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SLOW VISCOUS FLOW INSIDE A TORUS - THE RESISTANCE OF SMALL TORTUOUS BLOOD VESSELS*

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Abstract. The hydrodynamic resistance of a buckled microvessel in the form of a tightly would helix is approximated by studing the Stokes flow inside a torus. The unidirectional flow is driven by a constant tangential pressure gradient. The solution is obtained by an eigenfunction expansion in toroidal coordinates. The ratio of volume flow carried by the torus to that carried by a straight tube is computed as a function of the vessel radius: coil radius ratio. An asymptotic expansion for this flux ratio is also obtained. The results show that the resistance of a moderately curved vessel is slightly less than the resistance of a straight one, whereas the resistance of a greatly curved vessel is at most 3% greater than the straight one.

1. Introduction. Blood vessels of all sizes often follow a tortuous path. It is of physiological importance to determine to what extent this tortuosity influences the resistance of a vascular bed, particularly in the microcirculation where the Reynolds number is negligible. Buckling is often the cause of the varicosity. Capillaries that supply cardiac and skeletal muscle run parallel to and are periodically attached to the muscle fibers. When the muscle fibers shorten the capillaries buckle (1). Blood vessel growth can be another cause of buckling (2). Two common forms of the buckled shape are the helical mode and the sinusoidal mode. If the helix is tightly would such that adjacent coils are close to each other (small pitch angle), then flow in a single coil should be closely approximated by flow inside a torus. The flow is assumed to be driven by a constant tangential pressure gradient. As the flow circulates once around the torus, the pressure is less than the driving pressure, and the fluid can be thought of as being in the next coil of the helix. Stokes flow external to a torus has been previously considered (3) using toroidal coordinates, but not the internal Stokes flow considered here.

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2. Problem formulation and solution. We first consider an incompressible flow field described by the velocity components (u_R, u_θ, u_Z) and pressure $p(R, \theta, Z)$, with (R, θ, Z) cylindrical coordinates. We further assume the properties

$$\begin{aligned} &u_R = u_Z = 0; \qquad u_\theta = u_\theta(R, Z), \\ &\frac{\partial p}{\partial R} = \frac{\partial p}{\partial Z} = 0; \qquad \frac{\partial p}{\partial \theta} = \text{given constant.} \end{aligned} \tag{1}$$

This unidirectional, fully developed flow field identically satisfies the continuity equation, and the R and Z momentum equations in Stokes flow. This leaves only the θ-momentum equation to be satisfied.

$$\nabla^2 u_\theta - \frac{1}{R^2} u_\theta = \frac{1}{\mu R}\frac{\partial p}{\partial \theta}. \tag{2}$$

We now introduce toroidal coordinates (ξ, θ, η). A family of toroidal surfaces is described by constant values of ξ. The tube radius is $b \operatorname{csch} \xi_0$ and the coil radius (measured from the axis of revolution to the tube center) is $b \coth \xi_0$, where b is a scaling constant. It is convenient to scale all lengths relative to the tube radius; hence we set $b = \sinh \xi_0$. The ratio of coil radius to tube radius is $\zeta_0 = \cosh \xi_0$, which is the primary geometric parameter. The metrical coefficients are

$$h_1 = h_2 = \frac{1}{b}(\cosh \xi - \cos \eta); \; h_3 = \frac{1}{R} = \frac{1}{b}\frac{(\cosh \xi - \cos \eta)}{\sinh \xi} \tag{3}$$

Eq. (2) can then simply be written in toroidal coordinates

$$h_1 h_2 \left\{ \frac{\partial}{\partial \xi}\left(\frac{1}{h_3}\frac{\partial}{\partial \xi} \right) + \frac{\partial}{\partial \eta}\left(\frac{1}{h_3}\frac{\partial}{\partial \eta} \right) \right\} u_\theta - h_3 u_\theta = \frac{1}{\mu}\frac{\partial p}{\partial \theta} \tag{4}$$

A standard device (Hobson [4, p. 434]), useful in obtaining separable forms, is to write

$$u_\theta(\xi, \eta) = (\cosh \xi - \cos \eta)^{1/2} U(\xi, \eta) \tag{5}$$

Then after some reduction Eq. (4) takes the form

$$U_{\xi\xi} + (\coth \xi) U_\xi + U_{\eta\eta} + \left(\frac{1}{4} - \frac{1}{\sinh^2 \xi} \right) U = (\cosh \xi - \cos \eta)^{-3/2} \frac{b}{\mu \sinh \xi}\frac{\partial p}{\partial \theta}. \tag{6}$$

The velocity must be periodic in η, and by symmetry it must be an even function of η. We therefore look for solutions having the form

$$U_m(\xi, \eta) = V_m(\xi) \cos m\eta, \qquad m = 0, 1, 2 \ldots. \tag{7}$$

For the homogeneous problem we obtain

$$\frac{d}{d\zeta}\left\{ (\zeta^2 - 1)\frac{dV_m}{d\zeta} \right\} + \left(\frac{1}{4} - m^2 - \frac{1}{\zeta^2 - 1} \right) V_m = 0 \tag{8}$$

where $\zeta = \cosh \xi$. The solutions are the ring functions $P^1_{m-1/2}(\zeta)$ and $Q^1_{m-1/2}(\zeta)$, which are the associated Legendre functions of order one and of degree half an odd integer. The latter function is regular at infinity, which lies inside the torus (Hobson, p. 436). The homogeneous solution can be written

$$u_\theta^h(\zeta,\eta) = (\zeta - \cos\eta)^{1/2} \sum_{m=0}^{\infty} A_m Q^1_{m-1/2}(\zeta)\cos m\eta \tag{9}$$

with the A_m being constants to be determined. We use a similar form for the particular solution

$$u_\theta^p(\zeta,\eta) = (\zeta - \cos\eta)^{1/2} \sum_{m=0}^{\infty} G_m(\zeta)\cos m\eta. \tag{10}$$

The no-slip condition $u_\theta(\zeta_0,\eta) = 0$ can then be easily satisfied by choosing

$$A_m = -\frac{G_m(\zeta_0)}{Q^1_{m-1/2}(\zeta_0)}, \qquad m = 0,1,\ldots. \tag{11}$$

As it will turn the G_m satisfy an inhomogeneous Lagendre equation. We first note that the factor $(\zeta - \cos\eta)^{-1/2}$ can be developed in a cosine series (Hobson, p. 443)

$$(\zeta - \cos\eta)^{-1/2} = \frac{\sqrt{2}}{\pi} Q_{-1/2}(\zeta) + \frac{2\sqrt{2}}{\pi} \sum_{m=1}^{\infty} Q_{m-1/2}(\zeta)\cos m\eta \tag{12}$$

A cosine series expansion of the factor $(\zeta - \cos\eta)^{-3/2}$ appearing on the right hand side of Eq. (6) can then be obtained by differentiation of Eq. (12) with respect to ζ. Equating coefficients of $\cos m\eta$ then gives the inhomogeneous equation

$$\frac{d}{d\zeta}\left\{(\zeta^2 - 1)\frac{dG_m}{d\zeta}\right\} + \left(\frac{1}{4} - m^2 - \frac{1}{\zeta^2 - 1}\right)G_m = D_m \frac{Q^1_{m-1/2}(\zeta)}{\zeta^2 - 1} \tag{13}$$

with

$$D_0 = -\frac{2\sqrt{2}}{\pi\mu} b \frac{\partial p}{\partial\theta}$$

$$D_m = -\frac{4\sqrt{2}}{\pi\mu} b \frac{\partial p}{\partial\theta}, \qquad m = 1,2,\ldots, \tag{14}$$

where we have also used the relation

$$Q^1_{m-1/2}(\zeta) = (\zeta^2 - 1)^{1/2} \frac{dQ_{m-1/2}(\zeta)}{d\zeta}. \tag{15}$$

The particular solution of Eq. (13) can easily be found by trying

$$G_m(\zeta) = f_m(\zeta) Q^1_{m-1/2}(\zeta) \tag{16}$$

and noting that $Q^1_{m-1/2}(\zeta)$ is an homogeneous solution. We then obtain a first order equation for the derivative $g_m = df_m/d\zeta$

$$\frac{d}{d\zeta} g_m + H_m(\zeta) g_m = D_m (\zeta^2 - 1)^{-2} \tag{17}$$

where

$$H_m(\zeta) = \frac{2dQ^1_{m-1/2}(\zeta)/d\zeta}{Q^1_{m-1/2}(\zeta)} + \frac{2\zeta}{\zeta^2 - 1}. \tag{18}$$

The integrating factor for Eq. (17) is

$$\exp\left(\int H_m(\zeta)\, d\zeta\right) = (\zeta^2 - 1)\left[Q^1_{m-1/2}(\zeta)\right]^2$$

from which we obtain the first integral

$$(\zeta^2 - 1)\left[Q^1_{m-1/2}(\zeta)\right]^2 \frac{df_m}{d\zeta} = D_m \int_\infty^\zeta \frac{\left[Q^1_{m-1/2}(\sigma)\right]^2}{\sigma^2 - 1} d\sigma. \tag{19}$$

It can be verified a posteriori that the left hand side approaches zero for large ζ using the large argument asymptotic behavior of the ring function. The necessary relations are given in Section 3. The final expression for the velocity field can be written as

$$u_\theta(\zeta, \eta) = (\zeta - \cos\eta)^{1/2} \sum_{m=0}^{\infty} Q^1_{m-1/2}(\zeta)\{f_m(\zeta) - f_m(\zeta_0)\} \cos m\eta, \tag{20}$$

where f_m must be determined by quadrature of Eq. (19).

3. Calculation of flux ratio. The flux ratio is defined here as the ratio of the volume flow rate carried by the toroidal tube to that of a straight tube having a length equal to 2π times the coil radius, with the same pressure gradient applied in both cases. The flux ratio is then the relative hydrodynamic conductance, or the inverse of the relative hydrodynamic resistance. The flux Q carried by the toroidal tube is the integral of u_θ over the cross section whose differential area element is

$$\frac{\sinh^2 \xi_0}{(\cosh\xi - \cos\eta)^2} d\xi\, d\eta.$$

The integration over the angular variable η can be done with the use of the integral representation

$$Q^1_{m-1/2}(\zeta) = -2^{3/2}(\zeta^2 - 1)^{1/2} \int_0^\pi \frac{\cos m\eta\, d\eta}{(\zeta - \cos n)^{3/2}}. \tag{21}$$

We note that the volume flux carried by the straight tube is given by Poiseuille's relation

$$Q_S = -\frac{\pi}{8\mu\zeta_0} \frac{\partial p}{\partial\theta} \tag{22}$$

where μ is the viscosity of the fluid. Then the flux ratio $\tilde{Q} = Q/Q_s$ can be written as the summation of integrals

$$\tilde{Q}(\zeta_0) = \frac{128}{\pi^2} \zeta_0(\zeta_0^2 - 1)^{3/2} \sum_{m=0}^{\infty} d_n \int_{\zeta_0}^{\infty} \left[Q^1_{m-1/2}(\zeta)\right]^2 \times \{\tilde{f}_m(\zeta_0) - \tilde{f}_m(\zeta)\} \frac{d\zeta}{(\zeta^2 - 1)} \tag{23}$$

where

$$\tilde{f}_m = f_m/D_m, \qquad d_n = \begin{cases} 1, & m = 0 \\ 2, & m = 1, 2, 3 \ldots. \end{cases} \tag{24}$$

It is of interest to compute the first few terms of the asymptotic expansion of $\tilde{Q}$ for large ζ_0. The ring functions have the series representation (5)

$$Q^1_{m-1/2}(\cosh\xi) = -\sqrt{\pi}\,\frac{\Gamma(\frac{3}{2}+m)}{\Gamma(1+m)}e^{-(m+1/2)\xi}(1-e^{-2\xi})$$

$$\cdot F(\tfrac{3}{2}, \tfrac{3}{2}+m; 1+m; e^{-2\xi}), \tag{25}$$

$$m = 0, 1, 2, \ldots, \quad 0 < \xi \leqslant \infty,$$

where Γ is the gamma function and F is the Gaussian hypergeometric function given in terms of the hypergeometric series.

$$F(a, b,; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!}. \tag{26}$$

We also need to invert the relationship $\zeta = \cosh\xi$ to obtain

$$\xi = \log 2\zeta - \frac{1}{4}\zeta^{-2} - \frac{3}{32}\zeta^{-4} - \frac{5}{96}\zeta^{-6} + \ldots. \tag{27}$$

Then Eqs. (25)–(27) yield the large ζ approximations

$$Q^1_{-1/2}(\zeta) = -\frac{\pi}{2\sqrt{2\zeta}}\left\{1 + \frac{7}{16}\zeta^{-2} + \frac{337}{1024}\zeta^{-4} + \frac{4511}{16384}\zeta^{-6} + \ldots\right\},$$

$$Q^1_{1/2}(\zeta) = -\frac{3\pi}{8\sqrt{2\zeta}}\left\{\zeta^{-1} + \frac{19}{32}\zeta^{-3} + \frac{467}{1024}\zeta^{-5} + \ldots\right\}, \tag{28}$$

$$Q^1_{3/2}(\zeta) = \frac{-15\pi}{64}\frac{1}{\sqrt{2\zeta}}\left\{\zeta^{-2} + \frac{13}{16}\zeta^{-4} + \frac{41}{64}\zeta^{-6} + \ldots\right\}.$$

From Eqs. (19) and (25) we can approximate the $\tilde{f}_m$ for large argument

$$\tilde{f}_0 = \frac{1}{4\zeta^2}\left\{1 + \frac{17}{32}\zeta^{-2} + \frac{273}{768}\zeta^{-4} + \ldots\right\},$$

$$\tilde{f}_1 = \frac{1}{8\zeta^2}\left\{1 + \frac{61}{96}\zeta^{-2} + \frac{365}{12288}\zeta^{-4} + \ldots\right\}, \tag{29}$$

$$\tilde{f}_2 = \frac{1}{12\zeta^2} + \ldots.$$

Then with Eqs. (23), (28), and (29) we can obtain the first few terms of the asymptotic expansion of the flux ratio

$$\tilde{Q}(\zeta_0) = 1 + \frac{1}{48}\zeta_0^{-2} - \frac{13}{1024}\zeta_0^{-4} + \ldots. \tag{30}$$

The leading term shows that the Poiseulle limit is recovered as $\zeta_0 \to \infty$. The sign of the second term is quite unexpected given it implies a slightly curved tube has a smaller resistance than a straight one.[1] When the same problem is considered in two dimensions, i.e. an annulus, the flux ration is given in terms of elementary functions (6).

[1] Professor Van Dyke found a discrepancy in the coefficient of the ζ_0^{-4} term in Eq. (30) in an earlier version of the paper. The present value now agrees with his own independent calculations as well as the zero Dean number limit in (7).

$$\tilde{Q}_{an}(\zeta_0) = \frac{3}{4}\zeta_0^2 - \frac{3}{16}\zeta_0^4\left(1 - \frac{1}{\zeta_0^2}\right)^2 \log^2\left(\frac{1 + 1/\zeta_0}{1 - 1/\zeta_0}\right). \tag{31}$$

Expanding for large ζ_0 gives

$$\tilde{Q}_{an}(\zeta_0) = 1 - \frac{2}{15}\zeta_0^{-2} - \frac{1}{189}\zeta_0^{-4} - \dots \tag{32}$$

and it is seen that small curvature of a two dimensional channel increases the relative resistance. In the most extreme case of curvature where $\zeta_0 \to 1$, it is seen from Eq. (31) that $Q_{an} \to 3/4$. We have been unable to analytically determine the behavior of Q near $\zeta_0 = 1$ for the torus, and so a complete numerical evaluation of Eq. (20) was undertaken. The results of this computation are given in Table I and also graphically in Fig. 1. It is remarkable how little the flux ratio deviates from unity over the entire range $0 \leqslant \zeta_0^{-1} \leqslant 1$. The maximal increase in flux is about 1/2%, while the maximal decrease is about 3%.

Table I. Flux Ratio in a Toroidal Tube as a Function of Tube Radius/Coil Radius.

ζ_0^{-1}	$\tilde{Q}$	ζ_0^{-1}	$\tilde{Q}$
0.000	1.00000	0.550	1.00477
0.050	1.00005	0.600	1.00519
0.100	1.00021	0.650	1.00540
0.150	1.00046	0.700	1.00527
0.200	1.00081	0.750	1.00461
0.250	1.00125	0.800	1.00313
0.300	1.00176	0.850	1.00038
0.350	1.00234	0.900	0.99555
0.400	1.00296	0.950	0.98697
0.450	1.00360	0.990	0.97452
0.500	1.00421	0.998	0.97079

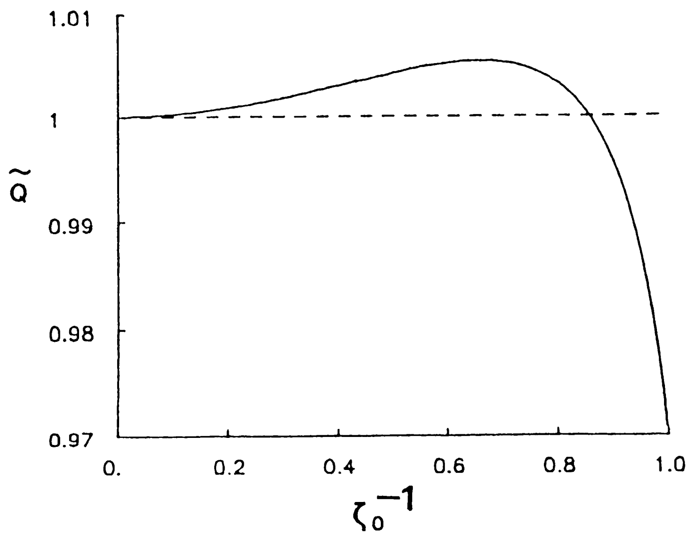

Fig. 1.

4. Concluding Remarks. The present calculations indicate that the hydrodynamic resistance of the tightly coiled helix form of a buckled microvessel is not significantly different than an straight vessel for all values of the tube radius: coil radius ratio. However, one might expect different results for other buckled forms not considered here, e.g. the sinusoid. The flow is fully three dimensional even at negligible Reynolds numbers when the curvature is not constant along the primary flow direction, in contrast to the unidirectional flow for the present configuration.

References

[1] R. F. Potter and A. C. Groom, *Capillary diameter and geometry in cardiac and skeletal muscle studied by means of corrosion casts*, Microvascular Research **25**, 68–84 (1983)

[2] A. M. Waxman, *Blood vessel growth as a problem in morphogenesis: a physical theory*, Microvascular Research **22**, 32–42 (1981)

[3] L. E. Payne and W. H. Pell, *On Stokes flow about a torus*, Mathematika **7**, 78–92 (1960)

[4] E. W. Hobson, *The theory of spherical and ellipsoidal harmonics*, Chelsea Publishing, New York, 1955

[5] I. A. Stegun, *Legendre functions*, in Handbook of Mathematical Functions, National Bureau of Standards Applied Mathematics Series 55, ed. M. Abramovitz and I. A. Stegun, 1968

[6] L. W. Schwartz, personal communication

[7] J. Larrain and C. F. Bonilla, *Theoretical analysis of the pressure drop in the laminar flow of fluid in a coiled pipe*, Trans. Soc. Rheol. **14**, 135–147 (1970)