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SLOW VARIATION AND UNIQUENESS OF SOLUTIONS TO THE FUNCTIONAL EQUATION IN THE BRANCHING RANDOM WALK

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Abstract

In this short communication, some of the recent results of Liu (1998) and Biggins and Kyprianou (1997), concerning solutions to a certain functional equation associated with the branching random walk, are strengthened. Their importance is emphasized in the context of travelling wave solutions to a discrete version of the KPP equation and the connection with the behaviour of the rightmost particle in the nth generation.

Keywords: Branching random walk; functional equations; martingales; KPP equation; travelling wave solutions

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1. Introduction: the functional equation in the branching random walk

A branching random walk is defined as follows. An initial ancestor resides at the origin of the real line and gives birth to a random number or children scattered on $\mathbb{R}$ according to the point process $Z^{(1)}$ with intensity measure μ. Each of these children has offspring in the same way, so that the positions of each family relative to the parent are given by an independent copy of $Z^{(1)}$, and so on. The generation that each realized individual u resides in is written as $|u|$, so that, for example, the set $\{u : |u| = n\}$ is the individuals in the nth generation. Let $\{\zeta_u\}_{|u|=1}$ be the points of $Z^{(1)}$ and $m(\theta)$ be the Laplace–Stieltjes transform $\int e^{-\theta\zeta}\mu(d\zeta)$. The term $m'(\theta)$ is understood to mean $-\int \zeta e^{-\theta\zeta}\mu(d\zeta)$ and it is assumed throughout that this integral is finite when $m(\theta)$ is finite. If $\theta \in \int\{\phi : m(\phi) < \infty\} \neq \emptyset$ then the derivative of $m(\theta)$ is bounded and equal to $m'(\theta)$.

Associated with the branching random walk is the additive martingale

$$W^{(n)}(\theta) := \sum_{|u|=n} e^{-\theta\zeta_u}/m(\theta)^n$$

(see Biggins (1977) and Kingman (1975) for further details of this martingale). This martingale has an almost sure limit $W(\theta)$ which serves as one possible solution to the following distributional identity:

$$\Lambda(\theta) \stackrel{d}{=} \sum_{|u|=1} \frac{e^{-\theta\zeta_u}}{m(\theta)} \Lambda_u(\theta), \tag{1}$$

where $\stackrel{d}{=}$ indicates equality in distribution and, given $Z^{(1)}$, Λ_u are independent copies of Λ. Taking Laplace transforms of (1) it can be seen that $E[\exp(-xW(\theta))]$ is a solution to the

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relation

$$\Psi(x) = E\left[\prod_{|u|=1} \Psi\left(x \frac{e^{-\theta\zeta_u}}{m(\theta)}\right)\right]. \tag{2}$$

This functional equation will become the object of investigation in what follows.

Solutions to distributional identities or functional equations of the type (1) and (2) respectively are known as smoothing transforms and often appear in the study of branching processes. Examples include Biggins (1977), Biggins and Kyprianou (1997), Cohn (1985) and Schuh (1982) (in connection with martingale convergence theorems for several types of branching processes) and Chauvin and Rouault (1997) (in connection with the construction of a Boltzmann–Gibbs measure for the study of overlaps in the branching random walk). Similar types of equations appear in a variety of other areas of study. To name but a few: Pakes (1992) and Baringhaus and Grübel (1995) in characterization theory and Kahane and Peyrière (1976) and Waymire and Williams (1996) in the study of random cascades. A generalized version of (2) has been studied by Durrett and Liggett (1983) and Liu (1997, 1998) from which further references to applications of the smoothing transform can be found.

We will refer to a non-trivial solution to the functional equation as a function $\Psi : [0, \infty] \longmapsto [0, 1]$, such that $\Psi \not\equiv 1$ and $\Psi \not\equiv 0$. For any such non-trivial solution let $L(x) = (1 - \Psi(x))/x$. Define the set $\mathcal{R}$ to be the set of functions for which L is monotone decreasing and define $\mathcal{L}$ to be the set of Laplace transforms of non-negative variables (note that $\mathcal{L} \subseteq \mathcal{R}$). Biggins and Kyprianou (1997) have shown that under the conditions

(i) the mean family size, $m = m(0) > 1$,

(ii) $\text{int}\{\phi : m(\phi) < \infty\}$ is non-empty,

(iii) $\theta \in \text{int}\{\phi : m(\phi) < \infty\}$,

(iv) $P(Z^{(1)}(\mathbb{R}) = \infty) = 0$,

(v) $\log m(\theta) - \theta m'(\theta)/m(\theta) > 0$,

within $\mathcal{L}$, there exists a unique non-trivial solution to the functional equation up to a multiplicative constant in the argument whose associated function L is slowly varying at the origin. In fact as the analysis stands in this paper this result can be stated with slightly more generality so that $\mathcal{L}$ may be replaced by $\mathcal{R}$. (See the presentation in Kyprianou (1996).) Using different techniques, Liu (1998) has also produced, under more stringent conditions, similar conclusions for solutions in $\mathcal{L}$.

The primary objective of this paper is to present new results concerning uniqueness and slow variation of solutions under the condition

(vi) $\log m(\theta) - \theta m'(\theta)/m(\theta) = 0$.

Biggins (1977) demonstrated that with condition (ii),

$$\text{int}\{\phi : m(\phi) < \infty\} \cap \{\phi : \log m(\phi) - \phi m'(\phi)/m(\phi) > 0\}$$

is an open interval, outside of which the martingale $W^{(n)}(\theta)$ converges almost surely to zero. We will refer to this interval as $(\vartheta_1, \vartheta_2)$. When (ii), (iii) and (vi) hold, θ takes one of the edge values ϑ_1 or ϑ_2.

2. A connection with travelling wave solutions to the KPP equation

One important motivation for the understanding of solutions to the functional equation under (vi) that has not yet been mentioned comes from the hypothesis concerning B_n, the rightmost individual in the nth generation: that there exist a sequence of constants c_n such that $(B_n - c_n)$ converges in distribution. This hypothesis is inspired by a similar result already established for branching Brownian motion. Bramson (1978) showed that there exist centring constants

$$c_t = \sqrt{2}t - 3.2^{-3/2}\log t + O(1),$$

such that for the rightmost individual at time t, B_t,

$$P(B_t - c_t \le x) \to \Phi(x)$$

where Φ is a travelling wave solution to the KPP equation of minimal wave speed $\sqrt{2}$.

In the branching random walk, solutions to the functional equation correspond to travelling wave solutions of a discrete-time diffusion equation. To see this, make the transformation $w(x) = \Psi(\mathrm{e}^{\theta x})$, where Ψ is a solution to (2), and note that $w(x + n\theta^{-1}\log m(\theta))$ is a solution to the recursion

$$u_{n+1}(x) = E\left[\prod_{|u|=1} u_n(x - \zeta_u)\right], \tag{3}$$

(so that w has wave speed $-\theta^{-1}\log m(\theta)$). Let us assume that the constants c_n do exist. (The recent results of McDiarmid (1995) would suggest that this assumption is not unreasonable; his conclusions indicate that when (ii), (iii) hold and (vi) holds at ϑ_1, $c_n \sim -n\vartheta_1^{-1}\log m(\vartheta_1) + O(\log n)$ as $n \uparrow \infty$.) It is not difficult to show in the style of McKean (1975) that $u_n(x) = P(B_n \le x)$ is a solution to (3) with initial conditions $u_0(x) = I(x \ge 0)$. Therefore defining $\Psi(\mathrm{e}^{x\vartheta_1}) = \lim_{n\uparrow\infty} P(B_n - c_n \le x)$ and assuming further that $c_{n+1} - c_n \to -\vartheta_1^{-1}\log m(\vartheta_1)$ as $n \uparrow \infty$, a simple calculation shows that Ψ satisfies the functional equation for $\theta = \vartheta_1$. In this light we may consider (3) a discrete version of the KPP equation and $-\vartheta_1^{-1}\log m(\vartheta_1)$ to play the role of a minimal wave speed. Indeed it can be checked by differentiating the function $-\theta^{-1}\log m(\theta)$ that when $m < \infty$, (ii) and (iii) are in force and (vi) holds at ϑ_1, it has a minimum on $(-\infty, 0)$ at ϑ_1.

We have shown that by assuming the extreme individual B_n behaves in a certain way we gain information about solutions to the functional equation under (vi). It would not seem unreasonable, therefore, that information can be transferred the other way around giving access to knowledge of the existence of the sequence $\{c_n\}$. This provides the main stimulus for producing results about slow variation and uniqueness for the functional equation under (vi).

3. Slow variation

Theorem 2 of Liu (1998) shows that when $m < \infty$, $m(\theta) < \infty$ and condition (vi) holds, any non-trivial solution in $\mathcal{L}$ to the functional equation has associated function L that is slowly varying at the origin. In this section we will improve on this result by removing the requirement that the mean family size is finite and including solutions in a broader class than $\mathcal{L}$.

Theorem 1. *When $m(\theta) < \infty$, and condition (vi) holds, any non-trivial solution to the functional equation in $\mathcal{R}$ has associated function L that is slowly varying at the origin.*

Proof. For simplicity let us define the weights $y_u = \mathrm{e}^{-\theta\zeta_u}/m^{|u|}(\theta)$. By ordering (arbitrarily) the set $\{|u|=1\}$ by $\prec$ we have that

$$\begin{aligned} L(x) &= \frac{1-\Psi(x)}{x} \\ &= E\left[\frac{1-\prod_{|u|=1}\Psi(xy_u)}{x}\right] \\ &= E\left[\sum_{|u|=1}\frac{1-\Psi(xy_u)}{x}\prod_{v\prec u}\Psi(xy_v)\right], \end{aligned} \tag{4}$$

where the third equality is the result of a telescoping sum. (This decomposition is reproduced from Biggins and Kyprianou (1997).)

Assuming that $\Psi \in \mathcal{R}$, suppose that there exists a $\beta_0 > 0$ and a subsequence of the reals $\{t_k\}$ such that $t_k \downarrow 0$ and $\lim_{k\uparrow\infty} L(t_k\beta_0)/L(t_k) \neq 1$. Define the monotone increasing function

$$\varphi_x(\beta) = \exp\left\{-\frac{L(x\beta)}{L(x)}\right\}.$$

Note that $\varphi_x(0) \geq 0$ and $\varphi_x(\infty) \leq 1$. An application of the Helly–Bray theorem (see for example Kolmogorov and Fomin (1970), p. 372, Theorem 5) yields that there exists a subsequence $\{x_k\} \subseteq \{t_k\}$ such that for all $\beta > 0$

$$\exp\{-g(\beta)\} := \lim_{k\uparrow\infty} \varphi_{x_k}(\beta)$$

exists for all $\beta > 0$, where g is a monotone decreasing function. By arrangement $g(\beta_0) \neq 1$. From the identity in (4) we have

$$\frac{L(x_k\beta)}{L(x_k)} = E\left[\sum_{|u|=1} y_u \frac{L(x_k y_u \beta)}{L(x_k)} \prod_{v\prec u}\Psi(x_k y_v)\right].$$

Taking limits as $k \to \infty$, a double application of Fatou's lemma yields

$$g(\beta) \geq E\left[\sum_{|u|=1} y_u g(\beta y_u)\right]. \tag{5}$$

Define as in Biggins and Kyprianou (1997), X such that $\theta^{-1}(X - \log m(\theta))$ has distribution $\exp\{-\theta\zeta - \log m(\theta)\}\mu(\mathrm{d}\zeta)$ and therefore mean $\log m(\theta) - \theta m'(\theta)/m(\theta)$. The decomposition (5) can be written as $g(\beta) \geq E[g(\beta\mathrm{e}^{-X})]$. With S_n as the random walk that has independent steps distributed like X we have $g(\beta\mathrm{e}^{-S_n})$ is a supermartingale that converges almost surely and in mean to some limit $\eta(\beta)$. Condition (vi) implies that the variable X has zero mean and thus S_n is transient such that $-\infty = \liminf_{n\uparrow\infty} S_n < \limsup_{n\uparrow\infty} S_n = \infty$. Therefore,

$$g(\infty) = \liminf_{n\uparrow\infty} g(\beta\mathrm{e}^{-S_n}) = \eta(\beta) = \limsup_{n\uparrow\infty} g(\beta\mathrm{e}^{-S_n}) = g(0^+)$$

and hence monotonicity implies that g is a constant function. Since $g(1) = 1$ we have that $g(\beta_0) = 1$ which is a contradiction and hence the result follows.

Replacing the condition (vi) by (v) in this theorem, it is not difficult to produce a stronger result than the counterparts in both Biggins and Kyprianou (1997) and Liu (1998).

Theorem 2. *When $m(\theta) < \infty$ and (v) holds, any non-trivial solution to the functional equation in $\mathcal{R}$ has associated function L that is slowly varying at the origin.*

Proof. The proof follows an almost identical path to the proof of Theorem 1. Construct the function g and variable X as before and note that this time (v) implies that $EX > 0$. We argue by contradiction as previously making note of the following; the supermartingale $g(\beta e^{-S_n})$ converges in mean such that $g(\beta) \geq E\eta(\beta) = g(0^+)$ for all $\beta > 0$ and hence as g is monotone decreasing with $g(1) = 1$, it must be the case that $g(\beta) = 1$ for all $\beta > 0$.

4. Uniqueness

Under the conditions (vi) and

(vii) $E[Z^{(1)}(\mathbb{R})^\alpha] < \infty$ and $E[W^{(1)}(\theta)^\alpha] < \infty$ for some $\alpha > 1$,

Theorem 4 of Liu (1998) demonstrates for each solution Ψ to the functional equation in $\mathcal{L}$, that there exists a positive constant c, such that

$$\lim_{x \downarrow 0} \frac{1 - \Psi(x)}{-cx \log x} = 1. \tag{6}$$

Define $\mathcal{S}$ to be the set of all functions whose associated function L is slowly varying. Note that under the conditions of Theorem 1,

$$\{\Psi \in \mathcal{L} : \Psi \text{ solves (2)}\} \subseteq \{\Psi \in \mathcal{R} : \Psi \text{ solves (2)}\} \subseteq \{\Psi \in \mathcal{S} : \Psi \text{ solves (2)}\}.$$

Lemmas 2.2, 5.1 and 5.2 of Biggins and Kyprianou (1997) together show that when $m(\theta) < \infty$ and Ψ is a non-trivial solution to the functional equation in $\mathcal{S}$,

$$\lim_{n \uparrow \infty} \sum_{|u|=n} y_u L(y_u) = \Delta \tag{7}$$

almost surely, where Δ a finite, non-negative variable with Laplace transform Ψ. This now implies that under the conditions of Theorem 1,

$$\{\Psi \in \mathcal{S} : \Psi \text{ solves (2)}\} = \{\Psi \in \mathcal{R} : \Psi \text{ solves (2)}\} = \{\Psi \in \mathcal{L} : \Psi \text{ solves (2)}\} \neq \emptyset,$$

where the inequality is guaranteed by Theorem 1 of Liu (1998). It is essentially the substitution of Liu's logarithmic asymptote into the limit (7) that will provide the route to uniqueness of the functional equation under (vi). This substitution however inevitably carries the moment conditions (vii) with it.

Before stating our theorem we should make a brief note about the martingale $W^{(n)}(\theta)$. It is not difficult to prove that the derivative of this martingale

$$\partial W^{(n)}(\theta) := \frac{d}{d\theta} W^{(n)}(\theta) = -\sum_{|u|=n} \left\{\zeta_u + n\frac{m'(\theta)}{m(\theta)}\right\} y_u$$

is also a martingale (see for example Biggins (1991) and Neveu (1988)). It is not clear however whether this martingale converges almost surely. Our uniqueness theorem provides a new result in this respect.

Theorem 3. *Under assumptions (i), (vi) and (vii), $\partial W(\theta) := \lim_{n\uparrow\infty} \partial W^{(n)}(\theta)$ exists and has Laplace transform that is is the unique solution in $\mathcal{S}$ (up to a multiplicative constant in the argument) to the functional equation.*

Note that from the previous discussion, this theorem also demonstrates that the Laplace transform of ∂W is also the unique solution in $\mathcal{R}$ and $\mathcal{L}$.

Proof. As we have already seen, any solution to the functional equation that is slowly varying must also be a Laplace transform. Hence for such a solution,

$$\lim_{n\uparrow\infty}\left|\frac{\sum_{|u|=n} y_u L(y_u)}{-c\sum_{|u|=n} y_u \log y_u} - 1\right| = \lim_{n\uparrow\infty}\left|\frac{-c\sum_{|u|=n} y_u \log y_u\{L(y_u)/(-c\log y_u)\}}{-c\sum_{|u|=n} y_u\log y_u} - 1\right|$$
$$\leq \lim_{n\uparrow\infty}\sup_{|u|=n}\left|\frac{L(y_u)}{-c\log y_u} - 1\right|. \tag{8}$$

The right-hand side of (8) tends to zero by Liu's asymptotic result because (from Lemma 2.2 of Biggins and Kyprianou (1997)) $\sup_{|u|=n} y_u$ tends almost surely to zero. Thus recalling (7) and that $\log m(\theta) - \theta m'(\theta)/m(\theta) = 0$ we have

$$\begin{aligned}\Delta &= -c\lim_{n\uparrow\infty}\sum_{|u|=n} y_u \log y_u\\ &= -c\lim_{n\uparrow\infty}\sum_{|u|=n} y_u\{-\theta\zeta_u - \log m(\theta)\}\\ &= -c\theta\lim_{n\uparrow\infty}\partial W^{(n)}(\theta).\end{aligned}$$

This shows that $\partial W^{(n)}(\theta)$ converges to a multiple of Δ which can be identified as $\partial W(\theta)$. Indeed as there can be only one limit $\partial W(\theta)$ then any solution to the functional equation in $\mathcal{S}$ must be unique up to a multiplicative constant in the argument.

This result complements the findings of Neveu (1988) and Lalley and Sellke (1987) who both showed for branching Brownian motion that there exists a travelling wave solution of minimal wave speed to the KPP equation, which may be expressed as the Laplace transform of the limit of a differentiated additive martingale.

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