Slip length for transverse shear flow over a periodic array of weakly curved menisci

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By exploiting the reciprocal theorem of Stokes flow we find an explicit expression for the first order slip length correction, for small protrusion angles, for transverse shear over a periodic array of curved menisci. The result is the transverse flow analogue of the longitudinal flow result of Sbragaglia & Prosperetti [*Phys. Fluids*, **19**, 043603, (2007)]. For small protrusion angles, it also generalizes the dilute-limit result of Davis & Lauga [*Phys. Fluids*, **21**, 113101 (2009)] to arbitrary no-shear fractions. While the leading order slip lengths for transverse and longitudinal flow over flat no-shear slots are well-known to differ by a factor of 2, the first order slip length corrections for weakly protruding menisci in each flow are found to be identical.

I. INTRODUCTION

Quantifying the hydrodynamic slip properties of superhydrophobic surfaces has been the focus of intense research activity in recent years owing to their use in significantly reducing friction factors in micro- and nano-fluidics applications[1,2]. These surfaces reduce friction due to the presence of free surfaces, or menisci, spanning interstitial grooves between protrusions in the substructure of the surface. Much theoretical[3–11], experimental[12–14] and numerical work[15–18] has been done to understand the friction properties of these surfaces. A paper by Philip[3], which solves a varierty of pertinent mixed boundary value problems, has become a well-known reference in this area but it only deals with flat menisci and under the assumption that they are shear-free. There has been recent efforts to quantify hydrodynamic slip in more general situations where, for example, the menisci are curved[5–8,14,18] and where the effect of a second subphase fluid is incorporated[10,11,19–21]. Special superhydrophobic microfluidic devices even exist with the capability of actively controlling the meniscus curvature to "tune" surfaces to have desired friction properties[22]. On the other hand, the role of interface curvature, in concert with surface immobilization effects due to surfactants and other contaminants, have been studied as mechanisms for understanding observed compromised slip properties for certain surfaces[23,24].

As research in the area grows, with new physical effects constantly added and novel surface geometries devised, it is desirable to have available a catalogue of explicit formulas quantifying slip in canonical flow scenarios, especially ones involving non-zero meniscus curvature.

This paper, which adds to this catalogue, concerns the problem shown in Figure 1: transverse shear flow in an (x, y) plane over a $2a$-periodic surface of no-shear menisci, protruding with angle θ into the flow (a "bubble mattress"), or into the groove if $\theta < 0$, with those menisci spanning gaps of length $2b$. This problem has been considered theoretically by Davis & Lauga[6] in the dilute limit $b/a \ll 1$ with a view to rationalizing the observed negation of any slip advantage[14] afforded by the no-shear menisci when those menisci protrude too prominently into the oncoming shear. When the menisci are flat, so that $\theta = 0$, Philip[3] found the transverse velocity field analytically as well as the following formula for the effective slip length:

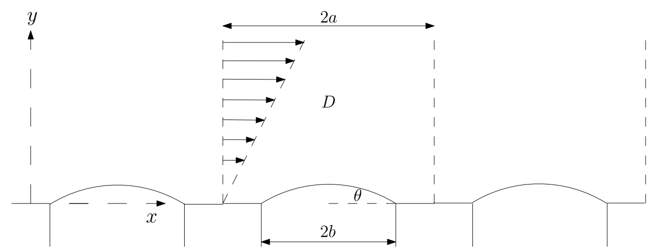

FIG. 1. Transverse shear flow in an (x, y) plane over a periodic array of weakly protruding menisci (or bubbles). The protrusion angle can be either positive or negative. The longitudinal flow problem, where the flow is into the page, was considered by Sbragaglia & Prosperetti[5].

$$\lambda_P^{\perp} = \frac{a}{\pi} \log \sec\left(\frac{\pi\xi}{2}\right), \qquad \xi = \frac{b}{a}, \tag{1}$$

The latter formula is valid for any no-shear fraction ξ.

Philip[3] also found an analytical solution for the problem of semi-infinite longitudinal shear flow $(0, 0, w_P(x, y))$ (where the flow is now into the page in Figure 1) over the same surface and found that the associated slip length $\lambda_P^{||} = 2\lambda_P^{\perp}$. Recently, the present author has demonstrated[11] using a reciprocal theorem approach that if the longitudinal slip length for shear flow over weakly protruding menisci, i.e. $\theta \ll 1$, is developed in the regular perturbation expansion

$$\lambda^{||} = \lambda_P^{||} + \theta\lambda_1^{||} + \dots \tag{2}$$

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then the first order slip length correction, $\lambda_1^{||}$, is given by the integral formula

$$\lambda_1^{||} = \frac{1}{2a}\int_{-b}^{b}\left(\frac{b^2-x^2}{2b}\right)\left(\frac{\partial w_P}{\partial x}(x,0)\right)^2 dx. \tag{3}$$

We emphasize that the integral on the right hand side depends only on Philip's known flat-meniscus solution $w_P(x,y)$ for which it is known that

$$\begin{aligned} w_P(x,0) &= \frac{2a}{\pi}\cosh^{-1}\left[\frac{\cos(\pi x/2a)}{\cos(\pi b/2a)}\right], \\ \frac{\partial w_P}{\partial x}(x,0) &= -\frac{\sin(\pi x/2a)}{[\cos^2(\pi x/2a)-\cos^2(\pi b/2a)]^{1/2}}. \end{aligned} \tag{4}$$

The same problem of longitudinal flow over weakly-protruding menisci was considered by Sbragaglia & Prosperetti[5] using a very different approach where the full first order flow perturbation and slip length correction were computed by solving an infinite linear system of so-called dual series equations. After a series of manipulations, and use of several special function identities, Sbragaglia & Prosperetti[5] report the result

$$\lambda_1^{||} = \frac{bF(\xi)}{2}, \tag{5}$$

with

$$F(\xi) = \xi\int_0^1 (1-s^2)\frac{[1-\cos(s\pi\xi)]ds}{\cos(s\pi\xi)-\cos(\pi\xi)}. \tag{6}$$

Since that work, the present author[11] has shown that exactly the same result follows on substitution of (4) into formula (3):

$$\lambda_1^{||} = \frac{1}{2a}\int_{-b}^{b}\left(\frac{b^2-x^2}{2b}\right)\frac{\sin^2(\pi x/2a)}{\cos^2(\pi x/2a)-\cos^2(\pi b/2a)}dx \tag{7}$$

which, after a change of integration variable, $x = bs$, and use of some trigonometric identities, retrieves (5). While both approaches reach the same final result (5), Sbragaglia & Prosperetti[5] did not derive the integral formula (3) expressing the first order correction in terms of Philip's known flat-meniscus solution w_P.

The purpose of this Letter is to show that exactly the same feature is true of the transverse flow problem; namely, that the first order slip length correction for transverse flow can also be found as an explicit integral dependent only on Philip's known flat-state transverse flow solution. The transverse problem was not considered by Sbragaglia & Prosperetti[5]; indeed, to the best of the author's knowledge, there has been no previous attempt to generalize the longitudinal analysis of Sbragaglia & Prosperetti[5] to the transverse flow scenario, probably owing to the much more complicated biharmonic nature of the governing field equation for the streamfunction in this case. The generalization is made here, by extending the reciprocity approach of recent work[11] – this avoids the need for a direct solution of the full first-order problem – with the surprising result that the first-order slip length correction for the transverse problem is identical to that for the longitudinal problem in the same geometry.

To proceed with the analysis we let $\{u_i, \sigma_{ij}\}$ represent Philip's solution for transverse shear flow, with unit shear rate, over a periodic array of flat no-shear slots as depicted in Figure 1 but with $\theta = 0$. Let $\{u_i', \sigma_{ij}'\}$ represent the solution for transverse shear flow, with the same unit shear rate, over a surface of weakly protruding menisci with protrusion angle θ and $|\theta| \ll 1$ (the menisci can protrude into the fluid or into the groove). Let D be the fluid domain in a single period window of this weakly protruding scenario; see Figure 1. Without loss of generality we assume zero pressure in the subphase gas so that, on the meniscus,

$$\sigma_{ij}'n_j = T\kappa n_i, \tag{8}$$

where T is the surface tension and κ is the meniscus curvature in the transverse (x,y)-plane. We follow Davis & Lauga[6] and assume the capillary number is sufficiently small that the meniscus can be assumed to be a circular arc of constant curvature. As $y \to \infty$,

$$\mathbf{u} \to \begin{pmatrix} y+\lambda_P^{\perp} \\ 0 \end{pmatrix}, \qquad \mathbf{u}' \to \begin{pmatrix} y+\lambda^{\perp} \\ 0 \end{pmatrix}, \tag{9}$$

where $\lambda^{\perp}$ is the quantity we wish to find. By the reciprocal theorem of Stokes flow[25]:

$$\oint_{\partial D}(u_i'\sigma_{ij}n_j - u_i\sigma_{ij}'n_j)ds = 0. \tag{10}$$

The periodicity in the x-direction of both flows precludes any contribution to this boundary integral from the period window sidewalls. Along the "edge" at infinity, i.e., $x \in [-a,a], y = H$ as $H \to \infty$, we find

$$\sigma_{ij}n_j, \sigma_{ij}'n_j \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{11}$$

hence the contribution to the boundary integral from this edge is

$$2a(\lambda^{\perp} - \lambda_P^{\perp}). \tag{12}$$

On use of the no-slip conditions satisfied by both u_i and u_i' on the lower no-slip walls of the period window (10) we find

$$2a(\lambda^{\perp} - \lambda_P^{\perp}) + \oint_{\text{meniscus}}(u_i'\sigma_{ij}n_j - u_i\sigma_{ij}'n_j)ds = 0. \tag{13}$$

Since θ is small it is easy to show that its curvature $\kappa = \mathcal{O}(\theta)$. Hence by the boundary condition (8) and the fact that, on the curved interface, $u_in_i = \mathcal{O}(\theta)$ where u_i is the velocity field of the flat-interface problem,

$$\oint_{\text{meniscus}} u_i \sigma'_{ij} n_j ds = o(\theta). \qquad (14)$$

By the no-slip conditions satisfied by both u_i and u'_i on the lower walls of the period window we then conclude that

$$2a(\lambda^\perp - \lambda_P^\perp) + \oint_{\text{meniscus}} u'_i \sigma_{ij} n_j ds = o(\theta). \qquad (15)$$

Now if we develop the perturbation expansions

$$\mathbf{u}' = \mathbf{u} + \theta \mathbf{u_1} + \ldots, \qquad \lambda^\perp = \lambda_P^\perp + \theta \lambda_1^\perp + \ldots \qquad (16)$$

then it follows from (15) that

$$2a\theta\lambda_1^\perp = -\oint_{\text{meniscus}} u_i \sigma_{ij} n_j ds + o(\theta). \qquad (17)$$

Therefore the first-order slip length correction for a weakly curved circular meniscus is given by an integral whose integrand depends only on the known flat-meniscus solution. Formula (17) is the analogue of the aforementioned result (3) for the longitudinal flow problem derived using similar reciprocity arguments[11].

It only remains to compute the integral on the right hand side of (17) correct to order θ. The problem for transverse flow over a periodic array of flat no-shear slots was solved by Philip[3] and rederived[9] in a convenient parametric form using a complex variable formulation of the Stokes flow problem. On introducing the complex variable $z = x + \mathrm{i}y$ (note: the use of z here is *not* to label the third axis orthogonal to the (x, y) plane), Crowdy[9] finds

$$\psi = \mathrm{Im}[\overline{z} f(z) + g(z)], \quad f(z) = \frac{\mathrm{i}}{4} h(z), \quad g(z) = -z f(z), \qquad (18)$$

where an explicit expression for $h(z)$ was given. Now, as shown in[9],

$$\sigma_{ij} n_j \mapsto 2\mu\mathrm{i}\frac{dH}{ds}, \qquad H(z, \overline{z}) \equiv f(z) + z\overline{f'(z)} + \overline{g'(z)}, \qquad (19)$$

where the arrow $\mapsto$ is used to denote the procedure of expressing a vector quantity (a_x, a_y) in its natural complex variable form $a_x + \mathrm{i}a_y$. Similarly,

$$\mathbf{u} \mapsto u + \mathrm{i}v = -f(z) + z\overline{f'(z)} + \overline{g'(z)}. \qquad (20)$$

On the flat meniscus, we can then write

$$u + \mathrm{i}v = H - 2f(z). \qquad (21)$$

It follows from (17) that

$$2a\theta\lambda_1^\perp = -\mathrm{Re}\left[2\mu\mathrm{i}\int_{\text{meniscus}} (\overline{H} - 2\overline{f}) dH\right], \qquad (22)$$

where we have used the fact that $\mathbf{a}.\mathbf{b} \mapsto \mathrm{Re}[(a_x - \mathrm{i}a_y)(b_x + \mathrm{i}b_y)]$. On use of the expression for $g(z)$ from (18),

$$H = f(z) - \overline{f(z)} + (z - \overline{z})\overline{f'(z)}. \qquad (23)$$

It is straightforward to show[5,11] that the meniscus can be parametrized by $z = x + \mathrm{i}\theta\eta(x)$ where

$$\eta(x) = \frac{b^2 - x^2}{2b}. \qquad (24)$$

Hence, on the meniscus,

$$\begin{aligned} H &= f(x) + \mathrm{i}\theta\eta(x) f'(x) - \overline{f(x)} + \mathrm{i}\theta\eta(x)\overline{f'(x)} \\ &\quad + 2\mathrm{i}\theta\eta(x)\overline{f'(x)} + o(\theta) = 4\mathrm{i}\theta\eta f'(x) + o(\theta), \end{aligned} \qquad (25)$$

where we have used the fact, established in[9], that $f(x) = \overline{f}(x)$ and, hence, $\overline{f'}(x) = f'(x)$. Equation (22) yields

$$\begin{aligned} 2a\theta\lambda_1^\perp &= \mathrm{Re}\left[4\mu\mathrm{i}\int_{-b}^{b} \overline{f(x)} dH\right] + o(\theta) \\ &= \theta\mathrm{Re}\left[-16\mu\int_{-b}^{b} \overline{f(x)} d[\eta f'(x)]\right] + o(\theta). \end{aligned} \qquad (26)$$

It follows that

$$\lambda_1^\perp = \frac{8}{a}\int_{-b}^{b} \eta(x) f'(x)^2 dx, \qquad (27)$$

where we have used integration by parts and the fact that η has simple zeros at $x = \pm b$ as seen from (24). (27) and (18) together imply

$$\lambda_1^\perp = \frac{8}{a}\int_{-b}^{b} \eta(x) f'(x)^2 dx = -\frac{1}{2a}\int_{-b}^{b} \eta(x) h'(x)^2 dx. \qquad (28)$$

The author has also shown elsewhere[9] that Philip's solution for longitudinal shear flow over flat slots can be written as $w_P = \mathrm{Im}[h(z)]$ where the analytic function $h(z)$ is precisely that appearing in (18). By the analyticity of $h(z)$, and the fact that $\partial w_P/\partial y = 0$ on the meniscus in the longitudinal flow problem,

$$\frac{dh}{dz} = \frac{\mathrm{i}\partial w_P}{\partial x} \qquad (29)$$

so that (3) can be written as

$$\lambda_1^{||} = \frac{1}{2a}\int_{-b}^{b} \eta(x)\left(\frac{\partial w_P}{\partial x}\right)^2 dx = -\frac{1}{2a}\int_{-b}^{b} \eta(x) h'(x)^2 dx. \qquad (30)$$

The conclusion, on combining (7), (28) and (30), is

$$\lambda_1^\perp = \lambda_1^{||} = \frac{1}{2a}\int_{-b}^{b} \eta(x)\frac{\sin^2(\pi x/2a)}{\cos^2(\pi x/2a) - \cos^2(\pi b/2a)} dx. \qquad (31)$$

That is, the first-order slip length correction, in small protrusion angle, is the *same* for both longitudinal and transverse shear over these superhydrophobic surfaces. Moreover, (31) provides an explicit integral formula for it. We are unable to suggest a physical reason for this surprising result, but it is clearly related to the fact that the same analytic function $h(z)$ appears in the flat-meniscus solutions for both longitudinal and transverse flow.

We can check this result in the dilute limit $\xi \ll 1$. An expansion of (31) for small ξ yields, after some manipulations,

$$\lambda_1^{\perp} = \lambda_1^{||} \sim \frac{b\xi}{6}. \qquad (32)$$

It has already been verified elsewhere[8] that the result (32) for longitudinal flow is consistent with a small-θ expansion of a formula derived by a conformal geometric approach valid for arbitrary protrusion angles θ. On the other hand, for transverse flow, Davis & Lauga[6] find the dilute-limit slip length in this case as a general function of θ in the form

$$\frac{\pi b^2}{a} \int_0^{\infty} A(s, \theta) ds, \qquad (33)$$

where an explicit form of $A(s, \theta)$ is given[6]. A small-θ expansion yields

$$A(s, \theta) = \frac{2s}{\sinh(2s\pi)} + \frac{2s^2\theta}{\cosh^2(s\pi)} + \mathcal{O}(\theta^2), \qquad (34)$$

implying, on substitution into (33), the first-order slip length correction

$$\frac{\pi b^2}{a} \int_0^{\infty} \frac{2s^2}{\cosh^2(s\pi)} ds = \frac{b\xi}{6}, \qquad (35)$$

which confirms the new result (32). Here we have used the fact that

$$\int_{-\infty}^{\infty} \frac{s^2}{\cosh^2(s\pi)} ds = \frac{1}{6\pi} \qquad (36)$$

which follows from an exercise in residue calculus (integrating $s^4/\cosh^2(s\pi)$ around the closed boundary contour of the channel region $-\infty < \mathrm{Re}[s] < \infty$ with upper walls at $\mathrm{Im}[s] = 0$ and $\mathrm{Im}[s] = 1$).

It is clear from this calculation that, in the small angle limit, the new result (31) generalizes the dilute-limit transverse slip length result of Davis & Lauga[6] (33) to *any* no-shear fraction.

At the same time, the formula in (31) is the direct analogue, for transverse shear flow, of the small-angle longitudinal slip length result found by Sbragaglia & Prosperetti[5].

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