

Sliding Mode and Active Disturbance Rejection Control to Stabilization of One-Dimensional Anti-Stable Wave Equations Subject to Disturbance in Boundary Input

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Abstract—In this technical note, we are concerned with the boundary stabilization of a one-dimensional anti-stable wave equation subject to boundary disturbance. We propose two strategies, namely, sliding mode control (SMC) and the active disturbance rejection control (ADRC). The reaching condition, and the existence and uniqueness of the solution for all states in the state space in SMC are established. The continuity and monotonicity of the sliding surface are proved. Considering the SMC usually requires the large control gain and may exhibit chattering behavior, we then develop an ADRC to attenuate the disturbance. We show that this strategy works well when the derivative of the disturbance is also bounded. Simulation examples are presented for both control strategies.

Index Terms—Boundary control, disturbance rejection, sliding mode control (SMC), wave equation.

I. INTRODUCTION

In the last few years, the backstepping method has been introduced to the boundary stabilization of some PDEs ([6], [13]). This powerful method can also be used to deal with the stabilization of wave equations with uncertainties in either boundary input or in observation ([4], [5]). Owing to its good performance in disturbance rejection and insensitivity to uncertainties, the sliding mode control (SMC) has also been applied to some PDEs, see [1], [2], [8], [9], [11], [12]. Other approaches that are proposed to deal with the disturbance include the active disturbance rejection control (ADRC) ([3]), and the adaptive control method for systems with unknown parameters ([6], [7]). In the ADRC approach, the disturbance is first estimated in terms of the output; and then canceled by its estimates. This is the way used in [4], [5], [7]. Compared with the SMC, the ADRC has not been used in distributed parameter systems.

Motivated mainly by [1], [11], we are concerned with the stabilization of the following PDEs:

$$\begin{cases} u_{tt}(x, t) = u_{xx}(x, t), & x \in (0, 1), t > 0, \\ u_x(0, t) = -qu_t(0, t), & t \geq 0, \\ u_x(1, t) = U(t) + d(t), & t \geq 0 \end{cases} \quad (1)$$

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where u is the state, U is the control input, $0 < q \neq 1$ is a constant number. The unknown disturbance d is supposed to be bounded measurable, that is, $|d(t)| \leq M_0$ for some $M_0 > 0$ and all $t \geq 0$. The system represents an anti-stable distributed parameter physical system ([7]).

We proceed as follows. The SMC for disturbance rejection is presented in Section II. The ADRC for disturbance attenuation is presented in Section III. Section IV presents some numerical simulations for both control methods.

II. SLIDING MODE CONTROLLER

A. Design of Sliding Surface

Following [13], we introduce a transformation:

$$w(x, t) = u(x, t) - \frac{q+c}{q^2-1} \int_0^x u_t(y, t) dy - \frac{q(q+c)}{q^2-1} \int_0^x u_x(y, t) dy. \quad (2)$$

This transformation brings system (1) into the following system:

$$\begin{cases} w_{tt}(x, t) = w_{xx}(x, t), & x \in (0, 1), t > 0, \\ w_x(0, t) = cw_t(0, t), & t \geq 0, \\ w_x(1, t) = -\frac{c^2-1}{q^{c+1}}(U(t) + d(t)) + \frac{q+c}{q^{c+1}}w_t(1, t), & t \geq 0 \end{cases} \quad (3)$$

where $0 < c \neq 1$ is the design parameter. The transformation (2) is invertible, that is

$$u(x, t) = w(x, t) + \frac{q+c}{c^2-1} \int_0^x w_t(y, t) dy - \frac{c(q+c)}{c^2-1} \int_0^x w_x(y, t) dy. \quad (4)$$

Let us consider systems (1) and (3) in the state space $\mathcal{H} = H^1(0, 1) \times L^2(0, 1)$ with inner product given by

$$\begin{aligned} \langle (f_1, g_1)^\top, (f_2, g_2)^\top \rangle &= \int_0^1 [f_1'(x)\overline{f_2'(x)} + g_1(x)\overline{g_2(x)}] dx \\ &+ f_1(1)\overline{f_2(1)}, \forall (f_1, g_1)^\top, (f_2, g_2)^\top \in \mathcal{H}. \end{aligned} \quad (5)$$

In this section, we consider \mathcal{H} as a real function space. In Section III, \mathcal{H} is considered as the complex space.

Define the energy of system (3): $E(t) = (1/2) \int_0^1 (w_t^2(x, t) + w_x^2(x, t)) dx + (1/2)w^2(1, t)$. Then

$$\dot{E}(t) = -cw_t^2(0, t) + w_x(1, t)w_t(1, t) + w(1, t)w_t(1, t).$$

It is seen that in order to make $E(t)$ non-increasing on the sliding surface S_W for system (3), which is a closed subspace of \mathcal{H} , it is natural to choose $S_W(t) = w(1, t) = 0$ (so $w_t(1, t) = 0$), i.e.

$$S_W = \{(f, g)^\top \in \mathcal{H} \mid f(1) = 0\}. \quad (6)$$

In this way, $\dot{E}(t) = -cw_t^2(0, t) \leq 0$ on S_W , and on S_W , system (3) becomes

$$\begin{cases} w_{tt}(x, t) = w_{xx}(x, t), & x \in (0, 1), t > 0, \\ w_x(0, t) = cw_t(0, t), & t \geq 0, \\ w(1, t) = 0, & t \geq 0. \end{cases} \quad (7)$$

It is well-known that for any initial value $(w(x, 0), w_t(x, 0))^\top \in S_W$, there exists a unique C_0 -semigroup $T_S(t)$ solution $(w(\cdot, t), w_t(\cdot, t))^\top = T_S(t)(w(\cdot, 0), w_t(\cdot, 0))^\top \in C[0, \infty; \mathcal{H})$ to

(7), where the norm in S_W is the induced norm of \mathcal{H} . Moreover, system (7) is exponentially stable in S_W , that is, there exist $M_S, \omega_S > 0$ independent of initial value such that

$$\|(w(\cdot, t), w_t(\cdot, t))^\top\|_{\mathcal{H}} \leq M_S e^{-\omega_S t} \|(w(x, 0), w_t(x, 0))^\top\|_{\mathcal{H}}. \quad (8)$$

Transforming S_W by (2) into the original system (1), that is

$$\begin{aligned} S_W(t) &\triangleq w(1, t) \\ &= -\frac{1+qc}{q^2-1}u(1, t) + \frac{q(q+c)}{q^2-1}u(0, t) \\ &\quad - \frac{q+c}{q^2-1} \int_0^1 u_t(x, t) dx \end{aligned} \quad (9)$$

we get the sliding surface S_U for system (1) as

$$S_U = \left\{ (f, g)^\top \in \mathcal{H} \mid -\frac{1+qc}{q^2-1}f(1) + \frac{q(q+c)}{q^2-1}f(0) - \frac{q+c}{q^2-1} \int_0^1 g(x) dx = 0 \right\} \quad (10)$$

on which the original system (1) becomes

$$\begin{cases} u_{tt}(x, t) = u_{xx}(x, t), & x \in (0, 1), t > 0, \\ u_x(0, t) = -qu_t(0, t), & t \geq 0, \\ u(1, t) = \frac{q(q+c)}{1+qc}u(0, t) - \frac{q+c}{1+qc} \int_0^1 u_t(y, t) dy, & t \geq 0 \end{cases} \quad (11)$$

which is exponentially stable by (8) and the equivalence between (7) and (11).

B. State Feedback Controller

To motivate the control design, we differentiate (9) formally with respect to t to obtain

$$\begin{aligned} \dot{S}_W(t) &= -\frac{1+qc}{q^2-1}u_t(1, t) + \frac{q(q+c)}{q^2-1}u_t(0, t) \\ &\quad - \frac{q+c}{q^2-1} \int_0^1 u_{tt}(y, t) dy \\ &= -\frac{q+c}{q^2-1}[U + d(t)] - \frac{1+qc}{q^2-1}u_t(1, t). \end{aligned} \quad (12)$$

This suggests us to design the boundary controller

$$U(t) = -\frac{1+qc}{q+c}u_t(1, t) + K \operatorname{sgn} \left(\frac{q+c}{q^2-1} S_W(t) \right) \quad \text{for } S_W(t) \neq 0 \quad (13)$$

where $K > M_0$ and $S_W(t)$ is defined by (9). Note that controller (13) deals with the worst case of disturbance by the high gain K . Under the feedback (13), the closed-loop system of (1) reads

$$\begin{cases} u_{tt}(x, t) = u_{xx}(x, t), & x \in (0, 1), t > 0, \\ u_x(0, t) = -qu_t(0, t), & t \geq 0, \\ u_x(1, t) = -\frac{1+qc}{q+c}u_t(1, t) + K \operatorname{sgn} \left(\frac{q+c}{q^2-1} S_W(t) \right) + d(t), & t \geq 0 \text{ for } S_W(t) \neq 0. \end{cases} \quad (14)$$

By the transformation (4), the corresponding controller U for system (3) is

$$\begin{aligned} U(t) &= -\frac{1+qc}{c^2-1}w_x(1, t) + \frac{(1+qc)^2}{(c^2-1)(q+c)}w_t(1, t) \\ &\quad + K \operatorname{sgn} \left(\frac{q+c}{q^2-1} w(1, t) \right) \text{ for } S_W(t) \neq 0. \end{aligned} \quad (15)$$

Then, $S_W(t)$ satisfies formally that

$$\begin{aligned} S_W(t) \dot{S}_W(t) &= -\frac{q+c}{q^2-1} K \operatorname{sgn} \left(\frac{q+c}{q^2-1} S_W(t) \right) S_W(t) \\ &\quad - \frac{q+c}{q^2-1} d(t) S_W(t) \\ &\leq -(K - M_0) \left| \frac{q+c}{q^2-1} \right| |S_W(t)| \\ &= -\eta |S_W(t)|, \quad \eta > 0. \end{aligned} \quad (16)$$

The closed-loop system of system (3) under the state feedback controller (15) is

$$\begin{cases} w_{tt}(x, t) = w_{xx}(x, t), & x \in (0, 1), t > 0, \\ w_x(0, t) = cw_t(0, t), & t \geq 0, \\ w_t(1, t) = -\frac{q+c}{q^2-1} K \operatorname{sgn} \left(\frac{q+c}{q^2-1} w(1, t) \right) - \frac{q+c}{q^2-1} d(t) \triangleq \tilde{d}(t), & t \geq 0 \text{ for } S_W(t) \neq 0. \end{cases} \quad (17)$$

Note that (16) is just the well-known ‘‘reaching condition’’ for system (3) but we do not know if \dot{S}_W makes sense for the initial value in the state space in present. This issue is not discussed in [1]. It would be studied in Section III.

C. Solution of Closed-Loop System

In this section, we investigate the well-posedness of the solution to (14). Since (14) and (17) are equivalent, and (17) takes a simpler form, we study the solution of (17) outside the sliding surface.

Define an operator $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ as follows:

$$\begin{cases} A(f, g)^\top = (g, f'')^\top, \quad \forall (f, g)^\top \in D(A), \\ D(A) = \{(f, g)^\top \in H^2(0, 1) \times H^1(0, 1) \mid f'(0) = cg(0), \\ g(1) = 0\}. \end{cases} \quad (18)$$

A direct computation shows that the adjoint operator of A is given by

$$\begin{cases} A^*(\varphi, \psi)^\top = (-\psi, -\varphi'')^\top, \quad \forall (\varphi, \psi)^\top \in D(A^*), \\ D(A^*) = \{(\varphi, \psi)^\top \in H^2(0, 1) \times H^1(0, 1) \mid \\ \varphi'(0) = -c\psi(0), \psi(1) = 0\}. \end{cases} \quad (19)$$

Take the inner product on both sides of (17) with $(\varphi, \psi)^\top \in D(A^*)$ to get

$$\begin{aligned} \frac{d}{dt} \left\langle \begin{pmatrix} w \\ w_t \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle_{\mathcal{H}} &= \left\langle \begin{pmatrix} w \\ w_t \end{pmatrix}, A^* \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &\quad + \left\langle \begin{pmatrix} -\delta'(x-1) + \delta(x-1) \\ 0 \end{pmatrix}, \tilde{d}(t), \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle_{D(A^*)' \times D(A^*)} \end{aligned}$$

where $D(A^*)'$ is the dual of $D(A^*)$ with the pivot space \mathcal{H} . Then system (17) can be written as

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} w \\ w_t \end{pmatrix} &= A \begin{pmatrix} w \\ w_t \end{pmatrix} + B \tilde{d}(t), \\ B &= \begin{pmatrix} -\delta'(x-1) + \delta(x-1) \\ 0 \end{pmatrix} \end{aligned} \quad (20)$$

and $\delta(\cdot)$ is the Dirac distribution.

Proposition 1: Suppose that d is bounded measurable in time. Then for any $(w(\cdot, 0), w_t(\cdot, 0))^\top \in \mathcal{H}$, there exists a $t_0 \geq 0$, depending on initial value, such that (17) admits a unique solution $(w, w_t)^\top \in C(0, t_0; \mathcal{H})$ and $w(1, t) = 0$ for all $t \geq t_0$. Moreover, $S_W(t) = w(1, t)$ is continuous and monotone in $[0, t_0]$.

Proof: We first suppose that $w(1, 0) \neq 0$. In this case, it follows from (17) that in the beginning of $t \geq 0$

$$\tilde{d}(t) = \begin{cases} -\left|\frac{q+c}{q^2-1}\right|K - \frac{q+c}{q^2-1}d(t), & w(1, 0) > 0, \\ \left|\frac{q+c}{q^2-1}\right|K - \frac{q+c}{q^2-1}d(t), & w(1, 0) < 0. \end{cases} \quad (21)$$

Let A be defined by (18). For any $(f, g)^\top \in D(A)$, it has

$$\left\langle A(f, g)^\top, (f, g)^\top \right\rangle_{\mathcal{H}} = -cg^2(0) \leq 0.$$

So A is dissipative. Since by argument below, $1 \in \rho(A)$, the resolvent set of A , it follows that A generates a C_0 -semigroup of contractions e^{At} on \mathcal{H} by the Lumer-Phillips theorem ([10, Theorem 4.3, p.14]).

Consider the observation problem of dual system of (20)

$$\begin{cases} \frac{d}{dt} \begin{pmatrix} w^* \\ w_t^* \end{pmatrix} = A^* \begin{pmatrix} w^* \\ w_t^* \end{pmatrix}, \\ y^* = B^* \begin{pmatrix} w^* \\ w_t^* \end{pmatrix}. \end{cases} \quad (22)$$

Then we can easily show that

$$\begin{aligned} \int_0^T |y^*(t)|^2 dt &\leq 2 \int_0^T [w_x^{*2}(1, t) + w^{*2}(1, t)] dt \\ &\leq 4(T+2)E(0). \end{aligned} \quad (23)$$

A straightforward computation shows that $B^*(I - A^*)^{-1}$ is bounded on \mathcal{H} . This together with (23) shows that B^* is admissible for the C_0 -semigroup e^{A^*t} generated by A^* ([14, Theorem 4.4.3, p.127]). Therefore, system (20) admits a unique solution which satisfies, for all $(\varphi, \psi)^\top \in D(A^*)$, that

$$\begin{aligned} &\frac{d}{dt} \left(\int_0^1 w_x(x, t) \varphi'(x) dx + w(1, t) \varphi(1) \right) \\ &= - \int_0^1 w_x(x, t) \psi'(x) dx + [\varphi'(1) + \varphi(1)] \tilde{d}(t), \\ &\frac{d}{dt} \int_0^1 w_t(x, t) \psi(x) dx \\ &= - \int_0^1 w_t(x, t) \varphi''(x) dx. \end{aligned} \quad (24)$$

Set $(\varphi, 4)^\top = (1, 0)^\top \in D(A^*)$ in the first identity of (24) to obtain

$$\dot{S}_W(t) = \tilde{d}(t) \text{ for } S_W(t) \neq 0. \quad (25)$$

This shows that S_W is continuous in the interval where $S_W \neq 0$. Moreover, (16) holds true. Therefore, there exists a $t_0 \geq 0$ such that $S_W(t)$ is monotone in $[0, t_0]$ and $S_W(t) = 0$ for all $t \geq t_0$. In particular, if $w(1, 0) = S_W(0) = 0$, then $t_0 = 0$. This completes the proof. ■

Returning to the original system (14) by the inverse transformation (4), we obtain the first main result of this technical note.

Theorem 1: Suppose that d is bounded measurable in time. Then for any $(u(\cdot, 0), u_t(\cdot, 0))^\top \in \mathcal{H}$, there exists some $t_0 \geq 0$, depending on initial value, such that (14) admits a unique solution $(u, u_t)^\top \in C([0, t_0]; \mathcal{H})$ and $S_U(t) = 0$ for all $t \geq t_0$. Moreover, $S_U(t)$ is continuous and monotone in $[0, t_0]$, where $S_U(t) = 0$ is the sliding surface of system (14) determined by

$$\dot{S}_U(t) = -\text{sgn}(S_U(t)) \left| \frac{q+c}{q^2-1} \right| K - \frac{q+c}{q^2-1} d(t). \quad (26)$$

Any solution of (14) in the state space \mathcal{H} will reach the sliding surface $S_U(t) = 0$ in finite time and remains on $S_U(t) = 0$ afterwards.

III. ACTIVE DISTURBANCE REJECTION CONTROL

It is well-known that the so-called chattering behavior is associated with the SMC, due to discontinuity of control. In this section, we shall use a direct approach to attenuate rather than to reject the disturbance. This is the key to the ADRC method in finite-dimensional systems ([3]). The idea is first to estimate the disturbance and then to cancel the estimate in the feedback-loop. Unlike the SMC which usually uses high gain control, the control effort in ADRC is found to be moderate.

In estimating the disturbance, we assume that the derivative of the disturbance is bounded: $|\dot{d}(t)| \leq R$ for some $R > 0$ and all $t \geq 0$. Again, by equivalence, we discuss (3) only for it has a simpler form.

The objective now is to design a continuous controller U which can stabilize system (3) in the presence of disturbance. In view of (15), this controller is designed as follows:

$$\begin{aligned} U(t) &= U_1(t) + U_2(t) \\ &= -\frac{1+qc}{c^2-1} w_x(1, t) \\ &\quad + \frac{(1+qc)^2}{(c^2-1)(q+c)} w_t(1, t) + U_2(t) \end{aligned} \quad (27)$$

where U_2 , also continuous, is to be designed in what follows. Under control (27), the closed-loop of system (3) becomes

$$\begin{cases} w_{tt}(x, t) = w_{xx}(x, t), & x \in (0, 1), t > 0, \\ w_x(0, t) = cw_t(0, t), & t \geq 0, \\ w_t(1, t) = -\frac{q+c}{q^2-1} d(t) - \frac{q+c}{q^2-1} U_2(t), & t \geq 0 \end{cases} \quad (28)$$

Introduce a variable $y(t) = w(1, t)$. Then the boundary condition at $x = 1$ in (28) gives that

$$\dot{y}(t) = -\frac{q+c}{q^2-1} d(t) - \frac{q+c}{q^2-1} U_2(t). \quad (29)$$

It is seen that (29) is an ODEs with state y and control U_2 . Then we are able to design an extended state observer to estimate both y and d as follows ([3]):

$$\begin{cases} \dot{\hat{y}}_\varepsilon(t) = -\frac{q+c}{q^2-1} \hat{d}_\varepsilon(t) - \frac{q+c}{q^2-1} U_2(t) - \frac{1}{\varepsilon} (\hat{y}_\varepsilon(t) - y(t)), \\ \dot{\hat{d}}_\varepsilon(t) = \frac{1}{\varepsilon^2} \frac{q^2-1}{q+c} (\hat{y}_\varepsilon(t) - y(t)) \end{cases} \quad (30)$$

where ε is the tuning small parameter. The errors $\tilde{y}_\varepsilon = \hat{y}_\varepsilon - y$, $\tilde{d}_\varepsilon = \hat{d}_\varepsilon - d$ satisfy

$$\begin{cases} \dot{\tilde{y}}_\varepsilon(t) = -\frac{q+c}{q^2-1} \tilde{d}_\varepsilon(t) - \frac{1}{\varepsilon} \tilde{y}_\varepsilon(t), \\ \dot{\tilde{d}}_\varepsilon(t) = \frac{1}{\varepsilon^2} \frac{q^2-1}{q+c} \tilde{y}_\varepsilon(t) - \dot{d}(t) \end{cases} \quad (31)$$

which can be rewritten as

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} \tilde{y}_\varepsilon(t) \\ \tilde{d}_\varepsilon(t) \end{pmatrix} &= \mathbf{A}_\varepsilon \begin{pmatrix} \tilde{y}_\varepsilon(t) \\ \tilde{d}_\varepsilon(t) \end{pmatrix} + \mathbf{B}_\varepsilon \dot{d}(t), \\ \mathbf{A}_\varepsilon &= \begin{pmatrix} -\frac{1}{\varepsilon} & -\frac{q+c}{q^2-1} \\ \frac{1}{\varepsilon^2} \frac{q^2-1}{q+c} & 0 \end{pmatrix}, \\ \mathbf{B}_\varepsilon &= \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \end{aligned} \quad (32)$$

A straightforward computation shows that the eigenvalues of the matrix \mathbf{A}_ε are

$$\lambda_1 = -\frac{1}{2\varepsilon} + \frac{\sqrt{3}}{2\varepsilon}i, \quad \lambda_2 = -\frac{1}{2\varepsilon} - \frac{\sqrt{3}}{2\varepsilon}i \quad (33)$$

and

$$e^{\mathbf{A}_\varepsilon t} \mathbf{B}_\varepsilon = - \begin{pmatrix} \frac{\lambda_1 \lambda_2 (e^{\lambda_2 t} - e^{\lambda_1 t})}{C_\varepsilon (\lambda_2 - \lambda_1)} \\ \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_2 - \lambda_1} \end{pmatrix}, \quad C_\varepsilon = \frac{1}{\varepsilon^2} \frac{q^2-1}{q+c}.$$

Since

$$\begin{pmatrix} \tilde{y}_\varepsilon(t) \\ \tilde{d}_\varepsilon(t) \end{pmatrix} = e^{\mathbf{A}_\varepsilon t} \begin{pmatrix} \tilde{y}_\varepsilon(0) \\ \tilde{d}_\varepsilon(0) \end{pmatrix} + \int_0^t e^{\mathbf{A}_\varepsilon(t-s)} \mathbf{B}_\varepsilon \dot{d}(s) ds$$

the first term above can be arbitrarily small as $t \rightarrow \infty$ by the exponential stability of $e^{\mathbf{A}_\varepsilon t}$, and the second term can also be arbitrarily small as $\varepsilon \rightarrow 0$ due to boundedness of \dot{d} and the expression of $e^{\mathbf{A}_\varepsilon t} \mathbf{B}_\varepsilon$. As a result, $(\tilde{y}_\varepsilon(t), \tilde{d}_\varepsilon(t))$ can be arbitrarily small as $t \rightarrow \infty, \varepsilon \rightarrow 0$.

Since \tilde{d}_ε is an approximation of d , we can design the controller for system (28) by cancelation/feedback as

$$U_2(t) = -\tilde{d}_\varepsilon(t) + M \frac{q^2 - 1}{q + c} w(1, t) \quad (34)$$

under which the overall closed-loop system of (28), (29), and (30) becomes

$$\begin{cases} w_{tt}(x, t) = w_{xx}(x, t), x \in (0, 1), t > 0, \\ w_x(0, t) = cw_t(0, t), t \geq 0, \\ w_t(1, t) = \frac{q+c}{q^2-1}(\tilde{d}_\varepsilon(t) - d(t)) - Mw(1, t), t \geq 0, \\ w(1, t) = y(t), t \geq 0, \\ \dot{y}(t) = \frac{q+c}{q^2-1}(\dot{\tilde{d}}_\varepsilon(t) - \dot{d}(t)) - Mw(1, t), t \geq 0, \\ \dot{\tilde{y}}_\varepsilon(t) = -\frac{1}{\varepsilon}(\dot{\tilde{y}}_\varepsilon(t) - \dot{y}(t)) - Mw(1, t), t \geq 0, \\ \dot{\tilde{d}}_\varepsilon(t) = \frac{1}{\varepsilon^2} \frac{q^2-1}{q+c}(\dot{\tilde{y}}_\varepsilon(t) - \dot{y}(t)), t \geq 0. \end{cases} \quad (35)$$

Using the error dynamics (31), we see that (35) is equivalent to

$$\begin{cases} w_{tt}(x, t) = w_{xx}(x, t), x \in (0, 1), t > 0, \\ w_x(0, t) = cw_t(0, t), t \geq 0, \\ w_t(1, t) = \frac{q+c}{q^2-1}\tilde{d}_\varepsilon(t) - Mw(1, t), t \geq 0, \\ \dot{\tilde{y}}_\varepsilon(t) = -\frac{1}{\varepsilon}\tilde{y}_\varepsilon(t) - \frac{q+c}{q^2-1}\tilde{d}_\varepsilon(t), t \geq 0, \\ \dot{\tilde{d}}_\varepsilon(t) = \frac{1}{\varepsilon^2} \frac{q^2-1}{q+c}\tilde{y}_\varepsilon(t) - \dot{d}(t), t \geq 0. \end{cases} \quad (36)$$

It is seen that in (36), the variable $(\tilde{y}_\varepsilon, \tilde{d}_\varepsilon)$ is independent of the “ w part”, and can be made as small as desired as $t \rightarrow \infty, \varepsilon \rightarrow 0$. Thus, we need to consider only the “ w part” that is rewritten as

$$\begin{cases} w_{tt}(x, t) = w_{xx}(x, t), x \in (0, 1), t > 0, \\ w_x(0, t) = cw_t(0, t), t \geq 0, \\ w_t(1, t) = \frac{q+c}{q^2-1}\tilde{d}_\varepsilon(t) - Mw(1, t), t \geq 0. \end{cases} \quad (37)$$

System (37) will be considered in the state Hilbert space $\mathcal{H} = H^1(0, 1) \times L^2(0, 1)$ also with the norm defined in (5). Define system operator A_c of (37) by

$$\begin{cases} A_c(f, g)^\top = (g, f'')^\top, \forall (f, g)^\top \in D(A_c), \\ D(A_c) = \{(f, g)^\top \in \mathcal{H} \mid \\ A_c(f, g)^\top \in \mathcal{H}, f'(0) = cg(0), g(1) = -Mf(1)\}. \end{cases} \quad (38)$$

A direct computation gives

$$\begin{cases} A_c^*(f, g)^\top = (-M(f'(1) + f(1)) - g, -f'')^\top, \\ \forall (f, g)^\top \in D(A_c^*), \\ D(A_c^*) = \{(f, g)^\top \in H^2(0, 1) \times H^1(0, 1) \mid A_c^*(f, g)^\top \in \mathcal{H}, \\ f'(0) = -cg(0), g(1) = 0\}. \end{cases} \quad (39)$$

Similar to (20), system (37) can be rewritten as an evolution equation in \mathcal{H} as

$$\frac{d}{dt}(w, w_t)^\top = A_c(w, w_t)^\top + \frac{q+c}{q^2-1} B \tilde{d}_\varepsilon \quad (40)$$

where B is defined in (20).

Proposition 2: Suppose that both d and \dot{d} are uniformly bounded in time. Then for any $(w(\cdot, 0), w_t(\cdot, 0))^\top \in \mathcal{H}$, there exists a unique solution $(w, w_t)^\top \in C(0, \infty; \mathcal{H})$ to (40). Moreover, $(w, w_t)^\top$ can reach arbitrary vicinity of zero as $t \rightarrow \infty, \varepsilon \rightarrow 0$ in (36).

Proof: We consider only the case of $c < 1$. The case of $c > 1$ can be treated similarly. Introduce an auxiliary system as follows:

$$\begin{cases} v_{tt}(x, t) = v_{xx}(x, t), x \in (0, 1), t > 0, \\ v_x(0, t) = v_t(1, t) = 0, t \geq 0 \end{cases} \quad (41)$$

which can be rewritten as an evolution equation in \mathcal{H}

$$\frac{d}{dt}(v, v_t)^\top = A_0(v, v_t)^\top \quad (42)$$

where

$$\begin{cases} A_0(f, g)^\top = (g, f'')^\top, \forall (f, g)^\top \in D(A_0), \\ D(A_0) = \{(f, g)^\top \in H^2(0, 1) \times H^1(0, 1) \mid A_0(f, g)^\top \in \mathcal{H}, \\ f'(0) = 0, g(1) = 0\}. \end{cases} \quad (43)$$

It is a trivial exercise to show that A_0 is skew self-adjoint: $A_0^* = -A_0$. Compute the eigenvalues $(\lambda_0, \lambda_n, \bar{\lambda}_n)$, and the eigenfunctions $(\psi_n, \lambda_n \psi_n)$ corresponding to λ_n of A_0 , to get

$$\begin{cases} \lambda_0 = 0, \psi_0 = 1, \\ \lambda_n = i(n - \frac{1}{2})\pi, \psi_n = \frac{\cos((n - \frac{1}{2})\pi x)}{(n - \frac{1}{2})\pi}, n = 1, 2, \dots \end{cases} \quad (44)$$

A direct computation shows that $1 \in \rho(A_0)$. Actually, we have

$$\begin{aligned} (I - A_0)^{-1} \begin{pmatrix} f \\ g \end{pmatrix} &= \begin{pmatrix} \Theta \\ \Theta - f \end{pmatrix}, \\ \Theta &= C_3(e^x + e^{-x}) \\ &\quad - \frac{1}{2} \int_0^x (e^{x-y} - e^{-(x-y)})(f(y) + g(y)) dy, \\ C_3 &= \frac{(e + e^{-1})^{-1}}{2} \int_0^1 (e^{1-y} - e^{-(1-y)})(f(y) + g(y)) dy \\ &\quad + (e + e^{-1})^{-1} f(1), \forall \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{H}. \end{aligned} \quad (45)$$

So $(I - A_0)^{-1}$ is compact on \mathcal{H} by the Sobolev embedding theorem. It follows from a general result in functional analysis that the eigenfunctions $(\psi_0, \lambda_0 \psi_0) \cup \{(\psi_n, \lambda_n \psi_n)\}_{n=1}^\infty \cup \{(\bar{\psi}_n, \bar{\lambda}_n \bar{\psi}_n)\}_{n=1}^\infty$ form an orthonormal basis for \mathcal{H} . Decompose \mathcal{H} as

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_1 + \mathcal{H}_2, \quad \mathcal{H}_1 \perp \mathcal{H}_2, \quad \mathcal{H}_1 = \text{span}\{(\psi_0, \lambda_0 \psi_0)\}, \\ \mathcal{H}_2 &= \text{span}\{(\psi_n, \lambda_n \psi_n)\}_{n=1}^\infty \cup \{(-\bar{\psi}_n, \bar{\lambda}_n \bar{\psi}_n)\}_{n=1}^\infty\}. \end{aligned}$$

By the norm defined in (5), $\mathcal{H}_1 = \text{span}\{(1, 0)\}$, and $\mathcal{H}_2 = \{f \in H^1(0, 1) \mid f(1) = 0\} \times L^2(0, 1)$. Define a map

$$\mathbb{P}(f(x), g(x)) = (f'(x), g(x)), \forall (f, g) \in \mathcal{H}_2.$$

Then \mathbb{P} is an isometric from \mathcal{H}_2 to $(L^2(0, 1))^2$. It is further seen that $\mathbb{P}(\psi_n, \lambda_n \psi_n) = F_n, \mathbb{P}(-\bar{\psi}_n, -\bar{\lambda}_n \bar{\psi}_n) = -\bar{F}_n$, where

$$F_n = \left(-\sin\left(n - \frac{1}{2}\right)\pi x, i \cos\left(n - \frac{1}{2}\right)\pi x \right), \quad n = 1, 2, \dots \quad (46)$$

Since $\{(\psi_n, \lambda_n \psi_n)\}_{n=1}^\infty \cup \{(-\bar{\psi}_n, \bar{\lambda}_n \bar{\psi}_n)\}_{n=1}^\infty$ form an orthonormal basis for \mathcal{H}_2 , it is equivalent to saying that $\{F_n\}_{n=1}^\infty \cup \{-\bar{F}_n\}_{n=1}^\infty$ form an orthonormal basis for $(L^2(0, 1))^2$.

Now we go back to system (40). When $0 < c < 1$, the eigenvalues $(\mu_0, \mu_n, \overline{\mu_n})$ and the eigenfunctions $(f_n, \mu_n f_n)$ corresponding to μ_n of A_c are found to be

$$\begin{cases} \mu_0 = -M, f_0(x) = e^{-Mx} + \frac{1-c}{1+c} e^{Mx}, \\ \mu_n = a + ib_n, f_n(x) = \frac{e^{(a+ib_n)x} + \frac{1-c}{1+c} e^{-(a+ib_n)x}}{a+ib_n}, \\ a = \frac{1}{2} \ln \frac{1-c}{1+c}, b_n = (n - \frac{1}{2}) \pi. \end{cases} \quad (47)$$

It is obvious that $(f_n, \lambda_n f_n) \in \mathcal{H}_2$. Let $G_n = \mathbb{P}(f_n, \mu_n f_n) = (f_n^T, \mu_n f_n)$ for $n \geq 1$. Then it is found that

$$\begin{aligned} G_n^T &= i \begin{pmatrix} -e^{ax} - \frac{1-c}{1+c} e^{-ax} & -e^{ax} + \frac{1-c}{1+c} e^{-ax} \\ -e^{ax} + \frac{1-c}{1+c} e^{-ax} & -e^{ax} - \frac{1-c}{1+c} e^{-ax} \end{pmatrix} F_n^T \\ &= iA(x)F_n^T, \\ \overline{G_n}^T &= iA(x)(-\overline{F_n})^T, n = 1, 2, \dots \end{aligned} \quad (48)$$

Hence, $\{(f_n, \mu_n f_n)\}_{n=1}^\infty \cup \{(\overline{f_n}, \overline{\mu_n f_n})\}_{n=1}^\infty$ form a Riesz basis for \mathcal{H}_2 . Hence $(1, 0) \cup \{(f_n, \mu_n f_n)\}_{n=1}^\infty \cup \{(\overline{f_n}, \overline{\mu_n f_n})\}_{n=1}^\infty$ form a Riesz basis for \mathcal{H} . Since $(f_0, \mu_0 f_0) \cup \{(f_n, \mu_n f_n)\}_{n=1}^\infty \cup \{(\overline{f_n}, \overline{\mu_n f_n})\}_{n=1}^\infty$ is ω -linearly independent and is quadratically close to $(1, 0) \cup \{(f_n, \mu_n f_n)\}_{n=1}^\infty \cup \{(\overline{f_n}, \overline{\mu_n f_n})\}_{n=1}^\infty$, it follows from the classical Bari's theorem that $(f_0, \mu_0 f_0) \cup \{(f_n, \mu_n f_n)\}_{n=1}^\infty \cup \{(\overline{f_n}, \overline{\mu_n f_n})\}_{n=1}^\infty$ form a Riesz basis for \mathcal{H} . This implies that A_c generates a C_0 -semigroup $e^{A_c t}$ on \mathcal{H} , and that the spectrum-determined growth condition is true for $e^{A_c t}$. Moreover, since $a = \ln(1-c)/(1+c) < 0$, $e^{A_c t}$ is exponentially stable.

Next, we show that B is admissible for $e^{A_c t}$, or equivalently, B^* is admissible for $e^{A_c^* t}$. To this end, we find the dual system of (40) to be

$$\begin{cases} v_{tt}^*(x, t) = v_{xx}^*(x, t) - M(v_{xt}^*(1, t) + v_t^*(1, t)), \\ x \in (0, 1), t > 0, \\ v_x^*(0, t) = c[v_t^*(0, t) + M(v_x^*(1, t) + v^*(1, t))], t \geq 0, \\ v_t^*(1, t) = -M(v_x^*(1, t) + v^*(1, t)), t \geq 0, \\ y_v^*(t) = \frac{q+c}{q^2-1}(v_x^*(1, t) + v^*(1, t)), t \geq 0. \end{cases} \quad (49)$$

Define functions

$$\begin{aligned} E^*(t) &= \frac{1}{2} \int_0^1 [v_x^{*2}(x, t) \\ &\quad + (v_t^*(x, t) + M(v_x^*(1, t) + v^*(1, t)))^2] dx \end{aligned} \quad (50)$$

and

$$\rho^*(t) = \int_0^1 x v_x^*(x, t) (v_t^*(x, t) + M(v_x^*(1, t) + v^*(1, t))) dx. \quad (51)$$

Similar to (23), we can show from (50) and (51) that

$$\begin{aligned} &\int_0^T [v_x^*(1, t) + v^*(1, t)]^2 dt \\ &\leq 2 \int_0^T [v_x^{*2}(1, t) + v^{*2}(1, t)] dt \\ &\leq C_T \left[E^*(0) + \frac{1}{2} v^{*2}(1, 0) \right] \text{ for some } C_T > 0. \end{aligned} \quad (52)$$

B^* is then admissible for $e^{A_c^* t}$ if we can show that $B^* A_c^{*-1}$ is bounded in \mathcal{H} . This is trivial since

$$A_c^{*-1} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \Pi(x) \\ -f(x) + f(1) \end{pmatrix},$$

$$\begin{aligned} \Pi(x) &= -c(f(1) - f(0))(2-x) + \int_0^1 g(x) dx \\ &\quad - \frac{1}{M} f(1) + \int_x^1 \int_0^y g(s) ds dy, \\ B^* A_c^{*-1} \begin{pmatrix} f \\ g \end{pmatrix} &= -\frac{1}{M} f(1). \end{aligned} \quad (53)$$

Since A_c generates an exponential stable C_0 -semigroup and B is admissible to $e^{A_c t}$, for any initial value $(w(\cdot, 0), w_t(\cdot, 0))^T \in \mathcal{H}_2$, it follows that there exists a unique solution to (40) provided that $\tilde{d} \in L_{loc}^2(0, \infty)$, which can be written as ([15])

$$\begin{aligned} \begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix} &= e^{A_c t} \begin{pmatrix} w(\cdot, 0) \\ w_t(\cdot, 0) \end{pmatrix} \\ &\quad + \frac{q+c}{q^2-1} \int_0^t e^{A_c(t-s)} B \tilde{d}_\varepsilon(s) ds. \end{aligned} \quad (54)$$

This is the first part of the theorem. Now we show the second part.

For any given $\varepsilon_0 > 0$, by assumption, we may assume that $|\tilde{d}_\varepsilon(t)| < \varepsilon_0$ for all $t > t_0$, $0 < \varepsilon < \varepsilon_1$ for some $t_0 > 0$ and $\varepsilon_1 > 0$. We can write (54) as

$$\begin{aligned} \begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix} &= e^{A_c t} \begin{pmatrix} w(\cdot, 0) \\ w_t(\cdot, 0) \end{pmatrix} \\ &\quad + \frac{q+c}{q^2-1} e^{A_c(t-t_0)} \int_0^{t_0} e^{A_c(t_0-s)} B \tilde{d}_\varepsilon(s) ds \\ &\quad + \frac{q+c}{q^2-1} \int_{t_0}^t e^{A_c(t-s)} B \tilde{d}_\varepsilon(s) ds. \end{aligned} \quad (55)$$

Since the admissibility of B implies that

$$\begin{aligned} \left\| \int_0^t e^{A_c(t-s)} B \tilde{d}_\varepsilon(s) ds \right\|_{\mathcal{H}}^2 &\leq C_t \|\tilde{d}_\varepsilon\|_{L_{loc}^2(0,t)}^2 \\ &\leq t^2 C_t \|\tilde{d}_\varepsilon\|_{L^\infty(0,t)}^2, \forall \tilde{d}_\varepsilon \in L^\infty(0, \infty) \end{aligned} \quad (56)$$

for some constant C_t that is independent of \tilde{d} , and since $e^{A_c t}$ is exponentially stable, it follows from Proposition 2.5 of [15] that

$$\begin{aligned} \left\| \int_{t_0}^t e^{A_c(t-s)} B \tilde{d}_\varepsilon(s) ds \right\| &= \left\| \int_0^t e^{A_c(t-s)} B (0 \diamond \tilde{d}_\varepsilon)(s) ds \right\| \\ &\leq L \|\tilde{d}_\varepsilon\|_{L^\infty(0, \infty)} \leq L \varepsilon_0 \end{aligned} \quad (57)$$

where L is a constant that is independent of \tilde{d} , and

$$(u \diamond v)(t) = \begin{cases} u(t), 0 \leq t \leq \tau, \\ v(t-\tau), t > \tau. \end{cases} \quad (58)$$

Suppose that $\|e^{A_c t}\| \leq L_0 e^{-\omega t}$ for some $L_0, \omega > 0$. By (55), (56), and (57), we have

$$\begin{aligned} \left\| \begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix} \right\| &\leq L_0 e^{-\omega t} \left\| \begin{pmatrix} w(\cdot, 0) \\ w_t(\cdot, 0) \end{pmatrix} \right\| \\ &\quad + L_0 C_{t_0} \frac{q+c}{q^2-1} e^{-\omega(t-t_0)} \|\tilde{d}_\varepsilon\|_{L^\infty(0,t_0)} + L \frac{q+c}{q^2-1} \varepsilon_0. \end{aligned} \quad (59)$$

As $t \rightarrow \infty$, the first two terms of (59) tend to zero. The result is then proved by the arbitrariness of ε_0 . ■

Remark 1: When $c > 1$, instead of (41), we choose the corresponding auxiliary system as follows:

$$\begin{cases} v_{tt}(x, t) = v_{xx}(x, t), x \in (0, 1), t > 0, \\ v_t(0, t) = v_t(1, t) = 0, t \geq 0. \end{cases} \quad (60)$$

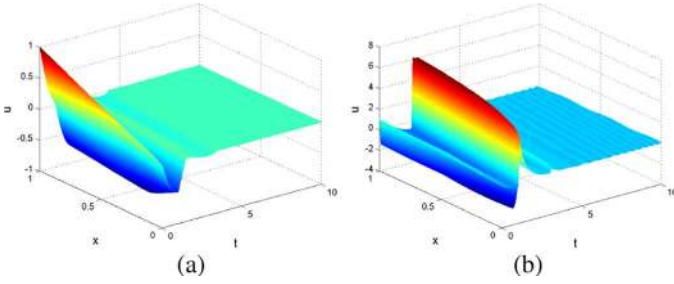


Fig. 1. Displacements for disturbance with unbounded derivative (a) SMC (b) ADRC.

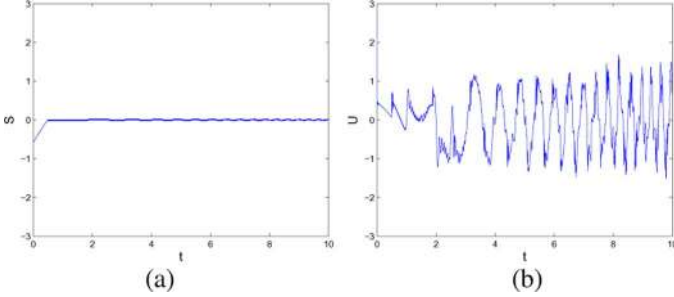


Fig. 2. SMC for disturbance with unbounded derivative (a) Sliding surface (b) Controller.

Going back to the original system, we have the second main result of this technical note.

Theorem 2: Suppose that d and \dot{d} are uniformly bounded measurable and $(u(\cdot, 0), u_t(\cdot, 0))^\top \in \mathcal{H}$. Let

$$S_U(t) = -\frac{1+qc}{q^2-1}u(1,t) + \frac{q(q+c)}{q^2-1}u(0,t) - \frac{q+c}{q^2-1} \int_0^1 u_t(x,t)dx. \quad (61)$$

Then the closed-loop system of (1) described by

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t), & x \in (0,1), t > 0, \\ u_x(0,t) = -qu_t(0,t), & t \geq 0, \\ u_x(1,t) = -\frac{1+qc}{q+c}u_t(1,t) + M\frac{q^2-1}{q+c}S_U(t) + d(t) - \hat{d}_\varepsilon(t), & t \geq 0, \\ \dot{\hat{y}}_\varepsilon(t) = -\frac{1}{\varepsilon}(\hat{y}_\varepsilon(t) - S_U(t)) - MS_U(t), & t \geq 0, \\ \dot{\hat{d}}_\varepsilon(t) = \frac{1}{\varepsilon^2}\frac{q^2-1}{q+c}(\hat{y}_\varepsilon(t) - S_U(t)), & t \geq 0 \end{cases} \quad (62)$$

admits a unique solution $(u, u_t)^\top \in C(0, \infty; \mathcal{H})$, and (u, u_t) can reach arbitrary vicinity of zero as $t \rightarrow \infty, \varepsilon \rightarrow 0$ in (62).

IV. NUMERICAL SIMULATION

In this section, we give some simulation results to illustrate the effects of both the SMC and ADRC.

Consider systems (14) with the equivalent control U_{eq} when $S_W(t) = 0$, and (62). Let the parameters be $q = 2, c = 1.2, \varepsilon = 1/50, K = 1.2, M = 5$, and the disturbance $d(t) = \sin t^2$. The initial conditions are

$$u(x,0) = x, u_t(x,0) = -x. \quad (63)$$

Note that $|d| < 1$ is bounded but $|\dot{d}|$ is unbounded.

We apply the finite difference method to compute the displacement. Fig. 1(a) and (b) show the displacements of system (14) and (62) respectively. Here the steps of space and time are taken as 0.001 and 0.0005, respectively. It is seen from Fig. 1 that system (14) converges

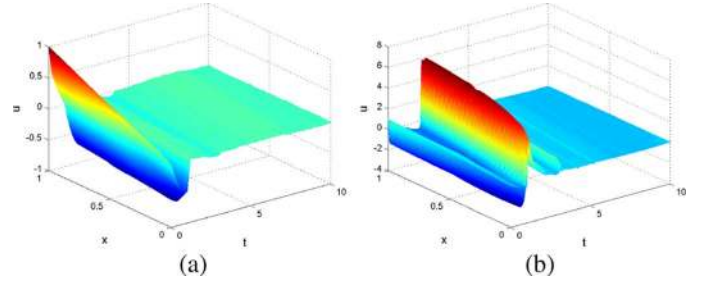


Fig. 3. Displacements for disturbance with bounded derivative (a) SMC (b) ADRC.

smoothly, but system (62) is oscillatory around the equilibrium before $t = 10$. It shows that ADRC is not adequate to deal with the disturbance with unbounded derivative. The corresponding control and sliding surface are plotted in Fig. 2 in this case.

If the disturbance is described as $d(t) = \sin t$. Then both d and \dot{d} are uniformly bounded. Take the steps of space and time by 0.005 and 0.001, respectively, which are larger than that in Fig. 1. We obtain the displacements of the system (14) and (62), which are shown in Fig. 3(a) and (b), respectively.

We point out that a chattering behavior is observed in Fig. 3(a) (see also the sliding surface in Fig. 2(a)), although it is convergent. On the other hand, the displacement in Fig. 3(b) is quite smooth. The results show that the ADRC yields more satisfactory performance than the SMC in dealing with the disturbance with bounded derivative.

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