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Slicing Convex Sets and Measures by a Hyperplane

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Abstract Given convex bodies $K_1, \dots, K_d$ in $\mathbb{R}^d$ and numbers $\alpha_1, \dots, \alpha_d \in [0, 1]$, we give a sufficient condition for existence and uniqueness of an (oriented) halfspace H with $\mathrm{Vol}(H \cap K_i) = \alpha_i \cdot \mathrm{Vol}\, K_i$ for every i. The result is extended from convex bodies to measures.

Keywords Convex bodies · Well separated families · Sections of convex sets and measures

1 Transversal Spheres

A well known result in elementary geometry states that there is a unique sphere which contains a given set of $d+1$ points in general position in $\mathbb{R}^d$. A similar thing happens with d-pointed sets and hyperplanes. What happens if we consider convex bodies instead of points?

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These questions are the main motivation for the present paper. The first result in this direction is due to Kramer and Németh [7]. They used the following, very natural definition.

A family $\mathcal{F}$ of connected sets in $\mathbb{R}^d$ is said to be *well separated*, if for any $k \le d+1$ distinct elements, $K_1, \ldots, K_k$, of $\mathcal{F}$ and for any choice of points $x_i \in K_i$, the set $\text{aff}\{x_1, \ldots, x_k\}$ is a $(k-1)$-dimensional flat. Here $[k]$ stands for the set $\{1, 2, \ldots, k\}$. It is well known (cf. [1, 4]), and also easy to check the following.

Proposition 1 *Assume $\mathcal{F} = \{K_1, \ldots, K_n\}$ is a family of connected sets in $\mathbb{R}^d$. The following conditions are equivalent*:

1. *The family $\mathcal{F}$ is well separated.*
2. *The family $\mathcal{F}' = \{\text{conv} K_1, \ldots, \text{conv} K_n\}$ is well separated.*
3. *For every pair of disjoint sets $I, J \subset [n]$ with $|I| + |J| \le d+1$, there is a hyperplane separating the sets K_i, $i \in I$ from the sets K_j, $j \in J$.*

By an elegant application of Brouwer's fixed point theorem, Kramer and Németh proved the following:

Theorem KN *Let $\mathcal{F}$ be a well separated family of $d+1$ compact convex sets in $\mathbb{R}^d$. Then there exists a unique Euclidean ball which touches each set and whose interior is disjoint from each member of $\mathcal{F}$.*

Denote by $B(x,r)$, resp. $S(x,r)$, the Euclidean ball and sphere of radius r and center x. We say that the sphere $S(x,r)$ *supports* a compact set K if $S(x,r) \cap K \ne \emptyset$ and either $K \subset B(x,r)$ or $K \cap \text{int}\, B(x,r) = \emptyset$. This definition is due to Klee et al. [6]. They proved the following:

Theorem KLH *Let $\mathcal{F} = \{K_1, K_2, \ldots, K_{d+1}\}$ be a well separated family of compact convex sets in $\mathbb{R}^d$, and let I, J be a partition of $[d+1]$. Then there is a unique Euclidean sphere $S(x,r)$ that supports each element of $\mathcal{F}$ in such a way that $K_i \subset B(x,r)$ for each $i \in I$ and $K_j \cap \text{int}\, B(x,r) = \emptyset$ for each $j \in J$.*

The case $I = \emptyset$ corresponds to Theorem KN. We are going to generalize these results. Let $Q^d = [0,1]^d$ denote the unit cube of $\mathbb{R}^d$. Given a well separated family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$, a sphere $S(x,r)$ is said to be *transversal* to $\mathcal{F}$ if it intersects every element of $\mathcal{F}$. Finally, a *convex body* in $\mathbb{R}^d$ is a convex compact set with nonempty interior.

Theorem 1 *Let $\mathcal{F} = \{K_1, \ldots, K_{d+1}\}$ be a well separated family of convex bodies in $\mathbb{R}^d$, and let $\alpha = (\alpha_1, \ldots, \alpha_{d+1}) \in Q^{d+1}$. Then there exists a unique transversal Euclidean sphere $S(x,r)$ such that $\text{Vol}(B(x,r) \cap K_i) = \alpha_i \cdot \text{Vol}(K_i)$ for every $i \in [d+1]$.*

Remark 1 The transversality of $S(x,r)$ only matters when α_i is equal to 0 or 1; otherwise the condition $\text{Vol}(B(x,r) \cap K_i) = \alpha_i \cdot \text{Vol}(K_i)$ plus convexity guarantees that $S(x,r)$ intersects K_i.

2 Transversal Hyperplanes and Halfspaces

In a similar direction, Cappell et al. [3] proved an analogous theorem for the case of supporting hyperplanes, which can be seen as spheres of infinite radius. Given a family $\mathcal{F}$ of sets in $\mathbb{R}^d$, a hyperplane will be called *transversal* to $\mathcal{F}$ if it intersects each member of $\mathcal{F}$. The following result is a special case of Theorem 3 of Cappell et al. [3] (cf. [2] as well):

Theorem C *Let $\mathcal{F} = \{K_1, \dots, K_d\}$ be a well separated family of compact convex sets in $\mathbb{R}^d$ with a partition I, J of the index set $[d]$. Then there are exactly two hyperplanes, H_1 and H_2, transversal to $\mathcal{F}$ such that both H_1 and H_2 have all K_i $(i \in I)$ on one side and all K_j $(j \in J)$ on the other side.*

Theorem C was also proved by Klee et al. [5] using Kakutani's extension of Brouwer's fixed point theorem. We are going to formulate this theorem in a slightly different way, more suitable for our purposes. So, we need to introduce new notation and terminology.

A halfspace H in $\mathbb{R}^d$ can be specified by its outer unit normal vector, v, and by the signed distance, $t \in \mathbb{R}$, of its bounding hyperplane from the origin. Thus, there is a one-to-one correspondence between halfspaces of $\mathbb{R}^d$ and pairs $(v, t) \in S^{d-1} \times \mathbb{R}$. We denote the halfspace $\{x \in \mathbb{R}^d : \langle x, v\rangle \le t\}$ by $H(v \le t)$. Analogously we write $H(v = t) = \{x \in \mathbb{R}^d : \langle x, v\rangle = t\}$, which is the bounding hyperplane of $H(v \le t)$. Furthermore, given a set $K \subset \mathbb{R}^d$, a unit vector v and a scalar t, we denote the set $H(v = t) \cap K$ by $K(v = t)$, analogously $K(v \le t) = H(v \le t) \cap K$.

Suppose next that $\mathcal{F} = \{K_1, \dots, K_d\}$ is a well separated family of convex sets in $\mathbb{R}^d$. Assume $a_1 \in K_1, \dots, a_d \in K_d$. The unit normal vectors to the unique transversal hyperplane containing these points are v and $-v$. We want to make the choice between v and $-v$ unique and depend only on $\mathcal{F}$. We first make it depend on $a_1, \dots, a_d$. Define $v = v(a_1, \dots, a_d)$ as the (unique) unit normal vector to $\operatorname{aff}\{a_1, \dots, a_d\}$ satisfying

$$\det \begin{vmatrix} v & a_1 & a_2 & \cdots & a_d \\ 0 & 1 & 1 & \cdots & 1 \end{vmatrix} > 0,$$

in other words, the points $v + a_1, a_1, a_2, \dots, a_d$, in this order, are the vertices of a positively oriented d-dimensional simplex. Clearly, with $-v$ in place of v the determinant would be negative. This gives rise to the map $v : K \longrightarrow S^{d-1}$ where $K = K_1 \times \cdots \times K_d$. This definition seems to depend on the choice of the a_i, but in fact, it does not. Write $H(v = t) = \operatorname{aff}\{a_1, \dots, a_d\}$.

Proposition 2 *Under the previous assumption, let $b_i \in K_i(v = t)$ for each i. Then $v(a_1, \dots, a_d) = v(b_1, \dots, b_d)$.*

Proof This is simple. The homotopy $(1 - \lambda)a_i + \lambda b_i$ $(\lambda \in [0, 1])$ moves the a_i to the b_i continuously, and keeps $(1-\lambda)a_i + \lambda b_i$ in $K_i(v = t)$. The affine hull of the moving points remains unchanged, and does not degenerate because $\mathcal{F}$ is well separated. So their outer unit normal remains v throughout the homotopy. □

The previous proposition is also mentioned by Klee et al. [5]. With this definition, a transversal hyperplane to $\mathcal{F}$ determines v and t uniquely. We call $H(v=t)$ *a positive transversal hyperplane to* $\mathcal{F}$, and similarly, $H(v \le t)$ is *a positive transversal halfspace to* $\mathcal{F}$.

Theorem 2 *Let $\mathcal{F} = \{K_1, \dots, K_d\}$ be a family of well separated convex bodies in $\mathbb{R}^d$, and let $\alpha = (\alpha_1, \dots, \alpha_d) \in Q^d$. Then there is a unique positive transversal halfspace, H, such that* $\mathrm{Vol}(K_i \cap H) = \alpha_i \cdot \mathrm{Vol}(K_i)$ *for every $i \in [d]$.*

Theorem C follows since the partition I, J gives rise to $\alpha, \beta \in Q^d$ via $\alpha_k = 1$ if $k \in I$, otherwise $\alpha_k = 0$, and $\beta_k = 1$ if $k \in J$, otherwise $\beta_k = 0$. By Theorem 2, there are unique positive transversal halfspaces $H(\alpha)$ and $H(\beta)$ with the stated properties. Their bounding hyperplanes satisfy the statement of Theorem C and they are obviously distinct. We mention, however, that Theorem C will be used in the proof of the unicity part of Theorem 2.

Remark 2 When all $\alpha_i = 1/2$, the existence of such a halfspace is guaranteed by Borsuk's theorem, even without the condition of convexity or $\mathcal{F}$ being well separated. (Connectivity of the sets implies that the halving hyperplane is a transversal to $\mathcal{F}$.) The case of general α_i, however, needs some extra condition as the following two examples show. If all K_i are equal, then each oriented hyperplane section cuts off the same amount from each K_i, so $\alpha_1 = \cdots = \alpha_d$ must hold. The second example consists of d concentric balls with different radii, and if the radius of the first ball is very large compared to those of the others and α_1 is too small, then a hyperplane cutting off α_1 fraction of the first ball is disjoint from all other balls. Thus no hyperplane transversal exists that cuts off an α_1 fraction of the first set.

Remark 3 Cappell et al. prove, in fact, a much more general theorem [3]. Namely, assume that $\mathcal{F}$ is well separated and consists of k strictly convex sets, $k \in \{2, \dots, d\}$ and let I, J be a partition of $[k]$. Then the set of all supporting hyperplanes separating the K_i $(i \in I)$ from the K_j $(j \in J)$ is homeomorphic to the $(d-k)$-dimensional sphere.

3 Extension to Measures

Borsuk's theorem holds not only for volumes but more generally for measures. Similarly, our Theorem 2 can and will be extended to *nice measures* that we are to define soon. We need a small piece of notation.

Let μ be a finite measure on the Borel subsets of $\mathbb{R}^d$ and let $v \in S^{d-1}$ be a unit vector. Define

$$t_0 = t_0(v) = \inf\{t \in \mathbb{R} : \mu(H(v \le t)) > 0\},$$
$$t_1 = t_1(v) = \sup\{t \in \mathbb{R} : \mu(H(v \le t)) < \mu(\mathbb{R}^d)\}.$$

Note that $t_0 = -\infty$ and $t_1 = \infty$ are possible.

Let $H(s_0 \le v \le s_1)$ denote the closed slab between the hyperplanes $H(v = s_0)$ and $H(v = s_1)$. Define the set K by

$$K = \bigcap_{v \in S^{d-1}} H(t_0(v) \le v \le t_1(v)).$$

K is called the *support* of μ. Note that K is convex (obviously) and $\mu(\mathbb{R}^d \setminus K) = 0$.

Definition 1 The measure μ is called *nice* if the following conditions are satisfied:

(i) $t_0(v)$ and $t_1(v)$ are finite for every $v \in S^{d-1}$,
(ii) $\mu(H(v = t)) = 0$ for every $v \in S^{d-1}$ and $t \in \mathbb{R}$,
(iii) $\mu(H(s_0 \le v \le s_1)) > 0$ for every $v \in S^{d-1}$ and for every s_0, s_1 satisfying $t_0(v) \le s_0 < s_1 \le t_1(v)$.

If μ is a nice measure, then its support is full-dimensional since, by (ii), it is not contained in any hyperplane.

The function $t \mapsto \mu(K(v \le t))$ is zero on the interval $(-\infty, t_0]$, is equal to $\mu(K)$ on $[t_1, \infty)$, strictly increases on $[t_0, t_1]$, and, in view of (iii), is continuous. Assume $\alpha \in [0, 1]$. Then there is a unique $t \in [t_0, t_1]$ with

$$\mu(K(v \le t)) = \alpha \cdot \mu(K).$$

Denote this unique t by $g(v)$; this way we defined a map $g : S^{d-1} \longrightarrow \mathbb{R}$. The following simple lemma is important and probably well known.

Lemma 1 *For fixed* $\alpha \in [0, 1]$ *the function* g *is continuous.*

Proof When $\alpha = 1$, $g(v)$ is the support functional of K, which is not only continuous but convex (when extended to all $v \in \mathbb{R}^d$). Similarly, g is continuous when $\alpha = 0$.

Assume now that $0 < \alpha < 1$. Let $v_0 \in S^{d-1}$ be an arbitrary point. In order to prove the continuity of g at v_0 we show first that $K(v = g(v))$ and $K(v_0 = g(v_0))$ have a point in common whenever v it is close enough to v_0.

Obviously, $K(v_0 = g(v_0))$ is a $(d-1)$-dimensional convex set lying in the hyperplane $H(v_0 = g(v_0))$. Then, for every small enough neighbourhood of v_0, and for each v in such a neighbourhood, the supporting hyperplane of K with unit normal v (and $-v$) is also a supporting hyperplane of $K(v_0 \ge g(v_0))$ (and $K(v_0 \le g(v_0))$).

Assume $s_v \le S_v$ and let $H(v = s_v)$ and $H(v = S_v)$ be the two supporting hyperplanes (with normal v) to $K(v_0 = g(v_0))$ which is a $(d-1)$-dimensional convex set. Since $K(v_0 = g(v_0))$ is a $(d-1)$-dimensional convex set, condition (iii) implies that $s_v < S_v$. It follows that

$$K(v \le s_v) \subset K(v_0 \le g(v_0)) \subset K(v \le S_v),$$

and so

$$\mu(K(v \le s_v)) \le \mu(K(v_0 \le g(v_0)) \le \mu(K(v \le S_v)).$$

As $\mu(K(v_0 \leq g(v_0)) = \alpha \cdot \mu(K)$, we have $s_v \leq g(v) \leq S_v$. Consequently, $K(v = g(v))$ and $K(v_0 = g(v_0))$ have a point, say $z = z(v)$, in common. This $z(v)$ is not uniquely determined but that does not matter.

It is easy to finish the proof now. Clearly $g(v) = \langle v, z(v)\rangle$ and $g(v_0) = \langle v_0, z(v)\rangle$ for all v in a small neighbourhood of v_0. Assume the sequence v_n tends to v_0. We claim that every subsequence, $v_{n'}$, of v_n contains a subsequence $v_{n''}$ such that $\lim g(v_{n''}) = g(v_0)$, which evidently implies the continuity of g at v_0.

For the proof of this claim observe first that, since $K(v_0 = g(v_0))$ is compact, $z(v_{n'})$ contains a convergent subsequence $z(v_{n''})$ tending to z_0, say. Taking limits gives $z_o \in K(v_0 = g(v_0))$. Then $g(v_{n''}) = \langle v_{n''}, z(v_{n''})\rangle \to \langle v_0, z_0\rangle = g(v_0)$. $\square$

Theorem 2 is extended to measures in the following way.

Theorem 3 *Suppose μ_i is a nice measure on $\mathbb{R}^d$ with support K_i for all $i \in [d]$. Assume the family $\mathcal{F} = \{K_1, \dots, K_d\}$ is well separated and let $\alpha = (\alpha_1, \dots, \alpha_d) \in Q^d$. Then there is a unique positive transversal halfspace, H, such that $\mu_i(K_i \cap H) = \alpha_i \cdot \mu_i(K_i)$, for every $i \in [d]$.*

Corollary 1 *Assume μ_i are finite measures on $\mathbb{R}^d$ satisfying conditions (i) and (ii) of Definition 1. Let K_i be the support of μ_i for all $i \in [d]$. Suppose the family $\mathcal{F} = \{K_1, \dots, K_d\}$ is well separated and let $\alpha = (\alpha_1, \dots, \alpha_d) \in Q^d$. Then there is a positive transversal halfspace, H, such that $\mu_i(K_i \cap H) = \alpha_i \cdot \mu_i(K_i)$, for every $i \in [d]$.*

The corollary easily follows from Theorem 3; we omit the simple details.

Theorem 2 is a special case of Theorem 3: when μ_i is the Lebesgue measure (or volume) restricted to the convex body K_i for all $i \in [d]$ and the family $\mathcal{F}$ is well separated. Also, Theorem C is a special case of Theorem 3: when μ_i and K_i are the same as above, and, for a given partition I, J of $[d]$, we set $\alpha_i = 1$ for $i \in I$, and $\alpha_j = 0$ for $j \in J$. Theorem 1 follows from Theorem 3 via "lifting to the paraboloid". This is explained in the last section.

4 Proof of Theorem 3

In the proof we will use Brouwer's fixed point theorem. We will define a continuous mapping from a topological ball to itself, such that a fixed point of this map yields a halfspace with the desired properties. Set $K = K_1 \times \cdots \times K_d$. Given a point $x = (x_1, \dots, x_d) \in K$ we consider the hyperplane $\operatorname{aff}\{x_1, \dots, x_d\}$. Since the family $\mathcal{F}$ is well separated, this hyperplane is well defined for each $x \in K$. Let $H(v \leq t)$ be the (unique) positive transversal halfspace whose bounding hyperplane is $\operatorname{aff}\{x_1, \dots, x_d\}$.

In Sect. 2 we defined the map $v : K \longrightarrow S^{d-1}$ which is the properly chosen unit normal to $\operatorname{aff}\{x_1, \dots, x_d\}$. Clearly, this function is continuous.

We prove existence first. We start with the case when $\alpha_i \in (0, 1)$ for every $i \in [d]$. We turn to the remaining case later by constructing a suitable sequence of halfspaces.

Let $g_i : S^{d-1} \longrightarrow \mathbb{R}$ be the function such that for each $v \in S^{d-1}$, $g_i(v)$ is the real number for which $\mu_i(K_i(v \le g_i(v))) = \alpha \cdot \mu_i(K_i)$ for each $i \in [d]$. Each g_i is a continuous function by Lemma 1. Let $h : S^{d-1} \longrightarrow K$ be the function sending $v \mapsto (s_1, \ldots, s_d)$ where s_i is the Steiner point of the $(d-1)$-dimensional section, $K_i(v = g_i(v))$ for each $i \in [d]$. As is well known, the family of sections $K_i(v = t)$ depend continuously (according to the Hausdorff metric) on the corresponding family of hyperplanes, $\{H(v = t)\}$ whenever every section is $(d-1)$-dimensional, which is obviously the case because $\alpha_i \in (0, 1)$. It is also well known that the function that assigns to a compact convex set its Steiner point is continuous. Hence, h is a continuous function.

It follows that

$$f := h \circ v : K \longrightarrow K$$

is a continuous function. As K is a compact convex set in $\mathbb{R}^d \times \cdots \times \mathbb{R}^d$ Brouwer's fixed point theorem implies the existence of a point $x \in K$ such that $f(x) = x$. Consider a fixed point, $x = (x_1, \ldots, x_d)$, of f. Then the halfspace $H(v \le t)$ whose bounding hyperplane is aff $\{x_1, \ldots, x_d\}$ is a positive transversal halfspace to $\mathcal{F}$ and it has the required properties.

Next we prove existence for vectors $\alpha = (\alpha_1, \ldots, \alpha_d) \in Q^d$ that may have 0, 1 components as well. Consider the sequence $\{\alpha^n\} \subset Q^d$ $\alpha^n = (\alpha_1^n, \ldots, \alpha_d^n)$ (defined for every $n \ge 2$), such that for every entry $\alpha_i = 0$ we define $\alpha_i^n = \frac{1}{n}$, for every entry $\alpha_i = 1$ we define $\alpha_i^n = 1 - \frac{1}{n}$, and for every entry $\alpha_i \notin \{0, 1\}$ we define $\alpha_i^n = \alpha_i$. Also, for every $n \ge 2$ we consider the unique positive transversal halfspace $H(v_n \le t_n)$ with $\mu_i(K_i(v_n \le t_n)) = \alpha_i^n \cdot \mu_i(K_i)$, for each i. The compactness of K implies that the set of all possible $(v, t) \in S^{d-1} \times \mathbb{R}$ such that the hyperplane $H(v = t)$ is transversal to $\mathcal{F}$ is compact. Thus there exists a convergent subsequence $\{(v_{n'}, t_{n'})\}$ which converges to a point $(v, t) \in S^{d-1} \times \mathbb{R}$. Clearly, $H(v \le t)$ is a positive transversal halfspace to $\mathcal{F}$ which satisfies $\mu_i(K_i(v \le t)) = \alpha_i \cdot \mu_i(K_i)$ for every i.

Next comes uniqueness. We start with the 0, 1 case, that is, when $\alpha = (\alpha_1, \ldots, \alpha_d)$ with all $\alpha_i \in \{0, 1\}$. Such an α defines a $\beta \in Q^d$ via $\beta_i = 1 - \alpha_i$ for every i. By the previous existence proof there is a unique positive transversal halfspace $H(v \le t)$ for α and another one $H(u \le s)$ for β. These two halfspaces are distinct, first because $u = v$ is impossible, and second because of the following fact which implies that $u \ne -v$.

Proposition 3 *For every pair of points* $(a_1, \ldots, a_d)$ *and* $(b_1, \ldots, b_d)$ *in* K, $v(a_1, \ldots, a_d)$ *and* $-v(b_1, \ldots, b_d)$ *are distinct.*

Proof Assume $v(a_1, \ldots, a_d) = -v(b_1, \ldots, b_d)$. Then the affine hulls of the a_i and the b_i are parallel hyperplanes. We use the same homotopy as in the proof of Proposition 2. As λ moves from 0 to 1, the moving points $(1-\lambda)a_i + \lambda b_i$ stay in K_i, and their affine hull remains parallel with aff$\{a_1, \ldots, a_d\}$. So the outer normal remains unchanged throughout the homotopy. A contradiction. □

The condition $\mu_i(K_i(v \le t)) = \alpha_i \mu_i(K_i)$ implies, in the given case, that all K_i $(i \in I)$ are in $H(v \le t)$ and all K_j $(j \in J)$ are in $H(v \ge t)$. Thus $H(v = t)$ is a transversal hyperplane satisfying the conditions of Theorem C with partition I, J where $I = \{i \in [d] : \alpha_i = 0\}$ and $J = \{j \in [d] : \alpha_j = 1\}$. The same way, $H(u = s)$ is a transversal hyperplane satisfying the conditions of Theorem C with the same partition J, I.

The uniqueness of $H(v \le t)$ follows now easily. If we had two distinct positive transversal halfspaces $H(v_1 \le t_1)$ and $H(v_2 \le t_2)$ for α, then we would have four distinct transversal hyperplanes with K_i $(i \in I)$ on one side and K_j $(j \in J)$ on the other side, contradicting Theorem C.

Now we turn to uniqueness for general α. Assume that there are two distinct positive transversal halfspaces $H(v_1 \le t_1)$ and $H(v_2 \le t_2)$ for α. Their bounding hyperplanes cannot be parallel. Define $M = H(v_1 \le t_1) \cap H(v_2 \le t_2)$ and $N = H(v_1 \ge t_1) \cap H(v_2 \ge t_2)$. The partition I, J of the index set $[d]$ is defined as follows: $i \in I$ if $M \cap \operatorname{int} K_i \ne \emptyset$ and $j \in J$ if $M \cap \operatorname{int} K_j = \emptyset$. Set $K_i' = M \cap K_i$ for every $i \in I$ and $K_j' = N \cap K_j$ for every $j \in J$. Let $\mathcal{F}'$ be the family consisting of all the convex bodies K_i' $(i \in I)$ and K_j' $(j \in J)$. It is quite easy to see that no member of $\mathcal{F}'$ is empty. Moreover, $\mathcal{F}'$ is evidently well separated. Given the partition I, J, define γ by $\gamma_i = 1$ for $i \in I$ and $\gamma_j = 0$ for $j \in J$. Then there are two transversal halfspaces (with respect to $\mathcal{F}'$), namely $H(v_k \le t_k)$ $k = 1, 2$ satisfying $\mu_i(K_i(v_k \le t_k)) = \gamma_i \mu_i(K_i)$ for every i. But every $\gamma_i \in \{0, 1\}$ and we just established uniqueness in the 0, 1 case. □

5 Proof of Theorem 2

We will use the well-known technique of lifting the problem from $\mathbb{R}^d$ to a paraboloid in $\mathbb{R}^{d+1}$, and then apply Theorem 3.

In this section we change notation a little. A point in $\mathbb{R}^d$ is denoted by $x = (x_1, \dots, x_d)$, a point in $\mathbb{R}^{d+1}$ is denoted by $\overline{x} = (x_1, \dots, x_d, x_{d+1})$. The projection of $\overline{x}$ is $\pi(\overline{x}) = (x_1, \dots, x_d)$, and the lifting of x is $\ell(x) = (x_1, \dots, x_d, |x|^2)$ where $|x|^2 = x_1^2 + \dots + x_d^2$. Clearly, $\ell(x)$ is contained in the paraboloid

$$P = \{\overline{x} \in \mathbb{R}^{d+1} : \overline{x} = (x_1, x_2, \dots, x_d, |x|^2)\}.$$

A set $K \subset \mathbb{R}^d$ lifts to $\ell(K) = \{\ell(x) \in P : x \in K\}$. Also, $\pi(\ell(K)) = K$.

A hyperplane is called *non-vertical* if $\pi(H) = \mathbb{R}^d$. The lifting gives a bijective relation between non-vertical hyperplanes in $\mathbb{R}^{d+1}$ (intersecting P) and $(d-1)$-dimensional spheres in $\mathbb{R}^d$ in the following way. Assume $S = S(u, r)$ is the sphere centered at u, with radius r in $\mathbb{R}^d$. Of course, $\ell(S) \subset P$, but more importantly,

$$\ell(S) = P \cap H,$$

where H is the hyperplane with equation $x_{d+1} = 2\langle u, x\rangle + r^2 - |u|^2$. Conversely, given a non-vertical hyperplane H with equation $x_{d+1} = 2\langle u, x\rangle + s$ where $s = r^2 - |u|^2$ with some $r > 0$,

$$\pi(H \cap P) = S(u, r).$$

As a first application of this lifting, here is a simple proof of a slightly stronger version of Theorem KLH (we can replace the convexity assumption by connectedness).

Consider a family of $d+1$ well separated connected compact sets in $\mathbb{R}^d$ and a partition of the sets into two classes. Lift the family into the paraboloid, and for each lifted set, consider its convex hull. This gives a $(d+1)$-element family of convex bodies in $\mathbb{R}^{d+1}$. The lifted family is well separated. This can be seen using Proposition 1: the lifting of the separating $(d-1)$-dimensional planes of the original family yield (vertical) separating hyperplanes of the corresponding lifting. Thus Theorem 3 applies to the lifted family (with the obviously induced partition) and gives a hyperplane H such $H \cap P$ projects onto a sphere S in $\mathbb{R}^d$ satisfying the requirements of Theorem 1. We omit the straightforward detail.

We apply Theorem 3 to the paraboloid lifting to obtain Theorem 1, in the same way. The family $\mathcal{F} = \{K_1, \dots, K_{d+1}\}$ lifts to the family $\ell(\mathcal{F}) = \{\ell(K_1), \dots, \ell(K_{d+1})\}$, and we define the measures μ_i via

$$\mu_i(C) = \operatorname{Vol} \pi(C \cap \ell(K_i)),$$

where C is a Borel subset of $\mathbb{R}^{d+1}$. Clearly, μ_i is finite and $\ell(\mathcal{F})$ is well separated. Its support is $\operatorname{conv} \ell(K_i)$. It is easy to see that μ_i is a nice measure by checking that it satisfies all three conditions.

Thus Theorem 3 applies and guarantees the existence of a unique positive transversal (to $\ell(\mathcal{F})$) halfspace $H \subset \mathbb{R}^{d+1}$ with $\mu_i(H \cap \ell(K_i)) = \alpha_i \cdot \mu_i(K_i)$ for each i. This translates to the ball $B = \pi(H \cap P)$ and sphere $S = \pi(H^0 \cap P)$ (where H^0 is the bounding hyperplane of H) as follows: S is a transversal sphere of the family $\mathcal{F}$ and $\operatorname{Vol}(B \cap K_i) = \alpha_i \cdot \operatorname{Vol} K_i$. Unicity of S follows readily. □

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