Sliding Mode Robot Control with Exponential Reaching Law

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Abstract— **In this paper, sliding mode control is applied on Multi Input / Multi Output (MIMO) nonlinear systems. A novel approach is proposed that allows chattering reduction on control input, while keeping high tracking performance of the controller in steady state regime. This approach consists of designing a nonlinear reaching law by using an exponential function that dynamically adapts to the variations of the controlled system. Experimental study was focused on a MIMO modular robot arm. Experimental results are presented to show the effectiveness of the proposed approach, regarding especially the chattering reduction on control input in steady state regime.**

***Index Terms*—Sliding Mode, Control, Chattering, Exponential Reaching Law, MIMO, Modular robot, Nonlinear.**

I. Introduction

MANY nonlinear control techniques can be found in literature; among them we find feedback linearization [1], Fuzzy feedback linearization [2], backstepping [3]; [4], forwarding control [5] or adaptive-backstepping [6] and sliding mode control [7] which belongs to the family of Variable Structure Controllers (VSC) [8]. Sliding mode control is based on the design of a high speed switching control law that drives the system's trajectory onto a user-chosen hyper plane in the state space, also known as sliding surface. Sliding mode control is an interesting approach thanks to its robustness and the simplicity of the derived control law. The key idea of the sliding mode theory is to bring the study of an n^{th} order system to that of a first order one, by considering only the sliding function and its derivative as the new state variables.

The robustness of sliding mode control can theoretically ensure perfect tracking performance despite parameters or model uncertainties. Thus, as far as robustness is concerned, sliding mode control is ahead of other nonlinear techniques. In [9], the performance of a sliding mode controller is studied using a hybrid controller applied to Induction Motors via Sampled Closed Representations. The results were very conclusive regarding the effectiveness of the sliding mode approach. The backstepping technique [3],[10] is also a well known nonlinear control approach based on the progressive construction of Lyapunov functions. However, backstepping control can only be applied to special classes of systems with a triangular dynamics structure, while sliding mode control can be applied to a more general class of nonlinear systems and has the ability to consider robustness issues for modeling uncertainties and disturbances. In addition, the ability to specify performance directly makes sliding mode control attractive from the design perspective.

Nonetheless, this approach isn't flawless; indeed, in real time applications, the switching control law in sliding mode is not instantaneous and the sliding surface is not rigorously known. This leads to a high control activity, known as chattering. In most systems, the chattering phenomenon is undesirable, because it can excite high frequency dynamics which could be the cause of severe damage. Thus, many alternatives have been proposed to overcome this phenomenon. Floquet et al. [11] proposed a higher order sliding mode control to reduce the chattering. This approach was also applied to trajectory tracking of robot by Hamerlain et al. [12]. Bartolini et al. [13] and [14] proposed a second order sliding mode control in order to eliminate the discontinuous term in the control input (also treated in [15]). Moura and Olgac [16] proposed a VSC with a non-sliding regime, thus eliminating high frequency oscillations. Camacho et al. [17] used a tuned sigmoid function instead of the *sign* function, in order to reduce chattering effects. An application of fuzzy sliding mode control applied to a 2 DOF can be found in [18] and in [19] the fuzzy sliding mode approach is applied to a six-phase induction machine. A neuro-fuzzy sliding mode applied to induction machine can also be found in [20]. Finally, a neural network sliding mode approach is proposed in [21] to control a robot manipulator. In this particular case, the nonlinear dynamics of the robot is approximated using a radial basis function neural network.

An interesting approach in literature for chattering reduction is to change the reaching law by making the discontinuous gain k a function of S. Gao and Hung [22] based their study on this approach to reduce or even eliminate chattering on control input. One of the reaching laws they studied is based on power rate reaching strategy, and uses the following reaching law:

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$$\dot{S} = -k \cdot |S|^{\alpha} sign(S) \quad , \quad 0 \le \alpha < 1 \tag{1}$$

However, in the above reaching law, the term $|S|^{\alpha}$ rapidly decreases because of the fractional power α, thus reducing the robustness of the controller near the sliding surface, and also increasing the reaching time.
In order to propose a solution to the above problems, this paper introduces a new reaching law containing an exponential term functions of the sliding surface *S*. This reaching law is able to deal with the chattering/tracking performance dilemma. The exponential term smoothly adapts to the variations of *S*.

The rest of the paper is organized as follows. Section II exposes the problem formulation and motivation. The proposed exponential reaching law (ERL) is introduced in section III. Section IV gives a general guideline for choosing ERL parameters for a system with uncertainties. Section V generalizes sliding mode control to MIMO systems. In section VI, the new approach is tested experimentally on a robot arm, and real time results are compared to conventional sliding mode approach. Section VII finally concludes the paper.

II. Problem Formulation and Motivation

A complete study of sliding mode theory can be found in [3]. In this section, we briefly present its basic theory in which we emphasize on the most important advantages and its major drawbacks. These limitations motivate our research for a new reaching law approach that will be introduced in the next section. To explain sliding mode approach, we consider the following second order nonlinear system:

$$\ddot{x} = f(x,\dot{x}) + b(x,\dot{x}) \cdot u \tag{2}$$

Where *f* and *b* are both nonlinear functions in terms of x and $\dot{x}$, and *b* is invertible. Let x_d be the reference trajectory and $e = x - x_d$ the tracking error which converges to zero. The first step in sliding mode control is to choose the switching function *S* in terms of the tracking error. The typical choice of *S* in this particular case is:

$$S = \lambda e + \dot{e} \tag{3}$$

When the sliding surface is reached, the tracking error converges to zero as long as the error vector stays on the surface. The convergence rate is in direct relation with the value of λ. Figure 1 shows how this mechanism takes place in the phase plane. From figure 1, it can be seen that there are two 'modes' in sliding mode approach. The first mode, called reaching mode, is the step in which the error vector $(e,\dot{e})$ is attracted to the switching surface *S*=0. In the second mode, also known as sliding mode, the error vector 'slides' on the surface until it reaches the equilibrium point $(0,0)$.

Having chosen at this stage the sliding surface, the next step would be to choose the control law *u* that will allow error vector $(e,\dot{e})$ to reach the sliding surface. To do so, the control law should be designed in such a way that the following condition, also named reaching condition, is met:

$$S \cdot \dot{S} < 0 \quad \forall t \tag{4}$$

In order to satisfy condition (4) $\dot{S}$ is typically chosen as follows:

$$\dot{S} = -k \cdot sign(S) \quad \forall t \, , \, k>0 \tag{5}$$

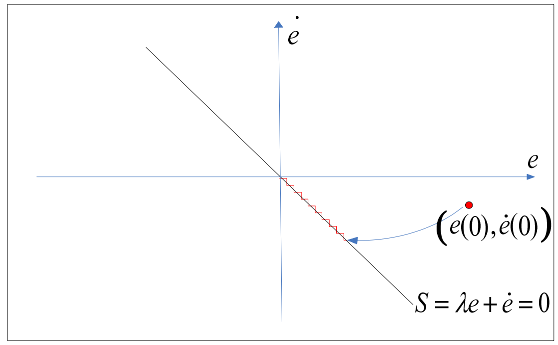


Fig. 1. Sliding mode mechanism in phase plane

Expression (5) is also called reaching law. Integrating equation (5) with respect to time yields the reaching time t_r, which is the required time for error vector $(e,\dot{e})$ to reach S:

$$t_r = \frac{|S(0)|}{k} \tag{6}$$

One can see from equation (6) that the reaching speed is increased with high values of *k*.
Taking into account the previous conditions, it is easy to show that the control input *u* has the following form [23]:

$$u = u_{eq} + u_{disc} \tag{7}$$

where

$$\begin{aligned} u_{eq} &= b^{-1}\left(\ddot{x}_d - \lambda \dot{e} - f\right) \\ u_{disc} &= -b^{-1}k \cdot sign\left(S\right) \end{aligned} \tag{8}$$

This control law shows that the control input contains the discontinuous term $b^{-1}k \cdot sign(S)$. This leads to the phenomenon of chattering. One can see that the chattering level is directly controlled by *k*. Therefore, the following dilemma arises: *In order to have a faster reaching time, a good robustness and tracking performance, k must be increased, however this will directly increase the chattering level on the control input.* In order to solve this dilemma, the interdependence between the reaching time and the chattering level should be removed. The exponential reaching law, presented in the next section, is designed to solve this problem.

III. Sliding Mode with Exponential Reaching Law (ERL)

The reaching law proposed in this paper is based on the choice of an exponential term that adapts to the variations of the switching function. This reaching law is given by:

$$\dot{S} = -\frac{k}{N(S)} \cdot sign(S), \; k > 0 \tag{9}$$

where

$$N(S) = \delta_0 + (1 - \delta_0) e^{-\alpha |S|^p} \tag{10}$$

δ_0 is a strictly positive offset less than 1, p is a strictly positive integer, and α is also strictly positive. Note that the ERL given by equation (9) does not affect the stability of the control, because $N(S)$ is always strictly positive. From the reaching law stated in equation (9), one can see that if $|S|$ increases, $N(S)$ approaches δ_0, and therefore $\mathrm{k}/\mathrm{N}(\mathrm{S})$ converges to k/δ_0, which is greater than *k*. This means that $\mathrm{k}/\mathrm{N}(\mathrm{S})$ increases in reaching phase, and consequently the attraction of the sliding surface will be faster. On the other hand, if $|S|$ decreases then $N(S)$ approaches 1 and $\mathrm{k}/\mathrm{N}(\mathrm{S})$ converges to *k*. This means that when the system approaches the sliding surface, $\mathrm{k}/\mathrm{N}(\mathrm{S})$ gradually decreases in order to limit the chattering. Therefore, the ERL allows the controller to dynamically adapt to the variations of the switching function by letting $\mathrm{k}/\mathrm{N}(\mathrm{S})$ to vary between *k* and k/δ_0.

Remark: If δ_0 is chosen to be equal to 1, the reaching law of equation (9) becomes identical to that of equation (5). Therefore, the conventional reaching law becomes a particular case of the proposed approach.

Proposition 1: For the same gain *k*, the ERL given by equation (9) ensures a reaching time always smaller than that of the conventional reaching law expressed in equation (5).

Proof: Let $t_r^{'}$ be the reaching time for expression (9). Using the same relation, one has:

$$\dot{S}\left[\delta_0 + (1-\delta_0) e^{-\alpha |S|^p}\right] = -k \cdot sign(S) \tag{11}$$

Integrating equation (11) between 0 and t_r', and noticing that $S(t_r') = 0$, yields:

$$t_r' = \frac{1}{k}\left(\delta_0 |S(0)| + (1-\delta_0) \int_0^{S(0)} sign(S) e^{-\alpha |S|^p} \cdot dS\right) \tag{12}$$

If $S \le 0$ for $t \le t_r'$, then

$$\int_0^{S(0)} sign(S) e^{-\alpha |S|^p} dS = -\int_0^{S(0)} e^{-\alpha |S|^p} dS = \int_0^{-S(0)} e^{-\alpha |S|^p} dS \tag{13}$$

On the other hand, if $S \ge 0$ for $t \le t_r'$, then

$$\int_0^{S(0)} sign(S) e^{-\alpha |S|^p} dS = \int_0^{S(0)} e^{-\alpha |S|^p} dS \tag{14}$$

Therefore, one can combine the last two expressions into the following:

$$\int_0^{S(0)} sign(S) e^{-\alpha |S|^p} dS = \int_0^{|S(0)|} e^{-\alpha |S|^p} dS \tag{15}$$

Thus, the expression of t_r' given by equation (12), can be rewritten as follows:

$$t_r' = \frac{1}{k}\left(\delta_0 |S(0)| + (1-\delta_0) \int_0^{|S(0)|} e^{-\alpha |S|^p} dS\right) \tag{16}$$

Now subtracting equation (6) from equation (16) yields:

$$t_r' - t_r = \frac{1}{k}\left(-(1-\delta_0)|S(0)| + (1-\delta_0) \int_0^{|S(0)|} e^{-\alpha |S|^p} dS\right) \tag{17}$$

which can also be written as

$$t_r' - t_r = \frac{(1-\delta_0)}{k}\left(\int_0^{|S(0)|} \left[e^{-\alpha |S|^p} - 1\right] dS\right) \tag{18}$$

However, the term $e^{-\alpha |S|^p} - 1$ is always negative, which implies that $t_r' - t_r \le 0$.

For the particular case of $\mathrm{p} = 1$, the expression of t_r' can be given by an analytical form. Indeed, considering equation (16) for $p = 1$ yields:

$$t_r' = \frac{1}{k}\left(\delta_0 |S(0)| + \frac{(1-\delta_0)}{\alpha}\left[1 - e^{-\alpha |S(0)|}\right]\right) \tag{19}$$

Proposition 1 shows that ERL increases the reaching speed of the sliding function, while keeping the same gain k (i.e. the same chattering level). Also, for the same reaching time, the gain *k* needed for reaching law of equation (9) is smaller than the *k* needed for equation (5). Therefore, for the same reaching speed, the proposed approach reduces chattering, which is a substantial asset over the conventional sliding mode control.

IV. Choice of ERL Parameters

This section gives a general idea about the role of ERL parameters and the way they can be chosen in the control design. It is shown how system uncertainties can affect the choice of ERL parameters to maintain the robustness of the controller. A similarity with the boundary layer approach is also observed.

A System without parameter uncertainties

In the case where the system has no parameter uncertainties, the most important factor for choosing ERL parameters is the desired reaching time t_{rd}. From equation (16), it can be shown (proof in Appendix 1) that the reaching time t'_r for ERL approach verifies:

$$t'_r \le \frac{\delta_0}{k}|S(0)| + \frac{(1-\delta_0)}{k\alpha^{1/p}} \tag{20}$$

Therefore, if we choose

$$\frac{\delta_0}{k}|S(0)| + \frac{(1-\delta_0)}{k\alpha^{1/p}} = t_{rd} \tag{21}$$

We can guaranty that the reaching time t'_r is less than the desired reaching time t_{rd}. Moreover, if we choose α such that

$$\alpha >> \left(\frac{1-\delta_0}{\delta_0\,|S(0)|}\right)^{1/p} \tag{22}$$

Equation (21) can be rewritten as follows:

$$k \approx \delta_0 \frac{|S(0)|}{t_{rd}} \tag{23}$$

Whereas in conventional sliding mode control,

$$k = \frac{|S(0)|}{t_{rd}} \tag{24}$$

Therefore, gain k can be tuned to a desired value with δ_0. Thus, without any parameter uncertainty, the choice of the ERL parameters is only bound by relations (22) and (23).

B System with bounded uncertainties

Considering now a system with bounded uncertainties will obviously add more constraints in choosing ERL parameters. For simplification purposes, consider system (2) with $b(x,\dot{x}) = 1$:

$$\ddot{x} = f(x,\dot{x}) + u \tag{25}$$

Where $f(x,\dot{x})$ includes modeling uncertainties. Let $\hat{f}(x,\dot{x})$ be the estimate of $f(x,\dot{x})$, and L_{MAX} be the superior bound of the error between f and $\hat{f}$:

$$L_{MAX} = \sup_t \left| f(x,\dot{x}) - \hat{f}(x,\dot{x}) \right| \tag{26}$$

With the same sliding function chosen as in (3), the conventional sliding mode control law is given by:

$$u(t) = -\lambda(\dot{x} - \dot{x}_d) + \ddot{x}_d - \hat{f}(x,\dot{x}) - k \cdot sign(S) \tag{27}$$

This yield

$$\dot{S} = \left(f(x,\dot{x}) - \hat{f}(x,\dot{x})\right) - k \cdot sign(S) \tag{28}$$

According to (28), in order for the sliding function to converge to zero, gain k must verify:

$$k > \left| f(x,\dot{x}) - \hat{f}(x,\dot{x}) \right|, \forall t \tag{29}$$

Since k is a constant in conventional sliding mode, (29) implies that

$$k > L_{MAX} \tag{30}$$

Condition (30) is aggressive in the sense that gain k is over dimensioned to insure the convergence of the sliding function. With ERL approach, (30) can be written as:

$$k > \delta_0 \cdot L_{MAX} + (1-\delta_0) \cdot e^{-\alpha|S|^p} \cdot L_{MAX} \tag{31}$$

From (31), one can see that k has to be at least greater than $\delta_0 \cdot L_{MAX}$. By choosing this minimum requirement for k, and solving for S in (31) gives the following:

$$|S| > \sqrt[p]{\frac{\ln\left(\frac{L_{MAX}(1-\delta_0)}{k - \delta_0 \cdot L_{MAX}}\right)}{\alpha}}, \; k > \delta_0 \cdot L_{MAX} \tag{32}$$

Relation (32) shows that in order to meet condition (31), sliding function S has to vary in a boundary of width W, given by:

$$W = \sqrt[p]{\frac{\ln\left(\frac{L_{MAX}(1-\delta_0)}{k - \delta_0 \cdot L_{MAX}}\right)}{\alpha}} \tag{33}$$

W is directly controlled with α.

At this stage a similarity can be drawn between ERL and conventional boundary layer approach widely discussed in scientific literature. Boundary layer approach consists of replacing discontinuous term $sign(S)$ with $sat(S/\phi)$:

$$sat(S/\phi) = \begin{cases} -1 & for\ S \le -\phi \\ S/\phi & for -\phi \le S \le \phi \\ 1 & for\ S \ge \phi \end{cases} \tag{34}$$

The boundary width for the *sat* function is given by:

$$W = \frac{\phi \cdot L_{MAX}}{k}, \; k > L_{MAX} \tag{35}$$

The width in this case is directly controlled by ϕ, similarly to α. However, gain k has still to be larger than L_{MAX}, and the reaching time for the boundary layer approach is not finite. Hence, the superiority of ERL approach lies in the fact that it introduces independent and tunable parameters that meet the reaching time, the bounded uncertainties condition and the boundary layer width for the latter, without having to over dimension gain k.

Combining the constraints in paragraphs A and B leads to the following relations which represent a general guideline on how ERL parameters can be chosen for the controller's design:

$$\begin{gathered} \frac{k}{\delta_0} > L_{MAX} \\ \alpha \ge \frac{\ln\left(\frac{L_{MAX}(1-\delta_0)}{k - \delta_0 \cdot L_{MAX}}\right)}{W^p} \; and \; \alpha >> \left(\frac{1-\delta_0}{\delta_0|S(0)|}\right)^{1/p} \end{gathered} \tag{36}$$

Figure 2 shows that in order to keep the same reaching time t_r, the ERL can change the concavity of the switching function in terms of time, by tuning the parameters k and δ_0. Note that if α is also chosen according to (22), then

$$\frac{k_1}{\delta_{01}} = \frac{k_2}{\delta_{02}} = \frac{k_3}{\delta_{03}} = k = \frac{|S(0)|}{t_r}, \text{with } \delta_{03} \le \delta_{02} \le \delta_{01}$$

This means that when δ_0 is decreased; gain k is decreased in the same proportion yielding therefore less chattering in sliding mode. Decrease of gain k can be graphically interpreted by smaller slopes of the switching function when the sliding surface is reached. Note that the conventional reaching law is obtained for $\delta_0 = 1$.

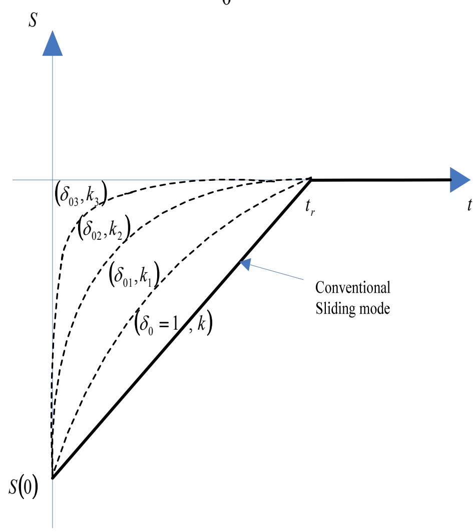


Fig. 2. Switching function with E.R.L. for different values of *k* and δ_0

V. Sliding Mode Control for MIMO Systems

In this section we extend the study of sliding mode control to Multi-Input/Multi-Output (MIMO) systems. We particularly focus on square systems of the form [14]:

$$x_i^{(ni)} = f_i(X) + \sum_{j=1}^{m} b_{ij}(X) u_j \ , i = 1, \ldots, m \tag{37}$$

Systems described by equation (37) are said square systems, because the number of control inputs u_j is equal to that of the independent output variables x_i and can be expressed in the following matrix form:

$$X_n = \Phi(X) + \mathrm{B}(X) \cdot U \tag{38}$$

where

$X_n = \left[x_1^{(n1)} \quad x_2^{(n2)} \ldots \quad x_i^{(ni)} \ldots \quad x_m^{(nm)} \right]^T$,

$\Phi = \left[f_1 \quad f_2 \ldots \quad f_i \ldots \quad f_m \right]^T$,

$\mathrm{B} = \left[b_{ij} \right], i = 1, \ldots, m \ and \ j = 1, \ldots, m$,

$U = \left[u_1 \quad u_2 \ldots \quad u_i \ldots \quad u_m \right]^T$, and

$$X = \Big[\underbrace{x_1 \quad x_1^{(1)} \quad x_1^{(n1-1)}}_{n1} \quad \underbrace{x_2 \quad x_2^{(1)} \quad x_2^{(n2-1)}}_{n2} \quad \cdots \quad \underbrace{x_i \quad x_i^{(1)} \quad x_i^{(ni-1)}}_{ni} \quad \cdots \quad \underbrace{x_m \quad x_m^{(1)} \quad x_m^{(nm-1)}}_{nm} \Big]^T$$

Note that

$\dim(\mathrm{X_n}) = \dim(\Phi) = \dim(\mathrm{U}) = (\mathrm{m} \times 1)$

and $\dim(\mathrm{X}) = \left(\left(\sum_{k=1}^{m} \mathrm{n_k} \right) \times 1 \right)$.

Having *m* independent output variables to control in this case, we therefore need to design *m* independent sliding functions for each of the output variables. Let X_d be the desired reference vector defined as follows:

$$X_d = \Big[\underbrace{x_{d1} \quad x_{d1}^{(1)} \quad x_{d1}^{(n1-1)}}_{n1} \quad \underbrace{x_{d2} \quad x_{d2}^{(1)} \quad x_{d2}^{(n2-1)}}_{n2} \quad \cdots \quad \underbrace{x_{di} \quad x_{di}^{(1)} \quad x_{di}^{(ni-1)}}_{ni} \quad \cdots \quad \underbrace{x_{dm} \quad x_{dm}^{(1)} \quad x_{dm}^{(nm-1)}}_{nm} \Big]^T$$

Let also $E_i = \Big[\underbrace{x_i - x_{di} \quad x_i^{(1)} - x_{di}^{(1)} \quad x_i^{(ni-1)} - x_{di}^{(ni-1)}}_{ni} \Big]^T$ be the i[th] error vector corresponding to the i[th] independent variable x_i. We can build the *m* sliding functions as follows:

$$S_i = \Lambda_i^T \cdot E_i \ , i = 1 \ldots m \tag{39}$$

where $\Lambda_i = \left[\lambda_{1,i}, \lambda_{2,i}, \ldots, \lambda_{ni,i} \right]^T$. Note that all Λ_i have to be chosen such that the sliding surfaces $S_i = 0$ are stable differential equations that allow the error vectors to converge to zero. Let us compute $\dot{S}_i$ from equation (39):

$$\begin{aligned} \dot{S}_i &= \Lambda_i^T \cdot \dot{E}_i \\ &= \sum_{k=1}^{ni-1} \lambda_{k,i} \left(x_i^{(k)} - x_{di}^{(k)} \right) + \lambda_{ni,i} \cdot \left(x_i^{(ni)} - x_d^{(ni)} \right) \quad i = 1, \ldots, m \end{aligned} \tag{40}$$

Let $v_i = \sum_{k=1}^{ni-1} \lambda_{k,i} \left(x_i^{(k)} - x_{di}^{(k)} \right) - \lambda_{ni,i} \cdot x_d^{(ni)}$ and consider the following notations that apply for the rest of the development in this section:

$$\Sigma = \left[S_1 \quad S_2 \ldots \quad S_m \right]^T, \dot{\Sigma} = \left[\dot{S}_1 \quad \dot{S}_2 \ldots \quad \dot{S}_m \right]^T$$

$$sign(\Sigma) = \left[sign(S_1) \quad sign(S_2) \ldots \quad sign(S_m) \right]^T,$$

$$V = \left[v_1 \quad v_2 \ldots \quad v_m \right]^T, \Gamma = diag\left(\lambda_{ni,i}, i = 1, \ldots m \right)$$

Equation (40) can therefore be written in the following matrix form:

$$\dot{\Sigma} = V + \Gamma \cdot X_n \tag{41}$$

Finally, the following control law is obtained

$$\mathrm{U} = -(\Gamma \cdot \mathrm{B})^{-1} (\mathrm{V} + \Gamma \cdot \Phi) - (\Gamma \cdot \mathrm{B})^{-1} \mathrm{K}(\Sigma) \cdot \mathrm{sign}(\Sigma) \tag{42}$$

Where

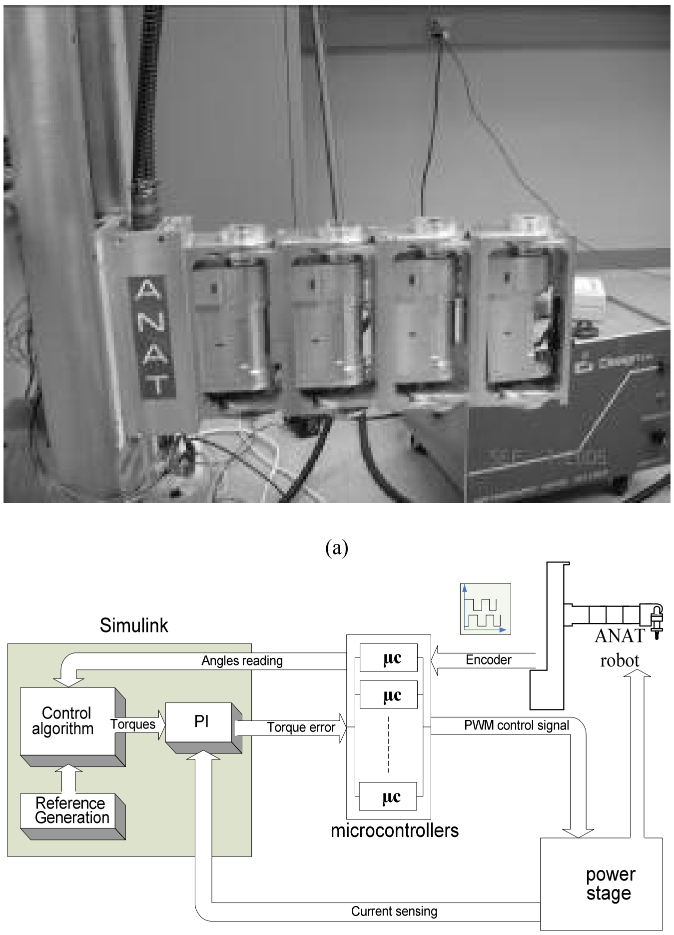
$K(\Sigma) = diag\left(\dfrac{k_i}{N_i(S_i)}, i = 1, \ldots, m \right)$ and

$N_i(S_i) = \delta_{0i} + (1 - \delta_{0i}) e^{-\alpha_i |S_i|^{pi}}$.

Note that the matrix $(\Gamma \cdot \mathrm{B})$ is invertible only if B is full rank.

VI. Case study: ERL Sliding Mode Applied on a Robotic Arm

As an application to sliding mode control on MIMO systems, the robot arm ANAT illustrated in figure 3 (a) is studied in this section with 3 DOF.
The real time controller was implemented in Simulink with Real Time Workshop (RTW) of Mathworks Inc. The real time target was chosen to be a National Instruments PCI 6024E digital card. Then, the control signals exiting from Simulink are applied to the ATMEGA 16 microcontrollers. PWM



(a)

(b)

Fig. 3. Real time setup: (a) ANAT robot arm, (b) Control scheme of the robot

equivalents are found and applied to the H-Bridge drives of the three actuators of the robot arm. In order to complete the feedback loop, current sensors located in the H-Bridge drives measure the current of each actuator and feed it back to Simulink for filtering and processing. Angular position loops are also fed to Simulink, via the microcontrollers which process the digital information of the actuators' encoders. Figure 3 (b) shows the complete control scheme applied on the robot.

The dynamics of the robot are given by the well known equation for rigid manipulators [24]:

$$\ddot{q} = -M(q)^{-1}F(q,\dot{q}) + M(q)^{-1}\tau \tag{43}$$

where M is the inertia matrix, symmetric and positive definite. So $M(q)^{-1}$ always exits. F is the centrifugal, Coriolis and gravity vector, q is the joint position vector, and τ is the torque input vector of the manipulator. First define a desired trajectory q_i^d, and define the tracking error for each joint as $e_i = q_i - q_i^d$, $i = 1,2,3$.

Now, comparing expression (43) with equation (38) in section IV gives the following equivalencies:

$\ddot{q} \leftrightarrow X_n$, $-M(q)^{-1}F(q,\dot{q}) \leftrightarrow \Phi(X)$, $M(q)^{-1} \leftrightarrow \mathrm{B}(X)$ *and* $\tau \leftrightarrow U$

And yields to the following control torques for the robot:

$$\tau = -M \cdot \left(\Lambda\dot{E} - \ddot{q}^d\right) + F - M \cdot K(\Sigma) \cdot sign(\Sigma) \tag{44}$$

Where $\Sigma = \begin{bmatrix} S_1 & S_2 & S_3 \end{bmatrix}^T$ is the sliding surface of the robot with $S_i = \lambda_i e_i + \dot{e}_i$, $i = 1,...,3$ the sliding surface of each DOF. $\Gamma = I_3$ in this case and

$$K(\Sigma) = diag\left(\frac{k_1}{N_1(S_1)}, \frac{k_2}{N_2(S_2)}, \frac{k_1}{N_2(S_2)}\right)$$

$$\dot{E} = \begin{bmatrix} \dot{e}_1 & \dot{e}_2 & \dot{e}_3 \end{bmatrix}^T, \Lambda = diag(\lambda_1, \lambda_2, \lambda_3)$$

$$\ddot{q}^d = \begin{bmatrix} \ddot{q}_1^d & \ddot{q}_2^d & \ddot{q}_3^d \end{bmatrix}^T$$

The experimental results below are obtained with a smooth fifth order polynomial reference trajectory:

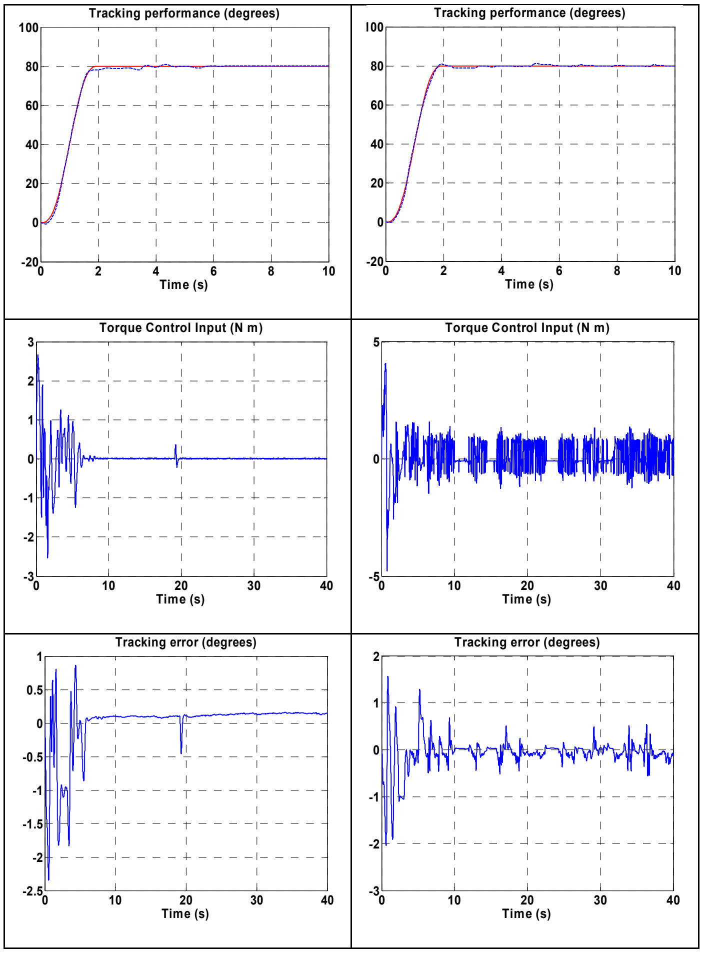
$$\begin{aligned} q^d{}_i(t) &= a_{qi5}(t-t_{0i})^5 + a_{qi4}(t-t_{0i})^4 + a_{qi3}(t-t_{0i})^3 + a_{qi2}(t-t_{0i})^2 \\ &\quad + a_{qi1}(t-t_{0i})^1 + a_{qi0} \ , \quad i = 1,2,3 \end{aligned} \tag{45}$$

Where

$$a_{qi5} = \frac{6\left(q_{if}^d - q_{i0}^d\right)}{t_1^5}, a_{qi4} = \frac{15\left(q_{if}^d - q_{i0}^d\right)}{t_1^4} \ a_{qi3} = \frac{10\left(q_{if}^d - q_{i0}^d\right)}{t_1^3},$$

$a_{qi2} = a_{qi1} = 0$, $a_{qi0} = q^d{}_{i0}$ and where q_{i0}^d and q_{if}^d are respectively the desired initial and final joint angles of link *i*, t_{0i} is the starting time of the reference trajectory for joint *i*, t_1 is the time required for the reference trajectory to reach q_{if}^d, starting from q_{i0}^d.

Appendix 1 gives the values of the parameters for the reference trajectory, and for all the other parameters of the controller. Note that in order to test the robustness of the controller, the dynamical parameters of the robot arm are not measured, but rather roughly estimated.



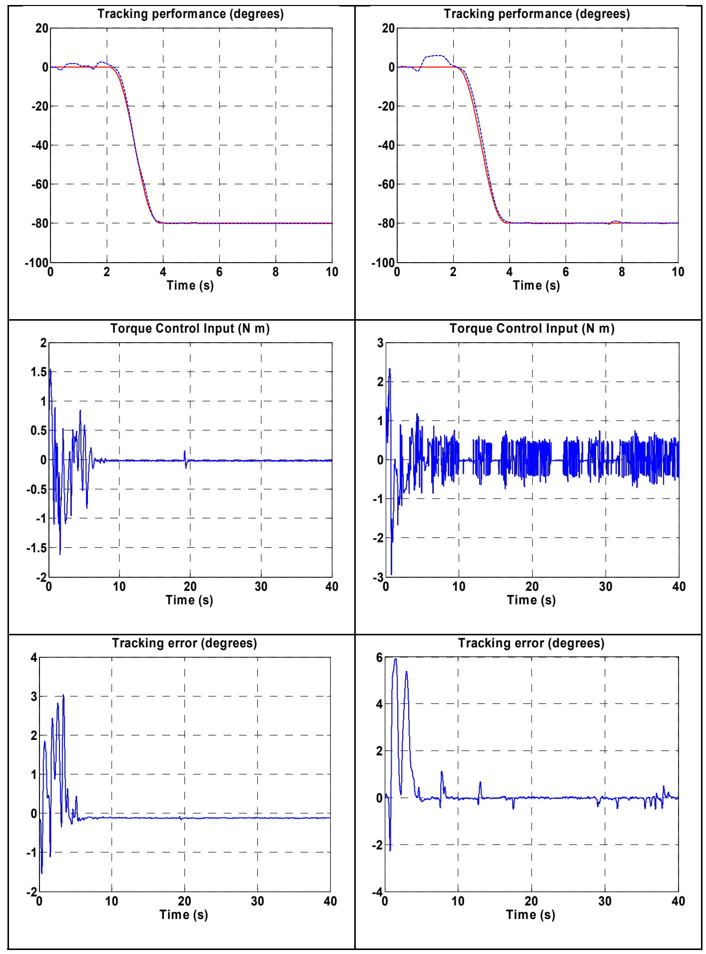
(a) ERL approach (b) conventional approach

Fig. 4. Experimental results for joint 1 (a) with reaching law and (b) with conventional law

Figures 4, 5 and 6 show experimental results for the three joints of ANAT arm. These figures compare the ERL approach, as shown in Fig. 4(a), Fig. 5(a) and Fig. 6(a), to that of the conventional sliding mode approach, as shown in Fig. 4(b), Fig. 5(b) and Fig. 6(b). These results show the effectiveness of the proposed approach, regarding particularly the chattering reduction on the torque input. The steady state error with ERL approach is due to the parameters uncertainties of the robot's model. However, it is bounded to be less than 0.1 degrees for all three axes, and it can also be directly controlled by the value of α according to condition given in (36). Therefore, with the ERL approach, the controller is able to reduce chattering on control input while maintaining a very good tracking performance of the desired trajectory, though the reaching time remains the same. This is not possible to achieve with conventional sliding mode approach. In the tracking performance figures (Fig.4 to Fig. 6), the solid line represents the reference trajectory, and the dashed line represents the actual trajectory of the joint.

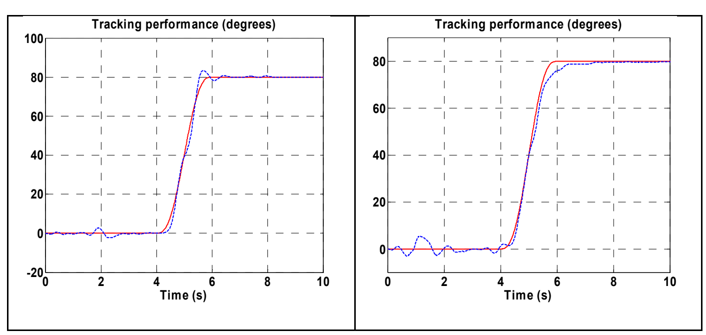
VII. CONCLUSION

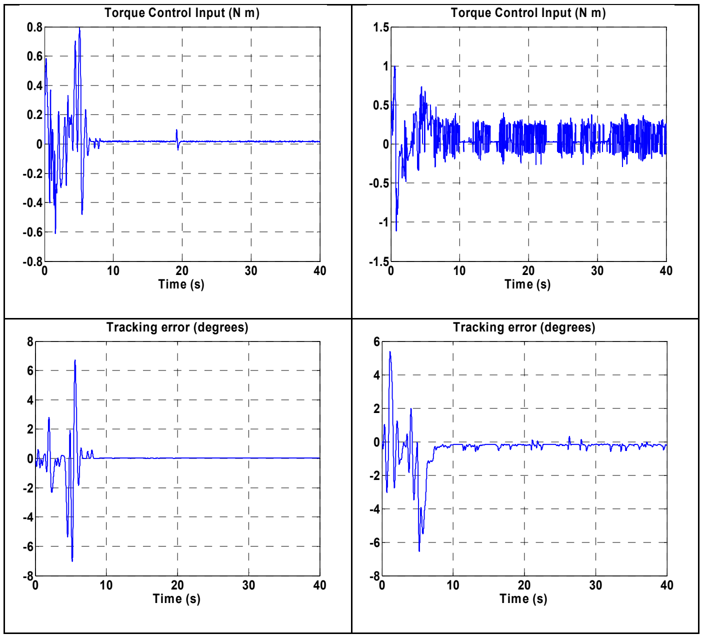
In this paper, sliding mode control is experimentally applied to MIMO nonlinear systems. The main contribution of this paper is to introduce an exponential reaching law (ERL) approach to the control mechanism, in order to control both the chattering and the tracking performances, which is impossible to achieve with the conventional sliding mode control approach. Experimental results on a robot arm with 3 DOF showed the superiority of the proposed approach over the conventional control, especially regarding the reduction of chattering levels on the control input.



(a) ERL approach (b) conventional approach

Fig. 5. Experimental results for joint 2 (a) with reaching law and (b) with conventional law





(a) ERL approach (b) conventional approach

Fig. 6. Experimental results for joint 3 (a) with reaching law and (b) with conventional law

APPENDIX I

ROBOT'S PARAMETERS

-Structure of $M(q,\dot{q})$ and $F(q,\dot{q})$:

$$M(1,1)=I_{zz1}+I_{zz2}+I_{zz3}+2m_3L^2\,c_{23}+2m_2L^2\,c_2+2m_3L^2\,c_3+2m_3L^2\,c_2+m_1L^2+2m_2L^2+3m_3L^2$$

$$M(2,1)=I_{zz2}+I_{zz3}+2m_3L^2\,c_3+m_3L^2\,c_2+m_2L^2\,c_2+m_3L^2\,c_{23}+2m_3L^2+m_2L^2$$

$$M(3,1)=I_{zz3}+m_3L^2+m_3L^2\,c_3+m_3L^2\,c_{23}\,,\; M(1,2)=M(2,1)$$

$$M(2,2)=I_{zz2}+I_{zz3}+2m_3L^2+m_2L^2+2m_3L^2\,c_3$$

$$M(3,2)=I_{zz3}+m_3L^2+m_3L^2\,c_3$$

$$M(1,3)=M(3,1)\,,\; M(2,3)=M(3,2)\,,\; M(3,3)=I_{zz3}+m_3L^2$$

$$F(1)=-L^2\begin{pmatrix} m_3\dot{q}_2^2\,s_2+m_2\dot{q}_2^2\,s_2+m_3\dot{q}_3^2\,s_3+m_3\dot{q}_2^2\,s_{23}+m_3\dot{q}_3^2\,s_{23}+ \\ 2m_3\dot{q}_3\dot{q}_1\,s_{23}+2m_2\dot{q}_1\dot{q}_2\,s_2+2m_3\dot{q}_1\dot{q}_2\,s_2+2m_3\dot{q}_2\dot{q}_3\,s_3+ \\ 2m_3\dot{q}_1\dot{q}_3\,s_3+2m_3\dot{q}_1\dot{q}_2\,s_{23}+2m_3\dot{q}_2\dot{q}_3\,s_{23} \end{pmatrix}$$

$$F(2)=L^2\begin{pmatrix} -2m_3\dot{q}_1\dot{q}_3\,s_3-m_3\dot{q}_3^2\,s_3-2m_3\dot{q}_3\dot{q}_2\,s_3+m_3\dot{q}_1^2\,s_{23}+ \\ m_3\dot{q}_1^2\,s_2+m_2\dot{q}_1^2\,s_2 \end{pmatrix}$$

$$F(3)=m_3L^2\left(2\dot{q}_1\dot{q}_2\,s_3+\dot{q}_1^2\,s_{23}+\dot{q}_2^2\,s_3+\dot{q}_1^2\,s_3\right)$$

Where

$s_i=\sin(q_i)$; $c_i=\cos(q_i)$; $s_{ij}=\sin(q_i+q_j)$; $c_{ij}=\cos(q_i+q_j)$;

-Kinematic parameters:

$L=0.1228\,m$

-Estimated dynamics parameters:

$m_1=m_2=m_3=3\,kg$; $I_{zz1}=I_{zz2}=I_{zz3}=0.0038\ kg\cdot m^2$

-Reference trajectory parameters:

$q_{10}^d=q_{20}^d=q_{30}^d=0$; $q_{1f}^d=80^o$, $q_{2f}^d=-80^o$, $q_{3f}^d=80^0$;

$t_{01}=0\,s$, $t_{02}=2\,s$, $t_{03}=6\,s$; $t_1=2\,s$

-Conventional reaching law parameters:

$\lambda_1=\lambda_2=\lambda_3=10$; $k_1=k_2=k_3=10$

-Exponential reaching law parameters:

$\lambda_1=\lambda_2=\lambda_3=10$; $k_1=k_2=k_3=1$; $\delta_{01}=\delta_{02}=\delta_{03}=0.1$;

$\alpha_1=\alpha_2=\alpha_3=20$; $p_1=p_2=p_3=1$

-Sampling time: $T_s=0.0003s$

PROOF OF RELATIONSHIP (20)

Using a symbolic software (MATHEMATICA) ,

$$\int_0^{|S(0)|} e^{-\alpha|S|^p}\,dS=\frac{\Gamma\left(1+\frac{1}{p}\right)-\frac{1}{p}\Gamma\left(\frac{1}{p},\alpha|S(0)|^p\right)}{\alpha^{1/p}}$$

Where $\Gamma(a)$ is the Euler gamma function and $\Gamma(a,z)$ is the incomplete gamma function defined as follows:

$$\Gamma(a)=\int_0^\infty t^{a-1}e^{-t}dt\ ;\ \Gamma(a,z)=\int_z^\infty t^{a-1}e^{-t}dt\le\Gamma(a),z\ge 0$$

It is straightforward that $\frac{1}{p}\cdot\Gamma\left(\frac{1}{p},\alpha|S(0)|^p\right)\le\frac{1}{p}\cdot\Gamma\left(\frac{1}{p}\right)$

On the other hand, using the properties of Γ , $\frac{1}{p}\Gamma\left(\frac{1}{p}\right)=\Gamma\left(1+\frac{1}{p}\right)$,then $\frac{1}{p}\cdot\Gamma\left(\frac{1}{p},\alpha|S(0)|^p\right)\le\Gamma\left(1+\frac{1}{p}\right)$.

This is expected since $\int_0^{|S(0)|} e^{-\alpha|S|^p}\,dS$ is always positive. This implies that $\Gamma\left(1+\frac{1}{p}\right)-\frac{1}{p}\Gamma\left(\frac{1}{p},\alpha|S(0)|^p\right)\le\Gamma\left(1+\frac{1}{p}\right)$

From Γ 's properties, $\Gamma(1+\frac{1}{p})\le 1$ for $p\ge 1$ with $\Gamma(1)=\Gamma(2)=1$, therefore $\int_0^{|S(0)|} e^{-\alpha|S|^p}\,dS\le\frac{1}{\alpha^{1/p}}$, and relation (20) is therefore straightforward.

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