Slice monogenic functions

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Abstract

In this paper we offer a new definition of monogenicity for functions defined on $\mathbb{R}^{n+1}$ with values in the Clifford algebra $\mathbb{R}_n$ following an idea inspired by the recent papers [6], [7]. This new class of monogenic functions contains the polynomials (and, more in general, power series) with coefficients in the Clifford algebra $\mathbb{R}_n$. We will prove a Cauchy integral formula as well as some of its consequences. Finally, we deal with the zeroes of some polynomials and power series.

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1 Introduction

In the past thirty years monogenic functions with values in a Clifford algebra $\mathbb{R}_n$ have been successfully and intensively studied. The literature is very rich of results and the studies on the topic are ongoing. However, one disappointment about monogenic functions is that the identity function or the powers of the variable considered are not monogenic functions. In the specific case of the Clifford algebra over two imaginary units, the algebra of quaternions $\mathbb{H}$, Cullen introduced a notion whose purpose was exactly to overcome this problem. Cullen's study (see [4]) is based on the so called intrinsic functions introduced by Rinehart for functions with values in an algebra. On the basis of Rinehart's work, Cullen defined a new class of regular functions and it can be shown that the class of quaternionic regular functions defined by Cullen contains all the power series of the form $\sum_n q^n a_n$ with real coefficients a_n. Recently, Gentili and Struppa adopted a definition of regularity (slice regularity), see [6], [7], where they prove that slice regular functions can be expanded into power series $\sum_n q^n a_n$, with coefficients $a_n \in \mathbb{H}$.

In this paper, widely inspired by [7], we further generalize the ideas therein to the case of functions defined on domains of the Euclidean space $\mathbb{R}^{n+1}$ having values in a Clifford algebra. We will say that a function $f : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ is slice monogenic if for any unit 1-vector I the restriction f_I of the function f to the complex plane $x + Iy$ is holomorphic. We show how slice

monogenic functions can be related to power series, we will prove a Cauchy integral formula as well as some of its consequences. Finally, we deal with the zeroes of polynomials and power series in the variable $\vec{x} \in \mathbb{R}^{n+1}$.

An important application of our ideas is a new way to define a functional calculus for an n-tuple of non commuting operators. In the last section of this paper we provide the algebraic foundations for such calculus that has been developed in [3].

Note that in [8] the authors used similar ideas to study the case of functions $f : \mathbb{R}_3 \to \mathbb{R}_3$: their study shows some interesting geometric peculiarities of $\mathbb{R}_3$ which leads the theory in a different direction from the one pursued in this paper.

2 Slice monogenic functions

Let $\mathbb{R}_n$ be the real Clifford algebra over n units $e_1, \ldots, e_n$ such that $e_i e_j + e_j e_i = -2\delta_{ij}$ (see e.g. [1], [2] or [5] for the basic notation). An element in the Clifford algebra will be denoted by $\sum_A e_A x_A$ where $A = i_1 \ldots i_r$, $i_\ell \in \{1, 2, \ldots, n\}$, $i_1 < \ldots < i_r$ is a multi-index and $e_A = e_{i_1} e_{i_2} \ldots e_{i_r}$; the number r of units in e_A is denoted by $|A|$. Elements in the linear space $\mathbb{R}_n^k$ generated by basis vectors e_A with $|A| = k$ are called k-vectors. An element $\underline{x} \in \mathbb{R}^n$ can be identified with a 1-vector in the Clifford algebra: $(x_1, x_2, \ldots, x_n) \mapsto \underline{x} = x_1 e_1 + \ldots + x_n e_n$ while real numbers will be identified with the 0-vectors, i.e. with elements in $\mathbb{R}_n^0$. A function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}_n$ is seen as a function $f(\underline{x})$ of $\underline{x}$. There are in the literature several ways to define a notion of generalized holomorphy for function with values in $\mathbb{R}_n$. The most successful is the so called monogenicity which has been intensively studied during the past thirty years (see for example [1], [2], [5]). A differentiable function f defined on an open set $U \subseteq \mathbb{R}^n$ is said to be monogenic if it is in the kernel of the Dirac operator

$$\partial_{\underline{x}} = \sum_{i=1}^n e_i \partial_{x_i}.$$

An important variation of the Dirac operator is the Weyl operator

$$\partial_{\vec{x}} = \partial_{x_0} + \sum_{i=1}^n e_i \partial_{x_i}$$

acting on functions $f : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ and where the variable in $\mathbb{R}^{n+1}$ is identified with $\vec{x} = x_0 + \underline{x}$. The theory of the functions in the kernel of the Dirac or of the Weyl operator are equivalent and the word monogenic is used for functions in the kernel of either of them. Despite the fact that several results on the holomorphic functions in one complex variable can be generalized to this setting, the theory of monogenic functions shows an unpleasant feature: neither the identity function $f(\underline{x}) = \underline{x}$ (or $f(\vec{x}) = \vec{x}$) or the powers of the variable $g(\underline{x}) = \underline{x}^n$ (or $g(\vec{x}) = \vec{x}^n$) are monogenic. To introduce our variation of "hyperholomorphic" functions, i.e. the theory of slice monogenic functions, we will denote by $\mathbb{S}$ the sphere of unit 1-vectors, i.e.

$$\mathbb{S} = \{\underline{x} = e_1 x_1 + \ldots + e_n x_n \mid x_1^2 + \ldots + x_n^2 = 1\}.$$

The 2-plane $\mathbb{R} + I\mathbb{R}$ passing through 1 and I will be denoted by L_I: it is a real subspace of $\mathbb{R}^{n+1}$ isomorphic to the complex plane. An element $\vec{x} \in L_I$ will be denoted by $x + Iy$.

Definition 2.1. *Let $U \subseteq \mathbb{R}^{n+1}$ be a domain and let $f : U \rightarrow \mathbb{R}_n$ be a real differentiable function. Let $I \in \mathbb{S}$ and let f_I the restriction of f to the complex line L_I. We say that f is a slice monogenic function (in short s-monogenic function) if for every $I \in \mathbb{S}$*

$$\frac{1}{2} \left(\frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + Iy) = 0.$$

However, when dealing with a notion of monogenicity (compare for example [1]) there are two possibilities for the position of the imaginary units so, in this case, it is possible to introduce an absolutely analogous notion of slice monogenicity on the right. The theory of right slice monogenic functions is equivalent to the theory of slice left monogenic functions. In the sequel, we will consider monogenicity on the left and, for simplicity, we will denote by $\overline{\partial}_I$ the operator $\frac{1}{2} \left(\frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right)$ and we will refer to the left slice monogenic function as s-monogenic functions. We will also introduce a notion of I-derivative by means of the operator:

$$\partial_I := \frac{1}{2} \left(\frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right).$$

Remark 2.2. The s-monogenic functions on $U \subseteq \mathbb{R}^{n+1}$ form a right module $\mathscr{M}(U)$. In fact it is trivial that if $f, g \in \mathscr{M}(U)$ then for every $I \in \mathbb{S}$ one has $\overline{\partial}_I f_I = \overline{\partial}_I g_I = 0$, thus $\overline{\partial}_I (f + g)_I = 0$. Moreover, for any $a \in \mathbb{R}_n$ we have $\overline{\partial}_I (f_I a) = (\overline{\partial}_I f) a = 0$. It is not true, in general, that the product of two s-monogenic functions is s-monogenic.

Remark 2.3. As noted in [7] monomials $\vec{x}^n a_n$, $a_n \in \mathbb{R}_n$ are left s-monogenic, thus also polynomials $\sum_{n=0}^{N} \vec{x}^n a_n$ are s-monogenic. Moreover, any power series $\sum_{n=0}^{+\infty} \vec{x}^n a_n$ is left s-monogenic in its domain of convergence.

Definition 2.4. *Let U be a domain in $\mathbb{R}^{n+1}$ and let $f : U \rightarrow \mathbb{R}_n$ be an s-monogenic function. Its s-derivative ∂_s is defined as*

$$\partial_s(f) = \begin{cases} \partial_I(f)(\vec{x}) & \vec{x} = x + Iy, \ y \neq 0 \\ \partial_x f(x) & x \in \mathbb{R}. \end{cases} \tag{1}$$

Note that the definition of s-derivative is well posed because it is applied only to s-monogenic functions. In fact, if a function f is s-monogenic, its s-derivative in the point $\vec{x}$ equals $\frac{\partial}{\partial x} f(\vec{x})$ while there can be problems if f is not s-monogenic. As pointed out in [7] this phenomenon is peculiar of the hypercomplex case, as the unit sphere of imaginary numbers has positive dimension, and does not appear in the complex case since the unit sphere is only made of two points. The following results can be proved as in [7].

Proposition 2.5. *1. ([7][Proposition 2.6]) The s-derivative $\partial_s(f)$ of an s-monogenic function f is an s-monogenic function. Moreover $\partial_s^n f(x + Iy) = \dfrac{\partial^n}{\partial x^n} f(x + Iy)$.*
2. The s-derivative of a power series $\sum_{n=0}^{+\infty} \vec{x}^n a_n$ equals $\sum_{n=0}^{+\infty} n \vec{x}^{n-1} a_n$ and has the same radius of convergence of the original series.

Now we are in need to select $n - 1$ unit vectors in $\mathbb{S}$ in order to have a basis of $\mathbb{R}_n$ containing the chosen 1-vector I. To this purpose we recall that if $\underline{a}$, $\underline{b}$ are two 1-vectors, then

$$\underline{a}\underline{b} = \langle \underline{a}, \underline{b} \rangle + \underline{a} \wedge \underline{b}, \tag{2}$$

where $\langle \underline{a}, \underline{b} \rangle$ denotes the scalar product of $\underline{a}$ and $\underline{b}$ and the wedge product is defined by $\underline{a} \wedge \underline{b} = \frac{1}{2}(\underline{a}\underline{b} - \underline{b}\underline{a})$.

Proposition 2.6. *Let $I = I_1 \in \mathbb{S}$. It is possible to choose $I_2, \ldots, I_n \in \mathbb{S}$ such that $I_1, \ldots, I_n$ form a basis for the Clifford algebra $\mathbb{R}_n$ satisfying the defining relations $I_r I_s + I_s I_r = -2\delta_{rs}$.*

Proof. First of all, note that since $\underline{x} \wedge \underline{y} = -\underline{y} \wedge \underline{x}$, formula (2) gives $\underline{x}\underline{y} + \underline{y}\underline{x} = 2\langle \underline{x}, \underline{y} \rangle$. Then it suffi-cient to select the vectors I_r such that $\langle I_s, I_r \rangle = 0$ and $\langle I_r, I_r \rangle = -1$, for $s = 1, \ldots, n, r = 2, \ldots, n$, $s \neq r$. Since $I_r = \sum_{\ell=1}^{n} x_{r\ell} e_\ell$ we have $\langle I_r, I_r \rangle = -(\sum_{\ell=1}^{n} x_{r\ell}^2)$ and $\langle I_s, I_r \rangle = -\sum_{\ell=1}^{n} x_{r\ell} x_{s\ell}$. By identifying each 1-vector I_r with its components $(x_1, \ldots, x_n) \in \mathbb{R}^n$ we conclude by the Gram Schmidt algorithm. $\qquad\square$

Lemma 2.7. *(Splitting Lemma) Let $U \subseteq \mathbb{R}^{n+1}$ be a domain and let $f : U \to \mathbb{R}_n$ be an s-monogenic function. For every choice of $I = I_1 \in \mathbb{S}$ let $I_2, \ldots, I_n$ be a completion to an orthonormal basis of $\mathbb{R}_n$. Then there exists 2^{n-1} holomorphic functions $F_A : U \cap L_I \to L_I$ such that for every $z = x + Iy$*

$$f_I(z) = \sum_{|A|=0}^{n-1} F_A(z) I_A, \quad I_A = I_{i_1} \ldots I_{i_s},$$

where $A = i_1 \ldots i_s$ is a subset of $\{2, \ldots, n\}$, with $i_1 < \ldots < i_s$, or, when $|A| = 0$, $I_\emptyset = 1$.

Example 2.8. To make clear the notation of the Lemma, we show an example before to prove the statement. Let us consider the case of $\mathbb{R}_4$-valued functions. A function f can be written as

$$f = f_0 + f_1 I_1 + f_2 I_2 + f_3 I_3 + f_4 I_4 + f_{12} I_{12} + f_{13} I_{13} + f_{14} I_{14} + f_{23} I_{23} +$$

$$f_{24} I_{24} + f_{34} I_{34} + f_{123} I_{123} + f_{124} I_{124} + f_{134} I_{134} + f_{234} I_{234} + f_{1234} I_{1234}$$

and grouping as prescribed in the statement of the Lemma, we obtain

$$f = (f_0 + f_1 I_1) + (f_2 + f_{12} I_1) I_2 + (f_3 + f_{13} I_1) I_3 + (f_4 + f_{14} I_1) I_4 +$$

$$+ (f_{23} + f_{123} I_1) I_{23} + (f_{24} + f_{124} I_1) I_{24} + (f_{34} + f_{134} I_1) I_{34} + (f_{234} + f_{1234} I_1) I_{234}.$$

Proof. The proof closely follows the proof of the analogue result in [7]. Given a function $f = \sum f_A I_A$ let us rewrite it by grouping its components as

$$\sum_{|A|=0}^{n-1} (f_A + f_{1A} I_1) I_A,$$

with obvious meaning of the subscript $1A$. Since f is s-monogenic we have $\left(\dfrac{\partial}{\partial x} + I_1 \dfrac{\partial}{\partial y}\right) f_{I_1}(x + I_1 y) = 0$ and so

$$\sum \left(\frac{\partial}{\partial x} + I_1 \frac{\partial}{\partial y}\right)(f_A + f_{1A} I_1) I_A = \left(\frac{\partial}{\partial x} f_A + I_1 \frac{\partial}{\partial y} f_A + \frac{\partial}{\partial x} f_{1A} I_1 - \frac{\partial}{\partial y} f_{1A}\right) I_A = 0.$$

Using the fact that the imaginary units commute with the real valued functions, we obtain:

$$\begin{cases} \dfrac{\partial}{\partial x} f_A - \dfrac{\partial}{\partial y} f_{1A} = 0 \\ \dfrac{\partial}{\partial y} f_A + \dfrac{\partial}{\partial x} f_{1A} = 0 \end{cases}$$

for all multi-indices A, thus all the functions $F_A = (f_A + f_{1A} I_1)$ satisfy the standard Cauchy-Riemann system and therefore they are holomorphic. $\qquad\square$

Proposition 2.9. If $B = B(0, R) \subseteq \mathbb{R}^{n+1}$ is a ball centered in 0 with radius $R > 0$, then $f : B \to \mathbb{R}_n$ is an s-monogenic function if and only if f has a series expansion of the form

$$f(\vec{x}) = \sum_{m \geq 0} \vec{x}^m \frac{1}{m!} \frac{\partial^m f}{\partial x^m}(0) \tag{3}$$

converging on B.

Proof. If a function admits a series expansion as in (3) it is obviously s-monogenic where the series converges. The converse needs the use of the Splitting lemma and mimics the proof of Theorem 2.7 in [7]. Let us consider an element $I = I_1 \in \mathbb{S}$ and the corresponding plane L_I. Let $\Delta \subset L_I$ be a disc with center in the origin and radius $r < R$. The function f_I restriction of f to the plane L_I can be split as $f_I(z) = \sum F_A(z) I_A$, $z = x + Iy$. Since every function F_A has values in L_I and is holomorphic, for any $z \in \Delta$ it admits an integral representation via the Cauchy formula, i.e.

$$F_A(z) = \frac{1}{2\pi I} \int_{\partial \Delta(0,r)} \frac{F_A(\zeta)}{\zeta - z} d\zeta,$$

thus

$$f_I(z) = \sum_{|A|=0}^{n-1} \left(\frac{1}{2\pi I} \int_{\partial \Delta(0,r)} \frac{F_A(\zeta)}{\zeta - z} d\zeta \right) I_A.$$

Now observe that as ζ, z commute being on the same plane L_I, we can expand the denominator in each integral in power series, as in the classical case:

$$F_A(z) = \frac{1}{2\pi I} \int_{\partial \Delta(0,r)} \sum_{m \geq 0} \left(\frac{z}{\zeta} \right)^m \frac{F_A(\zeta)}{\zeta} d\zeta$$

$$= \sum_{m \geq 0} z^m \int_{\partial \Delta(0,r)} \frac{F_A(\zeta)}{\zeta^{m+1}} d\zeta, = \sum_{m \geq 0} z^m \frac{1}{m!} \frac{\partial^m F_A}{\partial z^m}(0).$$

Plugging this expression into $f_I(z) = \sum F_A I_A$ we obtain:

$$f_I(z) = \sum_{|A|=0}^{n-1} \sum_{m \geq 0} z^m \frac{1}{m!} \frac{\partial^m F_A}{\partial z^m}(0) I_A = \sum_{|A|=0}^{n-1} \sum_{m \geq 0} z^m \frac{1}{m!} \frac{\partial^m f}{\partial z^m}(0)$$

and using the definition of s-derivative together with Proposition 2.5.1, we get

$$\sum_{|A|=0}^{n-1} \sum_{m \geq 0} z^m \frac{1}{m!} \frac{1}{2} \left(\frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right)^m f(0) = \sum_{|A|=0}^{n-1} \sum_{m \geq 0} z^m \frac{1}{m!} \frac{\partial^m}{\partial x^m} f(0).$$

Finally observe that the coefficients of the power series do not depend on the choice of the unit I, thus the statement holds for any $I \in \mathbb{S}$. $\square$

Remark 2.10. The interesting part of Proposition 2.9 is that, even though the definition of s-monogenic function depends on the direction of the unit vector I, the coefficients of the series expansion do not depend at all from a choice of I.

Remark 2.11. Note that the complex plane $\mathbb{C}$ can be seen both as $\mathbb{R}^2$ and as the Clifford algebra $\mathbb{R}_1$. It is immediate to note that the space of holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ coincides with the space of s-monogenic functions from $g : \mathbb{R}^2 \to \mathbb{R}_1$. For this reason in this paper we will consider the case of $n > 1$ (obviously, all the results we will prove are valid in the case $n = 1$, but for holomorphic functions in one complex variable usually stronger statements hold).

Note also that it is possible to interpret holomorphic functions of one complex variable as a (proper) subset of the space of s-monogenic functions as shown in the following proposition.

Proposition 2.12. *Let $I \in \mathbb{S}$ and let us identify L_I with $\mathbb{C}$. Any holomorphic function $f : \Delta(0, R) \subseteq \mathbb{C} \to \mathbb{C}$ can be extended (uniquely, up to a choice of an order for the elements in the basis of $\mathbb{R}_n$) to an s-monogenic function $\tilde{f} : B(0, R) \to \mathbb{R}_n$.*

Proof. The function $f(z)$ can be expanded in power series as $f(z) = f(x + Iy) = \sum_{n=0}^{+\infty}(x + Iy)^n a_n$, $a_n \in \mathbb{C} \cong L_I$. Suppose to embed $\mathbb{C}$ into the Clifford algebra $\mathbb{R}_n$ by identifying the imaginary unit $I \in \mathbb{C}$ with $I_1 \in \mathbb{R}_n$. Then we define $\tilde{f}(\vec{x}) = \sum_{n=0}^{+\infty} \vec{x}^n a_n$ which is, obviously, an s-monogenic function. $\square$

Proposition 2.13. *1. The product of two functions $f, g : B(0, R) \to \mathbb{R}_n$ whose series expansions have real coefficients is an s-monogenic function.*
2. The composition of an s-monogenic function $f : B(0, R) \to \mathbb{R}_n$ with an s-monogenic function $g : B(0, R') \to \mathbb{R}_n$ whose series expansion has real coefficients is an s-monogenic function where it is defined.

Proof. Suppose that $f(\vec{x}) = \sum_{m \geq 0} \vec{x}^m a_m$ and $g(\vec{x}) = \sum_{r \geq 0} \vec{x}^r b_r$, with $a_m, b_r \in \mathbb{R}$. Then $(fg)(\vec{x}) = \sum_{s \geq 0} \vec{x}^s (a_0 b_s + a_1 b_{s-1} + \ldots + a_s b_0)$ since the coefficients commute with the variable $\vec{x}$. Now consider $f(g(\vec{x}))$: we have $f(g(\vec{x})) = \sum_{m \geq 0} (\sum_{r \geq 0} \vec{x}^r b_r)^m a_m$. Since the coefficients b_r commute with the variables we can group them on the right and the statement follows. $\square$

Corollary 2.14. *Let $f : U \to \mathbb{R}_n$ be an s-monogenic function and $y_0 \in \mathbb{R}$. Then $f(\vec{x} - y_0)$ is an s-monogenic function in $U' = \{\vec{x}' = \vec{x} - y_0, \ \vec{x} \in U\}$.*

Proposition 2.15. *If $B = B(y_0, R) \subseteq \mathbb{R}^{n+1}$ is a ball centered in $y_0 \in \mathbb{R}$ with radius $R > 0$, then $f : B \to \mathbb{R}_n$ is an s-monogenic function if and only if f has a series expansion of the form*

$$f(\vec{x}) = \sum_{m \geq 0} (\vec{x} - y_0)^m \frac{1}{m!} \frac{\partial^m f}{\partial x^m}(y_0) \tag{4}$$

Proof. Consider the transformation of coordinates $\vec{z} = \vec{x} - y_0$. As the function $f(\vec{z})$ is s-monogenic in a ball centered in the origin with radius $R > 0$, we can apply Proposition 2.9. Using the inverse transformation $\vec{x} = \vec{z} + y_0$, we obtain the statement. $\square$

3 Cauchy integral formula and its consequences

A main result in the theory of monogenic functions is the analogue of the Cauchy integral formula. In order to state the result for s-monogenic functions we need some notation. Given an element $\vec{x} = x_0 + \underline{x} \in \mathbb{R}^{n+1}$ let us set

$$I_{\vec{x}} = \begin{cases} \dfrac{\underline{x}}{|\underline{x}|} & \text{if } \underline{x} \neq 0 \\ \text{any element of } \mathbb{S} & \text{otherwise.} \end{cases}$$

We have the following (compare with [7], Theorem 3.5):

Theorem 3.1. *Let $B = B(0, R)$ be a ball with center in 0 and radius $R > 0$ and let $f : B \to \mathbb{R}_n$ be an s-monogenic function. If $\vec{x} \in B$ then*

$$f(\vec{x}) = \frac{1}{2\pi} \int_{\partial \Delta_{\vec{x}}(0,r)} (\vec{\zeta} - \vec{x})^{-1} d\vec{\zeta}_{I_{\vec{x}}} f(\vec{\zeta})$$

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where $\vec{\zeta} \in L_{I_{\vec{x}}} \bigcap B$, $d\vec{\zeta}_{I_{\vec{x}}} := -d\vec{\zeta} I_{\vec{x}}$ and $r > 0$ is such that

$$\overline{\Delta_{\vec{x}}(0, r)} = \{x + I_{\vec{x}} y \mid x^2 + y^2 \leq r^2\}$$

contains $\vec{x}$ and is contained in B.

Proof. The proof is based on the Splitting Lemma. Consider the integral

$$\frac{1}{2\pi} \int_{\partial \Delta_{\vec{x}}(0,r)} (\vec{\zeta} - \vec{x})^{-1} d\vec{\zeta}_{I_{\vec{x}}} f(\vec{\zeta})$$

set $I_{\vec{x}} := I_1$, complete to a basis $I_1, \ldots, I_n$ of the Clifford algebra $\mathbb{R}_n$, and set $f_{I_{\vec{x}}} = \sum_A F_A I_A$. We have:

$$\frac{1}{2\pi} \int_{\partial \Delta_{\vec{x}}(0,r)} (\vec{\zeta} - \vec{x})^{-1} d\vec{\zeta}_{I_{\vec{x}}} f_{I_{\vec{x}}}(\vec{\zeta})$$

$$= \frac{1}{2\pi} \int_{\partial \Delta_{\vec{x}}(0,r)} (\vec{\zeta} - \vec{x})^{-1} d\vec{\zeta}_{I_{\vec{x}}} \sum_A F_A(\vec{\zeta}) I_A$$

$$= \sum_A \frac{1}{2\pi} \int_{\partial \Delta_{\vec{x}}(0,r)} (\vec{\zeta} - \vec{x})^{-1} d\vec{\zeta}_{I_{\vec{x}}} F_A(\vec{\zeta}) I_A = \sum_A F_A(\vec{x}) I_A = f(\vec{x}).$$

$\square$

Remark 3.2. Let $B_1 = B(0, R_1)$, $B_2 = B(0, R_2)$ be two balls centered in the origin and with radii $0 < R_1 < R_2$. The same argument used in the previous proof shows that if a function f is s-monogenic in a neighborhood of the annular domain $B_2 \setminus B_1$, then for any $\vec{x} \in B_2 \setminus B_1$, it is

$$f(\vec{x}) = \frac{1}{2\pi} \int_{\partial B_2 \cap L_{I_{\vec{x}}}} (\vec{\zeta} - \vec{x})^{-1} d\vec{\zeta}_{I_{\vec{x}}} f(\vec{\zeta}) - \frac{1}{2\pi} \int_{\partial B_1 \cap L_{I_{\vec{x}}}} (\vec{\zeta} - \vec{x})^{-1} d\vec{\zeta}_{I_{\vec{x}}} f(\vec{\zeta}).$$

Remark 3.3. The function $\mathscr{I}_{\vec{y}}(\vec{x}) = (\vec{x} - \vec{y})^{-1}$ corresponding to the Cauchy kernel is not s-monogenic on $\mathbb{R}^{n+1} \setminus \{\vec{y}\}$, unless $\vec{y} = y_0 \in \mathbb{R}$. In particular, the function $\mathscr{I}(\vec{x}) = \vec{x}^{-1} = \dfrac{\bar{\vec{x}}}{|\vec{x}|^2}$, where $\bar{\vec{x}} = x_0 - \underline{x}$, is s-monogenic in $\mathbb{R}^{n+1} \setminus \{0\}$. Note also that $\mathscr{I}_{y_0}(\vec{x})$ can be expanded in power series as $\mathscr{I}_{y_0}(\vec{x}) = \sum_{n \geq 0} y_0^n \vec{x}^{-n-1}$.

Theorem 3.4. *(Cauchy formula outside a ball) Let $B = B(0, R)$ and let $B^c = \mathbb{R}^{n+1} \setminus \overline{B}$. Let $f : B^c \to \mathbb{R}_n$ be an s-monogenic function with $\lim_{\vec{x} \to \infty} f(\vec{x}) = a$. If $\vec{x} \in B^c$ then*

$$f(\vec{x}) = a - \frac{1}{2\pi} \int_{\partial \Delta_{\vec{x}}(0,r)} (\vec{\zeta} - \vec{x})^{-1} d\vec{\zeta}_{I_{\vec{x}}} f(\vec{\zeta})$$

where $\vec{\zeta} \in L_{I_{\vec{x}}} \bigcap B^c$, $d\vec{\zeta}_{I_{\vec{x}}} = -d\vec{\zeta} I_{\vec{x}}$, $0 < R < r < |\vec{x}|$ and the complement of the set $\overline{\Delta_{\vec{x}}(0,r)}$ is contained in B^c and contains $\vec{x}$.

Proof. The proof is based on the Splitting Lemma. Let $\vec{x} \in \mathbb{R}^{n+1} \setminus \overline{B}$ and the corresponding imaginary unit $\vec{x}$. Consider $r' > r > R$, and the disks $\Delta = \Delta_{\vec{x}}(0, r)$, $\Delta' = \Delta_{\vec{x}}(0, r')$ on the plane $L_{I_{\vec{x}}}$ having radius r and r' respectively such that $\Delta' \ni \vec{x}$. Since f is s-monogenic on $\Delta' \setminus \Delta$ we can apply the Cauchy formula in $\overline{\Delta'} \setminus \Delta$ to compute $f(\vec{x})$. We obtain:

$$f(\vec{x}) = \frac{1}{2\pi} \int_{\partial \Delta' \setminus \partial \Delta} (\vec{\zeta} - \vec{x})^{-1} d\vec{\zeta}_{I_{\vec{x}}} f(\vec{\zeta}) = \frac{1}{2\pi} \int_{\partial \Delta'} (\vec{\zeta} - \vec{x})^{-1} d\vec{\zeta}_{I_{\vec{x}}} f(\vec{\zeta}) - \frac{1}{2\pi} \int_{\partial \Delta} (\vec{\zeta} - \vec{x})^{-1} d\vec{\zeta}_{I_{\vec{x}}} f(\vec{\zeta}).$$

Let us set $I_{\vec{x}} := I_1$ and complete to a basis $I_1, \ldots, I_n$ of the Clifford algebra $\mathbb{R}_n$. The Splitting Lemma gives $f_{I_{\vec{x}}} = \sum_A F_A I_A$ and we can write:

$$\frac{1}{2\pi} \int_{\partial \Delta'} (\vec{\zeta} - \vec{x})^{-1} d\vec{\zeta}_{I_{\vec{x}}} f(\vec{\zeta}) - \frac{1}{2\pi} \int_{\partial \Delta} (\vec{\zeta} - \vec{x})^{-1} d\vec{\zeta}_{I_{\vec{x}}} f(\vec{\zeta})$$

$$= \sum_A \frac{1}{2\pi} \int_{\partial \Delta'} (\vec{\zeta} - \vec{x})^{-1} d\vec{\zeta}_{I_{\vec{x}}} F_A(\vec{\zeta}) I_A - \sum_A \frac{1}{2\pi} \int_{\partial \Delta} (\vec{\zeta} - \vec{x})^{-1} d\vec{\zeta}_{I_{\vec{x}}} F_A(\vec{\zeta}) I_A.$$

Let us now consider a single component F_A at a time. By computing the integral on $\partial \Delta'$ in spherical coordinates, and letting $r' \to \infty$ we obtain that the integral equals $a_A = \lim_{r' \to \infty} F_A$, therefore:

$$F_A(\vec{x}) = a_A - \frac{1}{2\pi} \int_{\partial \Delta} (\vec{\zeta} - \vec{x})^{-1} d\vec{\zeta}_{I_{\vec{x}}} F_A(\vec{\zeta}) I_A = a_A - F_A(\vec{x}).$$

Taking the sum of the various components multiplied with the corresponding units I_A we get the statement with $a = \sum_A a_A I_A$. $\qquad \square$

Theorem 3.5. *(Cauchy estimates) Let $B = B(0, R)$ be the ball centered in 0 having radius $R > 0$. Let $f : B \to \mathbb{R}_n$ be an s-monogenic function, $I \in \mathbb{S}$ and $0 < r < R$. Set $\partial \Delta_I(0, r) = \{(x + Iy) \mid x^2 + y^2 = r^2\}$, $M_I = \max\{|f(\vec{x})| \mid \vec{x} \in \partial \Delta_I(0, r)\}$ and $M = \inf\{M_I \mid I \in \mathbb{S}\}$. Then*

$$\frac{1}{n!} \left| \frac{\partial^n f}{\partial x^n}(0) \right| \leq \frac{M}{r^n}, \quad n \geq 0.$$

Proof. The result easily follows from the Splitting Lemma and the corresponding proof in the complex case. $\qquad \square$

Using the previous result it is immediate to show the following

Theorem 3.6. *(Liouville) Let $f : \mathbb{R}^{n+1} \to \mathbb{R}_n$ be an entire s-monogenic function. If f is bounded then f is constant on $\mathbb{R}^{n+1}$.*

Proof. Suppose that $|f| \leq M$ on $\mathbb{R}^{n+1}$. By the previous theorem we have:

$$\frac{1}{n!} \left| \frac{\partial^n f}{\partial x^n}(0) \right| \leq \frac{M}{r^n}, \quad n \geq 0,$$

and letting $r \to +\infty$ we obtain $\frac{\partial^n f}{\partial x^n}(0) = 0$ for any $n > 0$ and this implies $f(\vec{x}) = c$, $c \in \mathbb{R}_n$. $\quad \square$

Corollary 3.7. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}_n$ be an entire s-monogenic function. If $\lim_{\vec{x} \to \infty} f$ exists then f is constant on $\mathbb{R}^{n+1}$.*

Theorem 3.8. *Let U be an open set in $\mathbb{R}^{n+1}$. If $f : U \to \mathbb{R}_n$ is s-monogenic then for every $I \in \mathbb{S}$ and for any closed, simple curve $\gamma_I \subset U \cap L_I$ we have*

$$\int_{\gamma_I} d\vec{x} \, f(\vec{x}) = 0.$$

Proof. It is an easy consequence of the Splitting Lemma and the analogue result for holomorphic functions. $\qquad \square$

In principle, the analogue of the main theorems which hold for holomorphic functions can be proved also in this setting, with minor changes in their proofs. We mention below some of them:

Theorem 3.9. *(Identity principle). Let U be an open set in $\mathbb{R}^{n+1}$ such that $U \cap \mathbb{R}$ has an accumulation point. Let $f : U \rightarrow \mathbb{R}_n$ be an s-monogenic function, and Z the set of its zeroes. If there is an imaginary unit I such that $L_I \cap Z$ has an accumulation point, then $f \equiv 0$ on U.*

Proof. Let us consider the restriction f_I of f to the line L_I. By the Splitting Lemma we have

$$f_I(z) = \sum_{|A|=0}^{n-1} F_A I_A$$

with F_A holomorphic for every multi-index A. Since $L_I \cap Z$ has an accumulation point, we deduce that all the functions F_A vanish, thus $f_I = 0$. In particular f_I vanishes in the points of U on the real axis. Any other plane $L_{I'}$ is such that $f_{I'}$ vanishes on $U \cap \mathbb{R}$ which has an accumulation point, thus its components $F_{A'}$ vanish on $U \cap \mathbb{R}$. This fact implies that also $f_{I'}$ vanish on $L_{I'}$, thus $f \equiv 0$ on U. $\qquad\square$

Corollary 3.10. *Let U be an open set in $\mathbb{R}^{n+1}$ such that $U \cap \mathbb{R}$ has an accumulation point. Let $f, g : U \rightarrow \mathbb{R}_n$ be s-monogenic functions. If there is an imaginary unit I such that $f = g$ on a subset of L_I having an accumulation point, then $f \equiv g$ on U.*

Corollary 3.11. *Let $B = B(x_0, R)$, $x_0 \in \mathbb{R}$, and $f, g : B \rightarrow \mathbb{R}_n$ be s-monogenic functions. If there exists $I \in \mathbb{S}$ such that $f = g$ on a subset of $L_I \cap B$ having an accumulation point, then $f \equiv g$ on B.*

Corollary 3.12. *Let $B = B(x_0, R)$, $x_0 \in \mathbb{R}$, and let $f : B \rightarrow \mathbb{R}_n$ be an s-monogenic function. Then $f \equiv 0$ on B if and only if $\partial_s^n f(0) = 0$ for all $n \in \mathbb{N}$.*

4   Laurent series

Proposition 4.1. *Let $f : B(0, R) \rightarrow \mathbb{R}_n$ be the s-monogenic function expressed by the series $\sum \vec{x}^m a_m$ converging on B. Then the function $f \cdot \mathscr{I}$ is s-monogenic on $\mathbb{R}^{n+1} \setminus B(0, 1/R)$ and it can be expressed by the series $\sum \vec{x}^{-m} a_m$ converging on $\mathbb{R}^{n+1} \setminus B(0, 1/R)$.*

Proof. Proposition 2.13 implies that $f \cdot \mathscr{I}$ is an s-monogenic function on $\mathbb{R}^{n+1} \setminus B(0, 1/R)$. The statement follows from the analogue result for holomorphic functions in one complex variable. $\qquad\square$

Theorem 4.2. *Let f be an s-monogenic function in an annular domain $A = \{\vec{x} \in \mathbb{R}^{n+1} \mid R_1 < |\vec{x}| < R_2\}$, $0 < R_1 < R_2$. Then f admits the following unique Laurent expansion*

$$f(\vec{x}) = \sum_{m=0}^{+\infty} \vec{x}^m a_m + \sum_{m=1}^{+\infty} \vec{x}^{-m} b_m \tag{5}$$

where $a_m = \frac{1}{m!} \partial_s^m f(0)$ and $b_m = \frac{1}{2\pi} \int_{\partial B(0, R_1') \cap L_{I_{\vec{x}}}} \vec{\zeta}^{m-1} d\vec{\zeta}_{I_{\vec{x}}} f(\vec{\zeta})$. The two series in (5) converge in the open ball $B(0, R_2)$ and $\mathbb{R}^{n+1} \setminus \overline{B(0, R_1)}$, respectively.

Proof. Let $\vec{x} \in A$, then there exist two positive real numbers R_1', R_2' such that $A' = \{\vec{x} \in \mathbb{R}^{n+1} \mid R_1' < |\vec{x}| < R_2'\} \subset A$, and $\vec{x} \in A'$. Using the Cauchy integral formula, we can write

$$f(\vec{x}) = \frac{1}{2\pi} \int_{\partial A' \cap L_{I_{\vec{x}}}} (\vec{\zeta} - \vec{x})^{-1} d\vec{\zeta}_{I_{\vec{x}}} f(\vec{\zeta}) = f_1(\vec{x}) + f_2(\vec{x})$$

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where

$$f_1(\vec{x}) = \frac{1}{2\pi} \int_{\partial B(0,R_2')\cap L_{I_{\vec{x}}}} (\vec{\zeta} - \vec{x})^{-1} d\vec{\zeta}_{I_{\vec{x}}} f(\vec{\zeta})$$

and

$$f_2(\vec{x}) = -\frac{1}{2\pi} \int_{\partial B(0,R_1')\cap L_{I_{\vec{x}}}} (\vec{\zeta} - \vec{x})^{-1} d\vec{\zeta}_{I_{\vec{x}}} f(\vec{\zeta}).$$

The first integral is associated to the first series in the Laurent expansion, thanks to Proposition 2.9. Let us consider the second integral. Using the Splitting Lemma, we can reason as in the case of functions in one complex variable, and consider the single components of $f_2(\vec{x})$. In $\mathbb{R}^{n+1} \setminus \overline{B(0,R_1')}$, we have

$$F_A(\vec{x}) = -\frac{1}{2\pi} \int_{\partial B(0,R_1')\cap L_{I_{\vec{x}}}} (\vec{\zeta} - \vec{x})^{-1} d\vec{\zeta}_{I_{\vec{x}}} F_A(\vec{\zeta}) = \frac{1}{2\pi} \int_{\partial B(0,R_1')\cap L_{I_{\vec{x}}}} \sum_{m\geq 0} \vec{x}^{-m-1} \vec{\zeta}^m d\vec{\zeta}_{I_{\vec{x}}} F_A(\vec{\zeta})$$

where we have used the fact that on the plane $L_{I_{\vec{x}}}$ the variables $\vec{\zeta}$ and $\vec{x}$ commute. Now, using the uniform convergence of the series we can write

$$F_A(\vec{x}) = \sum_{m\geq 0} \vec{x}^{-m-1} \frac{1}{2\pi} \int_{\partial B(0,R_1')\cap L_{I_{\vec{x}}}} \vec{\zeta}^m d\vec{\zeta}_{I_{\vec{x}}} F_A(\vec{\zeta}) = \sum_{m\geq 0} \vec{x}^{-m-1} b_{m+1,I_{\vec{x}};A}$$

where

$$b_{m+1,I_{\vec{x}};A} := \frac{1}{2\pi} \int_{\partial B(0,R_1')\cap L_{I_{\vec{x}}}} \vec{\zeta}^m d\vec{\zeta}_{I_{\vec{x}}} F_A(\vec{\zeta}).$$

We can write:

$$\tilde{f}_2(\vec{x}) = \sum_A F_A(\vec{x}) I_A = \sum_{m\geq 0} \sum_A \vec{x}^{-m-1} b_{m+1,I_{\vec{x}};A} I_A$$

however, $\tilde{f}_2(\vec{x})$ coincides with $f_2(\vec{x})$ on the plane $L_{I_{\vec{x}}}$, thus they coincide everywhere and the coefficients $b_{m+1,I_{\vec{x}};A}$ do not depend on the choice of the imaginary unit $I_{\vec{x}}$. The statement follows. $\square$

Remark 4.3. An analogue result holds for functions s-monogenic in an annular domain of the type $A = \{\vec{x} \in \mathbb{R}^{n+1} \mid R_1 < |\vec{x} - y_0| < R_2, \ y_0 \in \mathbb{R}\}$. In fact, it is sufficient to reason as in the proof of Proposition 2.15.

5 Some polynomial equations and series

As far as we know, in the literature there are no results about the zeroes of monogenic functions (here we mean in the sense of [1]) or about the zeroes of polynomials in the variables $\vec{x}$ or $\underline{x}$ (which, of course, are not interesting from the Clifford analysis point of view since they do not correspond to monogenic functions). As Clifford algebras are not division algebras, the Fundamental Theorem of Algebra does not hold, thus we cannot guarantee that a given polynomial in the variable $\vec{x}$ has a zero, not even if it is a degree one polynomial. However, if a given polynomial has a zero, it is interesting to characterize its vanishing set. As we will see in the sequel, the set of zeroes can contain $(n-1)$–spheres in $\mathbb{R}^{n+1}$. To describe those spheres it is useful to introduce the following notation: let $\vec{s} = s_0 + \sum_{i=1}^n s_i e_i$ and let

$$[\vec{s}] = \{\vec{x} \in \mathbb{R}^{n+1} \mid x_0 = s_0, \ |\vec{x}| = |\vec{s}|\}.$$

It is immediate to note that the relation $\vec{x} \sim \vec{s}$ if and only if $x_0 = s_0$, $|\vec{x}| = |\vec{s}|$ is an equivalence relation. An equivalence class contains only the element $\vec{s}$ when it is a real number, while it contains infinitely many elements when $\vec{s}$ is not real and corresponds to an $(n-1)$–dimensional sphere in $\mathbb{R}^{n+1}$. It is well known (see e.g. [9]) that in the skew field of quaternions, the equivalence class of a quaternion s is characterized by a quadratic equation. A first, yet interesting, result is that the same fact holds also in a (non division) Clifford algebra $\mathbb{R}_n$ if $\vec{s} \in \mathbb{R}^{n+1}$ and one looks for solutions in $\mathbb{R}_n^0 \oplus \mathbb{R}_n^1$:

Proposition 5.1. *Let $\vec{s} = s_0 + \sum_{i=1}^n s_i e_i \in \mathbb{R}^{n+1}$. Consider the equation*

$$\vec{x}^2 - 2Re[\vec{s}]\vec{x} + |\vec{s}|^2 = 0. \tag{6}$$

Then, $\vec{x} = x_0 + \underline{x}$, $x_0 \in \mathbb{R}_n^0 = \mathbb{R}$, $\underline{x} \in \mathbb{R}_n^1$ is a solution if and only if $\vec{x} \in [\vec{s}]$.

Proof. The result is immediate when $\vec{s} = s_0 \in \mathbb{R}$. Let us suppose that $\vec{s} \notin \mathbb{R}$. It is immediate that $\vec{x} \in [\vec{s}]$ is a solution. Conversely, let $\vec{x}$ be a solution, i.e. $(x_0 + \underline{x})^2 - 2Re[s](x_0 + \underline{x}) + |\vec{s}|^2 = 0$. A direct computation shows that this is possible if and only if $\underline{x} = 0$ or $x_0 = s_0$. The first possibility does not give any solution, while the second gives $|\vec{x}| = |\vec{s}|$, i.e. the equivalence class of $\vec{s}$. $\qquad\square$

Remark 5.2. The $(n-1)$-sphere corresponding to $[\vec{s}]$ contains elements of the type $s_0 + I|\vec{s}|$, with I varying in all the possible ways in $\mathbb{S}$.

Remark 5.3. It is not true, in general, that equation (6) characterizes an equivalence class if we consider generic vectors in $\mathbb{R}_n$. Consider for example $s = (1 - e_{123})$. Then s itself is not a solution of the equation.

Following [7], Theorem 5.1, we can prove the following theorem which establish that if the vanishing set of an s-monogenic function contains two different points on the same $(n-1)$–sphere, then it contains the whole sphere:

Theorem 5.4. *Let $\sum_{m\geq 0} \vec{x}^m a_m$ be the power series with radius of convergence R associated to a given s-monogenic function. If there are two different elements in a given equivalence class $[\vec{s}]$, both solutions to the equation*

$$\sum_{m\geq 0}^{+\infty} \vec{x}^m a_m = 0,$$

then all the elements in the equivalence class are solutions.

Proof. Suppose that there exist $x_0 + I_1 y_0$, $x_0 + I_2 y_0 \in [\vec{s}]$ and $I_2 \neq I_1 \in \mathbb{S}$, such that

$$\sum_{m\geq 0}^{+\infty} (x_0 + y_0 I_1)^m a_m = 0 \tag{7}$$

and

$$\sum_{m\geq 0}^{+\infty} (x_0 + y_0 I_2)^m a_m = 0. \tag{8}$$

For any fixed $m \in \mathbb{N}$ and any $I \in \mathbb{S}$ we have that

$$(x_0 + y_0 I)^m = \sum_{i=0}^m \binom{m}{i} x_0^{m-i} y_0^i I^i = \alpha_m + I\beta_m. \tag{9}$$

Equations (7), (8) and equality (9) give, by absolute convergence,

$$0 = \sum_{m\geq 0}(\alpha_m + I_1\beta_m)a_m - \sum_{m\geq 0}(\alpha_m + I_2\beta_m)a_m =$$

$$\sum_{m\geq 0}\left((\alpha_m + I_1\beta_m) - (\alpha_m + I_2\beta_m)\right)a_m =$$

$$\sum_{m\geq 0}\left(I_1\beta_m - I_2\beta_m\right)a_m = (I_1 - I_2)\left(\sum_{m\geq 0}\beta_m a_m\right).$$

Note that $I_1 - I_2$ is an invertible element, therefore $\sum_{m\geq 0}\beta_m a_m = 0$ and similarly, by (7) also $\sum_{m\geq 0}\alpha_m a_m = 0$. Using (9) we get that for any $I \in \mathbb{S}$

$$\sum_{m\geq 0}(x_0 + y_0 I)^m a_m = \sum_{m\geq 0}\alpha_m a_m + I\left(\sum_{m\geq 0}^{+\infty}\beta_m a_m\right) = 0.$$

$\square$

Remark 5.5. Whenever a polynomial $p(\vec{x})$ with complex coefficients (i.e. coefficients on a plane L_I) has two conjugate complex roots, all the points on the sphere defined by any of two points is a root of the polynomial.

Remark 5.6. In a Clifford algebra, the Euclidean algorithm is not allowed, in general. Thus, if a polynomial in the variable $\vec{x}$ factors as $p(\vec{x}) = (\vec{x} - \vec{a})q(\vec{x})$ then $\vec{x} = \vec{a}$ is a solution of $p(\vec{x}) = 0$ but knowing a solution $\vec{x} = \vec{a}$ is of the equation $p(\vec{x}) = 0$ does not allow to divide the polynomial by $(x - \vec{a})$.

5.1 The non commutative Cauchy kernel series

We conclude this section with the study of the noncommutative Cauchy kernel series

$$\sum_{n\geq 0}\vec{p}^n\vec{s}^{-1-n}, \tag{10}$$

for $\vec{p}\vec{s} \neq \vec{s}\vec{p}$, which is a fundamental tool to define a functional calculus for several noncommuting operators, see [3]. To motivate the results of this section we make the following observations. We consider the Cauchy integral formula given by Theorem 3.1 and we expand in power series the Cauchy kernel $(\vec{s} - \vec{p})^{-1}$. Now, let $T = T_0 + T_1 e_1 + ... + T_n e_n$ be a bounded linear operator where T_j, $j = 0, ..., n$ are linear bounded operators in the usual sense acting on a Banach space. To obtain a functional calculus we have to formally replace $\vec{p}$ by T in the non commutative Cauchy kernel series (10) and we have to get the correct resolvent operator for T. Thus it is necessary to sum the series $\sum_{n\geq 0}\vec{p}^n\vec{s}^{-1-n}$ in the case $\vec{p}\vec{s} \neq \vec{s}\vec{p}$ and, moreover, we have to observe that this sum does not depend on the commutativity of the real components of $\vec{p}$, see Remark 5.8. Indeed, the components of T in general do not commute.
The sum of $\sum_{n\geq 0}\vec{p}^n\vec{s}^{-1-n}$ allows us to define a new notion of resolvent operator for the $(n+1)$-tuple T of non commuting operators T_j and, as a consequence, a new eigenvalue equation.
The most important results of this section is Corollary 5.9, to prove it we begin with the following:

Theorem 5.7. *Let $\vec{p}, \vec{s} \in \mathbb{R}_n^0 \oplus \mathbb{R}_n^1$ be such that $\vec{p}\vec{s} \neq \vec{s}\vec{p}$. Then $-(\vec{p}-\overline{\vec{s}})^{-1}(\vec{p}^2 - 2\vec{p}\,Re[\vec{s}] + |\vec{s}|^2)$, is the inverse of the noncommutative Cauchy kernel series (10) which is convergent for $|\vec{p}| < |\vec{s}|$.*

Proof. Let us verify that

$$-(\vec{p} - \overline{\vec{s}})^{-1}(\vec{p}^2 - 2\vec{p}\,Re[\vec{s}] + |\vec{s}|^2) \sum_{n \geq 0} \vec{p}^n \vec{s}^{-1-n} = 1.$$

We therefore obtain

$$(-|\vec{s}|^2 - \vec{p}^2 + 2\vec{p}\,Re[\vec{s}]) \sum_{n \geq 0} \vec{p}^n \vec{s}^{-1-n} = \vec{s} + \vec{p} - 2\,Re[\vec{s}]. \tag{11}$$

Observing that $-|\vec{s}|^2 - \vec{p}^2 + 2\vec{p}\,Re[\vec{s}]$ commutes with $\vec{p}^n$ we can rewrite this last equation as

$$\sum_{n \geq 0} \vec{p}^n(-|\vec{s}|^2 - \vec{p}^2 + 2\vec{p}\,Re[\vec{s}])\vec{s}^{-1-n} = \vec{s} + \vec{p} - 2\,Re[\vec{s}].$$

Now the left hand side can be written as

$$\sum_{n \geq 0} \vec{p}^n(-|\vec{s}|^2 - \vec{p}^2 + 2\vec{p}\,Re[\vec{s}])s^{-1-n} =$$

$$= -\left(|\vec{s}|^2\vec{s}^{-1} + \vec{p}(-2\vec{s}\,Re[\vec{s}] + |\vec{s}|^2)\vec{s}^{-2} + \vec{p}^2(\vec{s}^2 - 2\vec{s}\,Re[\vec{s}] + |\vec{s}|^2)\vec{s}^{-3} + \vec{p}^3(\vec{s}^2 - 2\vec{s}\,Re[\vec{s}] + |\vec{s}|^2)\vec{s}^{-4} + ...\right).$$

Using the identity (6): $\vec{s}^2 - 2\vec{s}\,Re[\vec{s}] + |\vec{s}|^2 = 0$ we get

$$\sum_{n \geq 0} \vec{p}^n(-|\vec{s}|^2 - \vec{p}^2 + 2\vec{p}\,Re[\vec{s}])\vec{s}^{-1-n} = \vec{s} - 2\,Re[\vec{s}] + \vec{p}$$

which equals the right hand side of (11). $\qquad\square$

Remark 5.8. Observe that in the proof of Theorem 5.7 we have not used the the fact that the components of $\vec{p}$ commute.

A direct consequence of Theorem 5.7 is that we can explicitly write the sum of the noncommutative Cauchy kernel series:

Corollary 5.9. *Let $\vec{p}, \vec{s} \in \mathbb{R}_n^0 \oplus \mathbb{R}_n^1$ be such that $\vec{p}\vec{s} \neq \vec{s}\vec{p}$. Then*

$$\sum_{n \geq 0} \vec{p}^n \vec{s}^{-1-n} = -(\vec{p}^2 - 2\vec{p}\,Re[\vec{s}] + |\vec{s}|^2)^{-1}(\vec{p} - \overline{\vec{s}}),$$

for $|\vec{p}| < |\vec{s}|$.

Remark 5.10. We now observe that the expression $(\vec{p}^2 - 2\vec{p}\,Re[\vec{s}] + |\vec{s}|^2)^{-1}(\vec{p} - \overline{\vec{s}})$ involves an inverse which does not exist if we set $\vec{p} = \overline{\vec{s}}$; indeed, in this case we have $\overline{\vec{s}}^2 - 2\overline{\vec{s}}\,Re[\vec{s}] + |\vec{s}|^2 = 0$. On may wonder if the factor $(\vec{p} - \overline{\vec{s}})$ can be simplified. However, it can be shown that this is not possible and the function $(\vec{p}^2 - 2\vec{p}\,Re[\vec{s}] + |\vec{s}|^2)^{-1}(\vec{p} - \overline{\vec{s}})$ cannot be extended to a continuous function in $\vec{p} = \overline{\vec{s}}$.

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