Slope and G-Value characterization of Set-valued Functions and Applications to Non-differentiable Optimization Problems

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Abstract

In this paper we derive a generalizing concept of G-norms, which we call G-values, which is used to characterize minimizers of non-differentiable regularization functionals. Moreover, the concept is closely related to the definition of slopes as published in a recent book by Ambrosio, Gigli, Savaré. A paradigm of regularization models fitting in this framework is *robust* bounded variation regularization. Two essential properties of this regularization technique are documented in the literature and it is shown that these properties can also be achieved with metric regularization techniques, which also have the advantage that they attain a unique minimizers.

1 Introduction

In this work we are concerned with characterization of the minimizers of the functional

$$\mathcal{F}(u) := \int |u - f| + \alpha \|Du\| \,, \tag{1}$$

eq:l1l1

where $\|Du\|$ denotes the bounded variation semi-norm. Other functionals can be considered with the techniques presented below.

Recent attempts in characterization properties of the minimizers of this functional have been made by Chan & Esedoglu ChaEse05 [2] and in OshSch04 [5]. In the latter work we characterized minimizers of (eq:l1l1 1) using the G-norm introduced by

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Y. Meyer [4]. The results essentially apply if the zeros of $u_\alpha - f$ are sparse. This limits the applicability of the results. In this work we derive a general characterization of the minimizing elements. For this purpose we develop the concept of G-values, which is a generalization of Y. Meyer's G-norm to *set valued* functions. In general, for the functional (1) the characterization of minimizers is no longer possible by the G–norm as for instance for the Rudin-Osher-Fatemi model [6] (cf. Meyer [4]).

Moreover, we show a relation between G-value and *slopes* as introduced recently in [1].

The results of this paper allow to characterize minimizers of $\mathcal{F}$ in a functional analytical framework and as a byproduct we can generalize the results of Chan & Esedoglu [2]. Moreover, some of the results can easily be generalized to a wider class of metrical regularization techniques. This is shortly discussed and some numerical examples are presented.

2 Basic Facts on Minimizers and Notation

It is relatively easy to show that there exists a minimizer of $\mathcal{F}$ in $\mathtt{BV}$, the *space of functions of bounded variation* (cf. Evans & Gariepy [3]) and $\|Du\|$ denotes the *bounded variation semi norm*. A minimizer is denoted of $\mathcal{F}$ is denoted by u_α. Note that the minimizing elements does not have to be unique since the functional is *not* strictly convex.

For $v \in \mathtt{BV}$ we denote by

$$\begin{aligned} \psi_v(x) &= \begin{cases} \operatorname{sgn}(v(x) - f(x)) \text{ if } v(x) - f(x) \neq 0 \\ 0 \text{ if } v(x) - f(x) = 0 \end{cases} \in \\ \Psi_v &= \{\zeta \in L^\infty : \\ &\qquad \zeta = \operatorname{sgn}(v(x) - f(x)) \text{ if } v(x) \neq f(x)\,, \in [-1,1] \ \text{ else}\}\,. \end{aligned} \tag{2}$$

eq:psi

Moreover, let

$$\begin{aligned} \eta &: \mathbb{R} \times \mathtt{BV} \times \mathtt{BV} \to \mathbb{R} \cup \{+\infty\}\,. \\ &(t, v, h) \to \int (|v + th - f| - |v - f| - t\psi_v h) \end{aligned} \tag{3}$$

eq:r

Lemma 2.1. *Assume that $v, h \in \mathtt{BV}$, then*

$$\lim_{t \to 0} \frac{\eta(t, v, h)}{|t|} = \int_{v=f} |h| \ . \tag{4}$$

eq:nullset

Proof. The definition of η implies that

$$\left| \frac{\eta(t,v,h)}{|t|} - \int_{v=f} |h| \right| \leq 2 \int_{0<|v-f|\leq |th|} |h| \ .$$

The family of functions $g_{|t|}(x) := |h(x)|\, \chi_{0<|v-f|\leq |t||h|}(x)$ is monotonically decreasing in $|t|$ and thus by the monotone convergence theorem

$$\begin{aligned} \lim_{|t|\to 0} \int g_{|t|}(x) &= \int |h(x)| \lim_{|t|\to 0} \chi_{0<|v-f|\leq |th|}(x) \\ &= \int |h(x)|\, \chi_{M_0}(x) \\ &= 0\,, \end{aligned}$$

where M_0 is a set of measure 0. This gives the assertion. □

As a consequence of the above lemma we have that if $\{v = f\}$ has Lebesgue measure 0, then

$$\frac{|\eta(t,v,h)|}{|t|} \to 0 \ . \tag{5}$$ eq:consequence

Using this observation, we can reinterpret the results in OshSch04 [5], which read as follows:

le42a **Theorem 2.2.** *1. Let $\{0 = f\}$ have Lebesgue measure 0. Then $\|\psi_0\|_{G^s} \leq \alpha$ if and only if $u_\alpha \equiv 0$.*

2. Let $\{u_\alpha = f\}$ have Lebesgue measure 0. If $\|\psi_0\|_{G^s} > \alpha$, then

$$\|\psi_{u_\alpha}\|_G = \alpha \ \textit{and} \ -\int \psi_{u_\alpha} u_\alpha = \alpha \|Du_\alpha\| \ .$$

In the following we generalize the result of Theorem le42a 2.2 and neglect the assumption that $\{u_\alpha = f\}$ has Lebesgue measure zero. In this case we require instead of the G-norms the concept of G-values. This is outlined below and the relation to *slopes* is derived.

3 G-Values

Definition 3.1. *Let $\Psi : \mathbb{R}^n \to 2^{\mathbb{R}}$ be a set-valued function (here, as usual $2^{\mathbb{R}}$ denotes the power set of $\mathbb{R}$). Let*

$$\Psi := \{\psi : \mathbb{R}^n \to \mathbb{R} \textit{ is measurable and } \psi(x) \in \Psi(x) \textit{ almost everywhere}\} \ .$$

Note, that notationally we do not distinguish between the set Ψ and the function Ψ.

We define G-values of Ψ as follows:

$$G(\Psi) := \sup_{\{h \in C_0^\infty : \int |\nabla h| = 1\}} \inf_{\psi \in \Psi} \int \psi h \,. \tag{6}$$ eq:g-values

Note, that if Ψ is single valued and measurable than $G(\Psi)$ is the G-norm of Ψ. Later on, we also use G_β-values of Ψ defined by

$$G_\beta(\Psi) := \sup_{\{h \in C_0^\infty : \int |\nabla h| + \beta \int |h| = 1\}} \inf_{\psi \in \Psi} \int \psi h \,. \tag{7}$$ eq:g-values-b

If Ψ is single valued, then this is the dual norm of the closure of C_0^∞ with respect to the norm $\int |\nabla h| + \beta \int |h|$. For $\beta = 0$ this reduces to the dual of the closure of C_0^∞ with respect to the norm $\int |\nabla h|$.

A typical example for a set-valued function is

$$\partial |g| := \begin{cases} \operatorname{sgn}(g) \text{ if } & g \neq 0\,, \\ [-1,1] \text{ if } & g = 0\,. \end{cases}$$

The definition of G-values implies also that for every function $h \in C_0^\infty$

$$\inf_{\psi \in \Psi} - \int \psi h \leq G(\psi) \|D(-h)\| = G(\psi) \|Dh\| \,.$$

and consequently

$$-\sup_{\psi \in \Psi} \int \psi h \leq G(\psi) \|Dh\| \text{ and } \inf_{\psi \in \Psi} \int \psi h \leq G(\psi) \|Dh\| \,. \tag{8}$$ eq:norms

In the sequel we concentrate on $\Psi = \partial |g|$.

le:char **Lemma 3.2.** *For $g \in L^1$, $G(\partial |g|) \leq \alpha$ if and only if*

$$\left(\left| \int_{g \neq 0} sgn(g) h \right| - \int_{g=0} |h| \right)^+ \leq \alpha \int |\nabla h| \text{ for all } h \in C_0^\infty \,. \tag{9}$$ eq:gvalue1

Moreover,

$$G(\partial |g|) = \sup_{\{h \in \mathtt{BV} : \|Dh\| = 1\}} \inf_{\psi \in \Psi} \int \psi h \,.$$

Proof. Since $h \in \mathtt{BV}$ can be approximated by a sequence of functions $h_n \in C_0^\infty$ satisfying $h_n \to h$ in L^1 and $\int |\nabla h_n| \to \|Dh\|$ it follows that

$$\left| \int_{g \neq 0} \operatorname{sgn}(g) h_n \right| - \int_{g=0} |h_n| \to \left| \int_{g \neq 0} \operatorname{sgn}(g) h \right| - \int_{g=0} |h| \ .$$

Therefore ([eq:gvalue1] 9) holds for all $h \in \mathtt{BV}$ if it holds for all $h \in C_0^\infty$.

For $h \in C_0^\infty$ let

$$\psi_h := -\operatorname{sgn}(h)\chi_{g=0} + \operatorname{sgn}(g)\chi_{g \neq 0} \in \partial |g| \ .$$

Therefore,

$$\int \psi_h h = \int_{g \neq 0} \operatorname{sgn}(g) h - \int_{g=0} |h| \leq \int_{g \neq 0} \operatorname{sgn}(g) h - \int_{g=0} \psi h$$

for all $\psi \in \partial |g|$. Therefore

$$\begin{aligned} G(\partial |g|) &= \sup_{\{h \in C_0^\infty : \int |\nabla h| = 1\}} \left(\int_{g \neq 0} \operatorname{sgn}(g) h - \int_{g=0} |h| \right)^+ \\ &= \sup_{\{h \in C_0^\infty : \int |\nabla h| = 1\}} \max \left(\int_{g \neq 0} \operatorname{sgn}(g)(\pm h) - \int_{g=0} |h| \right)^+ \\ &= \sup_{\{h \in C_0^\infty : \int |\nabla h| = 1\}} \left(\left| \int_{g \neq 0} \operatorname{sgn}(g) h \right| - \int_{g=0} |h| \right)^+ \ . \end{aligned}$$

□

In particular Lemma [le:char] 3.2 shows that ([eq:norms] 8) holds for all $\mathtt{BV}$-functions.

4 Slopes

The main result of this section is to show that for functions $v \in W_0^{1,1}$ the concept of G-values is the same as the concept of a *slope* as defined in [AmbGigSav05] [1].

To see the relation we use the Banach space $\mathcal{B} = W_0^{1,1}$, the closure of C_0^∞ with respect to the norm

$$u \to \int |\nabla u| + \beta \int |u| \ .$$

The dual is denoted by $W_0^{1,1*}$, and the natural metric on $\mathcal{B}$ is

$$d(v,h) := \int |\nabla v - \nabla h| + \beta \int |v - h| \, ; .$$

The functional

$$\phi : W_0^{1,1} \to [0,\infty]$$
$$v \to |v|$$

is convex and lower semi continuous.

According to AmbGigSav05 [1] the *slope* is defined as

$$|\partial\phi|(v) = \min\left\{\|\zeta\|_{W_0^{1,1*}} : \zeta \in \partial\phi(v)\right\} ,$$

where

$$\partial\phi(v) = \{\zeta \in W_0^{1,1*} : \phi(h) - \phi(v) - \langle\zeta, h-v\rangle \geq 0 \text{ for all } h \in W_0^{1,1}\} ,$$

is the sub-gradient (here $\langle\cdot,\cdot\rangle$ denotes the dual pairing). Note that slope attains the minimum, since $|v|$ is lower semi continuous. This requires that $\beta > 0$. In the previous section we use in the definition of the G-value the infimum and compensated for the fact that there $|v|$ may not be lower semi continuous.

Note, we do not notationally distinguish between sub-differential of functions and operators. We also emphasize that a-priori we do not assume that $\partial\phi(v) \neq \emptyset$. We define

$$D(\partial\phi) := \{v \in W_0^{1,1} : \partial\phi(v) \neq \emptyset\} .$$

Note that $\zeta \in \partial\phi(v)$ requires $\nabla\cdot\vec{\zeta} = \zeta \in W_0^{1,1*}$, or in other words $\vec{\zeta} \in L^\infty$, and

$$\zeta := \begin{cases} \text{sgn}(v) & \text{if} \quad v \neq 0 \\ [-1,1] & \text{if} \quad v = 0 . \end{cases}$$

Therefore,

$$|\partial\phi|(v) := \inf\{\|\zeta\|_{W_0^{1,1*}} : \zeta = \text{sgn}(v) \text{ if } v \neq 0 \text{ and } \zeta \in [-1,1] \text{ if } v = 0\} .$$

This shows that

$$|\partial\phi|(v) = \inf_{\zeta\in\partial\phi(v)} \sup_{\{h\in C_0^\infty : \int|\nabla h| + \beta\int|h| \leq 1\}} \int \zeta h .$$

From Proposition 1.4.4. in AmbGigSav05 [1] it follows that

$$|\partial\phi|(v) = \mathcal{I}_{|\cdot|}(v) := \sup_{v\neq h\in W_0^{1,1}} \frac{\left(\int|v| - \int|h|\right)^+}{\int|\nabla(v-h)| + \beta\int|v-h|} . \qquad (10) \quad \text{eq:ambrosio}$$

In the following, we show that $\mathcal{I}_{|\cdot|}(v) = G(\partial |v|)$. We have that

$$\begin{aligned}
I_{|\cdot|}(v) &= |\partial\phi|(v) \\
&\geq \inf_{\zeta\in\partial\phi(v)} \sup_{\{h\in C_0^\infty : \int|\nabla h| + \beta\int|h|=1\}} \int_{v\neq 0} \operatorname{sgn}(v)h + \int_{v=0}\zeta h \\
&\geq \sup_{\{h\in C_0^\infty : \int|\nabla h| + \beta\int|h|=1\}} \left(\left| \int_{v\neq 0} \operatorname{sgn}(v)h \right| - \int_{v=0}|h| \right)^+ \\
&= G_\beta(\partial |v|) \,.
\end{aligned}$$

Let $G_\beta(\partial |v|) = \alpha$, then by definition for every $h \in C_0^\infty$

$$\begin{aligned}
\int (|v| - |h|) &= \int_{v\neq 0} |v| - \int_{v\neq 0} |h| - \int_{v=0} |v-h| \\
&\leq \int_{v\neq 0} |v| - \int_{v\neq 0} \operatorname{sgn}(v)h - \int_{v=0} |v-h| \\
&\leq \left(\int_{v\neq 0} \operatorname{sgn}(v)(v-h) - \int_{v=0} |v-h| \right)^+ \\
&\leq \alpha \left[\int |\nabla v - \nabla h| + \beta \int |v-h| \right] .
\end{aligned}$$

This shows that

$$\left(\int |v| - \int |h| \right)^+ \leq \alpha \int |\nabla v - \nabla h| + \int |v-h| \,,$$

and therefore,

$$\mathcal{I}_{|\cdot|}(v) \leq \alpha = G_\beta(\partial |v|) \,.$$

Important for our paper is that the results of AmbGigSav05 [1] can also be applied to the functional

$$\begin{aligned}
\tilde{\phi} : L^1 &\to [0,\infty] \,, \\
u &\to \|Du\|
\end{aligned}$$

where $\|Du\|$ is the bounded variation semi-norm of u if $u \in \mathtt{BV}$ and $+\infty$ else. We use the metric induced by the L^1-norm. In this case we have

$$\left|\partial\tilde{\phi}\right|(v) = \min\{\|\zeta\|_{L^\infty} : \zeta \in \partial\tilde{\phi}(v)\} \,.$$

$\zeta \in \partial\tilde{\phi}(v)$ satisfies

$$\tilde{\phi}(u) - \tilde{\phi}(v) - \langle \zeta, u-v \rangle \geq 0 \,,$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing between $L^\infty = L^{1*}$ and L^1. Formally, the inequality reads as follows

$$\tilde{\phi}(u) - \tilde{\phi}(v) + \int \nabla \cdot \left(\frac{\nabla v}{|\nabla v|} \right) (u - v) \geq 0 \; .$$

Note, that the sub-gradient could be empty, if there does not exist $\zeta \in L^\infty = L^{1*}$ which satisfies $\zeta = -\nabla \cdot \left(\frac{\nabla v}{|\nabla v|} \right)$.

Since the functional $\tilde{\phi}$ is weakly lower semi-continuous (cf. Evans & Gariepy EvaGar92 [3]), according to Proposition 1.4.4. in AmbGigSav05 [1]

$$\mathcal{I}_{\tilde{\phi}}(v) := \sup \frac{(\|Dv\| - \|Dh\|)^+}{\int |v - h|} = \left| \partial \tilde{\phi} \right| (v) \; . \tag{11}$$

eq:..

5 Properties of Minimizers

In the following we prove a similar result to (eq:ambrosio 10).

le:chara **Lemma 5.1.** *Assume that $f \in L^1$ and let u_α be a minimizer of $\mathcal{F}$, then $G(\partial |u_\alpha - f|) \leq \alpha$. In particular, if $f \in \mathtt{BV}$, then*

$$\int |u_\alpha - f| \leq \alpha \|D(u_\alpha - f)\| \; . \tag{12}$$

eq:gration1

Proof. From the definition of η it follows that for all $h \in \mathtt{BV}$

$$\int |u_\alpha + \varepsilon h - f| = \int |u_\alpha - f| + \varepsilon \int \psi_{u_\alpha} h + \eta(\varepsilon, u_\alpha, h) \; . \tag{13}$$

eq:defeta

Since u_α minimizes $\mathcal{F}$ it follows that for all $h \in \mathtt{BV}$

$$\begin{aligned} & \int |u_\alpha - f| + \alpha \|Du_\alpha\| \\ \leq & \int |u_\alpha + \varepsilon h - f| + \alpha \|D(u_\alpha + \varepsilon h)\| \\ \leq & \int |u_\alpha - f| + \varepsilon \int_{u_\alpha \neq f} \operatorname{sgn}(u_\alpha - f) h + \eta(\varepsilon, u_\alpha, h) \\ & \quad + \alpha \|Du_\alpha\| + \alpha |\varepsilon| \, \|Dh\| \; . \end{aligned} \tag{14}$$

eq:defetb

Therefore, we have

$$-\varepsilon \int_{u_\alpha \neq f} \operatorname{sgn}(u_\alpha - f) h \leq \eta(\varepsilon, u_\alpha, h) + \alpha |\varepsilon| \, \|Dh\| \; .$$

Let $\varepsilon > 0$. Dividing the above inequality by ε and taking $\varepsilon \to 0+$ together with (eq:nullset 4) shows that

$$-\int_{u_\alpha \neq f} \operatorname{sgn}(u_\alpha - f)h \leq \int_{u_\alpha = f} |h| + \alpha \|Dh\| \;. \tag{15}$$ eq:lower

Let $\varepsilon < 0$ and using the same argumentation, we get

$$-\int_{u_\alpha \neq f} \operatorname{sgn}(u_\alpha - f)h \geq -\int_{u_\alpha = f} |h| - \alpha \|Dh\| \;. \tag{16}$$ eq:upper

The last two inequalities show that

$$\left(\left| \int_{u_\alpha \neq f} \operatorname{sgn}(u_\alpha - f)h \right| - \int_{u_\alpha = f} |h| \right)^+ \leq \alpha \|Dh\| \;. \tag{17}$$ eq:general

From Lemma (le:char 3.2) the assertion follows.

(eq:gration1 12) follows from (eq:general 17) with $h = u_\alpha - f$. $\square$

We have shown that $|\partial\phi|\,(u_\alpha - f) = G(\partial\,|u_\alpha - f|)$ if $u_\alpha - f \in W_0^{1,1}$. Under this assumption Lemma (le:chara 5.1) follows from (eq:ambrosio 10).

le42 **Theorem 5.2.** $u_\alpha \equiv 0$ *if and only if*

$$G(\partial |f|) \leq \alpha \;. \tag{18}$$ eq:gvalue

Proof. 1. From (eq:general 17) and the fact that $G(\partial\,|f|) = G(\partial\,|-f|)$ it follows the first direction of the assertion if $u_\alpha \equiv 0$.

2. To prove the converse direction we use the convexity of $|\cdot|$, which shows that for all $v, h \in \mathtt{BV}$

$$|v + h - f| - |v - f| - \psi_v h - |h|\,\chi_{v=f} \geq 0 \text{ point-wise} \;. \tag{19}$$

Note that in the last equality we use the sub-gradient property. Therefore, from (eq:gvalue 18) it follows that for all $h \in \mathtt{BV}$

$$\begin{aligned}
\mathcal{F}(h) - \mathcal{F}(0) &= \int |h - f| - |-f| + \alpha \|Dh\| \\
&\geq -\int_{f \neq 0} \operatorname{sgn}(f)h + \int_{f=0} |h| + \alpha \|Dh\| \\
&\geq -\left| \int_{f \neq 0} \operatorname{sgn}(f)h \right| + \int_{f=0} |h| + \alpha \|Dh\| \\
&\geq 0 \;.
\end{aligned}$$

Thus 0 is a global minimizer of $\mathcal{F}$.

$\square$

co:chf **Corollary 5.3.** *$u_\alpha = f$ if and only if* $\left|\partial\tilde{\phi}\right|(f) \le \frac{1}{\alpha}$.

Proof. 1. If $u_\alpha = f$, then for all $h \in \mathtt{BV}$

$$\int |h - f| + \alpha\|Dh\| \ge \alpha\|Df\| ,$$

which shows that

$$\left|\partial\tilde{\phi}\right|(f) = \mathcal{I}_{\tilde{\phi}}(f) \le \frac{1}{\alpha} .$$

2. Using (11) [eq:..] again we find that

$$\|Df\| - \|D(f+h)\| \le \frac{1}{\alpha}\int |h| ,$$

and therefore

$$\int |f + h - f| + \alpha\|D(f+h)\| \ge \alpha\|Df\| .$$

Since this holds for all $h \in \mathtt{BV}$, we see that f is a minimizer.

□

th:meyerII **Theorem 5.4.** *Assume that $f \in L^1$ satisfies $G(|\partial f|) > \alpha$.*
If $u = u_\alpha$ minimizes $\mathcal{F}$, then

1. $u \in \mathtt{BV}$,

2.

$$\alpha\|Du\| \in \left\{ -\int \psi u : \psi \in \partial\,|u - f| \right\} , \tag{20}$$

eq:ass1

3.

$$G(\partial\,|u - f|) \le \alpha . \tag{21}$$

eq:ass2

Proof. From the assumption $G(|\partial f|) > \alpha$ it follows that $0 \ne u_\alpha \in \mathtt{BV}$.
From Lemma 5.1 [le:char] (21) [eq:ass2] follows.
From the definition of a minimizer u_α of $\mathcal{F}$ it follows that for every $0 \ne |\varepsilon| < 1$

$$\begin{aligned}
&\int |u_\alpha - f| \ + \ \alpha\|Du_\alpha\| \\
\le &\int |(1+\varepsilon)u_\alpha - f| \ + \ \alpha(1+\varepsilon)\|Du_\alpha\| \\
\le &\int |u_\alpha - f| \ + \varepsilon \int_{u_\alpha \ne f} \operatorname{sgn}(u_\alpha - f)u_\alpha \ + \ + |\varepsilon| \int_{u_\alpha = f} |u_\alpha| \ + \ \alpha(1+\varepsilon)\|Du_\alpha\| .
\end{aligned}$$

Taking $\varepsilon \to 0^{\pm}$, this shows that

$$-\int_{u_\alpha \neq f} \operatorname{sgn}(u_\alpha - f) u_\alpha - \int_{u_\alpha = f} |f| \leq \alpha \|Du_\alpha\| \,,$$
$$-\int_{u_\alpha \neq f} \operatorname{sgn}(u_\alpha - f) u_\alpha + \int_{u_\alpha = f} |f| \geq \alpha \|Du_\alpha\| \,.$$

Now, we note that

$$-\int_{u_\alpha \neq f} \operatorname{sgn}(u_\alpha - f) u_\alpha - \int_{u_\alpha = f} |f| = \inf_{\psi \in \partial |u_\alpha - f|} -\int \psi f \,,$$
$$-\int_{u_\alpha \neq f} \operatorname{sgn}(u_\alpha - f) u_\alpha + \int_{u_\alpha = f} |f| = \sup_{\psi \in \partial |u_\alpha - f|} -\int \psi f \,.$$

which shows ([eq:ass1] 20).

□

6 Relation to the Literature

Chan & Esedoglu [ChaEse05] [2] characterized minimizers of the functional ([eq:l1l1] 1) when $f = \chi_\Omega$ under the assumptions that

$$\|Df\| = \int f \nabla \cdot \vec{\phi} \text{ for some } \vec{\phi} \in C_0^1 \text{ satisfying } \left|\vec{\phi}(x)\right| \leq 1 \text{ and } \left|\nabla \cdot \vec{\phi}(x)\right| \leq C \,.$$

In this case we have for all $u \in L^1$

$$\frac{\|Df\| - \|Du\|}{\int |u - f|} \leq \frac{\int \nabla (f - u) \vec{\phi}}{\int |u - f|} \leq C \,.$$

That is $\left|\partial \tilde{\phi}\right|(f) \leq C$, and consequently, if $C \leq \frac{1}{\alpha}$, then $u_\alpha = f$.

From Theorem [th:meyerII] 5.4 we get even more, namely that $u_\alpha = f \in \mathtt{BV}$ is a global minimizer, if and only if

$$\alpha \|Df\| \leq \int |f| \qquad (22) \quad \text{[eq:more]}$$

Note that this is ([eq:ass1] 20) and ([eq:ass2] 21) is always satisfied, since $u_\alpha = f$. To convince ourselves that this assertion is true we repeat the converse direction of the proof of Theorem th:meyerII 5.4 and find

$$
\begin{aligned}
&\int |h| \; + \; \alpha\|D(u+h)\| \\
\geq &\int |u-f| \; - \inf_{\psi\in\partial|u-f|} - \int \psi h + \; \alpha\|D(u+h)\| \\
\geq &\int |u-f| \; - \; \alpha\|Dh\| + \; \alpha\|D(u+h)\| \\
\geq &\int |u-f| \; + \; \alpha\|Du\| \; .
\end{aligned}
$$

7 Metrical regularization

A minimizer $u_\alpha = f$ can be guaranteed to be a minimizer of functionals of the form

$$d(u, f) + \alpha\psi(u)\,,$$

where $d(\cdot,\cdot)$ is a metric on a Banach space $\mathcal{B}$ and $\psi(\cdot) : \mathcal{B} \to (-\infty, \infty]$ is a convex, lower semi continuous functional. From Proposition 1.4.4 in [1] (AmbGigSav05) we know that for $f \in \mathcal{B}$

$$|\partial\psi|\,(f) = \mathcal{T}_\psi(f) := \sup \frac{(\psi(f) - \psi(u))^+}{d(f, u)}\,.$$

This shows that

co:chfa **Corollary 7.1.** *$u_\alpha = f$ if and only if $|\partial\psi|\,(f) \le \frac{1}{\alpha}$.*

We have considered already the metric on L^1 and the convex functional $\tilde{\phi}(u) = \|Du\|$, which results in the functional $\mathcal{F}$. Another example of a metric is $d(f, g) = \sqrt{\int |f - g|^2}$. The functional

$$\begin{aligned} \tilde{\tilde{\phi}} : L^2 &\to [0, \infty]\,, \\ u &\to \|Du\| \end{aligned}$$

is convex and lower semi-continuous. Application of the Corollary 7.1 (co:chfa) shows that $u_\alpha = f$ if and only if $\left|\partial\tilde{\tilde{\phi}}\right| \le \frac{1}{\alpha}$. Note that in this case u_α satisfies the Euler equation

$$\frac{u - f}{\sqrt{\int (u - f)^2}} \in \alpha\nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right)\,.$$

This is variant of the Rudin-Osher-Fatemi functional where the minimizer satisfies similar analytical properties as the minimizers of the functional $\mathcal{F}$. Note however, that the functional is strictly convex and thus the minimizer is unique. For the numerical solution a non-local PDE has to be solved.

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