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# SLANT SUBMANIFOLDS OF LORENTZIAN SASAKIAN AND PARA SASAKIAN MANIFOLDS

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**Abstract.** In this paper we introduce the notion of slant submanifolds of a Lorentzian almost contact manifold and of a Lorentzian almost para contact manifold.

## 1. Introduction

Slant submanifolds of Kaehler manifolds were introduced by B. Y. Chen in [7] as a generalization of both invariant and anti-invariant submanifolds. Later J.L. Cabrerizo, A. Carriazo, L.M. Fernández and M. Fernández, [6], defined slant submanifolds of an almost contact manifold. Many authors have studied those submanifolds and certain generalizations, like semi-slant and generic submanifolds, in both complex and contact geometry. Also, B. Sahin has studied slant submanifolds of an almost product Riemannian manifold in [12].

Recently, it has been initiated the study of slant submanifolds of an indefinite Kaehler manifold [3] and of a Lorentzian para contact manifold [9]. Now, we define slant submanifolds of an odd dimensional Lorentzian metric manifold doted with an almost contact structure. In this paper we study slant submanifolds of both almost contact and almost para contact manifolds. We englobe this two structures because they can be treated in a similar way.

After some preliminaries, where we fix the notation, we introduce slant submanifolds. We present two sections with examples of slant submanifolds in a Lorentzian almost contact and in a Lorentzian almost para contact manifold. Finally, some characterization results are given and low dimensional slant submanifolds are studied.

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## 2. Preliminaries

Let $\widetilde{M}$ be a $(2n+1)$-dimensional Lorentzian metric manifold, if it is endowed with a structure $(\phi, \xi, \eta, g)$, where $\phi$ is a $(1,1)$ tensor, $\xi$ a vector field, $\eta$ a 1-form on $\widetilde{M}$ and $g$ is a Lorentz metric, satisfying

$$\begin{aligned} \phi^2 X = \varepsilon X + \eta(X)\xi, \qquad & g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \\ \eta(\xi) = -\varepsilon, \qquad & \eta(X) = \varepsilon g(X, \xi), \end{aligned} \tag{1}$$

for any vector fields $X, Y$ in $\widetilde{M}$, it is called *Lorentzian almost contact manifold* or *Lorentzian almost para contact manifold* for $\varepsilon = -1$ or 1, respectively. It follows that $g(\phi X, Y) = \varepsilon g(X, \phi Y)$ for any $X, Y$.

Let $\Phi$ denote the 2-form in $\widetilde{M}$ given by $\Phi(X,Y) = g(X, \phi Y)$, if $d\eta = \Phi$, $\widetilde{M}$ is called *normal contact Lorentzian manifold*.

If $\xi$ is a Killing tensor vector field, the (para) contact structure is called K-(para) contact. In such a case, we have

$$\widetilde{\nabla}_X \xi = \varepsilon \phi X. \tag{2}$$

Finally, it is called *Lorentzian Sasakian* (LS) or *Lorentzian para Sasakian* (LPS) if

$$(\widetilde{\nabla}_X \phi)Y = \varepsilon g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X. \tag{3}$$

Let $M$ be a submanifold of $(\widetilde{M}, \phi, \xi, \eta, g)$, the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{4}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{5}$$

for any $X, Y$ tangent vector fields and $N$ normal vector field, where $h$ is the second fundamental form of $M$, $A_N$ is the Weingarten endomorphism associated with $N$ and $\nabla^{\perp}$ is the connection in the normal bundle $TM^{\perp}$.

For every tangent vector field, $X$, we write

$$\phi X = TX + NX, \tag{6}$$

where $TX$ in the tangential component and $NX$ is the normal one. And for every normal vector field, $V$,

$$\phi V = tV + nV,$$

where $tV$ in the tangential component and $nV$ is the normal one.

## 3. Slant Submanifolds

A submanifold $M$ of a Lorentzian almost (para) contact manifold, $(\widetilde{M}, \phi, \xi, \eta, g)$, is said to be a *slant submanifold* if for any $x \in M$ and any $X \in T_x M$, the Wirtinger's angle, the angle between $\phi X$ and $TX$, is a constant $\theta \in [0, 2\pi]$. In such a case, $\theta$ it is called the *slant angle* of $M$ in $\widetilde{M}$. They englobe both *invariant* and *anti-invariant* submanifold for $\theta = 0$ and $\theta = \pi/2$, respectively. A slant submanifold is called *proper* if it is neither invariant nor anti-invariant. Slant submanifolds of a Lorentzian para contact manifold have been already defined by M.A. Khan, K. Singh and V. A. Khan in [9].

Span$\{\xi\}$ defines the time like vector field distribution. If $X$ is a space-like vector field, it is orthogonal to $\xi$, then

$$g(\phi X, \phi X) = g(X, X) \geq 0,$$

so $\phi X$ is also space-like, the same is valid for $TX$. For space-like vector fields the Cauchy-Schawrz inequality, $g(X, Y) \leq |X||Y|$, is verified. Therefore the Wirtinger angle, $\theta$, is given by:

$$\frac{g(\phi X, TX)}{|\phi X||TX|} = \cos\theta.$$

For a Lorentzian almost contact manifold we distinguish two important cases for which the submanifold turns to be anti-invariant.

**Proposition 3.1.** *Every submanifold $M$ of a Lorentzian contact metric manifold, $\widetilde{M}$, normal to $\xi$ is an anti-invariant submanifold.*

*Proof.* Let be $X, Y \in TM$,

$$d\eta(X, Y) = \frac{1}{2}(X\eta(Y) - Y\eta(X) - \eta([X, Y]), \tag{7}$$

so if $M$ is normal to $\xi$, $d\eta = 0$. But for a Lorentzian contact metric manifold $d\eta = \Phi$, then $g(TX, Y) = g(\phi X, Y) = \Phi(X, Y) = 0$. As a consequence, $T \equiv 0$ and $M$ is an anti-invariant submanifold. ∎

**Proposition 3.2.** *Every two dimensional submanifold of a Lorentzian almost contact manifold tangent to $\xi$ is anti-invariant.*

*Proof.* As $M$ is two dimensional, $TM =< \xi > \oplus < X >$. For this $X$, $g(\phi X, X) = 0$ and $g(\phi X, \xi) = 0$ so $\phi X = NX$ and $M$ is anti-invariant. ∎

Now, we characterize slant submanifolds of a Lorentzian almost contact manifold, the characterization for the para contact case already appeared in [9].

**Theorem 3.3.** *Let $M$ be a submanifold of a Lorentzian almost (para) contact metric manifold $\widetilde{M}$. Then, $M$ is slant if and only there exits a constant $\lambda \in [0,1]$ such*

$$T^2 = \lambda(\varepsilon I + \eta \otimes \xi). \tag{8}$$

*Moreover, in such a case, $\lambda = \cos^2\theta$, where $\theta$ is the slant angle.*

*Proof.* First, if $M$ is anti-invariant, $\phi X$ is normal, $TX = 0$ and it is equivalent to $T^2X = 0$.

If $M$ is not an anti-invariant slant submanifold, then for any $X \in TM$,

$$\cos\theta = \frac{g(\phi X, TX)}{|\phi X||TX|} = \frac{|TX|^2}{|\phi X||TX|} = \frac{|TX|}{|\phi X|}. \tag{9}$$

But also $TX \in \mathcal{D}$, so

$$\cos\theta = \frac{|T^2X|}{|\phi TX|} = \frac{|T^2X|}{|TX|}. \tag{10}$$

On the one hand, $g(T^2X, X) = g(\phi TX, X) = \varepsilon g(TX, \phi X) = \varepsilon g(TX, TX) = \varepsilon|TX|^2$ and using (9), (10) $g(T^2X, X) = \varepsilon|T^2X||\phi X| = \varepsilon|T^2X||\phi^2 X|$.

On the other hand, $g(T^2X, \phi^2X) = \varepsilon g(T^2X, X)$. So, $g(T^2X, \phi^2X) = |T^2X||\phi^2X|$ and, as they are space-like vector fields, it follows that they are colineal, that is $T^2X = \lambda\phi^2X = \lambda(\varepsilon X + \eta(X)\xi)$.

The reciprocal is just a simple computation, from (8) and (9), $\cos\theta = \sqrt{\lambda}$ is constant and $M$ is a slant submanifold. ∎

## 4. Examples in a Lorentzian Almost Contact Manifold

Beginning with the examples of slant submanifolds in complex geometry given by B. Y. Chen in [7] and proceeding in a similar way to [6] for almost contact manifolds, we can present examples of slant submanifolds of a Lorentzian Sasakian manifold.

In $\mathbf{R}^5$, $(x_1, x_2, y_1, y_2, z)$, we consider this vector field basis

$$\left\{2\frac{\partial}{\partial y^1}, 2\frac{\partial}{\partial y^2}, 2\left(\frac{\partial}{\partial x^1} + y^1\frac{\partial}{\partial z}\right), 2\left(\frac{\partial}{\partial x^2} + y^2\frac{\partial}{\partial z}\right), \frac{\partial}{\partial z}\right\},$$

and the following structure on $\mathbf{R}^{2n+1}$:

$$\phi_0\left(\sum_{i=1}^n X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i} + Z\frac{\partial}{\partial z}\right) = \sum_{i=1}^n Y_i\frac{\partial}{\partial x^i} - \sum_{i=1}^n X_i\frac{\partial}{\partial y^i} + \sum_{i=1}^n Y_i y^i\frac{\partial}{\partial z}$$

$$\xi = 2\frac{\partial}{\partial z},$$

$$\eta = \frac{1}{2}\left(dz - \sum_{i=1}^{n} y^i dx^i\right),$$

$$g = -\eta \otimes \eta + \frac{1}{4}\left(\sum_{i=1}^{n} dx^i \otimes dx^i + \sum_{i=1}^{n} dy^i \otimes dy^i\right).$$

Then, $(\mathbf{R}^5, \phi_0, \xi, \eta, g)$ is a Lorentzian Sasakian manifold, [4].

**Theorem 4.1.** *Let $S$ be a slant submanifold of $\mathbf{C}^2$ with Wirtinger angle different to $0$, and $\pi/2$ and equation $x(u', v') = (f_1(u', v'), f_2(u', v'), f_3(u', v'), f_4(u', v'))$, with $\partial/\partial u'$ and $\partial/\partial v'$ no null and orthogonal. Then, $y(u, v, t) = 2(f_1(u, v), f_2(u, v), f_3(u, v), f_4(u, v), t)$ defines a slant submanifold $M$ of $(\mathbf{R}^5, \phi_0, \xi, \eta, g)$.*

*Proof.* Let us consider,

$$e_1 = \frac{\partial}{\partial u} + \left(2f_3(u, v)\frac{\partial f_1}{\partial u} + 2f_4(u, v)\frac{\partial f_2}{\partial u}\right)\frac{\partial}{\partial t} \quad \text{and}$$
$$e_2 = \frac{\partial}{\partial v} + \left(2f_3(u, v)\frac{\partial f_1}{\partial v} + 2f_4(u, v)\frac{\partial f_2}{\partial v}\right)\frac{\partial}{\partial t}.$$

Then, $\{e_1, e_2, \xi\}$ is an orthogonal frame.

If $x(u', v')$ is a slant submanifold with slant angle $\theta$, for every tangent vector field $X = x_1 e_1 + x_2 e_2 + \eta(X)\xi$,

$$\begin{aligned} g(\phi X, e_1) &= x_2 g(\phi e_2, e_1) \\ &= x_2 g\left(-2\frac{\partial f_1}{\partial v}\frac{\partial}{\partial y^1} - 2\frac{\partial f_2}{\partial v}\frac{\partial}{\partial y^2} + 2\frac{\partial f_3}{\partial v}\frac{\partial}{\partial x^1} + 2\frac{\partial f_4}{\partial v}\frac{\partial}{\partial x^2} + 2\frac{\partial f_3}{\partial v} f_3 \frac{\partial}{\partial z}\right. \\ &\quad \left. +2\frac{\partial f_4}{\partial v} f_4 \frac{\partial}{\partial z}, \frac{\partial}{\partial u} + \left(2f_3(u, v)\frac{\partial f_1}{\partial u} + 2f_4(u, v)\frac{\partial f_2}{\partial u}\right)\frac{\partial}{\partial t}\right) \\ &= -x_2 G\left(JX', \frac{\partial}{\partial u'}\right), \end{aligned}$$

where $J$ is the complex structure and $X'$ is a vector field in $TS$ given by $X' = x_1\frac{\partial}{\partial u'} + x_2\frac{\partial}{\partial v'}$. In the same way, $g(\phi X, e_2) = -x_1 G\left(JX', \frac{\partial}{\partial v'}\right)$.

Therefore, $TX = \frac{g(\phi X, e_1)}{g(e_1, e_1)} e_1 + \frac{g(\phi X, e_2)}{g(e_2, e_2)} e_2$, and

$$\frac{|TX|}{\sqrt{|X^2| - \eta^2(X)}} = \frac{|PX'|}{|X'|} = \cos\theta,$$

with $PX'$ the tangent projection of $JX'$ and $\theta$ the slant angle of $S$. That is the Wirtinger angle is constant and hence $M$ is a slant submanifold. ■

From Theorem 4.1 and Examples 8.3, 8.4, 8.5 and 8.6 of [7] we obtain the following examples of slant submanifolds of the Lorentzian Sasakian manifold $(\mathbf{R}^5, \phi_0, \xi, \eta, g)$.

**Example 4.2.** For any $\theta$, $0 < \theta < \pi/2$,

$$x(u, v, t) = 2(u\cos\theta, u\sin\theta, v, 0, t)$$

defines a slant submanifold with slant angle $\theta$ in $(\mathbf{R}^5, \phi_0, \xi, \eta, g)$.

**Example 4.3.** For any positive constant $k$,

$$x(u, v, t) = 2(e^{ku}\cos u\cos v, e^{ku}\sin u\cos v, e^{ku}\cos u\sin v, e^{ku}\sin u\sin v, t)$$

defines a slant submanifold with slant angle $\cos^{-1}(k/\sqrt{1+k^2})$ in $(\mathbf{R}^5, \phi_0, \xi, \eta, g)$.

**Example 4.4.** For any positive constant $k$,

$$x(u, v, t) = 2(u, k\cos v, v, k\sin v, t)$$

defines a proper slant submanifold with slant angle $\cos^{-1}(k/\sqrt{1+k^2})$ in $(\mathbf{R}^5, \phi_0, \xi, \eta, g)$.

We can also present examples in higher dimensions.

**Example 4.5.** For any positive constant $k$,

$$x(u, v, w, z, t) = 2(u, v, k\sin w, \sin z, kw, kz, k\cos w, k\cos z, t)$$

defines a slant submanifold with slant angle $\cos^{-1} k$ in $(\mathbf{R}^9, \phi_0, \xi, \eta, g)$.

Moreover, we can construct an example that does not come from Theorem 4.1, it is inspired in Example 3.17 of [6].

**Example 4.6.** For any $\theta \in [0, 2\pi]$,

$$x(u, v, t) = 2(u, 0, v\cos\theta, v\sin\theta, 2uv\cos\theta + t)$$

defines a slant submanifold in $(\mathbf{R}^5, \phi_0, \xi, \eta, g)$ with slant angle $\theta$.

We can also present examples of slant submanifolds in manifolds doted with a Lorentzian $\beta$ Kenmotsu structure instead of a Lorentzian Sasakian one.

Given an almost Hermitian manifold $(\widetilde{N}^{2n}, J, G)$, the warped product $\widetilde{M}^{2n+1} = \mathbf{R} \times_f \widetilde{N}$ can be endowed with the following Lorentzian almost contact structure $(\phi, \xi, \eta, g_f)$,

$$g_f = -\pi^*(g_{\mathbf{R}}) + (f \circ \pi)^2 \sigma^*(G),$$

where $f > 0$ is a function on $\mathbf{R}$ and $\pi$ and $\sigma$ are the projections from $\mathbf{R} \times \widetilde{N}$ on $\mathbf{R}$ and $\widetilde{N}$, respectively, $\phi X = (J\sigma_* X)^*$ for any vector field $X$ on $\widetilde{M}$, and $\xi = \dfrac{\partial}{\partial t}$, where $t$ denotes the coordinate of $\mathbf{R}$.

In this case, $\widetilde{M}$ is not a Lorentzian Sasakian manifold, in fact if $(\widetilde{N}, J, G)$ is a Kaehlerian manifold then $(\widetilde{M}, \phi, \xi, \eta, g_f)$ is a Lorentzian $\beta$-Kenmotsu manifold [2], that is

$$(\nabla_X \phi)Y = \beta(-g_f(\phi X, Y)\xi - \eta(Y)\phi X).$$

**Theorem 4.7.** *Let $N$ be a slant submanifold of an almost Hermitian manifold $(\widetilde{N}, J, G)$. Then the warped product $M = \mathbf{R} \times_f N$ is a slant submanifold of the Lorentzian almost contact manifold $\widetilde{M} = \mathbf{R} \times_f \widetilde{N}$.*

*Proof.* For every tangent vector field $X$ of $M$, $TX = (P\sigma_* X)^*$ where $P$ denotes the part of $J$ tangent to $N$. Then, $T^2 X = (P^2 \sigma_* X)^* = -\cos^2\theta(\sigma_* X)^*$ because $N$ is a slant submanifold with slant angle $\theta$. Therefore, $T^2 X = \cos^2\theta(-X + \eta(X)\xi)$ and $M$ is a slant submanifold. ■

The same demonstration proves that $N$ immersed in $\widetilde{M} = \mathbf{R} \times_f \widetilde{N}$ is a slant submanifold orthogonal to the structure vector field $\xi$, in this case it is not anti-invariant and it is an even dimensional submanifold. There is no contradiction with Proposition 3.1 because $\widetilde{M}$ is not a Lorentzian contact metric manifold.

## 5. Examples in a Lorentzian Almost Para Contact Manifold

We can consider different almost para contact structures on $\mathbf{R}^{2n+1}$:

$$\phi_1\left(\sum_{i=1}^n X_i \frac{\partial}{\partial x^i} + Y_i \frac{\partial}{\partial y^i} + Z\frac{\partial}{\partial z}\right) = -\sum_{i=1}^n Y_i \frac{\partial}{\partial x^i} - \sum_{i=1}^n X_i \frac{\partial}{\partial y^i} + \sum_{i=1}^n Y_i y^i \frac{\partial}{\partial z}$$

$$\phi_2\left(\sum_{i=1}^n X_i \frac{\partial}{\partial x^i} + Y_i \frac{\partial}{\partial y^i} + Z\frac{\partial}{\partial z}\right) = \sum_{i=1}^n X_i \frac{\partial}{\partial x^i} - \sum_{i=1}^n Y_i \frac{\partial}{\partial y^i} + \sum_{i=1}^n X_i y^i \frac{\partial}{\partial z}$$

$$\phi_3\left(\sum_{i=1}^n X_i \frac{\partial}{\partial x^i} + Y_i \frac{\partial}{\partial y^i} + Z\frac{\partial}{\partial z}\right) = \sum_{i=1}^n Y_i \frac{\partial}{\partial x^i} + \sum_{i=1}^n X_i \frac{\partial}{\partial y^i} + \sum_{i=1}^n Y_i y^i \frac{\partial}{\partial z}$$

$$\xi = 2\frac{\partial}{\partial z},$$

$$\eta = -\frac{1}{2}\left(dz - \sum_{i=1}^n y^i dx^i\right),$$

$$g = -\eta \otimes \eta + \frac{1}{4}\left(\sum_{i=1}^{n} dx^i \otimes dx^i + \sum_{i=1}^{n} dy^i \otimes dy^i\right).$$

In [12], B. Sahin presented various examples of slant submanifolds of an almost product manifold, we use them for constructing examples in a Lorentzian almost para contact manifold.

First, consider in $\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$, with coordinates $(x_1, ..., x_{n_1}, y_1, ..., y_{n_2})$ and the almost product structure given by

$$J\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right) = \left(\frac{\partial}{\partial x^i}, -\frac{\partial}{\partial y^i}\right).$$

**Theorem 5.1.** *Let $S$ be a slant submanifold of $\mathbf{R}^2 \times \mathbf{R}^2$ with Wirtinger angle different to $0$, and $\pi/2$ and equation $x(u', v') = (f_1(u', v'), f_2(u', v'), f_3(u', v'), f_4(u', v'))$, with $\partial/\partial u'$ and $\partial/\partial v'$ no null and orthogonal. Then, $y(u, v, t) = 2(f_1(u, v), f_2(u, v), f_3(u, v), f_4(u, v), t)$ defines a slant submanifold of $(\mathbf{R}^5, \phi_2, \xi, \eta, g)$.*

*Proof.* The proof is similar to the Theorem 4.1, having into account that, in this case, $g(\phi e_i, e_i) \neq 0$. ■

From Theorem 5.1 and Examples 3.1, 3.2 and 3.3 of [12] we obtain the following examples of slant submanifolds of the Lorentzian almost para contact manifold $(\mathbf{R}^5, \phi_2, \xi, \eta, g)$.

**Example 5.2.** For any $\theta$, $0 < \theta < \pi/2$,

$$x(u, v, t) = 2(u\cos\theta, v\cos\theta, u\sin\theta, v\sin\theta, t)$$

defines a slant submanifold with slant angle $2\theta$ in $(\mathbf{R}^5, \phi_2, \xi, \eta, g)$.

**Example 5.3.** For any $u, v \in (0, \pi/2)$, and any positive constant $k \neq 1$,

$$x(u, v, t) = 2(u, v, -k\sin u, -k\sin v, k\cos u, k\cos v, t)$$

defines a proper slant submanifold with slant angle $\theta = \cos^{-1}\left(\frac{1-k^2}{1+k^2}\right)$ in $(\mathbf{R}^7, \phi_2, \xi, \eta, g)$.

**Example 5.4.** Consider a submanifold $M$ of $(\mathbf{R}^5, \phi_2, \xi, \eta, g)$ given by equation:

$$x(u, v, t) = 2(u + v, u + v, \sqrt{2}v, \sqrt{2}u, t).$$

Then $M$ is a slant submanifold with slant angle $\pi/3$.

Consider in $\mathbf{R}^4$ with coordinates $(x_1, x_2, y_1, y_2)$ and almost product structure

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad J\left(\frac{\partial}{\partial y^i}\right) = \frac{\partial}{\partial x^i}.$$

Again, like in Theorem 4.1 and 5.1, from a slant submanifold of $\mathbf{R}^4$ with this almost product structure we can obtain a slant submanifold of a Lorentzian almost para contact manifold.

**Theorem 5.5.** *Let $S$ be a slant submanifold of $\mathbf{R}^4$ with Wirtinger angle different to 0, and $\pi/2$ and equation $x(u', v') = (f_1(u', v'), f_2(u', v'), f_3(u', v'), f_4(u', v'))$, with $\partial/\partial u'$ and $\partial/\partial v'$ no null and orthogonal. Then, $y(u, v, t) = 2(f_1(u, v), f_2(u, v), f_3(u, v), f_4(u, v), t)$ defines a slant submanifold of $(\mathbf{R}^5, \phi_3, \xi, \eta, g)$.*

**Example 5.6.** For any $\theta$, $0 < \theta < \pi/2$,

$$x(u, v, t) = 2(u \cos\theta, u \sin\theta, v, 0, t)$$

defines a slant submanifold with slant angle $\theta$ in $(\mathbf{R}^5, \phi_3, \xi, \eta, g)$.

We can also obtain examples of slant submanifolds of a Lorentzian almost para contact manifold using warped products.

**Theorem 5.7.** *Let $N$ be a slant submanifold of a Riemannian almost product manifold $(\widetilde{N}, J, G)$. Then the warped product $M = \mathbf{R} \times_f N$ is a slant submanifold of the Lorentzian almost para contact manifold $\widetilde{M} = \mathbf{R} \times_f \widetilde{N}$.*

*Proof.* It was proved in [2] that this warped product is a Lorentzian almost para contact manifold. Here, the proof is similar to the Theorem 4.7, but in this case $P^2\sigma_* X = \cos^2\theta(\sigma_* X)$ using the characterization of slant submanifolds of an almost product manifold, [12]. Therefore, $T^2X = \cos^2\theta(X + \eta(X)\xi)$ and $M$ is a slant submanifold. ∎

Again, $N$ immersed in $\widetilde{M} = \mathbf{R} \times_f \widetilde{N}$ is a slant even dimensional submanifold orthogonal to the structure vector field $\xi$, and it not anti-invariant.

Finally we construct a slant surface tangent to $\xi$. Comparing with Proposition 3.2, this shows that the para contact case is completely different from the contact one: there exists both even and odd dimensional slant submanifolds tangent to $\xi$, including the two dimensional case. The main fact is that here $g(X, \phi X)$ could be different from 0.

**Example 5.8.** For any $\theta$, $0 < \theta < \pi/2$,

$$x(u, t) = 2(u \cos\theta, 0, u \sin\theta, 0, t)$$

defines a slant submanifold with slant angle $\cos^{-1}(\cos^2\theta - \sin^2\theta)$ in $(\mathbf{R}^5, \phi_2, \xi, \eta, g)$.

Let us consider the following orthonormal frame in $TM$

$$\left\{e_1 = \frac{\partial}{\partial u} + 2u \sin\theta \cos\theta \frac{\partial}{\partial t}, \xi = 2\frac{\partial}{\partial z}\right\}.$$

As, $g(\phi_2 e_1, e_1) = \cos^2\theta - \sin^2\theta$, it results $Te_1 = (\cos^2\theta - \sin^2\theta)e_1$, then for every $X \in TM$, $T^2X = (\cos^2\theta - \sin^2\theta)^2(X + \eta(X)\xi$, which proves the result.

## 6. Some Characterization Results

If the ambient manifold has certain structure, slant submanifolds can be characterized. We put these results in a relation with similar results known for the Sasakian and Lorentzian para Sasakian case.

**Theorem 6.1.** *Let $M$ be a proper slant submanifold of a Lorentzian contact manifold, tangent to $\xi$. Then the contact distribution $\mathcal{D}$ is not integrable.*

*Proof.* Let be $X \in \mathcal{D}$, if $\theta$ is the slant angle,

$$T^2TX = TT^2X = -\cos^2\theta TX,$$

so $TX$ also belongs to $\mathcal{D}$. From (7), $\eta([X, TX]) = -2g(TX, TX)$ which is not zero because $TX$ is space like. Then $[X, TX]$ is not orthogonal to $\xi$ and it does not belong to $\mathcal{D}$, so this distribution is not integrable. ■

Now we characterize slant submanifolds of both Lorentzian K-contact and Lorentzian K-para contact manifold in virtue of the behavior of $T^2 = Q$.

**Proposition 6.2.** *Let $M$ be a slant submanifold of a Lorentzian K-(para) contact manifold, $\widetilde{M}$, tangent to $\xi$, and slant angle $\theta$. Then,*

$$(\nabla_X Q)Y = \cos^2\theta(g(TX, Y)\xi + \varepsilon\eta(Y)TX),$$

*for each $X, Y$ tangent to $M$.*

*Proof.* On the one hand, $\nabla_X Y$ is tangent to $M$ so from (8)

$$Q\nabla_X Y = \cos^2\theta(\varepsilon\nabla_X Y + \eta(\nabla_X Y)\xi),$$

$$\nabla_X QY = \nabla_X(\cos^2\theta(\varepsilon Y + \eta(Y)\xi)) = \cos^2\theta(\varepsilon\nabla_X Y + \eta(Y)\nabla_X\xi + \nabla_X\eta(Y)\xi).$$

On the other hand from (2) and (6) it follows that $\nabla_X\xi = \varepsilon TX$, so

$$X\eta(Y) = \varepsilon Xg(Y, \xi) = \eta(\nabla_X Y) + g(Y, TX).$$

Therefore, $(\nabla_X Q)Y = \cos^2\theta(g(TX, Y)\xi + \varepsilon\eta(Y)TX)$. ■

This result and the following theorem were stated in [9] for slant submanifolds of a Lorentzian para Sasakian manifold.

**Theorem 6.3.** *Let $M$ be a submanifold of a Lorentzian K-(para) contact manifold, $\widetilde{M}$, tangent to $\xi$ is a slant submanifold if and only if the following conditions are verified:*

($i$) *The endomorphism $Q|_{\mathcal{D}}$ has just one eigenvalue at each point of $M$.*

($ii$) *There exists a function $\lambda : M \longrightarrow [0,1]$ such as*

$$(\nabla_X Q)Y = \lambda(g(TX,Y)\xi + \varepsilon\eta(Y)TX),$$

*for each $X, Y$ tangent to $M$.*

*Proof.* If $M$ is slant, then both the conditions are verified by Proposition 6.2. Reciprocally, let $\lambda_1(x)$ be the eigenvalue of $Q|_{\mathcal{D}}$ at each point $x \in M$ and $Y \in \mathcal{D}$ an unit eigenvector associated with $\lambda_1$.

For each $X \in TM$,

$$\begin{aligned} X(\lambda_1)Y + \lambda_1\nabla_X Y &= \nabla_X(\lambda_1 Y) = \nabla_X(QY) \\ &= Q\nabla_X Y + \lambda_1(g(TX,Y)\xi + \varepsilon\eta(Y)TX). \end{aligned} \tag{11}$$

But for both Lorentzian almost contact and almost para contact manifolds $Q$ is a selfadjoint endomorphism so

$$g(Q\nabla_X Y, Y) = g(\nabla_X Y, QY) = \lambda_1 g(\nabla_X Y, Y) = 0,$$

because $Y$ is an unitary vector field. Then, multiplying in (11) by $Y$, we deduce $X(\lambda_1) = 0$, so $\lambda$ is constant.

For each $X \in TM$, $X = \overline{X} - \varepsilon\eta(X)\xi$ with $\overline{X} \in \mathcal{D}$,

$$QX = Q\overline{X} = \lambda_1\overline{X} = \lambda_1(X + \varepsilon\eta(X)\xi), \tag{12}$$

because $Q|_{\mathcal{D}} = \lambda_1 I$. We can write, with $\lambda_1 = \varepsilon\cos^2\theta$, $T^2X = \cos^2\theta(\varepsilon X + \eta(X)\xi)$. Then from (12), in virtue of Theorem 3.3, it is proved that $M$ is a slant submanifold. ■

For low dimensional submanifolds being slant can be characterized by $\nabla T$. We distinguish the almost contact and almost para contact cases. Similar results have been obtained for slant submanifolds of a Sasakian manifold, [6].

**Theorem 6.4.** *Let $M$ be a three dimensional submanifold of a Lorentzian K-contact manifold tangent to $\xi$. The following statemets are equivalent:*

($i$) *$M$ is slant,*

($ii$) $(\nabla_X T)Y = -\cos^2\theta(g(X,Y)\xi + \eta(Y)X)$, *for any $X, Y \in TM$.*

($iii$) $(\nabla_X Q)Y = \cos^2\theta(g(TX,Y)\xi - \eta(Y)TX)$, *for any $X, Y \in TM$.*

*Proof.* From Theorem 6.3, the first and third enunciates are equivalent, and its just a simple computation that $ii)$ implies $iii)$. So it only rests to prove that every slant submanifold verifies this expresion of $(\nabla_X T)Y$. For a three dimensional submanifold, let us consider $\{e_1, e_2, \xi\}$ and orthogonal frame on $TM$, with $e_2 = \sec\theta T e_1$. First,

$$(\nabla_X T)e_1 = \nabla_X Te_1 - T\nabla_X e_1 = \nabla_X \cos\theta e_2 - T(\omega_1^1(X)e_1 + \omega_1^2(X)e_2 - \omega_1^3(X)\xi) =$$

$$= \cos\theta(\omega_2^1(X)e_1 + \omega_2^2(X)e_2 - \omega_2^3(X)\xi) - \omega_1^2(X)Te_2),$$

where $\omega_i^j(X) = g(\nabla_X e_i, e_j)$. Taking into account that $\omega_i^i = 0$, $\omega_i^j = -\omega_j^i$ and $Te_2 = \cos\theta e_1$, it follows $(\nabla_X T)e_1 = -\cos\theta\omega_2^3(X)\xi$. But,

$$\omega_2^3(X) = g(\nabla_X e_2, \xi) = -g(e_2, \nabla_X \xi) = g(e_2, TX),$$

because for a $K$ contact manifold $\nabla_X \xi = -TX$, and thefore:

$$(\nabla_X T)e_1 = -\cos^2\theta g(e_1, X)\xi. \tag{13}$$

Analogously,

$$(\nabla_X T)e_2 = -\cos^2\theta g(e_2, X)\xi. \tag{14}$$

Finally,

$$(\nabla_X T)\xi = -T\nabla_X \xi = T^2 X = \cos^2\theta(-X + \eta(X)\xi). \tag{15}$$

Then, for $Y \in TM$, $Y = Y^1 e_1 + Y^2 e_2 + \eta(Y)\xi$, and

$$(\nabla_X T)Y = Y^1(\nabla_X T)e_1 + Y^2(\nabla_X T)e_2 + \eta(Y)(\nabla_X T)\xi,$$

so using (13), (14) and (15)

$$\begin{aligned}(\nabla_X T)Y &= -\cos^2\theta(Y^1 g(e_1, X)\xi + Y^2 g(e_2, X)\xi + \eta(Y)(X - \eta(X)\xi)) \\ &= -\cos^2\theta(g(X, Y)\xi + \eta(Y)X),\end{aligned}$$

this is the form stated in $ii)$, which finishes the proof. ∎

For the para contact case we can consider a two dimensional slant submanifold, now we obtained a quite different expresion for $\nabla T$.

**Theorem 6.5.** *Let $M$ be a two dimensional submanifold of a Lorentzian K-para contact manifold tangent to $\xi$. The following statemets are equivalent:*

($i$) *$M$ is slant,*

($ii$) $(\nabla_X T)Y = \cos^2\theta(g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi$*, for any $X, Y \in TM$.*

(*iii*) $(\nabla_X Q)Y = \cos^2\theta(g(TX,Y)\xi + \eta(Y)TX$, *for any* $X, Y \in TM$.

*Proof.* Again it is only necessary to prove that $i)$ implies $ii)$. For a surface tangent to $\xi$ we consider $\{e_1, \xi\}$ a orthogonal basis in $TM$. If it is a slant submanifold $Te_1 = \cos\theta e_1$.

$$(16)\qquad \begin{aligned}(\nabla_X T)e_1 &= \nabla_X Te_1 - T\nabla_X e_1 = \nabla_X \cos\theta e_1 - T(\omega_1^1(X)e_1 - \omega_1^2(X)\xi)\\ &= \cos\theta\nabla_X e_1 = \cos\theta(\omega_1^1(X)e_1 - \omega_1^2(X)\xi) = \cos^2\theta g(e_1,X)\xi,\end{aligned}$$

because $\omega_1^1 = 0$ and

$$\omega_1^2(X) = g(\nabla_X e_1, \xi) = -g(e_1, \nabla_X\xi) = -g(e_1, TX) = -g(Te_1, X) = -\cos\theta g(e_1, X),$$

where we have use that, for a $K$ para contact manifold, $\nabla_X\xi = TX$. Moreover,

$$(17)\qquad (\nabla_X T)\xi = -T\nabla_X\xi = -T^2X = -\cos^2\theta(X + \eta(X)\xi).$$

Then, for $Y \in TM$, $Y = Y^1e_1 - \eta(Y)\xi$, and

$$(\nabla_X T)Y = Y^1(\nabla_X T)e_1 - \eta(Y)(\nabla_X T)\xi,$$

so using (16) and (17)

$$\begin{aligned}(\nabla_X T)Y &= \cos^2\theta(Y^1 g(e_1,X)\xi + \eta(Y)(X + \eta(X)\xi))\\ &= \cos^2\theta(g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi),\end{aligned}$$

this is the form stated in $ii)$, which finishes the proof. ■

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