SLICE DIRAC OPERATOR OVER OCTONIONS*

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ABSTRACT

The slice Dirac operator over octonions is a slice counterpart of the Dirac operator over quaternions. It involves a new theory of stem functions, which is the extension from the commutative $O(1)$ case to the non-commutative $O(3)$ case. For functions in the kernel of the slice Dirac operator over octonions, we establish the representation formula, the Cauchy integral formula (and, more in general, the Cauchy–Pompeiu formula), and the Taylor as well as the Laurent series expansion formulas.

1. Introduction

The purpose of this article is to initiate the study of the slice Dirac operator over octonions. The Dirac operator for quaternions,

$$(1.1) \qquad D = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3},$$

has its root in mathematical physics, quantum mechanics, special relativity, and engineering (see [1, 2, 21]) and it plays a key role in the Atiyah–Singer index theorem (see [5]). It may be called the Dirac operator since it factorizes the 4-dimensional Laplacian. However, we note that in the literature (1.1) is often called the generalized Cauchy–Riemann operator or Cauchy-Fueter operator, see e.g., [6, 23, 32], even though it was originally introduced in a paper by Moisil, see [24].

Based on the Dirac operator for quaternions in (1.1), we shall introduce what we call the slice Dirac operator over octonions, using the slice technique. This technique was used by Gentili and Struppa for quaternions in [15, 16] and for octonions in [17] based on Cullen's approach [11]. This technique makes it possible to extend some properties of holomorphic functions in one complex variable to the high dimensional and non-commutative case of quaternions. It has found significant applications especially in operator theory [3, 9, 10], differential geometry [14], geometric function theory [26, 27] and it can be generalized to other higher dimensional settings like Clifford algebras [7, 8] and real alternative algebras [18, 19, 20, 28].

The heart of the slice technique comes from the slice structure of quaternions $\mathbb{H}$, namely the fact that $\mathbb{H}$ can be expressed as a union of complex half planes as

$$\mathbb{H} = \bigcup_{I \in \mathbb{S}} \mathbb{C}_I^+,$$

where $\mathbb{S}$ denotes the set of imaginary units in $\mathbb{H}$, and $\mathbb{C}_I^+$ is the upper half plane

$$\{x + yI : x \in \mathbb{R}, y \geq 0\}.$$

From this decomposition, it is then natural to say that quaternions have a **book** structure since $\mathbb{C}_I^+$ plays the role of a page in a book for any $I \in \mathbb{S}$. The real axis $\mathbb{R}$ plays the role of the edge of the book in which all the pages of the book intersect, i.e.,

$$\mathbb{C}_I^+ \cap \mathbb{C}_J^+ = \mathbb{R}$$

for any $I \neq J$.

The book structure for quaternions plays the same role as the sheaf or fiber bundle structure in differential geometry.

It is remarkable that the topology in the book structure is no longer the Euclidean one. Indeed, the distance compatible with the topology is given by the Euclidean one in a plane, otherwise the distance between any two points from distinct half planes is measured through the path of light via the real axis.

Following Fueter's construction [13], when one considers an open set O in the upper half complex plane $\mathbb{C}^+$ minus the real line and a holomorphic function $f(x + \iota y) = F_1(x, y) + \iota F_2(x, y)$ on O, one may define a function defined over the quaternions using the book structure. In fact, if we consider $q = x + Iy$, $y > 0$, for some suitable I, we may set

$$f(q) = f(x + Iy) = F_1(x, y) + I F_2(x, y).$$

Note that $\bar{q} = x - Iy$, $y > 0$ and so, by definition,

$$f(\bar{q}) = f(x - Iy) = F_1(x, y) - I F_2(x, y).$$

Note also that the pair (F_1, F_2) satisfies the Cauchy–Riemann system and thus $f(x + Iy)$ is in the kernel of the Cauchy–Riemann operator $\partial_x + I \partial_y$. If one is willing to extend the definition to the points of the real line, there is a problem since if $q \in \mathbb{R}$ then $q = x + I0$ and the imaginary unit I is no longer unique.

To solve this problem, one may consider a weaker notion of book structure and observe that

$$\mathbb{H} = \bigcup_{I \in \mathbb{S}} \mathbb{C}_I,$$

in other words, we may consider $\mathbb{H}$ as the union of complex planes.

Following a slight modification of the Fueter construction, see [25], we consider an open set O in the complex plane symmetric with respect to the real axis (possibly intersecting the real axis) and a holomorphic function $f(x + \iota y) = F_1(x, y) + \iota F_2(x, y)$ on O. If F_1, F_2 are an even-odd pair in the second variable, namely if they satisfy

$$(1.2) \qquad \begin{cases} F_1(x + \iota y) = F_1(x - \iota y) \\ F_2(x + \iota y) = -F_2(x - \iota y) \end{cases} \qquad \forall \, x + \iota y \in O,$$

we may define a function on an open set in $\mathbb{H}$ (suitably constructed using O). Note that these conditions immediately imply that

$$f(x + Iy) = f(x + (-I)(-y))$$

so that f is well defined. Moreover, the fact that F_2 is odd in the second variable implies that $F_2(x, 0) = 0$, thus f is well defined also at real points. This second approach is the one that we will generalize to the octonionic case.

To this end, we set

$$F \equiv \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad z = x + iy \equiv \begin{pmatrix} x \\ y \end{pmatrix},$$

and we consider

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in O(1)$$

where $O(1)$ is identified with the group of matrices $\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \}$. Then we have

$$gz = \begin{pmatrix} x \\ -y \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so (1.2) can be rewritten as

$$(1.3) \qquad \begin{pmatrix} F_1(z) \\ F_2(z) \end{pmatrix} = g^{-1} \begin{pmatrix} F_1(gz) \\ F_2(gz) \end{pmatrix}.$$

Thus, following [29], we impose that

$$F(z) = g^{-1} F(gz), \quad \forall\, g \in O(1),$$

and any F satisfying this condition is called a **stem function**.

We can regard this construction as the commutative stem function theory since F is invariant under the commutative group $O(1)$.

As we shall see, the significant property of the slice regular function in $\mathbb{H}$ (non-commutative counterpart of holomorphic functions, i.e., holomorphic maps depending on the parameter $I \in \mathbb{S}$) is given by the representation formula, which demonstrates that any slice regular function is completely determined by its evaluation at any two distinct half planes, or **pages** in this description.

In order to extend the slice theory for the Cauchy–Riemann operator over quaternions into the slice theory for the slice Dirac operator over octonions, we need to introduce a modified theory of stem functions. It turns out that the corresponding notion of a stem function is invariant under the non-commutative group $O(3)$. It will result in a new form of the representation formula, expressed in terms of a quaternionic matrix.

We point out that the non-commutative and non-associative setting of octonions, the object of this paper, has found significant applications in the universal model of M-theory, in which the universe is given by the product of the 4-dimensional Minkowski space with a G_2-manifold of very small scale. Here the exceptional Lie group G_2 is an automorphism group of octonions (see [4, 22]).

We conclude this introduction with a remark about our definition of intrinsic and stem functions. Rinehart [29] studied the intrinsic functions as self-mappings of an associative algebra. In contrast, our intrinsic functions have distinct dimensions for their definition and target domains, and are constructed in the non-associative setting; see also [12, 30]. Fueter [13] initiated the study of stem functions for complex-valued functions in his construction of radially holomorphic functions on the space of quaternions; see [18] for its recent development. However, their considerations are all restricted to the commutative $O(1)$ setting. In this paper we initiate the study in the non-commutative $O(3)$ setting. It is interesting to note that the procedure we followed may lead to further generalizations to higher dimensional algebras.

The structure of this paper is as follows: In Section 2, we recall some important properties of octonionic algebra $\mathbb{O}$. In Section 3, we introduce the book structure in the octonionic algebra in terms of quaternionic subspaces and the stem function for the non-commutative group $O(3)$; we also provide the representation formula which can be written via a quaternionic matrix. In Section 4 we introduce the slice Dirac operator and a splitting property for slice Dirac functions. Section 5 contains the Cauchy–Pompeiu integral formula for slice functions and the Cauchy integral formula for slice Dirac-regular functions. Finally, in Section 6 we give the expansion of slice Dirac-regular as a Taylor series as well as a Laurent series.

2. The algebra of octonions

The algebra of octonions $\mathbb{O}$ is a real, alternative, non-commutative and non-associative division algebra (see for example [31]). It is isomorphic to $\mathbb{R}^8$ as a real inner product vector space and it can be equipped with the standard orthogonal basis: $e_0 = 1, e_1, \ldots, e_7$.

The multiplication between elements in the basis $e_0, e_1, \ldots, e_7$ is defined by

$$e_i e_j = -\delta_{ij} + \varepsilon_{ijk} e_k, \quad \forall\, i, j, k \in \{1, 2, \ldots, 7\}.$$

Here δ_{ij} is the Kronecker symbol and

$$\varepsilon_{ijk} = \begin{cases} (-1)^{\sigma(\pi)} & \text{if } (i, j, k) \in \pi(\Sigma), \\ 0 & \text{otherwise,} \end{cases}$$

where π is a permutation, $\sigma(\pi)$ its sign, and

$$\Sigma = \{(1, 2, 3), (1, 4, 5), (2, 4, 6), (3, 4, 7), (5, 3, 6), (6, 1, 7), (7, 2, 5)\}.$$

The octonionic algebra $\mathbb{O}$ also can be generated from the quaternion algebra $\mathbb{H}$ by the famous Cayley–Dickson process. Let $\{e_0 = 1, e_1, e_2, e_3 := e_1 e_2\}$ be a basis of $\mathbb{H}$. Then every $x \in \mathbb{O}$ can be expressed as $x = a + e_4 b$, where $a, b \in \mathbb{H}$, and e_4 is a fixed imaginary unit in $\mathbb{O}$ not belonging to $\mathbb{H} = \{e_0, e_1, e_2, e_3\}$. The addition and multiplication are defined as follows: for any $x = a + e_4 b$, $y = c + e_4 d \in \mathbb{O}$,

$$x + y := (a + c) + e_4(b + d),$$

$$xy := ac - d\overline{b} + e_4(\overline{a}d + cb).$$

These two definitions of the octonionic algebra $\mathbb{O}$ are equivalent by setting

$$e_5 := e_4 e_1, \quad e_6 := e_4 e_2, \quad e_7 := e_4 e_3.$$

Every $x \in \mathbb{O}$ can be written as

$$x = x_0 + \sum_{k=1}^{7} e_k x_k, \quad \forall\, x_k \in \mathbb{R}.$$

We can introduce its conjugate

$$\overline{x} := x_0 - \sum_{k=1}^{7} e_k x_k,$$

and then set

$$|x|^2 := x\overline{x} = \sum_{k=0}^{7} x_k^2.$$

The modulus is multiplicative, i.e.,

$$|xy| = |x||y|, \quad \forall\, x,\, y \in \mathbb{O}.$$

In the sequel, given $x \in \mathbb{O}$, we introduce a left multiplication operator

$$L_x : \mathbb{O} \to \mathbb{O},$$

defined as

$$L_x z = xz, \quad \forall\, z \in \mathbb{O}.$$

In general, for any x, $y \in \mathbb{O}$, $L_x L_y \neq L_{xy}$, but equality may hold when suitable assumptions hold:

THEOREM 2.1 (Artin's Theorem, [31]): *The subalgebra generated by any two elements of an alternative is associative. In particular, for all $r \in \mathbb{R}$, and for all $x \in \mathbb{O}$,*

(1) $L_x L_{\overline{x}} = L_{x\overline{x}}$,
(2) $L_r L_x = L_{rx}$.

3. Stem function in the octonionic setting

Let $\mathbb{O}$ be the algebra of octonions. The set of its imaginary units is a sphere of dimension six,

$$\mathbb{S}^6 := \{x \in \mathbb{O} : x^2 = -1\}.$$

Let

$$\mathbb{I} := (1, I, J, K) \in \mathbb{O}^4,$$

with the triple I, J, K satisfying

$$I, J \in \mathbb{S}^6, \quad I \perp J, \quad K = IJ.$$

The set of all such row vectors $\mathbb{I}$ is denoted by $\mathcal{N}$. For any $\mathbb{I} := (1, I, J, K) \in \mathcal{N}$, we consider the algebra of quaternions generated by it, i.e.,

$$\mathbb{H}_{\mathbb{I}} = \operatorname{span}_{\mathbb{R}}\{1, I, J, K\}.$$

LEMMA 3.1: *Let $\mathbb{I} \in \mathcal{N}$ and $e \in \mathbb{S}^6$ such that $e \perp \mathbb{H}_{\mathbb{I}}$. Then $a(eb) = e(\bar{a}b)$ for every $a, b \in \mathbb{H}_{\mathbb{I}}$.*

Proof. Since the Cayley–Dickson process does not depend on the choice of the orthonormal basis $\mathbb{I}$, the result follows directly from the definition of the product of octonions. ∎

We can endow the octonionic algebra with a structure that we still call a **book** structure

$$\mathbb{O} = \bigcup_{\mathbb{I} \in \mathcal{N}} \mathbb{H}_{\mathbb{I}},$$

as we prove in the following result:

PROPOSITION 3.2: *The octonionic algebra has the structure*

$$\mathbb{O} = \bigcup_{\mathbb{I} \in \mathcal{N}} \mathbb{H}_{\mathbb{I}}.$$

Proof. Any $x \in \mathbb{O}$ can be written as the sum of its real part x_0 and its imaginary part $\mathrm{Im}(x) = \sum_{k=1}^{7} e_k x_k$. Therefore, it can be further expressed as

$$x = x_0 + Iy$$

with $x_0,\ y \in \mathbb{R}$ and $I = \mathrm{Im}(x)/|\mathrm{Im}(x)|$. We have that

$$I^2 = \frac{1}{|\mathrm{Im}(x)|^2} \left(\sum_{k=1}^{7} e_k x_k \right)\left(\sum_{k=1}^{7} e_k x_k \right) = -\frac{1}{|\mathrm{Im}(x)|^2} \sum_{k=1}^{7} x_k^2 = -1,$$

thus $I \in \mathbb{S}^6$. Now we can choose $J,\ K \in \mathbb{S}^6$ such that $\mathbb{I} := (1,\ I,\ J,\ K) \in \mathcal{N}$. Hence $x_0 + Iy \in \mathbb{H}_{\mathbb{I}}$. ∎

We note that, in general, any $x \in \mathbb{O}$ belongs to more than one quaternionic space, as the following example shows.

Example 3.3: Let $\{1, e_1, \ldots, e_7\}$ be a standard basis of $\mathbb{O}$ and consider

$$x = 1 + e_1 + e_2 + e_3 + e_4.$$

According to Proposition 3.2, we have $x = 1 + 2I$ where $I = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$, so $x \in \mathbb{H}_{\mathbb{I}}$, $\mathbb{I} = (1,\ I,\ J,\ K)$ where $J,\ K$ are any two elements orthogonal to I such that $\mathbb{I} \in \mathcal{N}$. Take now

$$I' = e_1,\ J' = \frac{e_2 + e_3 + e_4}{\sqrt{3}},\ K' = I'J' = e_1 \frac{e_2 + e_3 + e_4}{\sqrt{3}}.$$

It is easy to see that

$$I',\ J',\ K' \in \mathbb{S}^6,\quad I'^2 = J'^2 = K'^2 = -1,\quad J'K' = -K'J',\quad K'I' = -I'K'.$$

Since $x = 1 + I' + \sqrt{3}J'$, we have $x \in \mathbb{H}_{\mathbb{I}'}$, where $\mathbb{I}' := (1,\ I',\ J',\ K')$.

Let $O(4)$ be the group of orthogonal transformations of $\mathbb{R}^4$, and let $O(3)$ be its subgroup keeping the real axis invariant. Therefore, any $g \in O(3)$ can be regarded as a matrix in the form

$$g = \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix},$$

where P is an orthogonal transformation of $\mathbb{R}^3$. The transformation $g : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ can be naturally extended to a map (still denoted by g) $g : \mathbb{O}^4 \longrightarrow \mathbb{O}^4$ via

$$ga = g \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix},$$

for any $a = (a_0, a_1, a_2, a_3) \in \mathbb{O}^4$.

Definition 3.4: Let Ω be an open subset of $\mathbb{R}^4$. If $F : \Omega \to \mathbb{O}^4$ is a $O(3)$-**intrinsic function**, i.e., for any $x \in \Omega$ and for any $g \in O(3)$ such that $gx \in \Omega$, it satisfies

(3.1)
$$F(x) = g^{-1} F(gx),$$

then F is called an $\mathbb{O}$-**stem function** on Ω.

Remark 3.5: We point out that it is not reductive to assume that $\Omega \subseteq \mathbb{R}^4$ is $O(3)$-intrinsic, otherwise in the previous definition we may consider the subset Ω' of Ω such that $x \in \Omega'$ if and only if $gx \in \Omega'$ for any $g \in O(3)$. But this is equivalent to assuming that Ω' is $O(3)$-intrinsic.

We also recall that in the quaternionic case, the stem function is complex intrinsic, namely it is invariant under the commutative group $O(1)$. In other words, $f(\bar{z}) = \overline{f(z)}$ where $\bar{z}$ denotes the complex conjugation. In our setting, the stem function is intrinsic under the non-commutative group $O(3)$ and this set is evidently non-empty since it contains, e.g., $F_i(x) = x_i c$, $i = 1, 2, 3, 4$ where c is a constant in $\mathbb{O}$.

With the book structure, we can define a slice function by lifting a stem function. In fact, if an $\mathbb{O}^4$-valued function F defined on $\mathbb{R}^4$ is an $O(3)$-intrinsic function, then there exists a slice function $f : \mathbb{O} \to \mathbb{O}$ such that the following

diagram commutes for all $\mathbb{I} = (1, I, J, K) \in \mathcal{N}$:

$$
\begin{array}{ccc}
\mathbb{O} & \xrightarrow{\;f\;} & \mathbb{O} \\
\phi_{\mathbb{I}} \big\uparrow & & \big\uparrow \widetilde{\phi}_{\mathbb{I}} \\
\mathbb{R}^4 & \xrightarrow{\;F\;} & \mathbb{O}^4
\end{array}
$$

where

$$\phi_{\mathbb{I}}(x) = x_0 + I x_1 + J x_2 + K x_3 =: \mathbb{I} x^T, \quad \forall\, x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4,$$

and

$$\widetilde{\phi}_{\mathbb{I}}(y) = \mathbb{I} y^T, \quad \forall\, y \in \mathbb{O}^4.$$

Here we denote by x^T the transpose of the row vector $x = (x_0, x_1, x_2, x_3)$ and similarly for y^T.

Given an open subset Ω of $\mathbb{R}^4$, we consider the axially symmetric open set in $\mathbb{O}$ generated by Ω, defined as

$$[\Omega] := \{ q = \mathbb{I} x^T \in \mathbb{O} : \mathbb{I} \in \mathcal{N}, x \in \Omega \}.$$

If Ω is a domain, i.e., a connected open subset of $\mathbb{R}^4$, then $[\Omega]$ is an axially symmetric domain.

For any $x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$, we consider the three involutions

$$
\begin{aligned}
\alpha(x) &= (x_0, x_1, -x_2, -x_3), \\
\beta(x) &= (x_0, -x_1, x_2, -x_3), \\
\gamma(x) &= (x_0, -x_1, -x_2, x_3).
\end{aligned}
$$

Let $\mathbb{I} \in \mathcal{N}$ be fixed arbitrarily. In virtue of the identification of $\mathbb{H}_{\mathbb{I}}$ with $\mathbb{R}^4$, the map α can be identified with the map

$$\alpha_{\mathbb{I}} : \mathbb{H}_{\mathbb{I}} \longrightarrow \mathbb{O}$$

defined by

$$\alpha_{\mathbb{I}}(\mathbb{I} x^T) := \mathbb{I} \alpha(x)^T.$$

For the simplicity of the notation we still keep the notation of α instead of $\alpha_{\mathbb{I}}$ whenever there is no confusion. The same convention is also suitable to other notations in the sequel such as $\beta, \gamma, \mathcal{F}, V, V_\alpha, \mathbb{V}, \mathcal{V}, P_\alpha, \mathbb{P}_\alpha, \mathcal{P}_\alpha, A_\alpha$, and B_α, which also rely on $\mathbb{I} \in \mathcal{N}$.

Definition 3.6: For any open subset Ω of $\mathbb{R}^4$, we define the symmetrized set $\widetilde{[\Omega]}$ as

$$\widetilde{[\Omega]} := \{\mathbb{I}x^T \in \mathbb{O} : \mathbb{I} \in \mathcal{N}, x \in \Omega \text{ such that } \alpha(x), \beta(x), \gamma(x) \in \Omega\}.$$

It is easy to check that

$$\widetilde{[\Omega]} \subset [\Omega].$$

Observe that $\widetilde{[\Omega]}$ can be empty, also in the case when $[\Omega]$ is not.

Definition 3.7: Let $[\Omega]$ be an open set in $\mathbb{O}$. Any $\mathbb{O}$-stem function $F : \Omega \to \mathbb{O}^4$ induces a function

$$f = \mathcal{L}(F) : [\Omega] \to \mathbb{O}$$

defined by

$$f(q) = \mathbb{I}F(x)^T$$

for any $q \in [\Omega]$ with $q = \mathbb{I}x^T$ for some $\mathbb{I} \in \mathcal{N}$. We say that f is a (left) slice function (induced by F).

Since, in general, any element in $\mathbb{O}$ may belong to more than one $\mathbb{H}_\mathbb{I}$, we need to prove the following:

PROPOSITION 3.8: *Definition 3.7 is well-posed.*

Proof. Assume that $q \in \mathbb{O}$ can be written in two ways as

$$q = \mathbb{I}x^T = \mathbb{I}'x'^T;$$

we have to show that

$$f(\mathbb{I}x^T) = f(\mathbb{I}'x'^T).$$

We divide the proof into various cases.

CASE 1: Assume that $\mathbb{I} \neq \mathbb{I}'$ but $\mathbb{H}_\mathbb{I} = \mathbb{H}_{\mathbb{I}'}$. Then there exist $g \in O(3)$ such that $\mathbb{I}' = \mathbb{I}g$. This means that

$$gx'^T = x^T.$$

Since F is an $\mathbb{O}$-stem function, we have

$$F(x'^T) = F(g^{-1}x^T) = g^{-1}F(x^T)$$

so that

$$\mathbb{I}'F(x'^T) = \mathbb{I}'g^{-1}F(x^T)$$

which yields $\mathbb{I}'F(x'^T) = \mathbb{I}F(x^T)$ and the assertion follows.

CASE 2: If $\mathbb{H}_{\mathbb{I}} \neq \mathbb{H}_{\mathbb{I}'}$, we claim that $\mathbb{H}_{\mathbb{I}}$ and $\mathbb{H}_{\mathbb{I}'}$ intersect at $\mathbb{C}_I$ for some $I \in \mathbb{S}^6$. Indeed, since $q = \mathbb{I}x^T = \mathbb{I}'x'^T$, there exists $y_0, y_1 \in \mathbb{R}$ and $I \in \mathbb{S}^6$ such that $q = y_0 + Iy_1 \in \mathbb{C}_I$ and the claim follows. Therefore, we can choose $J_1, J_1' \in \mathbb{S}^6$ respectively such that

$$\mathbb{H}_{\mathbb{I}} = \mathbb{H}_{\mathbb{I}_1}, \quad \mathbb{H}_{\mathbb{I}'} = \mathbb{H}_{\mathbb{I}_1'}$$

where

$$\mathbb{I}_1 = (1, I, J_1, IJ_1)$$

and

$$\mathbb{I}_1' = (1, I, J_1', IJ_1').$$

Since $q = \mathbb{I}x^T = \mathbb{I}'x'^T \in \mathbb{C}_I$, it can be written as

$$q = \mathbb{I}x^T = \mathbb{I}_1 y^T = \mathbb{I}_1' y^T = \mathbb{I}'x'^T$$

for some $y = (y_0, y_1, 0, 0) \in \mathbb{R}^4$. The computation in Case 1 then shows that

$$\mathbb{I}F(x)^T = \mathbb{I}_1 F(y)^T = \mathbb{I}_1' F(y)^T = \mathbb{I}'F(x')^T.$$

In conclusion, Definition 3.7 is well-posed. ∎

Definition 3.9: Let $[\Omega]$ be an open set in $\mathbb{O}$. We set

$$S([\Omega]) := \{f : [\Omega] \to \mathbb{O} \mid f = \mathcal{L}(F), \ F : \Omega \to \mathbb{O}^4 \text{ is an } \mathbb{O}\text{-stem function}\}.$$

In other words, $S([\Omega])$ denotes the collection of slice functions on $[\Omega]$.

Now we provide the representation formula of slice functions in terms of a quaternion matrix:

THEOREM 3.10: *Let f be a slice function on an axially symmetric set $[\Omega]$ in $\mathbb{O}$. Let $q \in \mathbb{O}$ and let $q = \mathbb{I}x^T$, for $\mathbb{I} \in \mathcal{N}$ and $x \in \mathbb{R}^4$. Then for any*

$$p := \mathbb{I}'x^T$$

with $\mathbb{I}' \in \mathcal{N}$ the following formula holds:

$$(3.2) \quad f(p) = \begin{pmatrix} 1 & I' & J' & K' \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -I & -I & I & I \\ -J & J & -J & J \\ -K & K & K & -K \end{pmatrix} \begin{pmatrix} f(q) \\ f(\alpha(q)) \\ f(\beta(q)) \\ f(\gamma(q)) \end{pmatrix}.$$

Proof. Since $[\Omega]$ is an axially symmetric set, we have $\alpha(q), \beta(q), \gamma(q) \in [\Omega]$ for any $q \in [\Omega]$. By definition,

$$
(3.3) \qquad
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
F_0(x) \\
IF_1(x) \\
JF_2(x) \\
KF_3(x)
\end{pmatrix}
=
\begin{pmatrix}
f(q) \\
f(\alpha(q)) \\
f(\beta(q)) \\
f(\gamma(q))
\end{pmatrix}
$$

so that

$$
(3.4) \qquad
\begin{aligned}
\begin{pmatrix}
F_0(x) \\
IF_1(x) \\
JF_2(x) \\
KF_3(x)
\end{pmatrix}
&=
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}^{-1}
\begin{pmatrix}
f(q) \\
f(\alpha(q)) \\
f(\beta(q)) \\
f(\gamma(q))
\end{pmatrix} \\
&=
\frac{1}{4}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
f(q) \\
f(\alpha(q)) \\
f(\beta(q)) \\
f(\gamma(q))
\end{pmatrix}.
\end{aligned}
$$

Thanks to Artin Theorem, see Theorem 2.1, we get

$$
(3.5) \qquad
\begin{pmatrix}
F_0(x) \\
F_1(x) \\
F_2(x) \\
F_3(x)
\end{pmatrix}
=
\frac{1}{4}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
-I & -I & I & I \\
-J & J & -J & J \\
-K & K & K & -K
\end{pmatrix}
\begin{pmatrix}
f(q) \\
f(\alpha(q)) \\
f(\beta(q)) \\
f(\gamma(q))
\end{pmatrix}.
$$

By the definition of slice functions, for any $\mathbb{I}' = (1, I', J', K') \in \mathcal{N}$ we then have

$$
\begin{aligned}
&f(x_0 + I'x_1 + J'x_2 + K'x_3) \\
&= F_0(x) + I'F_1(x) + J'F_2(x) + K'F_3(x) \\
&= \begin{pmatrix} 1 & I' & J' & K' \end{pmatrix}
\frac{1}{4}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
-I & -I & I & I \\
-J & J & -J & J \\
-K & K & K & -K
\end{pmatrix}
\begin{pmatrix}
f(q) \\
f(\alpha(q)) \\
f(\beta(q)) \\
f(\gamma(q))
\end{pmatrix}. \qquad \blacksquare
\end{aligned}
$$

Remark 3.11: The representation formula can be briefly expressed as

$$f(\mathbb{I}'x^T) = \mathbb{I}'(M_\mathbb{I}\mathcal{F}(q)),$$

where $\mathbb{I}' = (1, I', J', K') \in \mathcal{N}$, $q = \mathbb{I}x^T$,

$$\mathcal{F}(q) = (f(q), f(\alpha(q)), f(\beta(q)), f(\gamma(q)))^T,$$

and

$$M_\mathbb{I} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -I & -I & I & I \\ -J & J & -J & J \\ -K & K & K & -K \end{pmatrix}.$$

This representation is very useful to prove further properties of slice functions.

Moreover, notice that $2M_\mathbb{I}$ is an orthogonal matrix with elements in $\mathbb{H}_\mathbb{I}$, i.e.,

$$2M_\mathbb{I} \in O(\mathbb{H}_\mathbb{I}).$$

The following result shows that the slice function $f(\mathbb{I}x^T)$ is a linear function of $\mathbb{I}$.

THEOREM 3.12: *Let f be a slice function on an axially symmetric set $[\Omega]$. Then the octonionic-valued vector function $M_\mathbb{I}\mathcal{F}(q)$ depends only on x but not on $\mathbb{I}, \mathbb{I}'$ and $f(\mathbb{I}'x) = \mathbb{I}'(M_\mathbb{I}\mathcal{F}(q))$ is a linear function in $\mathbb{I}'$.*

Proof. By construction, $M_\mathbb{I}\mathcal{F}(q)$ is independent of $\mathbb{I}'$. Theorem 3.10 shows that

$$f(\mathbb{I}'x^T) = \mathbb{I}'(M_\mathbb{I}\mathcal{F}(q))$$

holds for any $\mathbb{I}$, which implies that $M_\mathbb{I}\mathcal{F}(q)$ is independent of $\mathbb{I}$. (As an alternative, one can prove the assertion noting that (3.5) shows that

$$M_\mathbb{I}\mathcal{F}(q) = \begin{pmatrix} F_0(x) \\ \vdots \\ F_3(x) \end{pmatrix}$$

and so $M_\mathbb{I}\mathcal{F}(q)$ is independent of $\mathbb{I}$.) Moreover, the linearity in $\mathbb{I}'$ is immediate. ∎

Remark 3.13: Also, the representation formula for quaternionic slice regular functions can be written in matrix form. In fact, for any $I, J \in \mathbb{S}$ where $\mathbb{S}$ is the set of imaginary units of quaternions, and for any $x, y \in \mathbb{R}$, the representation formula can be written as

$$(3.6) \qquad f(x + Jy) = \frac{1}{2}\begin{pmatrix} 1 & J \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -I & I \end{pmatrix}\begin{pmatrix} f(x + Iy) \\ f(x - Iy) \end{pmatrix}.$$

4. Slice Dirac operator

In this section, we introduce the slice Dirac operator in $\mathbb{O}$ and establish the corresponding splitting lemma. We begin by recalling the Dirac operator (1.1) introduced in Section 1:

$$D = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3} = (\, 1 \quad i \quad j \quad k \,) \begin{pmatrix} \frac{\partial}{\partial x_0} \\ \vdots \\ \frac{\partial}{\partial x_3} \end{pmatrix} = (\, 1 \quad i \quad j \quad k \,)\mathsf{D}$$

where $\mathsf{D} = (\frac{\partial}{\partial x_0} \ \frac{\partial}{\partial x_1} \ \frac{\partial}{\partial x_2} \ \frac{\partial}{\partial x_3})^T$, and its conjugate operator

$$\overline{D} = \frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3} = (\, 1 \quad -i \quad -j \quad -k \,)\mathsf{D}.$$

For any fixed $\mathbb{I} = (1, I, J, K) \in \mathcal{N}$, we define the **slice Dirac operator** in $\mathbb{O}$ as

$$(4.1) \qquad D_{\mathbb{I}} = \frac{\partial}{\partial x_0} + I\frac{\partial}{\partial x_1} + J\frac{\partial}{\partial x_2} + K\frac{\partial}{\partial x_3} = (\, 1 \quad I \quad J \quad K \,)\mathsf{D}.$$

In the sequel, the restriction $f|_{\mathbb{H}_{\mathbb{I}}}$ of a function f to $\mathbb{H}_{\mathbb{I}}$ shall be denoted by $f_{\mathbb{I}}$:

$$f_{\mathbb{I}} = f|_{\mathbb{H}_{\mathbb{I}}}.$$

We now introduce a main definition:

Definition 4.1: Let $[\Omega]$ be an axially symmetric domain in $\mathbb{O}$ and let $f \in S([\Omega]) \cap C^1([\Omega])$ so that $f = \mathcal{L}(F)$, $f(q) = \mathbb{I}F(x)^T$, where $q = \mathbb{I}x^T$, $F = [F_0, F_1, F_2, F_3]$. If F satisfies

$$(4.2) \qquad \begin{pmatrix} \partial_{x_0} & -\partial_{x_1} & -\partial_{x_2} & -\partial_{x_3} \\ \partial_{x_1} & \partial_{x_0} & -\partial_{x_3} & \partial_{x_2} \\ \partial_{x_2} & \partial_{x_3} & \partial_{x_0} & -\partial_{x_1} \\ \partial_{x_3} & -\partial_{x_2} & \partial_{x_1} & \partial_{x_0} \end{pmatrix} \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

then f is called a **(left) slice Dirac-regular function** in $[\Omega]$.

We denote the set of slice Dirac-regular functions on the axially symmetric set $[\Omega]$ by $SR([\Omega])$.

PROPOSITION 4.2: *Let $[\Omega]$ be an axially symmetric domain in $\mathbb{O}$ and let $f \in S([\Omega]) \cap C^1([\Omega])$. Then f is (left) slice Dirac-regular if and only if*

$$(4.3) \qquad D_{\mathbb{I}}f(q) = 0, \quad \forall\, q \in [\Omega] \cap \mathbb{H}_{\mathbb{I}} =: \Omega_{\mathbb{I}}$$

for all $\mathbb{I} \in \mathcal{N}$.

Proof. Let f be slice Dirac-regular and let $q \in [\Omega] \cap \mathbb{H}_\mathbb{I}$, $q = \mathbb{I}x^T$. Then using (4.2) we have

$$
\begin{aligned}
D_\mathbb{I}f(q) =& D_\mathbb{I}\mathbb{I}F(x^T) = D_\mathbb{I}(F_0 + IF_1 + JF_2 + KF_3) \\
=& (\partial_{x_0}F_0 - \partial_{x_1}F_1 - \partial_{x_2}F_2 - \partial_{x_3}F_3) \\
& + I(\partial_{x_1}F_0 + \partial_{x_0}F_1 - \partial_{x_3}F_2 + \partial_{x_2}F_3) \\
& + J(\partial_{x_2}F_0 + \partial_{x_3}F_1 + \partial_{x_0}F_2 - \partial_{x_1}F_3) \\
& + K(\partial_{x_3}F_0 - \partial_{x_2}F_1 + \partial_{x_1}F_2 + \partial_{x_0}F_3) \\
=& 0.
\end{aligned}
$$

(4.4)

Conversely, let us assume that the slice function f is such that (4.3) holds for all $\mathbb{I} \in \mathcal{N}$. Let us fix an arbitrary $\mathbb{I} \in \mathcal{N}$ and $q = \mathbb{I}x^T$ and let us impose that $f(q) = \mathbb{I}F(x^T)$ satisfies (4.3). Computations as in (4.4) show that (4.2) holds, by arbitrarity of $\mathbb{I}$. ∎

Remark 4.3: We note that (4.3) is well-defined. Indeed, for any $q \in \mathbb{O}$ there exist $\mathbb{I} \in \mathcal{N}$ such that $q \in \mathbb{H}_\mathbb{I}$ and $q = (1, I, J, K)(x_0, x_1, x_2, x_3)^T$. It can be also written as $q = ((1, I, J, K)g^{-1})(y_0, y_1, y_2, y_3)^T$ for any $g \in O(3)$ and $y^T = gx^T$. By the chain rule, it can be directly shown that

$$
(1, I, J, K)(\partial_{x_0}, \partial_{x_1}, \partial_{x_2}, \partial_{x_3})^T f(q) = ((1, I, J, K)g^{-1})(\partial_{y_0}, \partial_{y_1}, \partial_{y_2}, \partial_{y_3})^T f(q)
$$

which implies the claim.

Example 4.4: Consider the function $F = (F_0, F_1, F_2, F_3) : \mathbb{R}^4 \to \mathbb{O}^4$ defined by

$$
\begin{cases}
F_0(x) = 3x_0, \\
F_1(x) = x_1, \\
F_2(x) = x_2, \\
F_3(x) = x_3,
\end{cases}
$$

where $x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$. It is evident that F is an $\mathbb{O}$-stem function since (3.1) holds true. Then it induces a slice function $f : \mathbb{O} \to \mathbb{O}$ given by

$$
f(x_0 + Ix_0 + Jx_2 + Kx_3) = 3x_0 + Ix_1 + Jx_2 + Kx_3,
$$

for any $\mathbb{I} = (1, I, J, K) \in \mathcal{N}$. It is easy to verify that f is a solution of equations (4.3) by direct calculation, which means that f is a slice Dirac-regular function on $\mathbb{O}$.

Example 4.5: We construct a function $F = (F_0, F_1, F_2, F_3) : \mathbb{R}^4 \to \mathbb{O}^4$ via

$$\begin{cases} F_0(x) = S(x_0, r), \\ F_1(x) = x_1 h(x_0, r), \\ F_2(x) = x_2 h(x_0, r), \\ F_3(x) = x_3 h(x_0, r), \end{cases}$$

where

$$x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \quad \text{and} \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Here $S, h : \mathbb{R}^2 \to \mathbb{O}$ are functions satisfying differential equations

$$(4.5) \qquad \begin{cases} y\partial_y h(x, y) + 3h(x, y) = \partial_x S(x, y), \\ y\partial_x h(x, y) = -\partial_y S(x, y). \end{cases}$$

It can be directly verified that (3.1) holds true so that F is an $\mathbb{O}$-stem function. This stem function F induces a slice function $f : \mathbb{O} \to \mathbb{O}$ defined by

$$f(x_0 + Ix_0 + Jx_2 + Kx_3) = S(x_0, r) + Ix_1 h(x_0, r) + Jx_2 h(x_0, r) + Kx_3 h(x_0, r)$$

for any $\mathbb{I} = (1, I, J, K) \in \mathcal{N}$. Since S, h satisfy equations (4.5), it is easy to verify that F is a solution of equations (4.2). This means that f is a slice Dirac-regular function on $\mathbb{O}$.

An explicit example for F is given by

$$\begin{cases} F_0(x) = \frac{3}{5}x_0^5 - 2x_0^3 r^2 + \frac{3}{5}x_0 r^4, \\ F_1(x) = x_1(x_0^4 - \frac{6}{5}x_0^2 r^2 + \frac{3}{35}r^4), \\ F_2(x) = x_2(x_0^4 - \frac{6}{5}x_0^2 r^2 + \frac{3}{35}r^4), \\ F_3(x) = x_3(x_0^4 - \frac{6}{5}x_0^2 r^2 + \frac{3}{35}r^4). \end{cases}$$

The restriction of a slice Dirac-regular function to a quaternionic space $\mathbb{H}_{\mathbb{I}}$ satisfies the following splitting property:

LEMMA 4.6: *Let f be a slice Dirac-regular function defined on an axially symmetric domain $[\Omega]$. Then for any $\mathbb{I} \in \mathcal{N}$ and any $e_4 \in \mathbb{S}^6$ with $e_4 \perp \mathbb{H}_{\mathbb{I}}$, there exist two functions $G_1, G_2 : \Omega_{\mathbb{I}} \to \mathbb{H}_{\mathbb{I}}$ with $D_{\mathbb{I}}G_1 = 0$, $\overline{D}_{\mathbb{I}}G_2 = 0$ such that*

$$(4.6) \qquad f(q) = G_1(q) + e_4 G_2(q), \quad \forall\, q \in \Omega_{\mathbb{I}}.$$

Proof. Since $e_4 \perp \mathbb{H}_\mathbb{I}$, there exist (unique) functions $G_1, G_2 : \Omega_\mathbb{I} \to \mathbb{H}_\mathbb{I}$ such that

$$f = G_1 + e_4 G_2$$

on $\Omega_\mathbb{I}$. Hence, using Lemma 3.1, it follows that

$$\begin{aligned} 0 = D_\mathbb{I} f &= D_\mathbb{I} G_1 + D_\mathbb{I}(e_4 G_2) \\ &= D_\mathbb{I} G_1 + e_4(\overline{D}_\mathbb{I} G_2), \end{aligned}$$

which implies

$$D_\mathbb{I} G_1 = \overline{D}_\mathbb{I} G_2 = 0,$$

and the assertion follows. ∎

Remark 4.7: We note that, in principle, one could have written

$$f = G_1 + G_2 e_4$$

and the condition of being slice Dirac regular would translate into

$$D_\mathbb{I} G_1 = 0, \quad G_2 D_\mathbb{I} = 0,$$

obtaining that G_2 is right regular.

5. Cauchy integral formula

In this section, we present the Cauchy integral theory for the slice Dirac operator.

Throughout, we let Ω be an open subset in $\mathbb{R}^4$ and recall the notations

$$[\Omega] := \{q = \mathbb{I}x^T \in \mathbb{O} : \mathbb{I} \in \mathcal{N}, x \in \Omega\}, \quad \Omega_\mathbb{I} = [\Omega] \cap \mathbb{H}_\mathbb{I}.$$

We shall consider the function $f : [\Omega] \longrightarrow \mathbb{O}$ and its restrictions $f_\mathbb{I} := f|_{\mathbb{H}_\mathbb{I}}$.

We let

$$\mathbf{n}(\xi) = n_0 + I n_1 + J n_2 + K n_3$$

denote the unit exterior normal to the boundary $\partial \Omega_\mathbb{I}$ at ξ. We consider the Cauchy kernel in $\mathbb{H}_\mathbb{I}$ defined by

$$(5.1) \qquad V(\xi - q) = \frac{1}{2\pi^2} \frac{\overline{\xi - q}}{|\xi - q|^4}, \quad \forall\, \xi, q \in \mathbb{H}_\mathbb{I},$$

and we finally let

$$dm = dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$$

be the Lebesgue volume element in $\mathbb{R}^4$, and dS the induced surface element.

THEOREM 5.1: *Let $f : [\Omega] \longrightarrow \mathbb{O}$ be a slice function on a bounded axially symmetric set $[\Omega] \subset \mathbb{O}$. Suppose that $f_{\mathbb{I}} \in C^1(\overline{\Omega}_{\mathbb{I}})$ and $\partial\Omega_{\mathbb{I}}$ is piecewise smooth for some given $\mathbb{I} \in \mathcal{N}$. Then for all $q \in \Omega_{\mathbb{I}}$, we have*

$$(5.2) \quad f_{\mathbb{I}}(q) = \int_{\partial\Omega_{\mathbb{I}}} V(\xi - q)(\mathbf{n}(\xi)f_{\mathbb{I}}(\xi))dS(\xi) - \int_{\Omega_{\mathbb{I}}} V(\xi - q)(D_{\mathbb{I}}f_{\mathbb{I}}(\xi))dm(\xi).$$

Proof. The classical divergence theorem shows that

$$\int_{\Omega_{\mathbb{I}}} \frac{\partial}{\partial x_j}\mu \, dm = \int_{\partial\Omega_{\mathbb{I}}} n_j\mu \, dS, \quad j = 0, 1, 2, 3$$

for any real-valued function $\mu \in C^1(\Omega_{\mathbb{I}}) \cap C(\overline{\Omega}_{\mathbb{I}})$. Thus for the octonion-valued function $f_{\mathbb{I}} \in C^1(\overline{\Omega}_{\mathbb{I}})$, we have

$$(5.3) \qquad \int_{\Omega_{\mathbb{I}}} \frac{\partial}{\partial x_j}f_{\mathbb{I}} \, dm = \int_{\partial\Omega_{\mathbb{I}}} n_j f_{\mathbb{I}} \, dS, \quad j = 0, 1, 2, 3.$$

Let $e_4 \in \mathbb{S}^6$ with $e_4 \perp \mathbb{H}_{\mathbb{I}}$, and let G_1, G_2 be the $\mathbb{H}_{\mathbb{I}}$-valued functions defined on $\Omega_{\mathbb{I}}$ such that $f_{\mathbb{I}} = G_1 + e_4 G_2$. Hence for any map $V : \Omega_{\mathbb{I}} \to \mathbb{H}_{\mathbb{I}}$ such that $V \in C^1(\Omega_{\mathbb{I}}) \cap C(\overline{\Omega}_{\mathbb{I}})$, we have

$$
\begin{aligned}
&\int_{\Omega_{\mathbb{I}}} (VD_{\mathbb{I}})G_1 + V(D_{\mathbb{I}}G_1)dm \\
(5.4) \quad &= \int_{\Omega_{\mathbb{I}}} \frac{\partial}{\partial x_1}(VG_1) + \frac{\partial}{\partial x_2}(VIG_1) + \frac{\partial}{\partial x_3}(VJG_1) + \frac{\partial}{\partial x_3}(VKG_1)dm \\
&= \int_{\partial\Omega_{\mathbb{I}}} V\mathbf{n}G_1 dS,
\end{aligned}
$$

where we have used associativity in $\mathbb{H}_{\mathbb{I}}$. Similarly, we have

$$(5.5) \qquad \int_{\Omega_{\mathbb{I}}} (\overline{V}\,\overline{D_{\mathbb{I}}})G_2 + \overline{V}(\overline{D_{\mathbb{I}}}G_2)dm = \int_{\partial\Omega_{\mathbb{I}}} \overline{V}\overline{\mathbf{n}}G_2 dS.$$

The equalities (5.4) and (5.5) hold, in particular, when V is the Cauchy kernel in (5.1). We now fix $q \in \Omega_{\mathbb{I}}$ and note that

$$V(\xi - q) = -\frac{1}{4\pi^2}\overline{D}_\xi \frac{1}{|\xi - q|^2},$$

where D_ξ denotes the Dirac operator with respect to the variable ξ. Indeed,

$$
\begin{aligned}
(5.6) \quad V(\xi - q) &= \frac{-1}{4\pi^2}\Big(\frac{\partial}{\partial x_0} - I\frac{\partial}{\partial x_1} - J\frac{\partial}{\partial x_2} - K\frac{\partial}{\partial x_3}\Big)\frac{1}{|\xi - q|^2} \\
&= \frac{1}{2\pi^2}\frac{\overline{\xi - q}}{|\xi - q|^4}.
\end{aligned}
$$

Straightforward calculations show that

$$(5.7) \qquad (VD_\xi)(q) = (D_\xi V)(q) = 0$$

for any $\xi \neq q$.

Take a sufficient small ε such that the ball $B_\varepsilon(q)$ centered at q and with radius ε is contained in $\Omega_{\mathbb{I}}$. From (5.4), (5.5), (5.7), the equation $f_{\mathbb{I}} = G_1 + e_4 G_2$, and Lemma 3.1, we have

$$(5.8) \qquad \int_{\Omega_{\mathbb{I}} \backslash B_\varepsilon(q)} V(D_\xi f_{\mathbb{I}}) dm = \int_{\partial(\Omega_{\mathbb{I}} \backslash B_\varepsilon(q))} V(\mathbf{n} f_{\mathbb{I}}) dS.$$

Hence we can calculate this integral as follows:

$$
\begin{aligned}
\int_{\Omega_{\mathbb{I}} \backslash B_\varepsilon(q)} & V(\xi - q)(D_\xi f_{\mathbb{I}}) dm \\
(5.9) \qquad & = \int_{\partial\Omega_{\mathbb{I}}} V(\xi - q)(\mathbf{n} f_{\mathbb{I}}) dS - \int_{\{|\xi-q|=\varepsilon\}} V(\xi - q)(\mathbf{n} f_{\mathbb{I}}) dS \\
& := I_{\partial\Omega_{\mathbb{I}}} - I_\varepsilon
\end{aligned}
$$

By Equation (5.6) and Artin's Theorem 2.1, we can evaluate the limit of I_ε:

$$
\begin{aligned}
\lim_{\varepsilon \to 0} I_\varepsilon & = \frac{1}{2\pi^2} \lim_{\varepsilon \to 0} \int_{\{|\xi-q|=\varepsilon\}} \frac{\overline{\xi - q}}{|\xi - q|^4} \frac{\xi - q}{|\xi - q|} f_{\mathbb{I}}(\xi) dS \\
(5.10) \qquad & = \frac{1}{2\pi^2} \lim_{\varepsilon \to 0} \int_{\{|\xi-q|=\varepsilon\}} \frac{1}{|\xi - q|^3} f_{\mathbb{I}}(\xi) dS \\
& = f_{\mathbb{I}}(q).
\end{aligned}
$$

Let $\varepsilon \to 0$ in (5.9); we get

$$f_{\mathbb{I}}(q) = \int_{\partial\Omega_{\mathbb{I}}} V(\xi - q)(\mathbf{n}(\xi) f_{\mathbb{I}}(\xi)) dS - \int_{\Omega_{\mathbb{I}}} V(\xi - q)(D_\xi f_{\mathbb{I}}(\xi)) dm,$$

as desired. ∎

COROLLARY 5.2: *Let f be a slice Dirac-regular function on a bounded axially symmetric set $[\Omega]$. Suppose that $f_{\mathbb{I}} \in C^1(\overline{\Omega}_{\mathbb{I}})$ and $\partial\Omega_{\mathbb{I}}$ is piecewise smooth for some given $\mathbb{I} \in \mathcal{N}$. Then*

$$(5.11) \qquad f_{\mathbb{I}}(q) = \int_{\partial\Omega_{\mathbb{I}}} V(\xi - q)(\mathbf{n}(\xi) f_{\mathbb{I}}(\xi)) dS, \quad \forall\, q \in \Omega_{\mathbb{I}}$$

and

$$(5.12) \qquad \int_{\partial\Omega_{\mathbb{I}}} V(\xi - q)(\mathbf{n}(\xi) f_{\mathbb{I}}(\xi)) dS = 0, \quad \forall\, q \notin \overline{\Omega}_{\mathbb{I}}.$$

Proof. If $q \in \Omega_{\mathbb{I}}$, then (5.11) follows from Theorem 5.1 since $D_\xi f_{\mathbb{I}}(\xi) = 0$. For $q \notin \overline{\Omega}_{\mathbb{I}}$, the integral at the left hand side of (5.12) is a proper integral so that after a limit process, (5.8) becomes

$$\int_{\Omega_{\mathbb{I}}} V(\xi - q)(D_\xi f_{\mathbb{I}})dm = \int_{\partial\Omega_{\mathbb{I}}} V(\xi - q)(\mathbf{n} f_{\mathbb{I}})dS$$

Since f is slice Dirac-regular in $[\Omega]$, the left hand side vanishes and we obtain (5.12). ∎

Using the representation formula, we can introduce another kernel which allows us to write a Cauchy formula of more general validity.

Denote $M_{n \times m}(\mathbb{O})$ the set of octonion matrices of m rows and n columns where n, m are positive integers. Given an octonion matrix $A \in M_{n \times m}(\mathbb{O})$, the left multiplication operator

$$L_A : M_{m \times k}(\mathbb{O}) \to M_{n \times k}(\mathbb{O})$$

defined as

$$L_A B := AB, \quad \forall B \in M_{m \times k}(\mathbb{O}).$$

In general, $L_A L_B \neq L_{AB}$.

Definition 5.3: For some $\mathbb{I}, \mathbb{I}' \in \mathbb{S}^6$ and for any fixed $\xi, q \in \mathbb{H}_{\mathbb{I}}$ with $\xi \neq q$, we introduce an operator $\mathcal{V} := \mathcal{V}(\xi, q, \mathbb{I}')$. This operator $\mathcal{V} : \mathbb{O} \to \mathbb{O}$ is called the slice Cauchy kernel, defined by

(5.13) $$\mathcal{V}(\xi, q, \mathbb{I}') = L_{\mathbb{I}'} L_{M_{\mathbb{I}}} L_{\mathbb{V}(\xi - q)},$$

where

$$\mathbb{V}(\xi - q) = (V(\xi - q), V(\xi - \alpha(q)), V(\xi - \beta(q)), V(\xi - \gamma(q)))^T.$$

THEOREM 5.4: Let f be a slice function on a bounded axially symmetric set $[\Omega]$, suppose that $f_{\mathbb{I}} \in C^1(\overline{\Omega}_{\mathbb{I}})$ and $\partial\Omega_{\mathbb{I}}$ is piecewise smooth for some given $\mathbb{I} \in \mathcal{N}$. Then for any $p \in [\Omega]$, there exists $\mathbb{I}' \in \mathcal{N}$ such that $p = \mathbb{I}' x^T$ with $x \in \mathbb{R}^4$ and

(5.14) $$f(p) = \int_{\partial\Omega_{\mathbb{I}}} \mathcal{V}(\xi, q, \mathbb{I}')(\mathbf{n}(\xi) f_{\mathbb{I}}(\xi))dS - \int_{\Omega_{\mathbb{I}}} \mathcal{V}(\xi, q, \mathbb{I}')(D_{\mathbb{I}} f_{\mathbb{I}}(\xi))dm,$$

where $q = \mathbb{I} x^T \in \Omega_{\mathbb{I}}$ and $\mathcal{V}$ is as in (5.13).

Proof. By the representation formula in Theorem 3.10, for any $q = \mathbb{I}x^T \in \Omega_{\mathbb{I}}$ and any $\mathbb{I}' \in \mathcal{N}$ we have

$$(5.15) \qquad f(p) = f(\mathbb{I}'x^T) = \mathbb{I}'(M_{\mathbb{I}}\mathcal{F}(q)).$$

Since $[\Omega]$ is an axially symmetric set, it follows that $\alpha(q), \beta(q), \gamma(q) \in \Omega_{\mathbb{I}}$ for any $q \in \Omega_{\mathbb{I}}$. Theorem 5.1 gives

$$(5.16) \qquad \mathcal{F}(q) = \int_{\partial\Omega_{\mathbb{I}}} \mathbb{V}(\xi - q)(\mathbf{n}(\xi)f_{\mathbb{I}}(\xi))dS - \int_{\Omega_{\mathbb{I}}} \mathbb{V}(\xi - q)(D_{\mathbb{I}}f_{\mathbb{I}}(\xi))dm.$$

Substituting (5.16) into (5.15) and moving the integral out, we finally get

$$f(\mathbb{I}'x^T) = \int_{\partial\Omega_{\mathbb{I}}} \mathbb{I}'(M_{\mathbb{I}}(\mathbb{V}(\xi-q)(\mathbf{n}(\xi)f_{\mathbb{I}}(\xi))))dS - \int_{\Omega_{\mathbb{I}}} \mathbb{I}'(M_{\mathbb{I}}(\mathbb{V}(\xi-q)(D_{\mathbb{I}}f_{\mathbb{I}}(\xi))))dm,$$

and (5.13) allows us to conclude. ∎

6. Slice Dirac-regular power series

In this section, we provide the Taylor series for the slice Dirac-regular function and the Laurent series for the slice Dirac-regular function near an isolated singularity.

For any $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, we set $n = |\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $\alpha! = \alpha_1!\alpha_2!\alpha_3!$,

$$\partial_\alpha = \frac{\partial^n}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}},$$

and for any $q = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}$ denote

$$V_\alpha(q) = \partial_\alpha V(q),$$

where

$$V(q) = \frac{1}{2\pi^2} \frac{\bar{q}}{|q|^4}.$$

It is well known that $V_\alpha(q)$ is left and right Dirac-regular except at zero since ∂_α commutes with the Dirac operator D.

Note that the monomials $f(q) = q^n$ are not Dirac-regular. Their Dirac-regular counterparts are the homogeneous left and right Dirac-regular polynomials P_α, defined by

$$P_\alpha(q) = \frac{\alpha!}{n!} \sum (x_{\beta_1} - i_{\beta_1}x_0) \cdots (x_{\beta_n} - i_{\beta_n}x_0),$$

where $q = x_0 + ix_1 + jx_2 + kx_3$. Here the sum runs over all $\frac{n!}{\alpha!}$ different orderings of α_1 1's, α_2 2's and α_3 3's and $i_{\beta_l} \in \{i, j, k\}$ for any $l = 1, 2, \ldots, n$.

The polynomials P_α are homogeneous of degree n, while V_α is homogeneous of degree $-n-3$ (see [6]).

Let U_n be the right quaternionic vector space of homogeneous Dirac-regular functions of degree $n \in \mathbb{N}$. Then, the polynomials P_α ($\alpha \in \mathbb{N}^3$) are Dirac-regular and form a basis for U_n.

THEOREM 6.1: *Let f be a slice Dirac-regular function in the unit ball $B \subset \mathbb{O}$ centered at the origin and let $f \in C^1(\overline{B})$. For any $q \in B$, there exist $\mathbb{I} \in \mathcal{N}$ such that $q \in \mathbb{H}_\mathbb{I}$, and*

$$(6.1) \qquad f(q) = \sum_{n=0}^{+\infty} \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha|=n}} P_\alpha(q) \frac{\partial_\alpha f_\mathbb{I}(0)}{\alpha!},$$

where the power series is uniformly convergent over $B_\mathbb{I}$.

Proof. Let $q \in B$; then there exists $\mathbb{I} \in \mathcal{N}$ such that $q \in \mathbb{H}_\mathbb{I}$. Moreover, there exists a closed ball B_ρ with $\rho < 1$ such that $q \in B_\rho$. By Lemma 4.6, we can pick $e_4 \in \mathbb{S}^6$ with $e_4 \perp \mathbb{H}_\mathbb{I}$, and write

$$f_\mathbb{I}(q) = G_1(q) + e_4 G_2(q),$$

where G_1 and G_2 are $\mathbb{H}_\mathbb{I}$-valued Dirac-regular and conjugate Dirac-regular, respectively. The integral formula (5.11) gives

$$
\begin{aligned}
(6.2) \qquad f_\mathbb{I}(q) &= \int_{\partial B_\mathbb{I}} V(\xi - q)(\mathbf{n}(\xi) f_\mathbb{I}(\xi)) dS \\
&= \int_{\partial B_\mathbb{I}} V(\xi - q)\mathbf{n}(\xi) G_1(\xi) dS + e_4 \left(\int_{\partial B_\mathbb{I}} \overline{V(\xi - q)} \ \overline{\mathbf{n}(\xi)} G_2(\xi) dS \right) \\
&:= I_1 + e_4 I_2.
\end{aligned}
$$

By Theorem 28 in [32], we can expand $V(\xi - q)$ in a power series for any $|q| < |\xi|$:

$$
\begin{aligned}
(6.3) \qquad V(\xi - q) &= \sum_{n=0}^{+\infty} \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha|=n}} (-1)^n P_\alpha(q) V_\alpha(\xi) \\
&= \sum_{n=0}^{+\infty} \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha|=n}} (-1)^n V_\alpha(\xi) P_\alpha(q),
\end{aligned}
$$

and the right-hand side converges uniformly in any region $\{(\xi, q) : |q| \le r|\xi|\}$ with $r < 1$. Since $q \in B_\rho$ and $\xi \in \partial B$, we have $|q| \le r|\xi|$ with $r < 1$. Using the

rightmost expression in (6.3), we get

$$
(6.4) \qquad
\begin{aligned}
I_2 &= \int_{\partial B_{\mathbb{I}}} \sum_{n=0}^{+\infty} \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha|=n}} (-1)^n \overline{V_\alpha(\xi) P_\alpha(q)} \; \overline{\mathbf{n}(\xi)} G_2(\xi) dS \\
&= \sum_{n=0}^{+\infty} \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha|=n}} \overline{P_\alpha(q)} (-1)^n \int_{\partial B_{\mathbb{I}}} \overline{V_\alpha(\xi)} \; \overline{\mathbf{n}(\xi)} G_2(\xi) dS.
\end{aligned}
$$

Using the first expression in (6.3) and repeating the procedure, we have

$$
(6.5) \qquad I_1 = \sum_{n=0}^{+\infty} \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha|=n}} P_\alpha(q)(-1)^n \int_{\partial B_{\mathbb{I}}} V_\alpha(\xi) \mathbf{n}(\xi) G_2(\xi) dS.
$$

Substituting (6.4) and (6.5) into (6.2) we obtain

$$
f_{\mathbb{I}} = \sum_{n=0}^{+\infty} \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha|=n}} P_\alpha(q)\left((-1)^n \int_{\partial B_{\mathbb{I}}} V_\alpha(\xi)(\mathbf{n}(\xi) f_{\mathbb{I}}(\xi)) dS\right).
$$

Differentiating both sides of the integral formula (6.2), we have

$$
\partial_\alpha f_{\mathbb{I}}(q) = (-1)^n \int_{\partial B_{\mathbb{I}}} V_\alpha(\xi - q)(\mathbf{n}(\xi) f_{\mathbb{I}}(\xi)) dS.
$$

In particular, letting $q \to 0$ we conclude that

$$
\partial_\alpha f_{\mathbb{I}}(0) = (-1)^n \int_{\partial B_{\mathbb{I}}} V_\alpha(\xi)(\mathbf{n}(\xi) f_{\mathbb{I}}(\xi)) dS. \qquad \blacksquare
$$

Remark 6.2: We point out that although the polynomial functions $P_\alpha : \mathbb{H}_{\mathbb{I}} \to \mathbb{O}$ are homogeneous left and right Dirac-regular polynomials, they cannot extend to a slice Dirac-regular function on the whole $\mathbb{O}$ in general. For example, we consider the special case when $n = 2$ and set

$$
f(q) := \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha|=2}} P_\alpha(q).
$$

For any $q = x_0 + Ix_1 + Jx_2 + Kx_3 \in \mathbb{H}_{\mathbb{I}}$, by direct calculation we have

$$
f(q) = F_0(x) + IF_1(x) + JF_2(x) + KF_3(x),
$$

where

$$
\begin{aligned}
F_0(x) &= -6x_0^2 + (x_1 + x_2 + x_3)^2 + x_1^2 + x_2^2 + x_3^2, \\
F_1(x) &= -2x_0(x_1 + x_2 + x_3) - 2x_0 x_1, \\
F_2(x) &= -2x_0(x_1 + x_2 + x_3) - 2x_0 x_2, \\
F_3(x) &= -2x_0(x_1 + x_2 + x_3) - 2x_0 x_3.
\end{aligned}
$$

Notice that F_0 does not satisfy the compatibility conditions (3.1). Therefore, not all P_α can be extended to a slice Dirac-regular function on the whole $\mathbb{O}$.

Remark 6.3: We still do not know if the series in (6.1) is convergent uniformly on the whole unit ball B, besides on the subsets $B_\mathbb{I}$. Our proof on $B_\mathbb{I}$ depends on the explicit formula of the kernel V and the associativity of quaternions. This technique obviously fails in the setting of octonions and to consider the uniform convergence over B, one should follow a different approach. In fact, for any $f \in C^1(\overline{B})$, one needs the estimate

$$
|f(q) - f(q')| \le |f(\mathbb{I}x^T) - f(\mathbb{I}'x^T)| + |f(\mathbb{I}'x^T) - f(\mathbb{I}'x'^T)|,
$$

and it is problematic to show that $|f(\mathbb{I}x^T) - f(\mathbb{I}'x^T)|$ is small enough.

We now study the power series at any point $q_0 \in \mathbb{O}$ for slice Dirac-regular functions. With the same approach used in Theorem 6.1, one can show that

$$
\tag{6.6}
f(q) = \sum_{n=0}^{+\infty} \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha|=n}} P_\alpha(q - q_0) \frac{\partial_\alpha f_\mathbb{I}(q_0)}{\alpha!}
$$

for any $q_0 \in \mathbb{O} \cap \mathbb{H}_\mathbb{I}$ and

$$
q \in B_\mathbb{I}(q_0, R, \mathbb{O}) := B(q_0, R, \mathbb{O}) \cap \mathbb{H}_\mathbb{I}.
$$

Here $B(q_0, R, \mathbb{O}) \subset \mathbb{O}$ denotes the ball of radius R centered at q_0.

Let $B(x_0, R) \subset \mathbb{R}^4$ be the ball of radius R centered at x_0 and denote by $\widetilde{B}(x_0, R) \subset \mathbb{O}$ the symmetrized set of $B(x_0, R)$.

We set

$$
\mathbb{P}_\alpha(q - q_0) = (P_\alpha(q - q_0),\, P_\alpha(\alpha(q) - q_0),\, P_\alpha(\beta(q) - q_0),\, P_\alpha(\gamma(q) - q_0))^T
$$

and consider the operator

$$
\mathcal{P}_\alpha(q, q_0, \mathbb{I}') = L_{\mathbb{I}'} L_{M_\mathbb{I}} L_{\mathbb{P}_\alpha(q - q_0)}.
$$

THEOREM 6.4:   Assume that $f \in C^1(\overline{B(q_0, R, \mathbb{O})})$ with $R > 0$ and let $q_0 = \mathbb{I}x_0^T \in \mathbb{H}_\mathbb{I}$. If f is slice Dirac-regular on $B(q_0, R, \mathbb{O})$, then

$$f(\mathbb{I}'x^T) = \sum_{n=0}^{+\infty} \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha|=n}} \mathcal{P}_\alpha(q, q_0, \mathbb{I}')\frac{\partial_\alpha f(q_0)}{\alpha!}$$

for any $q = \mathbb{I}x^T \in B_\mathbb{I}(q_0, R, \mathbb{O}) \cup \widetilde{B}(x_0, R)$.

Proof. By virtue of (6.6), f admits the power series expansion

$$f(q) = \sum_{n=0}^{+\infty} \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha|=n}} P_\alpha(q - q_0)\frac{\partial_\alpha f(q_0)}{\alpha!}$$

for any $q \in B_\mathbb{I}(q_0, R, \mathbb{O})$ and $q_0 = \mathbb{I}x_0^T \in \mathbb{H}_\mathbb{I}$. If $q \in \widetilde{B}(x_0, R)$, then

$$\alpha(q), \beta(q), \gamma(q) \in \widetilde{B}(x_0, R)$$

so that

$$\mathcal{F}(q) = \sum_{n=0}^{+\infty} \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha|=n}} \mathbb{P}_\alpha(q - q_0)\frac{\partial_\alpha f(q_0)}{\alpha!}.$$

By the representation formula, we have

$$f(\mathbb{I}'x^T) = \mathbb{I}'(M_\mathbb{I}\mathcal{F}(q)).$$

Combining these two formulas and taking the sum we conclude that

$$f(\mathbb{I}'x^T) = \sum_{n=0}^{+\infty} \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha|=n}} \mathcal{P}_\alpha(q, q_0, K)\frac{\partial_\alpha f(q_0)}{\alpha!}. \qquad \blacksquare$$

Finally we study the Laurent power series. We need to introduce some notation. Let $q_0 \in \mathbb{O}$ and $0 \leq R_1 < R_2 \leq +\infty$. We consider the spherical shell in $\mathbb{O}$

$$B(q_0, R_1, R_2, \mathbb{O}) = \{q \in \mathbb{O} : R_1 < |q - q_0| < R_2\}$$

and the spherical shell in $\mathbb{R}^4$,

$$B(x_0, R_1, R_2) = \{x_0 \in \mathbb{R}^4 : R_1 < |x - x_0| < R_2\}.$$

We let $\widetilde{B}(x_0, R_1, R_2)$ denote the symmetrized set of $B(x_0, R_1, R_2)$.

Let $f \in C^1(\overline{B(q_0, R_1, R_2, \mathbb{O})})$ and, for $q_0 \in \mathbb{H}_\mathbb{I}$, we set

$$S_i = \{q \in \mathbb{H}_\mathbb{I} : |q - q_0| = R_i\}, \quad i = 1, 2,$$

and the formulas

$$A_\alpha = (-1)^n \int_{S_2} V_\alpha(q - q_0)(\mathbf{n}(\xi)f(\xi))dS,$$

$$B_\alpha = (-1)^n \int_{S_1} P_\alpha(q - q_0)(\mathbf{n}(\xi)f(\xi))dS.$$

THEOREM 6.5: Let $q_0 = \mathbb{I}x_0^T \in \mathbb{H}_\mathbb{I}$. Let f be a slice Dirac-regular function on a spherical shell $B(q_0, R_1, R_2, \mathbb{O})$ and $f \in C^1(\overline{B(q_0, R_1, R_2, \mathbb{O})})$. Then

$$f(\mathbb{I}'x^T) = \sum_{n=0}^{+\infty} \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha| = n}} [\mathcal{P}_\alpha(q, q_0, \mathbb{I}')A_\alpha + \mathcal{V}_\alpha(q, q_0, \mathbb{I}')B_\alpha]$$

for any $q = \mathbb{I}x^T \in B_\mathbb{I}(q_0, R_1, R_2, \mathbb{O}) \cup \widetilde{B}(x_0, R_1, R_2)$.

Proof. Let $q_0 = \mathbb{I}x_0^T \in \mathbb{H}_\mathbb{I}$. The integral formula in Theorem 5.1 implies

$$f(q) = \int_{S_2} V(\xi - q)(\mathbf{n}(\xi)f(\xi))dS - \int_{S_1} V(\xi - q)(\mathbf{n}(\xi)f(\xi))dS.$$

For any $\xi \in S_2$, we have $|\xi - q_0| > |q - q_0|$. Therefore, the same approach as in the proof of Theorem 6.1 shows that

$$\int_{S_2} V(\xi - q)(\mathbf{n}(\xi)f(\xi))dS = \sum_{n=0}^{+\infty} \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha| = n}} P_\alpha(q - q_0)A_\alpha.$$

For any $\xi \in S_1$, we have $|\xi - q_0| < |q - q_0|$. Now we use the second series in (6.3) and repeat the procedure in the proof Theorem 6.1 to deduce that

$$-\int_{S_1} V(\xi - q)(\mathbf{n}(\xi)f(\xi))dS$$

$$= \int_{S_1} V(q - \xi)(\mathbf{n}(\xi)f(\xi))dS$$

(6.7)
$$= \int_{S_1} \sum_{n=0}^{+\infty} \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha| = n}} (-1)^n V_\alpha(q - q_0)P_\alpha(\xi - q_0)(\mathbf{n}(\xi)f(\xi))dS$$

$$= \sum_{n=0}^{+\infty} \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha| = n}} V_\alpha(q - q_0)B_\alpha.$$

This means that

$$f(q) = \sum_{n=0}^{+\infty} \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha|=n}} [P_\alpha(q - q_0)A_\alpha + V_\alpha(q - q_0)B_\alpha].$$

For any $q \in B_{\mathbb{I}}(q_0, R_1, R_2, \mathbb{O}) \cap \widetilde{B}(x_0, R_1, R_2)$, we have

$$\alpha(q), \beta(q), \gamma(q) \in B_{\mathbb{I}}(q_0, R_1, R_2, \mathbb{O}) \cap \widetilde{B}(x_0, R_1, R_2)$$

so that

$$\mathcal{F}(q) = \sum_{n=0}^{+\infty} \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha|=n}} [\mathbb{P}_\alpha(q - q_0)A_\alpha + \mathbb{V}_\alpha(q - q_0)B_\alpha].$$

By the representation formula

$$f(\mathbb{I}'x^T) = \mathbb{I}'(M_{\mathbb{I}}\mathcal{F}_{\mathbb{I}}(q)).$$

We obtain the stated result

$$f(\mathbb{I}'x^T) = \sum_{n=0}^{+\infty} \sum_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha|=n}} [\mathcal{P}_\alpha(q, q_0, K)A_\alpha + \mathcal{V}_\alpha(q, q_0, K)B_\alpha]. \qquad \blacksquare$$

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