

## Slow Drag in a Granular Medium

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We have studied the drag force acting on an object moving with low velocity through a granular medium. Although the drag force is a dynamic quantity, its behavior in this regime is dominated by the inhomogeneous distribution of stress in static granular media. We find experimentally that the drag force on a vertical cylinder is linearly dependent on the cylinder diameter, quadratically dependent on the depth of insertion, and independent of velocity. An accompanying analytical calculation based on the static distribution of forces arrives at the same result, demonstrating that the local theory of stress propagation in static granular media can be used to predict this bulk dynamic property. [S0031-9007(98)08142-3]

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Granular materials have many unusual properties of current interest, ranging from the inhomogeneous nature of stress distribution in granular piles [1,2] to the unusual dynamic properties of granular motion [3]. Little is known, however, about an apparently simple quantity: the drag force ( $F_d$ ) resisting a solid object moving slowly through a granular medium. The properties of  $F_d$  are determined by the complex nature of stress propagation in the bulk of the medium since the drag originates not only in the grains immediately in front of the object but also in the successive layers of grains supporting them. As for a viscous fluid,  $F_d$  is a fundamental property of a granular material and is important to any industrial process involving objects moving through granular matter.

In order to appreciate the complex nature of the drag force, one must consider that, in contrast with fluids, any force applied to a granular medium is resisted inhomogeneously and anisotropically by granular “chains” of force—narrow regions of strained grains surrounded by the bulk of the medium which is relatively unstrained [1,4]. Furthermore, in order for an object to move within a granular medium, all of the grains impeding its motion must be displaced from its path, requiring a large-scale reorganization of the material. The inhomogeneity of the force propagation and the requirement for grain reorganization give the force a stick-slip character, i.e., the force between the grains and the moving object fluctuates as strain builds and is released by reorganization of the grains. Therefore, the magnitude of  $F_d$  is determined by exactly how the inhomogeneously distributed force from the object acts to move the grains impeding its motion.

Previous studies of the granular drag force have been quite limited, although long experience in soil physics suggests that, due to its frictional origin, the velocity dependence will be much weaker than that in fluids [5]. There has been no theoretical effort to calculate the drag force and measurements have been performed only on media which were fluidized by vibrations [6] or when the dragged object moved so rapidly that the grains in

front were effectively fluidized [7]. These last studies by Wieghardt found that  $F_d$  depended nonmonotonically on the object’s velocity (first decreasing and then increasing with increasing velocity) and was proportional to the  $\frac{5}{2}$  power of the depth of insertion and the square root of its horizontal dimension.

While there has only been limited direct study of  $F_d$ , other workers have measured the related property of shear stress propagation in rotating shear cells. These studies were, however, intentionally restricted to high enough shear rates so as to avoid stick-slip fluctuations in the force—thus probing the same high velocity regime where the grains did not settle into static configurations. Indeed, Tardos *et al.* [8] observed the same nonmonotonic velocity dependence as Wieghardt, confirming the equivalence of the measurement regimes. Although these measurements can be successfully analyzed within the framework of a Coulomb stress analysis, since the grains are constantly in motion, connection with recent work on stress propagation in static grains [1] is difficult.

We have studied the drag force resisting an object traveling through a granular material in the low velocity regime, i.e., when the grains are not fluidized by the motion. While the drag force in this regime is a dynamic quantity, we find that it is determined by the static properties of the medium. It therefore offers a new bridge between recent developments in understanding static stress propagation in granular media [1,2,4] and the dynamic properties [3].

In our experiments a vertical cylinder of diameter  $d_c$  is extended a distance  $H$  into a bucket of granular material and held fixed while the bucket is slowly rotated (see Fig. 1). The granular medium consisted of glass spheres with diameters  $d_g = 0.41 \pm 0.05$ ,  $0.88 \pm 0.03$ , and  $3.0 \pm 0.05$  mm where the larger two sizes had polished smooth surfaces and the smallest were unpolished with visibly rough surfaces. The rotating bucket is constructed with an open central tube through which a concentric shaft is mounted on low friction

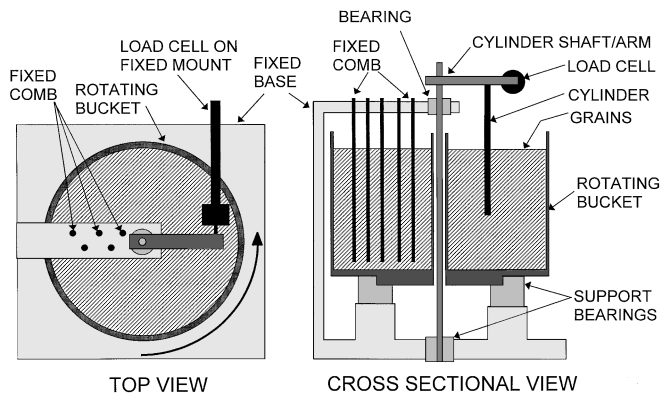


FIG. 1. Schematic drawings of the apparatus used in the experiments as described in the text (not to scale). The rotating bucket was 25.5 cm in diameter and 23.5 cm deep.

bearings. The cylinder which extends into the medium is mounted on an arm fixed to this shaft which is mechanically isolated from the rotation of the bucket except through the torque on the cylinder from the granular medium. The arm holding the cylinder is held fixed (i.e., prevented from rotating with the bucket) by the sensor of a load cell which measures the stopping force which is equivalent to  $F_d$ . In order to prevent the cylinder from repeatedly traveling through a trough created by previous rotations, the granular environment is randomized by mixing the medium with a fixed “comb” consisting of a series of thin steel rods inserted almost to the complete depth of the bucket and held fixed during the rotation. This apparatus allows us to measure  $F_d$  while independently varying the cylinder diameter, grain diameter ( $d_g$ ), velocity ( $v$ ), and depth to which the cylinder is submerged. The largest uncertainty in the determination of the force arose from fluctuations in  $F_d$ , which led to a typical uncertainty of  $\pm 1\%$  and no more than  $\pm 5\%$ . We restrict our velocities to the regime  $v < \sqrt{2gd_g}/10$  to insure that we are indeed in the low velocity regime as discussed below.

The data from this apparatus are shown in Figs. 2, 3, and 4. As the inset in Fig. 2 illustrates, the raw data reflect the stick-slip nature of  $F_d$  in the static regime: The drag force on the cylinder consists of a series of linear increases followed by sudden drops corresponding to reorganization of the grains directly impeding motion. In Fig. 3 we show the velocity dependence of  $F_d$  for a variety of values of  $H$ ,  $d_c$  and  $d_g$ . To within the uncertainty of the measurements ( $\sim 1\%$ ), we find that the drag force is independent of velocity. While the velocity dependence is expected to be weak, previous studies of the shear stress at higher velocities [8] did measure a decrease of  $\sim 10\%$  in the stress with a decade increase in velocity at their lowest velocities. By contrast, in the low velocity regime we observe *no* change in  $F_d$  over a factor of  $\sim 30$  variation in velocity.

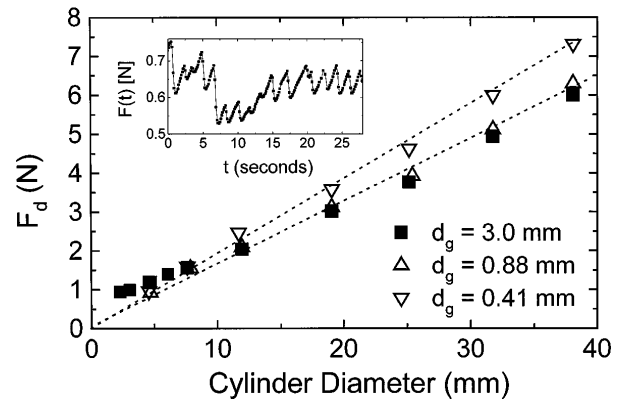


FIG. 2. The dependence of the drag force ( $F_d$ ) on the diameter of the cylinder ( $d_c$ ) for  $v = 1.5$  mm/s. The dashed lines are guides to the eye. The inset shows the fluctuations in the force with  $d_g = 0.88$  mm,  $d_c = 4.7$  mm,  $H = 48$  mm, and  $v = 0.041$  mm/s.

As seen in Fig. 2, we find  $F_d$  to be proportional to  $d_c$  and independent of  $d_g$  for the two larger grain sizes and increasing by only  $\sim 15\%$  for the smallest grains. This increase is not surprising given the frictional origin of the drag force and the significantly rougher surfaces of the smallest grains. The only deviation from linearity is seen for the largest grains ( $d_g = 3.0$  mm) and smallest cylinders where the force is larger than would be expected assuming linearity. This deviation might be expected (and can be understood within our calculations below), since even when  $d_c \rightarrow 0$ , the cylinder must push grains aside as it moves relative to the medium, and must therefore approach a constant value for small  $d_c$  ( $< d_g$ ). The depth dependence of  $F_d$  is shown for several combinations of

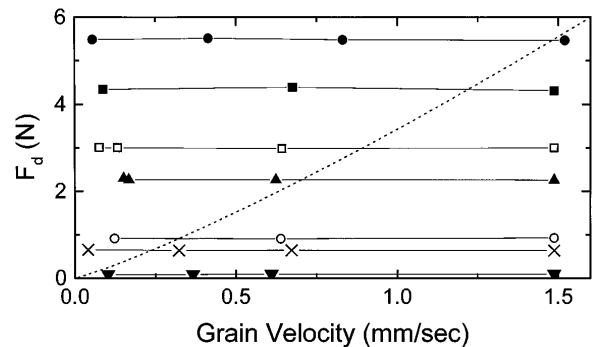


FIG. 3. The velocity dependence of the measured drag force ( $F_d$ ) for various parameters (given in mm).  $\bullet$ :  $d_c = 38$ ,  $d_g = 0.41$ ,  $H = 44$ ;  $\blacksquare$ :  $d_c = 4.7$ ,  $d_g = 3.0$ , and  $H = 95$ ;  $\square$ :  $d_c = 19$ ,  $d_g = 3.0$ , and  $H = 51$ ;  $\blacktriangle$ :  $d_c = 4.7$ ,  $d_g = 0.88$ , and  $H = 88$ ;  $\circ$ :  $d_c = 4.7$ ,  $d_g = 3.0$ , and  $H = 45$ ;  $\times$ :  $d_c = 4.7$ ,  $d_g = 0.88$ , and  $H = 48$ ; and  $\blacktriangledown$ :  $d_c = 4.7$ ,  $d_g = 0.88$ , and  $H = 18$ . The dashed line shows for comparison the velocity dependence of the expected viscous drag force on a 100-mm-long rod with  $d_c = 4.7$  mm moving through Glycerin at 20 °C (multiplied by 10000 so that it could be plotted on the same scale).

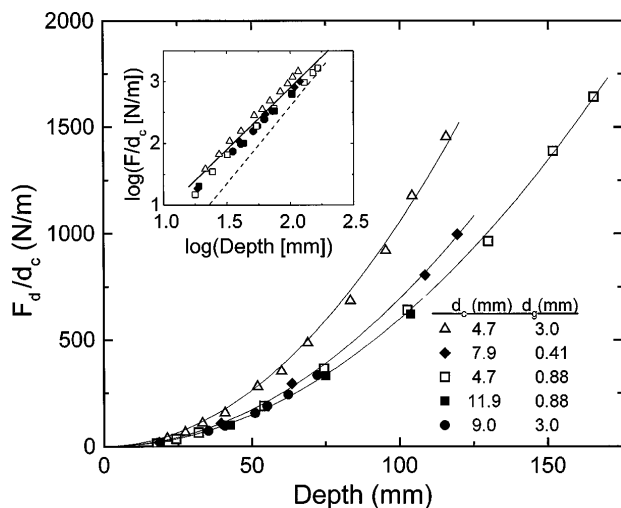


FIG. 4. The depth ( $H$ ) dependence of the drag force ( $F_d$ ). The data were all taken at  $v = 1.5$  mm/sec with different grain and cylinder diameters and normalized to the cylinder size. Note that we obtain a larger normalized force when  $d_c \sim d_g$  (open triangles), since  $F_d$  does not approach zero when  $d_c < d_g$ . The solid lines are one parameter fits of the form  $F_d/d_c = \eta g \rho H^2$ , which, using  $\rho = 2.5$  g/cm<sup>3</sup> gives  $\eta = 4.32$  ( $\triangle$ ),  $\eta = 2.83$  ( $\blacklozenge$ ), and  $\eta = 2.43$  ( $\blacksquare$ ,  $\square$ , and  $\bullet$ ). The inset shows the same data on a logarithmic scale where the solid line has a slope of 2. The  $\frac{5}{2}$  power dependence observed by Wieghardt is shown by the dashed line.

$d_c$  and  $d_g$  in Fig. 4. As shown by the solid lines in the figure, and by the log-log plot shown in the inset,  $F_d$  is convincingly quadratic in depth in each case. The slight deviation for the smallest  $H$  is expected since the force in that regime is entirely due to the grains near the top surface which have much higher dilatancy. Our experimental data can be summarized by the empirical relation  $F_d \propto d_c H^2$  which is explicitly independent of grain size and the velocity of motion.

The above data led us to consider the open theoretical question of how to calculate  $F_d$ . We consider a model corresponding to our experiment in which a stationary vertical cylinder extends to a depth  $H$  below the surface of a slowly moving granular medium. The complexity of the problem arises from the anisotropic and inhomogeneous nature of force propagation which is intrinsic to granular media, i.e., the fact that the drag consists not of a uniform resistance, but a strongly fluctuating force associated with the force chains.

The cylinder cannot move relative to the medium until all of the grains opposing the motion are reorganized, which requires the grains in the bulk to slip relative to each other. Such reorganization therefore occurs when the local transverse force between the grains exceeds the local static friction at some point along each force chain, requiring that the external force with which the cylinder acts on the granular medium exceeds a threshold value  $F_T$ . As the motion of the cylinder relative to the

granular medium is blocked until the grains reorganize, a linearly increasing horizontal force  $f(t)$  builds up between the cylinder and the granular medium. When the force exceeds  $F_T$ , the grains suddenly reorganize near the cylinder, and such reorganizations result in repeated sharp drops in  $f(t)$  as the cylinder moves through the medium, giving a stick-slip character to the force of the sort seen in the raw data of the inset to Fig. 2. In the low velocity regime, the cylinder repeatedly encounters a static configuration of grains which require approximately the same threshold force,  $F_T$ , to reorganize. Therefore, the drag force (i.e., the average external force required to hold the cylinder stationary) is proportional to  $F_T$ . We also expect  $F_d$  to be independent of the velocity, since the distance between the consecutive static configurations blocking the cylinder motion depends only on characteristics of the granular medium, such as packing fraction and intergrain friction. This velocity independence will be limited to the regime of velocities smaller than the velocity of the grains rearranging in front of the cylinder  $v \ll v_{\text{rearrange}} \cong \sqrt{2gd_g} \sim 0.1$  m/s for 1 mm grains). At higher velocities, the grains are fluidized, i.e., they do not settle after the cylinder breaks a given grain configuration.

To calculate  $F_d$ , we assume that the system is made up of  $h = H/d_g$  discrete horizontal layers of the granular medium. In a given layer  $j$ , the grain configuration is stable until the local force between a pair of grains within a layer is larger than a critical force  $F_c$ . Physically,  $F_c$  is the force necessary to make a grain slip relative to another somewhere in the layer, and it is proportional to the static friction between the grains. In turn, the static friction is proportional to the local normal component of the force acting at the bead contact, controlled by the average pressure, which for large containers is known to increase linearly with the depth  $j$  [9]. Consequently, we obtain  $F_c = Agm_g j$  where  $A$  is a constant depending on the surface properties, morphology, and packing of the grains,  $g$  is the gravitational acceleration, and  $m_g$  is the mass of the individual grains [10].

Given this expression for  $F_c$ , we calculate the probability that in a given layer  $j$ , the inhomogeneous force distribution will lead the local force to exceed  $F_c$ . For this calculation, we use the  $q$  model of Coppersmith *et al.* [1] which was originally applied to the force chains resultant from gravitational force on the grains. The  $q$  model assumes that the physical origin of the force chains lies in the inhomogeneity of packing. In the model, the total force acting on a given particle is transmitted unevenly to  $N$  adjacent particles such that a fraction  $q_{ij}$  of the force acting on particle  $i$  will be transmitted to particle  $j$ . In the  $q$  model  $q_{ij}$  is a random variable with the constraint of force balance on each particle. We extend the model to calculate the probability distribution of intergrain forces within each layer, assuming that the source of the force

on the layer is an object of transverse size  $d_c$  exerting a horizontal force  $F/h$ . The probability  $p_j$  that the grains within a particular layer will meet the criteria to reorganize is obtained by integrating over the entire layer the probability that a single local force between grains exceeds  $F_c$ , leading to [11]

$$p_j(h, F) = 1.5[Fd_g/(Aghjm_g d_c)]^2. \quad (1)$$

For the cylinder to move relative to the medium, the grains in all layers must simultaneously meet the criteria to reorganize, thus the probability of relative motion is given by  $\prod_{j=1}^h p_j$ . The expectation value of  $F(\langle F \rangle \equiv F_d)$  can therefore be obtained by setting  $\prod_{j=1}^h p_j = 1$ , this condition being satisfied when  $F = \sqrt{2/3}e^2 Am_g H^2 d_c / d_g^3$ , leading to

$$F_d = \eta g \rho H^2 d_c, \quad (2)$$

where  $\rho$  is the density of the individual solid grains, and  $\eta = A\sqrt{32\pi^2/27}e^2$  [12].

Equation (2) is in agreement with the experimental measurements of the  $H$  and  $d_c$  dependence of  $F_d$  and provides the equivalent of a granular Stokes Law for an object moving through a granular medium. The calculations are valid for any object of transverse size  $d_c$ , not just a cylinder, so the shape dependence comes in the value of  $\eta$ , and not in the functional form of the drag force [13]. This theoretical result can be easily extended to objects fully immersed in a granular medium and also to the important case of grains deep in a silo where the pressure is depth independent.

Since we find that  $F_d$  is not time independent but rather has strong fluctuations (Fig. 2, inset), this study also suggests that the drag can provide a direct measure of force fluctuations inside the bulk of granular media, a subject of considerable current interest [3] which has heretofore been largely limited to probes at the containers' walls. While the cylinder itself may be affecting the grain configuration and therefore the bulk granular properties, the very limited impact of the shape of the cylinder [13] suggests that the insertion of the cylinder does not drastically alter the granular properties. The incorporation of the  $q$  model into our analysis allows us to systematically treat non-mean-field dynamic quantities such as the fluctuation distribution within the framework of an analytical theory, thereby building a bridge between the newly developed understanding of force propagation in static granular media and the bulk dynamic behavior.

The measurement of the drag force also offers a new tool for probing other physical properties of granular media, such as the development of correlations due to the presence of interstitial liquid [14].

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  - [9] R. L. Brown and J. C. Richards, *Principles of Powder Mechanics* (Pergamon Press, Oxford, 1970). Note that while the pressure is depth independent for depth much larger than the container width, our rotating bucket has an aspect ratio near unity.
  - [10] Note that the increase of  $F$  with depth implies that the force acting on the bottom portion of the cylinder is larger than that acting on its upper regions.
  - [11] R. Albert (unpublished); see also <http://www.nd.edu/~granular>
  - [12] Note that Eq. (2) could also be derived from mean-field arguments, assuming that the rod is opposed by a force that linearly increases with the depth. However, such a mean field calculation would not be *a priori* reliable since it does not account for important aspects of the physics such as the strong fluctuations in the force distribution or the stick-slip character of the motion. A quadratic depth dependence was in fact also observed by Tardos *et al.* [8] in their shear experiments at all velocities where it could be justified in terms of a mean-field calculation due to the high velocities probed.
  - [13] Measurements of the force on a half-cylinder (bisected along a plane through the axis and normal to the grain flow) show identical results with  $F_d$  increased by  $\sim 5\%$ .
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