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Slow viscous flow past a rotating sphere

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Introduction. Keller and Rubinow (1) have considered the force on a spinning sphere which is moving through an incompressible viscous fluid by employing the method of matched asymptotic expansions to describe the asymmetric flow. Childress (2) has investigated the motion of a sphere moving through a rotating fluid and calculated a correction to the drag coefficient. Brenner (3) has also obtained some general results for the drag and couple on an obstacle which is moving through the fluid. The present paper is concerned with a similar problem, namely the axially symmetric flow past a rotating sphere due to a uniform stream of infinity. It is shown that leading terms for the flow consist of a linear superposition of a primary Stokes flow past a non-rotating sphere together with an antisymmetric secondary flow in the azimuthal plane induced by the spinning sphere. For $a^3n^2 > 6U\nu$, where n is the angular velocity of the sphere, U the speed of the uniform stream, and a the radius of the sphere, there is in the azimuthal plane a region of reversed flow attached to the rear portion of the sphere. The structure of the vortex is described and is shown to be confined to the rear portion of the sphere. A similar phenomenon occurs for a sphere rotating about an axis oblique to the direction of the uniform stream but the analysis will be given in a separate paper.

Equations of motion. The fluid motion to be considered is the steady flow due to a uniform stream of velocity $-U\hat{k}$ at infinity past a sphere of radius a rotating with angular velocity $n\hat{k}$. Let (z, ϖ, ϕ) be cylindrical polar coordinates with origin at the centre of the sphere and $\hat{\phi}$ be the unit vector directed perpendicular to the azimuthal plane ϕ = constant and in the sense of ϕ increasing. Then an axially symmetric fluid motion may be defined by

$$\mathbf{q} = \operatorname{curl}\left\{-\frac{\psi}{\varpi}\hat{\phi}\right\} + \frac{v}{\varpi}\hat{\phi}; \tag{1}$$

ψ is the stream function, v/ϖ is the swirl component of velocity and $\mathbf{q}$ is the fluid velocity. The scaled equations satisfied by ψ and v are

$$\frac{-2R_1^2}{R\varpi^2} V\frac{\partial V}{\partial z} + R\varpi\,\partial\left\{\frac{\psi, D^2(\psi)/\varpi^2}{\partial(z,\varpi)}\right\} = D^4(\psi), \tag{2}$$

$$\frac{R}{\varpi}\frac{\partial(\psi, v)}{\partial(z,\varpi)} = D^2(v),$$

where the Stokes operator $\qquad D^2 \equiv \dfrac{\partial^2}{\partial z^2} + \dfrac{\partial^2}{\partial\varpi^2} - \dfrac{1}{\varpi}\dfrac{\partial}{\partial\varpi}$

and the parameters R and R_1 are defined by $R = Ua/\nu$ and $R_1 = a^2n/\nu$.

The boundary conditions to be satisfied by ψ and v are given by

$$\left.\begin{aligned} \psi = \frac{\partial \psi}{\partial r} = 0, \quad r = 1, \quad \psi \sim \tfrac{1}{2}\varpi^2 \quad \text{as} \quad r \to \infty, \\ V = \varpi^2, \quad r = 1, \quad \frac{V}{\varpi} \to 0 \quad \text{as} \quad r \to \infty, \end{aligned}\right\} \tag{3}$$

where $z = r\cos\sigma$, $\varpi = r\sin\sigma$ define spherical polar coordinates. Due to the coupled non-linearity of (2) the solution of (2) subject to (3) will only be discussed for small values of R and R_1. In this case there is a formal solution of the scaled equations (2) given by

$$\psi = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \chi_p^q R_1^{2q} R^p, \quad V = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} V_p^q R_1^{2q} R^p, \tag{4}$$

where χ_p^2 is defined by

$$\chi_p^q = \psi_p^q + \frac{R_1^2}{R}\psi_{p-1}^{q+1}. \tag{5}$$

The expansion (4) will not be uniformly valid at infinity but for the present choice of parameters will be sufficiently accurate to discuss the flow in a vicinity of the sphere. In particular, the effort of the rotation of the sphere will be dominant in a neighbourhood of the sphere and the case of most interest in the present paper is for choices of the parameter R and R_1 such that $R_1^2/R > 6$. For example, if $R_1^2 = O(10^{-n})$, $R = O(10^{-m})$, $m > n$ so that $R_1^2R^{-1} = O(10^{m-n})$, m, n being positive integers, but in order that convection terms like

$$R_1^4R^{-1}\varpi\frac{\partial\{\psi_{-1}^1, D^2(\psi_{-1}^1)/\varpi^2\}}{\partial(z,\varpi)} \tag{6}$$

are uniformly small near the sphere, it is necessary that $R_1^4R^{-1} \ll 1$ which is satisfied in a numerical sense if $m = n+1$ and $n \geqslant 3$, say. The first few terms in the expansions are given by

$$\left.\begin{aligned} \psi_0^0 &= \left(\tfrac{1}{2}r^2 - \tfrac{3}{4}r + \frac{1}{4r}\right)(1-\beta^2), \quad \beta = \cos\sigma, \\ \psi_{-1}^1 &= \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2r^2} - \frac{1}{r}\right)\beta(1-\beta^2), \\ V_0^0 &= \frac{1-\beta^2}{r}, \quad V_1^0 = \left(\frac{3}{4r} - \frac{1}{2} - \frac{1}{4r^2}\right)\beta(1-\beta^2). \end{aligned}\right\} \tag{7}$$

The procedure is straightforward and higher-order terms may be constructed in a similar manner.

However, for $R_1^2R^{-1} = O(10)$ it is not necessary to consider any higher-order terms as the dominant effects in the neighbourhood of the boundary are represented by

$$\chi_0^0 = \psi_0^0 + \frac{R_1^2}{R}\psi_{-1}^1.$$

It is noted at this point that, if the more refined method of matched asymptotic expansions were employed to determine v and ψ, higher-order terms would be introduced into the Stokes expansion but the leading terms would still be represented by χ_0^0. Now the stream function for the leading terms may be written in the form

$$\psi = \frac{(r-1)^2}{4r}\left[2r + 1 + \frac{3\cos\sigma}{r\cos\sigma_0}\right]\sin^2\sigma, \tag{8}$$

where $R_1^2/6R = \alpha/6 = \sec\sigma_0$. It follows that the sphere and z-axis are streamlines and also the curve C defined by

$$2r+1+\frac{3\cos\sigma}{r\cos\sigma_0} = 0. \tag{9}$$

This curve C encloses a vortex or a region of reversed flow (in the azimuthal plane) exterior to the rear portion of the sphere and meets the sphere at the point $P(1, \pi-\sigma_0)$, $0 \leqslant \sigma_0 \leqslant \frac{1}{2}\pi$ and the axis at the point

$$r = \tfrac{1}{4}(1+4\alpha)^{\frac{1}{2}}-\tfrac{1}{4}, \quad \sigma = \pi. \tag{10}$$

(a)

(b)

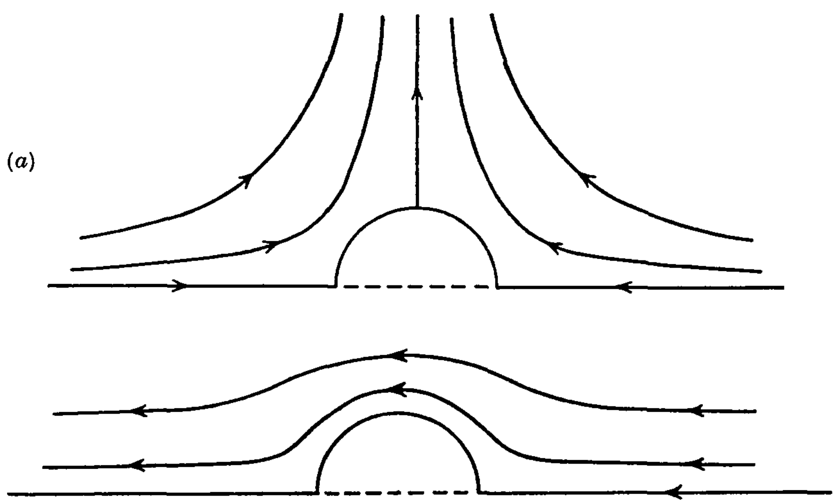

Fig. 1. (a) Secondary flow streamlines in the azimuthal plane.
(b) Primary Stokes flow streamlines.

The fluid velocity along the axis is positive for $1 < r < \frac{1}{4}(1+4\alpha)^{\frac{1}{2}}-\frac{1}{4}$, $\sigma = \pi$, and this latter end-point is a stagnation point for the flow. There is also a stagnation point inside the region bounded by C and exterior to the sphere defined by the equations

$$(1-3\beta^2)-4r\beta(2r+1) = 0, \quad r(4r^2+r+1)+\alpha\beta = 0. \tag{11}$$

The vorticity on the boundary arising from the motion in the azimuthal plane is given by

$$\zeta = \frac{3}{2}\left(1+\frac{R_1^2}{6R}\cos\sigma\right)\sin\sigma \tag{12}$$

and for $R_1^2 > 6R$ is easily seen to vanish at $\sigma = \pi-\sigma_0$. This is indicative of a reversed flow and also demonstrates that the reversed flow region cannot advance beyond the equatorial plane $\sigma = \frac{1}{2}\pi$ and is confined to the rear portion of the sphere (Fig. 1). Now the first term ψ_0^0 of χ_0^0 is the stream function for well-known Stokes flow past a non-rotating sphere. The motion is symmetrical fore and aft of the sphere and there is no separated flow. The rotation of the sphere produces a body force in the fluid whose z-component is given by

$$\frac{R_1^2}{R\varpi}V_0^0\frac{\partial V_0^0}{\partial z} = -\frac{2R_1^2}{Rr^4}\sin^3\sigma\cos\sigma. \tag{13}$$

This force represents a symmetric push and pull in the fluid and is directed against the Stokes flow for $\frac{1}{2}\pi < \sigma < \pi$. Again the secondary flow produced by the sphere spinning, and driven by the term

$$\frac{2R_1^2}{R\varpi^2} V_0^0 \frac{\partial V_0^0}{\partial z},$$

is antisymmetric about the equatorial plane $z = 0$ (which is a stream surface). To the degree of approximation considered, superposition of the primary Stokes flow

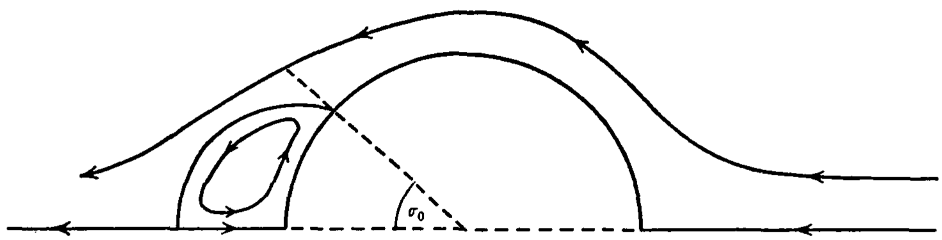

Fig. 2. Diagram of the streamlines illustrating region of reversed flow at the rear stagnation point.

and the secondary flow is possible and the net effect is to produce a flow in the azimuthal plane with eddies behind the sphere, provided that $R_1^2 > 6R$. Diagrams of the primary, secondary and superposed flows are given in Fig. 2.

Flow at large distances (*azimuthal component*). The equation for the azimuthal component of velocity at large distances from the sphere can be obtained by replacing ψ by $\frac{1}{2}\varpi^2$ in (2) to yield the Oseen-type equation

$$D^2(V) + R\frac{\partial V}{\partial z} = 0. \tag{14}$$

The solution
$$V = \exp\left\{-\frac{Rr}{2}(1+\beta)\right\}\left(\frac{1}{r}+\frac{R}{2}\right)(1-\beta^2) \tag{15}$$

satisfies the differential equation and outer boundary conditions exactly and for small R and finite r behaves like $(1-\beta^2)/r + O(R)$, so that to zero order in R the inner boundary conditions are satisfied exactly. It is clear that the vorticity of the azimuthal or swirl component of velocity decays exponentially everywhere in the fluid except in the wake region where it decays algebraically on the downstream axis.

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