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Slowly oscillating continuity in abstract metric spaces

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Abstract. In this paper, we investigate slowly oscillating continuity in cone metric spaces. It turns out that the set of slowly oscillating continuous functions is equal to the set of uniformly continuous functions on a slowly oscillating compact subset of a topological vector space valued cone metric space.

1. Introduction

The cone metric concept plays a very important role in computer sciences, and especially in image processing. A fundamental task in image processing is comparing images and computing some measure of the distance between images. The choice of a suitable definition of distance is not at all easy; what is close in one metric can be very far in another one. This naturally leads to an environment in which many possible metrics can be considered simultaneously and cone metric spaces lend themselves to this requirement. One specific instance of this is in the analysis of the structural similarity (SSIM) index of images (see [5, 6, 22]). SSIM is used to improve the measuring of visual distortion between images and is also used in fractal-based approximation using entropy maximization and sparsity constraints (see [30]). In both of these contexts the difference between two images is calculated using multiple criteria, which lead in a natural way to consider vector-valued distances.

Using the idea of continuity of a real function via sequences, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: slowly oscillating continuity ([7]), quasi-slowly oscillating continuity, Δ-quasi-slowly oscillating continuity ([8, 9, 17]), ward continuity ([10]), δ-ward continuity ([11]), statistical ward continuity, lacunary statistical ward continuity ([12, 13]). Investigation of some of these kinds of continuities lead some authors to find certain characterizations of uniform continuity of a real function via sequences in the above manner ([32, Theorem 8], [12, Theorem 6], [4, Theorem 1], [14, Theorem 3.8]).

Slowly oscillating continuity concept was introduced by Çakalli in [7] for real functions, and further investigation of slowly oscillating continuity was done by Çanak and Dik in [17], and by Vallin in [32].

The aim of this paper is to investigate slowly oscillating continuity in topological vector space valued cone metric spaces.

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2. Preliminaries

The concept of cone metric which has been known since the middle of 20th century (see, e.g., [23, 26, 31, 33]) was studied by Long-Guang and Xian in [24]. Beg, Abbas, and Nazir ([3]), Beg, Azam, and Arshad ([2]) replaced the set of an ordered Banach space by a locally convex Hausdorff topological vector space in the definition of a cone metric and a generalized cone metric space. Finally, Du ([19]) showed that the Banach contraction principles in general metric spaces and in topological vector space valued cone metric spaces are equivalent (See also [1, 16, 18, 20, 25, 27]). Let W be a topological vector space with its zero vector θ. A nonempty subset K of W is called a convex cone if $K + K \subseteq K$ and $\lambda K \subseteq K$ for $\lambda \geq 0$. A convex cone K is said to be pointed if $K \cap (-K) = \{\theta\}$. For a given cone $K \subseteq W$, we can define a partial ordering $\precsim_K$ with respect to K by

$$x \precsim_K y \Leftrightarrow y - x \in K.$$

$x \prec_K y$ will stand for $x \precsim_K y$ and $x \neq y$, while $x \ll_K y$ stands for $y - x \in \overset{\circ}{K}$, where $\overset{\circ}{K}$ denotes the interior of K. Throughout this paper we will use $x \precsim y$, $x \prec y$, and $x \ll y$ instead of $x \precsim_K y$, $x \prec_K y$, and $x \ll_K y$ respectively, when no confusion arises. In the following, unless otherwise specified, we always suppose that W is a locally convex Hausdorff topological vector space with its zero vector θ, K is a proper, closed and convex pointed cone in W with $\overset{\circ}{K} \neq \emptyset$ and $\precsim_K$ is a partial ordering with respect to K.

Now we recall the concept of a topological vector space valued cone metric.

Definition 2.1. Let X be a nonempty set and a vector-valued function

$$p : X \times X \to K$$

be a function satisfying the following:
(CM1) $\theta \precsim p(x, y)$ for all $x, y \in X$ and $p(x, y) = \theta$ if and only if $x = y$;
(CM2) $p(x, y) = p(y, x)$ for all $x, y \in X$;
(CM3) $p(x, z) \precsim p(x, y) + p(y, z)$ for all $x, y, z \in X$.
Then the function p is called a topological vector space valued cone metric (TVS-cone metric for short) on X, and (X, p) is said to be a topological vector space valued cone metric space (TVS-cone metric space for short).

Throughout this paper, $\mathbf{N}$ and X denote the set of positive integers, and a TVS-cone metric space with p, respectively. We will use boldface letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ... for sequences $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, $\mathbf{z} = (z_n)$, ... of points in X.

A sequence (x_n) of points in X is called quasi-Cauchy if for each $c \in \overset{\circ}{K}$ there exists an $n_0 \in \mathbf{N}$ such that $c - p(x_{n+1}, x_n) \in \overset{\circ}{K}$ for $n \geq n_0$.

3. Slowly oscillating continuity

A sequence (x_n) of points in X is slowly oscillating if for every $c \in \overset{\circ}{K}$ there exist a positive real number δ and a positive integer $N = N(c)$ such that $c - p(x_m, x_n) \in \overset{\circ}{K}$ for $n \geq N(c)$ and $n \leq m \leq (1 + \delta)n$. Trivially, Cauchy sequences are slowly oscillating not only in the real case but also in the TVS-cone metric space setting. It is easy to see that any slowly oscillating sequence of points in X is quasi-Cauchy, and therefore any Cauchy sequence is quasi-Cauchy. The converses are not always true. There are quasi-Cauchy sequences which are not Cauchy. There are quasi-Cauchy sequences which are not slowly oscillating. Any subsequence of a Cauchy sequence is Cauchy. The analogous property fails for quasi-Cauchy sequences, and slowly oscillating sequences as well.

Let $p : X \times X \to K$ be a TVS-cone metric, $F \subset X$, and f be a function on F into X. Then f is called uniformly continuous on F if given $c \in \overset{\circ}{K}$ there is a $\delta_c \in \overset{\circ}{K}$ such that for all x and y with $p(x, y) \ll \delta_c$ we have $p(f(x), f(y)) \ll c$.

A function f defined on a subset E of X is called slowly oscillating continuous if it preserves slowly oscillating sequences of points in E, i.e. $(f(x_n))$ is slowly oscillating whenever (x_n) is a slowly oscillating sequence of points in E.

Recently in [32] it was proved that a slowly oscillating continuous function is uniformly continuous on **R**. We see in this section that is also the case that any slowly oscillating continuous function on a connected subset of X is uniformly continuous.

In connection with slowly oscillating sequences and convergent sequences the problem arises to investigate the following types of continuity of functions on X where $c(X)$ denotes the set of all convergent sequences and $w(X)$ denotes the set of all slowly oscillating sequences of points in X.

(w) Slowly oscillating continuity of f on X,

(wc) $(x_n) \in w(X) \Rightarrow (f(x_n)) \in c(X)$,

(c) $(x_n) \in c(X) \Rightarrow (f(x_n)) \in c(X)$,

(cp1) Sequential continuity of f on X,

(d) $(x_n) \in c(X) \Rightarrow (f(x_n)) \in w(X)$,

(u) Uniform continuity of f on X.

It is clear that (c) is equivalent to $(cp1)$. It is easy to see that (wc) implies (d); (wc) implies (w); and (w) implies (d). Now we give a proof of the implication (w) implies (c) in the following:

Theorem 3.1. *If f is slowly oscillating continuous on a subset E of X, then it is continuous on E in the ordinary sense.*

Proof. Let (x_n) be a convergent sequence with the limit x_0. Then the limit of the sequence $(x_1, x_0, x_2, x_0, ..., x_0, x_n, x_0, ...)$ is also x_0. Then the sequence $(x_1, x_0, x_2, x_0, ..., x_0, x_n, x_0, ...)$ is slowly oscillating hence, by the hypothesis, the sequence $(f(x_1), f(x_0), f(x_2), f(x_0), ..., f(x_0), f(x_n), f(x_0), ...)$ is also slowly oscillating and therefore quasi-Cauchy. Now it follows from this that for any $c \in \overset{\circ}{K}$ there exists an n_0 such that $c - p(f(x_n), f(x_0)) \in \overset{\circ}{K}$ for $n \geq n_0$. Thus the sequence $(f(x_1), f(x_2), ..., f(x_n), ...)$ also converges to $f(x_0)$. This completes the proof of the theorem. □

A counterexample for a special case is provided in [7].

Theorem 3.2. *If a function f defined on a subset E of X is uniformly continuous on E, then it is slowly oscillating continuous on E.*

Proof. Let f be uniformly continuous on E. To prove that f is slowly oscillating continuous on E, take any slowly oscillating sequence (x_n), and $c \in \overset{\circ}{K}$. Uniform continuity of f on E implies that there exists a $\delta_c \in \overset{\circ}{K}$, depending on c, such that $c - p(f(x), f(y)) \in \overset{\circ}{K}$ for all $x, y \in E$ with $\delta_c - p(x, y) \in \overset{\circ}{K}$. For this $\delta_c \in \overset{\circ}{K}$ there exists an $N = N(\delta_c)$ and a δ such that $\delta_c - p(x_m, x_n) \in \overset{\circ}{K}$ for $n \geq N(\delta_c)$ and $n \leq m \leq (1+\delta)n$. Hence $c - p(f(x_m), f(x_n)) \in \overset{\circ}{K}$ if for $n \geq N(\delta_c)$ and $n \leq m \leq (1+\delta)n$. Now it follows that $(f(x_n))$ is slowly oscillating, which completes the proof of the theorem. □

A subset E of X is called slowly oscillating compact if any sequence of points in E has a slowly oscillating subsequence. We see that any compact subset of **R** is slowly oscillating compact. Union of two slowly oscillating compact subsets of **R** is slowly oscillating compact. Any subset of a slowly oscillating compact set is also slowly oscillating compact and so intersection of any slowly oscillating compact subsets of **R** is slowly oscillating compact.

Theorem 3.3. *Slowly oscillating continuous image of any slowly oscillating compact subset of X is slowly oscillating compact.*

Proof. Let f be a slowly oscillating continuous function on X and E be a slowly oscillating compact subset of X. Take any sequence $\mathbf{y} = (y_k)$ of points in $f(E)$. Write $y_k = f(x_k)$ for each $k \in \mathbf{N}$. As E is slowly oscillating compact, the sequence $\mathbf{x} = (x_k)$ has a slowly oscillating subsequence $z = (z_k) = (x_{n_k})$. Since f is slowly oscillating continuous, $f(\mathbf{z}) = (f(z_k))$ is slowly oscillating. The sequence $f(\mathbf{z}) = (f(z_k))$ is a subsequence of the sequence $\mathbf{y}$. Hence $f(E)$ is slowly oscillating compact, so the proof is completed. □

We add one more compactness defining as saying that a subset E of X is called Cauchy compact if any sequence of points in E has a Cauchy subsequence. We see that any Cauchy compact subset of X is also slowly oscillating compact and slowly oscillating continuous image of any Cauchy compact subset of X is Cauchy compact.

Corollary 3.4. *For any regular subsequential method G, if a subset E of X is G-sequentially compact, then it is slowly oscillating compact.*

It is a well known result that continuity of a function on a compact subset of a metric space coincides with uniform continuity. It is also true for the slowly oscillating continuity of a function defined on a slowly oscillating compact subset of X, i. e. slowly oscillating continuity coincides with uniform continuity on E.

Theorem 3.5. *Let E be a slowly oscillating compact subset of X, and f be a function from E to X. If f is slowly oscillating continuous, then it is uniformly continuous.*

Proof. Now assume that f is not uniformly continuous on E so that there exist a $c_0 \in \overset{\circ}{K}$ and sequences (x_n), (y_n) in E such that

$$\frac{c}{n} - p(x_n, y_n) \in \overset{\circ}{K}$$

and

$$c_0 - p(f(x_n), f(y_n)) \notin \overset{\circ}{K} \qquad (2.1)$$

for all $n \in \mathbf{N}$ and for all $c \in \overset{\circ}{K}$. Since E is slowly oscillating compact, there is a slowly oscillating subsequence (x_{n_k}) of (x_n). It is clear that the corresponding sequence (y_{n_k}) is also slowly oscillating, since

$$p(y_{n_k}, x_{n_k}) + p(x_{n_k}, x_{n_m}) + p(x_{n_m}, y_{n_m}) - p(y_{n_k}, y_{n_m}) \in \overset{\circ}{K}.$$

It follows from (2.1) that the sequences $(f(x_{n_k}))$ and $(f(y_{n_k}))$ are not slowly oscillating. This contradiction completes the proof of the theorem. □

Corollary 3.6. *Let E be a slowly oscillating compact subset of X, and f be a function from E to X. Then f is uniformly continuous if and only if it is slowly oscillating continuous.*

Since totally boundedness coincides with slowly oscillating compactness ([13, Theorem 3]), and boundedness coincides with totally boundedness in the real space we have the following:

Corollary 3.7. *A real valued function defined on a bounded subset of $\mathbf{R}$ is uniformly continuous if and only if it is slowly oscillating continuous.*

We extend the definition of uniform convergence of a sequence of functions given in [28] for cone metric spaces to TVS-cone metric spaces in the following.

Definition 3.8. Let E be a subset of X, and f_n and f be functions from E to X for all $n \in N$. A sequence of functions (f_n) is said to converge uniformly to f if any given $c \in \overset{\circ}{K}$ there exists a positive integer N such that for all $x \in E$ and all $n \geq N$ we have $p(f_n(x), f(x)) \ll c$.

It is a well known result that uniform limit of a sequence of continuous functions is continuous. This is also true in case of slowly oscillating continuity, i.e. uniform limit of a sequence of slowly oscillating continuous functions is slowly oscillating continuous.

Theorem 3.9. *If (f_n) is a sequence of slowly oscillating continuous functions defined on a subset E of X and (f_n) is uniformly convergent to a function f, then f is slowly oscillating continuous on E.*

Proof. Let $c \in \overset{\circ}{K}$. Then there exists a positive integer N such that $\frac{c}{3} - p(f_n(x), f(x)) \in \overset{\circ}{K}$ for all $x \in E$ whenever $n \geq N$. There exists a positive integer N_1, depending on c, and greater than N and a $\delta > 0$ such that $\frac{c}{3} - p(f_N(x_m), f_N(x_n)) \in \overset{\circ}{K}$ if $n \geq N_1$ and $n \leq m \leq (1+\delta)n$. Now if $n \geq N_1$ and $n \leq m \leq (1+\delta)n$, then

$$p(f(x_m), f_N(x_m)) + p(f_N(x_m), f_N(x_n)) + p(f_N(x_n), f(x_n)) - p(f(x_m), f(x_n)) \in \overset{\circ}{K}.$$

Thus it implies that $p(f(x_m), f(x_n)) \ll c$ if $n \geq N_1$ and $n \leq m \leq (1+\delta)n$. This completes the proof of the theorem. □

Theorem 3.10. *Let X be complete. The set of all slowly oscillating continuous functions on a subset E of X is a closed subset of the set of all continuous functions on E.*

Proof. Let us denote the set of all slowly oscillating continuous functions on E by $SOS(E)$ and f be any element in the closure of $SOS(E)$. Then there exists a sequence of points in $SOS(E)$ such that $\lim_{k\to\infty} f_k = f$. To show that f is slowly oscillating continuous on E take any slowly oscillating sequence (x_n) of points in E. Let $c \in \overset{\circ}{K}$. Since (f_k) converges to f, there exists an N such that for all $x \in E$ and for all $n \geq N$, $\frac{c}{3} - p(f(x), f_n(x)) \in \overset{\circ}{K}$. As f_N is slowly oscillating continuous, there exists an N_1, greater than N, a positive real number δ such that for all $n \geq N_1$, $\frac{c}{3} - p(f_N(x_m), f_N(x_n)) \in \overset{\circ}{K}$ if $n \geq N(1)$ and $n \leq m \leq (1+\delta)n$. Hence for all $n \geq N_1$ and $n \leq m \leq (1+\delta)n$ we have

$$p(f(x_m), f_N(x_m)) + p(f(x_n), f_N(x_n)) + p(f_N(x_m), f_N(x_n)) - p(f(x_m), f(x_n)) \in \overset{\circ}{K}.$$

Thus it implies that $p(f(x_m), f(x_n)) \ll c$ if $n \geq N_1$ and $n \leq m \leq (1+\delta)n$. This completes the proof of the theorem. □

Corollary 3.11. *Let X be complete. Then the set of all slowly oscillating continuous functions on a subset E of X is a complete subspace of the set of all continuous functions on E.*

4. Conclusion

This paper is mainly devoted to investigate slowly oscillating continuity in topological vector space valued cone metric spaces, including not only an improvement and a generalization of results given in the papers, [7, 12, 13], as it has been presented in a more general setting, i.e. in a topological vector space valued cone metric space which is more general than the metric space setting, but also an investigation of some further results which are also new for the metric space setting. So that one may expect it to be a more useful tool in the field of cone metric space theory in modeling various problems occurring in many areas of science, computer science, information theory, and image processing. We note that the results in this paper are valid both in Banach space valued cone normed spaces ([29]) and in topological vector space valued cone normed spaces as any topological vector space valued cone normed space is a topological vector space valued cone metric space with the induced topological vector space valued cone metric $p(x, y) = |||x - y|||$. For a further study, we suggest to investigate slowly oscillating sequences of fuzzy points, and characterizations of uniform continuity for the fuzzy functions defined on a fuzzy cone metric space. However due to the change in settings, the definitions and methods of proofs will not always be analogous to those of the present work (see [15, 21] for the definitions and related concepts in fuzzy setting).

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