2$, which implies that $RadTM$ is invariant. On the other hand, we can see that $S(TM) = span\{z_3, z_4\}$ is a slant distribution with slant angle $\frac{\Pi}{4}$. Thus M is a screen slant lightlike submanifold of R_2^9. By direct computation, the screen transversal bundle $S(TM^\perp)$ is spanned by

$$W_1 = sinu_3\partial x_2 + sinu_4\partial x_3 - cosu_3\partial y_2 + cosu_4\partial y_4 + (sinu_3 y_2 + sinu_4 y_3)\partial z,$$
$$W_2 = cosu_3\partial x_2 + sinu_3\partial y_2 + cosu_3 y_2\partial z$$

and lightlike transversal bundle *ltr(TM)* spanned by

$$N_1 = \frac{1}{2}(-sin\theta\partial x_1 + cos\theta\partial y_1 + \partial y_4) - sin\theta y_1\partial z,$$

$$N_2 = \frac{1}{2}(-cos\theta\partial x_1 + \partial x_4 - sin\theta\partial y_1) + (-cos\theta y_1 + y_4)\partial z.$$

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