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In this work we study the effect of static disorder on the growth of commutators—a probe of information scrambling in quantum many-body systems—in a variety of contexts. We find generically that disorder slows the onset of scrambling and, in the case of a many-body localized (MBL) state, partially halts it. In the MBL state, we show using a fixed point Hamiltonian that operators exhibit slow logarithmic growth under time evolution and compare the result with the expected growth of commutators in (de)localized noninteracting disordered models. Finally, using a scaling argument, we state a conjecture on the growth of commutators in a weakly interacting diffusive metal.

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Introduction. Understanding the nature of thermalization in closed quantum systems is one of the great challenges of modern many-body physics [1–4], especially in light of many recent experiments probing thermalization in isolated quantum many-body systems [5–12]. Of particular interest are the time scales for various aspects of thermalization, from early-time relaxation to scrambling at intermediate times to the late-time buildup of complexity [13]. Given a many-body Hamiltonian with local interactions, relaxation describes the initial decay of local perturbations as measured by simple autocorrelation functions. Scrambling describes the slower spreading of quantum information across all the degrees of freedom of the system, rendering such information invisible to local probes [14–16]. Scrambling is distinct from relaxation, with the time needed to scramble information over a set of degrees of freedom typically scaling in some way with the number of said degrees of freedom.

It is interesting to study the effects of static disorder on the process of thermalization because disorder is common in experimental systems and because it can lead to qualitatively new physics. In the limit of noninteracting particles, weak disorder in low dimensions or sufficiently strong disorder in three or more dimensions causes localization [17,18], which completely arrests thermalization. However, noninteracting particles, being integrable, already fail to thermalize, even when they remain delocalized. Thus it is particularly interesting to study scrambling in interacting disordered systems. It is known that the noninteracting delocalized limit can remain metallic in the presence of interactions, and recent work has shown that the localized limit is also stable to interactions [19–23], resulting in many-body localization [24,25].

To probe scrambling in these systems, we study the growth of commutators of local operators. The study of such commutators is closely related to the physics of classical chaos [26] and diagnoses a quantum version of the butterfly effect, whereby a small local perturbation eventually spreads over the entire system [27]. To set up the precise computations, consider two local unitary operators, V and W, along with time evolution specified by a many-body Hamiltonian H. Define the squared commutator $C(t)$ to be [28]

$$C(t) = \langle [W(t),V]^\dagger [W(t),V]\rangle = 2(1 - \mathrm{Re}[F(t)]), \qquad (1)$$

where $W(t) = e^{iHt} W e^{-iHt}$ is a Heisenberg operator and where $F(t)$ is a so-called out-of-time-order (OTO) correlator

$$F(t) = \langle W^\dagger(t) V^\dagger W(t) V\rangle. \qquad (2)$$

Here the average $\langle \cdots \rangle$ is taken over any quantum state ρ of interest; a natural choice, which we focus on below, is to take a product state or some short-range correlated state.

The OTO correlator $F(t)$ is our primary object of study. The physical picture is this: W is meant to correspond to a simple local perturbation which grows in size and complexity under time evolution. The commutator of $W(t)$ with other simple operators V diagnoses the growth of $W(t)$. The squared commutator is studied to prevent unwanted cancellations and to diagnose typical matrix elements of the commutator. The OTO correlator is a cousin of the Loschmidt echo, but generally probes different physics (see, e.g., [29]). We will be particularly interested in the disorder average of F as well as the disorder average of $|F|^2$. We emphasize that the system (ensemble of systems) only really scrambles if, for a given disorder realization (typical disorder realization), the OTO correlator becomes small and remains small for an extended period of time.

OTO correlators first appeared many years ago in the context of semiclassical methods in superconductivity [30], and they have received renewed attention in the context of the AdS/CFT correspondence where they were shown to diagnose quantum chaos in black hole physics [27,31,32]. Very recently, it has been shown how to measure OTO correlators [33–35] and hence scrambling. The broad relevance of scrambling in quantum many-body dynamics has also been emphasized [33]; for example, scrambling diagnoses the growth of quantum chaos [30–32] and the spread of entanglement [16].

In the spirit of investigating scrambling across a wide variety of physical systems, we study the OTO correlator $F(t)$ in simple disordered local many-body models. In the presence of interactions the physics of OTO correlators is typically distinct from the physics of time-ordered correlation functions [36], although for noninteracting particles everything is

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determined from two-point functions via Wick's theorem. The basic consequence of locality is approximate causality: If W and V are local operators separated in space by a distance d, then $F(t)$ remains close to 1, meaning V and $W(t)$ commute, until the operator $W(t)$ has grown in space to size d. Our results are stated in terms of the the "operator radius" of $W(t)$, denoted $R_W(t)$, which is defined as the distance d such that $F(t)$ significantly deviates from 1 for operators V located less than or equal to d away from the location of W. Because of Eq. (1), this operator radius also defines the region within which the square commutator $C(t)$ is significantly different from zero. Physically, the operator radius $R_W(t)$ defines the region of space over which information in the initially localized perturbation W has spread after time t.

In a localized noninteracting particle state, commutators simply do not grow beyond the localization length ξ, so $R_W(t) \sim \xi$. In a noninteracting diffusive metal, commutators grow in space diffusively, with $R_W(t) \sim \sqrt{Dt}$ and D the diffusion constant, but ultimately become small again at late time (with recurrences in finite size systems). Including interactions in the single-particle localized state gives a many-body localized state, and using a simple fixed-point model for the many-body localized state we show that $R_W(t) \sim \xi \log(\Delta t)$ for nonconserved local operators, where Δ encodes the typical strength of interactions. This is consistent with Lieb-Robinson bounds [37–40]. Finally, we give a scaling argument that in a diffusive metal weak interactions lead to a ballistic growth, $R_W(t) \sim v_B t$, with a small "butterfly velocity" [41,42], $v_B \sim \sqrt{D\Gamma}$ where Γ is a small interaction-induced inelastic scattering rate.

Noninteracting particles. To orient the discussion, we first recall the behavior of commutators in noninteracting particle models. As stated above, it is sufficient to study just the anticommutator, $A_{\boldsymbol{r}\boldsymbol{r}'}(t)$, of the underlying fermion field for noninteracting models, $A_{\boldsymbol{r}\boldsymbol{r}'}(t) = \langle\{c_{\boldsymbol{r}}(t), c^\dagger_{\boldsymbol{r}'}\}\rangle$, where $c_{\boldsymbol{r}}$ ($c^\dagger_{\boldsymbol{r}}$) represents a fermionic annihilation (creation) operator at site $\boldsymbol{r}$, satisfying the usual anticommutation algebra: $\{c_{\boldsymbol{r}}, c^\dagger_{\boldsymbol{r}'}\} = \delta_{\boldsymbol{r}\boldsymbol{r}'}$. When A is small, then commutators of localized unitary operators built from the $c_{\boldsymbol{r}}$ will also be small. As a concrete model, consider the tight-binding model [17] of noninteracting (NI) fermions on an infinite d-dimensional lattice, with nearest-neighbor hopping and a static random on-site potential. In one and two dimensions, an infinitesimal disorder induces localization of the eigenstates [17,18]; in three dimensions a critical amount of disorder is required to drive localization. The anticommutator in any noninteracting fermion model is simply

$$A_{\boldsymbol{r}\boldsymbol{r}'}(t) = \sum_\alpha e^{-iE_\alpha t}\phi_\alpha(\boldsymbol{r})\phi^*_\alpha(\boldsymbol{r}'), \tag{3}$$

where $\phi_\alpha(\boldsymbol{r})$ are the single-particle energy eigenstates and E_α are the single-particle energies. Note that $A_{\boldsymbol{r}\boldsymbol{r}'}(t)$ is state independent, and is simply the single-particle propagator, i.e., the amplitude for a particle to move from site $\boldsymbol{r}$ to site $\boldsymbol{r}'$ in time t. We will consider two cases, the localized state and the delocalized state, and leave a discussion of critical points for future work.

In the case of the localized phase, the anticommuator never grows large. Because A is a sum over single-particle states, it follows that if every single-particle state is localized, then A remains exponentially small. Ignoring the oscillating (time-dependent) phases, which can only make A smaller, we may estimate the size of A by assuming that every state ϕ_α is exponentially localized around some site $\boldsymbol{r}_\alpha$, so that the sum over α becomes [43]

$$A_{\boldsymbol{r}\boldsymbol{r}'} \lesssim \sum_\alpha e^{-|\boldsymbol{r}-\boldsymbol{r}_\alpha|/\xi - |\boldsymbol{r}'-\boldsymbol{r}_\alpha|/\xi} e^{i\Theta_\alpha(\boldsymbol{r},\boldsymbol{r}')}, \tag{4}$$

where $\Theta_\alpha(\boldsymbol{r},\boldsymbol{r}')$ represents the phases associated with the overlap of $\phi_\alpha(\boldsymbol{r})\phi^*_\alpha(\boldsymbol{r}')$. The disorder average of $A_{\boldsymbol{r}\boldsymbol{r}'}$ (denoted $\overline{A_{\boldsymbol{r}\boldsymbol{r}'}}$) is zero as a result of averaging over the phases. However, assuming that the $\boldsymbol{r}_\alpha$ are distributed roughly uniformly in space, it immediately follows that $\overline{|A_{\boldsymbol{r}\boldsymbol{r}'}|^2} \lesssim e^{-|\boldsymbol{r}-\boldsymbol{r}'|/\xi}$.

In the case of the delocalized phase, the resulting many-body state is a diffusive metal. Conduction of charge and heat take place, and the density-density response function exhibits a diffusion pole [44]. Within the Born approximation, we have $\overline{A_{\boldsymbol{r}\boldsymbol{r}'}} = A^{\text{clean}}_{\boldsymbol{r}\boldsymbol{r}'}\exp(-\frac{t}{2\tau})$, with the simplifying assumption of an energy-independent scattering rate τ. However, as is well known, this quantity is not a good measure of the fermion motion. A better picture of the dynamics is obtained by $\overline{|A_{\boldsymbol{r}\boldsymbol{r}'}|^2}$. This calculation is closely related to the disorder average of the density-density response function [44], and exhibits diffusive behavior, $\overline{|A_{\boldsymbol{r}\boldsymbol{r}'}(t)|^2} \sim \exp(-\frac{|\boldsymbol{r}-\boldsymbol{r}'|^2}{2Dt})/t^{d/2}$.

Many-body localized state. We now consider the effects of interactions, first on the localized state. Assuming that the interacting state is a many-body localized state, we study a standard "fixed-point" Hamiltonian [45–49] of N spin-1/2 of the form

$$H_{\text{MBL}} = \sum_{\boldsymbol{r}_1} J_{\boldsymbol{r}_1} Z_{\boldsymbol{r}_1} + \frac{1}{2}\sum_{\boldsymbol{r}_1\neq\boldsymbol{r}_2} J_{\boldsymbol{r}_1\boldsymbol{r}_2} Z_{\boldsymbol{r}_1} Z_{\boldsymbol{r}_2} + \frac{1}{3!}\sum_{\boldsymbol{r}_1\neq\boldsymbol{r}_2\neq\boldsymbol{r}_3} J_{\boldsymbol{r}_1\boldsymbol{r}_2\boldsymbol{r}_3} Z_{\boldsymbol{r}_1} Z_{\boldsymbol{r}_2} Z_{\boldsymbol{r}_3} + O(Z^4), \tag{5}$$

where $Z_{\boldsymbol{r}}$ is the z Pauli operator of spin $\boldsymbol{r}$. The couplings $J_{\boldsymbol{r}_1}, J_{\boldsymbol{r}_1\boldsymbol{r}_2}, \dots$, are assumed to be drawn at random from Gaussian distributions of mean zero and variance $\Delta_n^2(\boldsymbol{r}_1, \dots, \boldsymbol{r}_n)$ for the n-spin coupling $J_{\boldsymbol{r}_1,\dots,\boldsymbol{r}_n}$. The variances Δ_n^2 are assumed to decay with n and with the separation between the $\boldsymbol{r}_i$. Such a Hamiltonian can be viewed as arising from H_{NI} by adding interactions and taking the limit $w \to 0$. As discussed in [45,48], this fixed-point Hamiltonian is expected to be sufficient to describe the coarse-grained physics of an entire many-body localized phase.

The time evolution of any local spin operator is given by precession about the z axis in an effective field,

$$h_{\boldsymbol{r}} = J_{\boldsymbol{r}} + \sum_{\boldsymbol{r}_1\neq\boldsymbol{r}} J_{\boldsymbol{r}\boldsymbol{r}_1} Z_{\boldsymbol{r}_1} + \frac{1}{2}\sum_{\boldsymbol{r}_2\neq\boldsymbol{r}_1\neq\boldsymbol{r}} J_{\boldsymbol{r}\boldsymbol{r}_1\boldsymbol{r}_2} Z_{\boldsymbol{r}_1} Z_{\boldsymbol{r}_2} + O(Z^3), \tag{6}$$

which is itself an operator that depends on the z components of all other spins. The time evolution of any operator $O_{\boldsymbol{r}}$ on site $\boldsymbol{r}$ is given by $O_{\boldsymbol{r}}(t) = e^{ith_{\boldsymbol{r}}Z_{\boldsymbol{r}}} O_{\boldsymbol{r}} e^{-ith_{\boldsymbol{r}}Z_{\boldsymbol{r}}}$. A basis of local operators is provided by the Pauli operators $\{X_{\boldsymbol{r}}, Y_{\boldsymbol{r}}, Z_{\boldsymbol{r}}\}$; in terms of these operators, the $Z_{\boldsymbol{r}}$ are exactly conserved in time, while $X_{\boldsymbol{r}}$ and $Y_{\boldsymbol{r}}$ rotate into each other at a rate determined by $h_{\boldsymbol{r}}$.

To build intuition, assume first that $J_{\boldsymbol{r}_1\boldsymbol{r}_2\boldsymbol{r}_3}$ and all higher-order terms are zero. Then a significantly nonzero commutator will develop between two operators $O_{\boldsymbol{r}_1}(t)$ and $O_{\boldsymbol{r}_2}$ after a time t of order $1/J_{\boldsymbol{r}_1\boldsymbol{r}_2}$. In a many-body localized phase, where $J_{\boldsymbol{r}_1\boldsymbol{r}_2}$ is expected to decay exponentially with distance ($J_{\boldsymbol{r}_1\boldsymbol{r}_2} \sim \Delta e^{-|\boldsymbol{r}_1-\boldsymbol{r}_2|/\xi}$), it follows that a nonzero commutator will develop only after a time exponentially long in the distance between the operators. Phrased in terms of the operator radius, we have $R_W(t) \sim \xi \log(\Delta t)$, representing a slow logarithmic growth of nonconserved operators. This feature is also responsible for the logarithmic growth of entanglement in the MBL phase [47,50]; these are in fact related statements [16]. Note, however, that without including higher spin interactions, the commutator is exactly periodic in time with period $2\pi/J_{\boldsymbol{r}_1\boldsymbol{r}_2}$, so a fixed realization of disorder only weakly scrambles.

Now consider including the neglected multispin interactions. It is useful to define an effective $J^{\text{eff}}_{\boldsymbol{r}_1\boldsymbol{r}_2} = \frac{\partial h_{\boldsymbol{r}_1}}{\partial Z_{\boldsymbol{r}_2}}$ by

$$J^{\text{eff}}_{\boldsymbol{r}_1\boldsymbol{r}_2} = J_{\boldsymbol{r}_1\boldsymbol{r}_2} + \sum_{\boldsymbol{r}_3} J_{\boldsymbol{r}_1\boldsymbol{r}_2\boldsymbol{r}_3} Z_{\boldsymbol{r}_3} + \frac{1}{2}\sum_{\boldsymbol{r}_3\boldsymbol{r}_4} J_{\boldsymbol{r}_1\boldsymbol{r}_2\boldsymbol{r}_3\boldsymbol{r}_4} Z_{\boldsymbol{r}_3} Z_{\boldsymbol{r}_4} + \cdots , \tag{7}$$

because this quantity is an operator that depends on the environment of other spins and can lead to dephasing.

For concreteness, consider the squared commutator $C(t)$ introduced in Eq. (1) above with the identification, $W \equiv X_{\boldsymbol{r}} = S^+_{\boldsymbol{r}} + S^-_{\boldsymbol{r}}$ and $V \equiv X_{\boldsymbol{r}'} = S^+_{\boldsymbol{r}'} + S^-_{\boldsymbol{r}'}$, where $S^{\pm}_{\boldsymbol{r}} = (X_{\boldsymbol{r}} \pm iY_{\boldsymbol{r}})/2$ are spin ladder operators. With this special choice, W and V are unitary and Hermitian. The time development of $S^{\pm}_{\boldsymbol{r}}$ is simple: $S^{\pm}_{\boldsymbol{r}}(t) = e^{\pm i h_{\boldsymbol{r}} t} S^{\pm}_{\boldsymbol{r}}$, so the OTO correlator is

$$F(t) = \left\langle \left(S^+_{\boldsymbol{r}} e^{i h_{\boldsymbol{r}} t} + S^-_{\boldsymbol{r}} e^{-i h_{\boldsymbol{r}} t}\right) X_{\boldsymbol{r}'} \left(S^+_{\boldsymbol{r}} e^{i h_{\boldsymbol{r}} t} + S^-_{\boldsymbol{r}} e^{-i h_{\boldsymbol{r}} t}\right) X_{\boldsymbol{r}'} \right\rangle, \tag{8}$$

$$F(t) = \left\langle S^+_{\boldsymbol{r}} S^-_{\boldsymbol{r}} e^{i J^{\text{eff}}_{\boldsymbol{r}\boldsymbol{r}'} t} + S^-_{\boldsymbol{r}} S^+_{\boldsymbol{r}} e^{-i J^{\text{eff}}_{\boldsymbol{r}\boldsymbol{r}'} t} \right\rangle. \tag{9}$$

In the last equality we have used $X_{\boldsymbol{r}'} h_{\boldsymbol{r}}(\ldots, Z_{\boldsymbol{r}'}, \ldots) = h_{\boldsymbol{r}}(\ldots, -Z_{\boldsymbol{r}'}, \ldots) X_{\boldsymbol{r}'}$, $(S^{\pm}_{\boldsymbol{r}})^2 = 0$, and $X^2_{\boldsymbol{r}'} = 1$ to simplify $F(t)$. Noting that $J^{\text{eff}}_{\boldsymbol{r}\boldsymbol{r}'}$ must remain inside the expectation value since it is an operator, the physics of $F(t)$ will thus be controlled by an average of a phase $e^{\pm i J^{\text{eff}}_{\boldsymbol{r}\boldsymbol{r}'} t}$ over different spin configurations. Such an average will generically lead to dephasing.

We now give some quantitative formulas for the case where $J_{\boldsymbol{r}_1\boldsymbol{r}_2\boldsymbol{r}_3}$ is nonzero, with mean zero and variance $\Delta_3^2(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3)$, but all higher-order multispin interactions are set to zero. The general case involves a trivial extension of the reported formulas and should only enhance dephasing. With only $J_{\boldsymbol{r}_1\boldsymbol{r}_2\boldsymbol{r}_3}$ nonzero, the effective coupling is $J^{\text{eff}}_{\boldsymbol{r}\boldsymbol{r}'} = J_{\boldsymbol{r}\boldsymbol{r}'} + \sum_{\boldsymbol{s}\neq \boldsymbol{r},\boldsymbol{r}'} J_{\boldsymbol{r}\boldsymbol{r}'\boldsymbol{s}} Z_{\boldsymbol{s}}$.

There are two sources of randomness in $J^{\text{eff}}_{\boldsymbol{r}\boldsymbol{r}'}$: the random couplings themselves and the quantum operators $Z_{\boldsymbol{s}}$. We first analyze the effects of the quantum operators. For illustrative purposes, assume for the remainder of the calculation that the $Z_{\boldsymbol{s}}$ are uncorrelated. We believe similar results will hold for generic short-range entangled states because it should be possible to use a coarse-graining procedure to produce local spin blocks which are effectively decoupled. Since the many-body localized state is a stable phase, this coarse-graining procedure should not alter the basic physics. We emphasize that we are not in general considering a thermal state, because thermal states are not privileged in models which fail to thermalize.

Viewing the J's as fixed and again assuming the spins are in a product state, the quantum expectation value and quantum variance of $J^{\text{eff}}_{\boldsymbol{r}\boldsymbol{r}'}$ are

$$\left\langle J^{\text{eff}}_{\boldsymbol{r}\boldsymbol{r}'} \right\rangle = J_{\boldsymbol{r}\boldsymbol{r}'} + \sum_{\boldsymbol{s}} J_{\boldsymbol{r}\boldsymbol{r}'\boldsymbol{s}} \langle Z_{\boldsymbol{s}} \rangle, \tag{10}$$

$$\left\langle \left(J^{\text{eff}}_{\boldsymbol{r}\boldsymbol{r}'}\right)^2 \right\rangle - \left\langle J^{\text{eff}}_{\boldsymbol{r}\boldsymbol{r}'} \right\rangle^2 = \sum_{\boldsymbol{s}} J^2_{\boldsymbol{r}\boldsymbol{r}'\boldsymbol{s}} (1 - \langle Z_{\boldsymbol{s}} \rangle^2). \tag{11}$$

If we compute the quantum average of $e^{i J^{\text{eff}}_{\boldsymbol{r}\boldsymbol{r}'} t}$ by keeping only the first two cumulants, then we find

$$\left\langle e^{i J^{\text{eff}}_{\boldsymbol{r}\boldsymbol{r}'} t} \right\rangle = \exp\left(it\left\langle J^{\text{eff}}_{\boldsymbol{r}\boldsymbol{r}'} \right\rangle - \frac{t^2}{2}\left\{ \sum_{\boldsymbol{s}} J^2_{\boldsymbol{r}\boldsymbol{r}'\boldsymbol{s}} (1 - \langle Z_{\boldsymbol{s}} \rangle^2) \right\} \right). \tag{12}$$

Let us now consider the disorder average of $F(t)$. Taking again $W = X_{\boldsymbol{r}_1}$ and $V = X_{\boldsymbol{r}_2}$, the first moment is trivial,

$$\overline{F(t)} = \exp\left[-\tfrac{1}{2}\Delta_2^2(\boldsymbol{r}_1,\boldsymbol{r}_2)t^2 + \cdots\right], \tag{13}$$

where we have shown only the contribution from $J_{\boldsymbol{r}_1\boldsymbol{r}_2}$ and $\cdots$ denotes higher-order terms. In chaotic systems, $F(t)$ can typically be expanded at early times like $F(t) = 1 - \epsilon e^{\lambda t} + \cdots$; the analog of this early time expansion in the many-body localized state is

$$\overline{F(t)} = 1 - \tfrac{1}{2}\Delta_2^2(\boldsymbol{r}_1,\boldsymbol{r}_2)t^2 + \cdots . \tag{14}$$

By comparing the chaotic and localized early-time expansions we learn that while scrambling grows exponentially fast in chaotic systems, it only grows polynomially fast in localized systems. Furthermore, for operators at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ both ϵ and Δ_2 are expected to be exponentially small in $|\boldsymbol{r}_1 - \boldsymbol{r}_2|$, leading to a scrambling time linear in $|\boldsymbol{r}_1 - \boldsymbol{r}_2|$ for chaotic systems and exponential in $|\boldsymbol{r}_1 - \boldsymbol{r}_2|$ for localized systems.

The second moment of $F(t)$ is more interesting, since we do not obtain trivial dephasing from disorder averaging the two-spin interaction. The key point is that the second moment involves two quantum averages over ρ, and hence can be thought of as a single quantum average over a two-copy system in the state $\rho \otimes \rho$. Then, analogous to the standard replica trick, the disorder average couples observables in the two copies together. The result is

$$\overline{|F(t)|^2} = 2\exp\left[-2\Delta_2^2(\boldsymbol{r}_1,\boldsymbol{r}_2)t^2\right] \times \operatorname{tr}\left\{ \rho\otimes\rho S^+_{\boldsymbol{r}} S^-_{\boldsymbol{r}} \otimes S^-_{\boldsymbol{r}} S^+_{\boldsymbol{r}} \prod_{\boldsymbol{s}\neq \boldsymbol{r}_1,\boldsymbol{r}_2} G^+_{\boldsymbol{s}} \right\} + \operatorname{tr}\left\{ \rho\otimes\rho S^+_{\boldsymbol{r}} S^-_{\boldsymbol{r}} \otimes S^+_{\boldsymbol{r}} S^-_{\boldsymbol{r}} \prod_{\boldsymbol{s}\neq \boldsymbol{r}_1,\boldsymbol{r}_2} G^-_{\boldsymbol{s}} \right\}$$

$$+\mathrm{tr}\left\{\rho\otimes\rho S_r^- S_r^+\otimes S_r^- S_r^+ \prod_{s\neq r_1,r_2} G_s^-\right\}, \tag{15}$$

$$\text{with}\quad G_s^{\pm}=\exp\left[-\frac{\Delta_3^2(s)t^2}{2}(Z_s\otimes I\pm I\otimes Z_s)^2\right] \tag{16}$$

and $\Delta_3(s)=\Delta_3(r_1,r_2,s)$. If the Z_s are uncorrelated and if $Z_s=1$ with probability q_s, then

$$\overline{|F(t)|^2}\sim\prod_{s\neq r_1,r_2}\left[q_s^2+(1-q_s)^2+2q_s(1-q_s)e^{-2\Delta_3^2(s)t^2}\right], \tag{17}$$

where we have dropped the faster decaying terms. One can obtain similar expressions for $\overline{F^2}$ and other higher moments of F [43]. Provided all the disorder averaged moments of F decay at late time, the commutator $C(t)$ will concentrate in probability around a late time value of 2.

The physics of $\overline{|F(t)|^2}$ is as follows. At early times, the exponentials in Eq. (17) are close to 1 and $\overline{|F(t)|^2}$ is also close to 1. As time passes, more and more of the exponentials decay towards zero and hence the product in Eq. (17) decays due to the multiplication of many numbers smaller than 1. To say more, we must specify the form of the variance, which we take to be

$$\Delta_3(s)=\Delta_3\exp\left(-\frac{|r_1-r_2|}{\xi}-\frac{|r_1-s|}{\xi}-\frac{|r_2-s|}{\xi}\right). \tag{18}$$

A little geometry shows that contours of constant u in the equation $|r_1-s|+|r_2-s|=u|r_1-r_2|$ are ellipsoids, $|r_1-r_2|^2u^2(u^2-1)=4(u^2-1)s_\parallel^2+4u^2s_\perp^2$, where $s_\parallel$ and $s_\perp$ denote the parallel and perpendicular components of s relative to r_1-r_2. Note that $u\geqslant 1$ is required to have a solution. The d-dimensional volume of the ellipsoid is

$$\mathrm{vol}\sim\frac{|r_1-r_2|^d u(u^2-1)^{(d-1)/2}}{2^d}. \tag{19}$$

The equation $\Delta_3(s)t=1$ denotes the rough boundary within which the exponentials in Eq. (17) have substantially decayed; in terms of the u parameter just discussed, the solution is

$$u(t)=\frac{\xi}{|r_1-r_2|}\log(\Delta_3 t)-1. \tag{20}$$

This gives a complex pattern of decay of $\overline{|F(t)|^2}$; to illustrate the basic physics, we make the simplifying assumption that $q_s=1/2$ and focus on late times, which gives

$$\overline{|F(t)|^2}\sim\exp[-a\xi^d\log^d(\Delta_3 t)], \tag{21}$$

a quasipolynomial decay in general spatial dimension d and a is a constant.

This result was derived for a particular kind of uncorrelated Gaussian disorder. For generic short-range correlated disorder, we expect that the central limit theorem applied to sufficiently large spin blocks will lead to an effective disorder distribution which is approximately Gaussian and uncorrelated. It would be interesting to extend our results to cases in which the initial state is correlated with the disorder, e.g., a thermal state, or where the disorder itself has long-range correlations. It would also be interesting to investigate the effects of bounded vs unbounded disorder distributions.

Interacting diffusive metal. We now turn to the effects of interactions on the disordered but delocalized metallic state. In the noninteracting limit of a diffusive metal, commutators grow diffusively and then decay as a power law at late time. Now we sketch a simple argument that including interactions significantly modifies this behavior, leading to a commutator which grows ballistically, albeit with a small velocity in the limit of weak interactions, and which remains nonzero at late times.

We first note that ballistic growth is the generic case, and the fastest growth allowed by the Lieb-Robinson bound [37,41]. In the noninteracting limit, all operator growth is tied to the motion of particles (i.e., Wick's theorem); physically it is the statement that energy, charge, and entanglement are only carried by the single-particle modes. When we include interactions, then the transport of energy, charge, and entanglement decouple and we expect more generic behavior for the motion of entanglement even if charge motion remains diffusive [51]. Since operators must grow for entanglement to be generated, we also expect to obtain ballistic growth for generic operators. This is a physical argument, but below we sketch a simple ansatz giving ballistic growth.

Now the question is how to estimate the butterfly velocity v_B. First, the butterfly velocity should vanish in the limit that the interaction-induced inelastic scattering rate Γ goes to zero. Second, the butterfly velocity must be constructed from a ratio of the relevant length and time scales, including the mean free path ℓ, the elastic scattering rate γ, and the inelastic rate Γ. Assuming no other scales are relevant, dimensional analysis gives $v_B\sim\ell\gamma f(\Gamma/\gamma)$. Further assuming that v_B depends on ℓ and γ only through the diffusion constant $D\sim\ell^2\gamma$ then fixes the scaling function to be $f(x)\sim\sqrt{x}$ and gives $v_B\sim\sqrt{D\Gamma}$.

To better understand this form for v_B, let us imagine a pertubative calculation of the squared anticommutator (or the commutator of some local bosonic operators) in the presence of interactions. The bare result is just $|A|^2$ which when disorder averaged gives the previously discussed diffusive form. Interactions lead to other terms involving integrals over powers of A and other Green's functions. Assuming these interaction terms can be resummed [52] at early time to give exponential growth at roughly the inelastic rate Γ, then the interacting early time growth of the anticommutator will be

$$C(r,t)\sim\exp(\Gamma t)\exp(-r^2/2Dt). \tag{22}$$

Solving for $C(R_W(t),t)\sim 1$ gives $R_W^2\sim D\Gamma t^2$, which is ballistic growth with butterfly velocity $v_B\sim\sqrt{D\Gamma}$. Based on this scaling argument, we conjecture that a full perturbative calculation will yield the same result.

Discussion. In this work we studied the growth of operators under time evolution in disordered models using squared (anti-)commutators and OTO correlators. Two directions for future work are a systematic perturbative calculation of scrambling in the interacting diffusive metal and a study of the behavior of scrambling at the transition from a many-body localized phase to an ergodic phase [53,54]. Another interesting direction concerns scrambling in glassy models, including long-range models [35,55], where we may study the interplay of glassy physics and scrambling.

Our study has focused on various disorder averaged OTO correlators, but one could ask about rare-region effects [53,54,56,57] as well. For example, can rare thermalized regions in the localized phase effectively give a shortcut to faster scrambling? Alternatively, in one dimension, rare localized regions in the ergodic phase should slow the growth of operators, leading to slower scrambling.

Experimentally, the effects of static disorder can be induced, for example, using laser speckle [58] or by modulating an optical lattice with incommensurate wavelengths, and experiments observing some of the physics of many-body localization have recently been carried out [7–11]. There have been recent proposals focusing on measurements to probe the collective dephasing [59,60]; we have shown that OTO correlators also access the slow logarithmic growth of dephasing which is characteristic of the many-body localized state. Some of the experimental methods for adding disorder are compatibile with the time-reversal requirements of [33], so measurements of scrambling might be possible. It would be particularly interesting to make measurements of OTO correlators at the transition between localized and ergodic states, where the growth of operators may diagnose the onset of ergodicity across the transition.

Note added. Recently, other studies [61–64] of OTO correlators in many-body localized states appeared.

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