SMALL BALL PROBABILITY ESTIMATES FOR LOG-CONCAVE MEASURES

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ABSTRACT. We establish a small ball probability inequality for isotropic log-concave probability measures: there exist absolute constants $c_1, c_2 > 0$ such that if X is an isotropic log-concave random vector in $\mathbb{R}^n$ with ψ_2 constant bounded by b and if A is a non-zero $n \times n$ matrix, then for every $\varepsilon \in (0, c_1)$ and $y \in \mathbb{R}^n$,

$$\mathbb{P}\left(\|Ax - y\|_2 \leq \varepsilon \|A\|_{\rm HS}\right) \leq \varepsilon^{\left(\frac{c_2}{b}\frac{\|A\|_{\rm HS}}{\|A\|_{\rm op}}\right)^2},$$

where $c_1, c_2 > 0$ are absolute constants.

1. INTRODUCTION

Recently, there is an increasing interest in extending results for independent random variables, which are known from probability theory, to the setting of log-concave probability measures. A Central Limit Theorem for isotropic log-concave measures was established by B. Klartag in [12] for these measures (see also [7] for an alternative proof and [13], [6] for related developments). A "large deviation inequality" for isotropic log-concave measures was proved in [27]. In all these questions the main effort is put in trying to replace the notion of independence by the "geometry" of convex bodies, since a log-concave measure should be considered as the measure-theoretic equivalent of a convex body. Most of these recent results make heavy use of tools from the asymptotic theory of finite-dimensional normed spaces.

The purpose of this paper is to add a "small ball probability" estimate in this setting. The motivation for us was a question of N. Tomczak-Jaegermann initiated by results in [16]. In this paper the authors, motivated by questions on random polytopes, proved the following "small ball probability" estimate.

Theorem 1.1 ([16]). *Let A be a non-zero $n \times n$ matrix and let $X = (\xi_1, \ldots, \xi_n)$ be a random vector, where ξ_i are independent subgaussian random variables with* $\mathrm{Var}(\xi_i) \geq 1$ *and subgaussian constants bounded by β. Then, for any $y \in \mathbb{R}^n$, one has*

$$\mathbb{P}\left(\|AX - y\|_2 \leq \frac{\|A\|_{\rm HS}}{2}\right) \leq 2\exp\left(-\frac{c_0}{\beta^4}\left(\frac{\|A\|_{\rm HS}}{\|A\|_{\rm op}}\right)^2\right),$$

where $c_0 > 0$ is a universal constant.

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It is pointed out in [16] that, in the special case where the ξ_i's are independent standard Gaussian random variables, one can obtain the following stronger result.

Theorem 1.2 ([16])**.** *Let A be a non-zero $n\times n$ matrix and let $X=(\xi_1,\dots,\xi_n)$ be a random vector, where the ξ_i's are independent standard Gaussian random variables. Then, for any $\varepsilon\in(0,c_1)$ and any $y\in\mathbb{R}^n$, one has*

$$\mathbb{P}\left(\|AX-y\|_2\le\varepsilon\|A\|_{\rm HS}\right)\le\varepsilon^{\left(c_2\frac{\|A\|_{\rm HS}}{\|A\|_{\rm op}}\right)^2},$$

where $c_1,c_2>0$ are universal constants.

The proof of Theorem 1.2 makes use of the affirmative answer to the B-conjecture by Cordero-Erausquin, Fradelizi and Maurey (see [5]). The "B-Theorem" has been already applied for small ball probability estimates in [15] and [14].

The main result of this paper extends the previously mentioned results to the setting of log-concave probability measures, answering a question posed to us by N. Tomczak-Jaegermann.

Theorem 1.3. *Let X be an isotropic* log-*concave random vector in $\mathbb{R}^n$, which has subgaussian constant b. Let A be a non-zero $n\times n$ matrix. For any $y\in\mathbb{R}^n$ and $\varepsilon\in(0,c_1)$, one has*

$$\mathbb{P}\left(\|AX-y\|_2\le\varepsilon\|A\|_{\rm HS}\right)\le\varepsilon^{\left(\frac{c_2}{b}\frac{\|A\|_{\rm HS}}{\|A\|_{\rm op}}\right)^2},$$

where $c_1,c_2>0$ are universal constants.

Theorem 1.3 depends on a reverse Hölder inequality for the negative moments of the Euclidean norm with respect to a log-concave probability measure μ with density f. Let $-n<p<\infty$, $p\neq0$ and $I_p(f):=\left(\int_{\mathbb{R}^n}\|x\|_2^pf(x)dx\right)^{\frac{1}{p}}$. A result of O. Guédon (see [10]) implies that for $p\in(-1,0)$ one has

$$I_p(f)\ge c_pI_2(f),$$

where the constant c_p depends only on p.

Actually, Guédon's result is more general and holds even if we replace the Euclidean norm by any other norm. Moreover, the result is sharp and can be achieved for a 1-dimensional density.

In order to reveal the role of the dimension we introduce the quantity $q_*(f)$:

$$q_*(f):=\max\{k\ge1:k_*(Z_k(f))\ge k\},$$

where $k_*(Z_k(f))$ is the Dvoretzky number of the L_k-centroid body of f (see Section 2 for precise definitions). Then, one can show the following.

Theorem 1.4. *Let f a* log-*concave density in $\mathbb{R}^n$ with center of mass at the origin. Then for every $k\le c_1q_*(f)$ one has*

$$I_{-k}(f)\ge c_2I_2(f)\ ,$$

where $c_1,c_2>0$ are absolute constants.

The paper is organized as follows. In §2 we gather some background material needed in the rest of the paper. In the next section we study a family of convex bodies associated to a log-concave measure. This family was introduced by K. Ball in [1]. In §4 we establish a volumetric relation between any marginal of a log-concave measure and the corresponding projection of its associated generalized centroid body. Precisely, we prove an L_q-version of the Rogers-Shephard inequality. This is one of the main steps towards the proof of Theorem 1.4. Part of the material of §3 and §4 are adaptations to the case of *log*-concave measures of tools and results of [27]. This connection is clarified in the propositions 4.3- 4.5. In §5 we give an exact formula (Proposition 5.4 in the main text) relating the negative moments of the norm of the polar L_q centroid body on the sphere with the negative moments of the Euclidean norm with respect to the measure. This can be seen as a transfer principle permitting the use of known concentration results on the sphere. We stress the fact that all the results up to §5 are valid for an arbitrary log-concave measure and not just merely for an isotropic one. This special class of measures is treated in §6. The proof of Theorem 1.3 is completed in §6: It is based on Proposition 5.4 which is based on the L_q Rogers-Shephard inequality. In the last section, we also discuss the sharpness of the estimate in Theorem 1.3 and its connections with the well-known "Hyperplane Conjecture" in Convex Geometry.

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2. Background material

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\mu_{n,k}$. We write P_F for the orthogonal projection onto the subspace F. We also write $\widetilde{A}$ for the homothetic image of volume 1 of a compact set $A \subseteq \mathbb{R}^n$, i.e. $\widetilde{A} := \frac{A}{|A|^{1/n}}$.

A set V is called star-shaped if for every $x \in V$ and $\lambda \in [0,1]$ $\lambda x \in V$. We define the gauge function of V as

$$\|x\|_V := \min\{\lambda > 0 : x \in \lambda V\}\ .$$

A convex body is a compact convex subset C of $\mathbb{R}^n$ with non-empty interior. We say that C is symmetric if $-x \in C$ whenever $x \in C$. We say that C has center of mass at the origin if $\int_C \langle x, \theta\rangle dx = 0$ for every $\theta \in S^{n-1}$. The support function $h_C : \mathbb{R}^n \to \mathbb{R}$ of C is defined by $h_C(x) = \max\{\langle x, y\rangle : y \in C\}$. The mean width of C is defined by

$$W(C) = \int_{S^{n-1}} h_C(\theta)\sigma(d\theta).$$

The radius of C is the quantity $R(C) = \max\{\|x\|_2 : x \in C\}$, and the polar body C° of C is

$$C^\circ := \{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \text{ for all } x \in C\}.$$

Whenever we write $a \simeq b$, we mean that there exist universal constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. The letters $c, c', c_1, c_2 > 0$ etc. denote universal positive constants which may change from line to line. Also, if $K, L \subseteq \mathbb{R}^n$ we will write $K \simeq L$ if there exist universal constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$.

Let $A = (a_{i,j})_{1\leq i,j\leq n}$ be a $n\times n$ matrix. We write $\|A\|_{\rm HS}$ for the Hilbert-Schmidt norm of A:

$$\|A\|_{\rm HS}^2 := \sum_{i,j} a_{i,j}^2,$$

and $\|A\|_{\rm op}$ for the operator norm of A:

$$\|A\|_{\rm op} := \max_{\theta\in S^{n-1}} \|A\theta\|_2.$$

Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be an integrable function. We say that f has center of mass at the origin if

$$\int_{\mathbb{R}^n} \langle x, y\rangle f(x)dx = 0 \quad \text{for all } y \in \mathbb{R}^n. \tag{2.1}$$

Given f and $y \in \mathbb{R}^n$ we write f_y for the function $f_y(x) := f(x + y)$.

Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be an integrable function with $\int_{\mathbb{R}^n} f(x)dx = 1$. For every $1 \leq p \leq \infty$ and $\theta \in S^{n-1}$ we consider the quantities

$$h_{Z_p(f)}(\theta) := \left(\int_{\mathbb{R}^n} |\langle x, \theta\rangle|^p f(x)dx\right)^{1/p} \tag{2.2}$$

If $h_{Z_p(f)}(\theta) < \infty$ for every $\theta \in S^{n-1}$, we define the L_p-centroid body $Z_p(f)$ of f to be the centrally symmetric convex body that has support function $h_{Z_p(f)}$.

L_p-centroid bodies were introduced in [18] (see also [19]), where a generalization of Santaló's inequality was proved. In [18] and [19] a different normalization (and notation) was used. Here, we follow the normalization (and notation) that appeared in [25], since it fits better in a probabilistic setting. These bodies played a crucial role in [27] and [7]. If K is a compact set of volume 1, we will write $Z_p(K)$ instead of $Z_p(\mathbf{1}_K)$.

Note that for $1 \leq p \leq q \leq \infty$ one has $Z_p(f) \subseteq Z_q(f)$. If $f := \mathbf{1}_A$ for some compact set $A \subseteq \mathbb{R}^n$, then $Z_\infty(f) = {\rm co}\{A, -A\}$. Note that if $T \in SL_n$ then for all $p > 0$ one has

$$Z_p(f \circ T^{-1}) = TZ_p(f). \tag{2.3}$$

We refer to [27] for additional information on Z_p-bodies.

A random variable ξ is called subgaussian if there exists a constant $0 < \beta < \infty$ such that

$$\|\xi\|_{2k} \leq \beta\|\gamma\|_{2k} \;\; k = 1, 2, \dots,$$

where γ is a standard Gaussian random variable.

Let μ be a probability measure in $\mathbb{R}^n$ with density $f \geq 0$ and let $\alpha \geq 1$. We say that μ (or f) is ψ_α with constant b_α if for every $p \geq \alpha$ one has

$$Z_p(f) \subseteq b_\alpha p^{1/\alpha} Z_\alpha(f),$$

or, equivalently, if for every $\theta \in S^{n-1}$ and $t > 0$,

$$\mu\left(\left\{x \in \mathbb{R}^n : |\langle x, \theta\rangle| \geq t \left(\int_{\mathbb{R}^n} |\langle x, \theta\rangle|^\alpha f(x)dx\right)^{1/\alpha}\right\}\right) \leq 2\exp\left(-\frac{t}{b_\alpha}\right)^\alpha.$$

For $-n < p \le \infty$ we define the quantities $I_p(f)$ as

$$I_p(f) := \left(\int_{\mathbb{R}^n} \|x\|_2^p f(x)dx \right)^{1/p}. \tag{2.4}$$

We say that a function $f : \mathbb{R}^n \to [0, \infty]$ is log-concave if, for every $x, y \in \mathbb{R}^n$ and $\lambda \in (0, 1)$,

$$f(\lambda x + (1-\lambda)y) \ge f(x)^\lambda f(y)^{1-\lambda}.$$

Note that if f is log-concave and finite then, $I_p(f) < \infty$ for $-n < p < \infty$ and $\left(\int_{\mathbb{R}^n} |\langle x, \theta\rangle|^p f(x)dx\right)^{1/p} < \infty$ for $p > 0$.

It is well known that the level sets of a log-concave function are convex sets. Also, if $K \subseteq \mathbb{R}^n$ is a convex body, the Brunn-Minkowski inequality implies that the measure μ with $d\mu := \mathbf{1}_{\frac{K}{|K|^{1/n}}}(x)dx$ is a log-concave probability measure in $\mathbb{R}^n$.

We refer to the books [31], [23] and [28] for basic facts from the Brunn-Minkowski theory and the asymptotic theory of finite dimensional normed spaces.

3. Keith Ball's Bodies

K. Ball introduced a way to "pass" from a log-concave function to a convex body (see [1]). In this section we focus on the interaction between K. Ball's bodies $K_p(f)$ of some function f and the L_q-centroid bodies $Z_q(f)$ of this function.

Let f be an integrable and bounded function in $\mathbb{R}^n$ and let $p > 0$. We define a set $K_p(f)$ by

$$K_p(f) := \left\{ x \in \mathbb{R}^n : p \int_0^\infty f(rx) r^{p-1} dr \ge f(0) \right\}. \tag{3.1}$$

Note that $0 \in K_p(f)$ and if $x \in K_p(f)$ and $\lambda \in [0, 1]$, $\lambda x \in K_p(f)$. So $K_p(f)$ is a star shaped set and we can write

$$\|x\|_{K_p(f)} := \left(\frac{p}{f(0)} \int_0^\infty f(rx) r^{p-1} dr \right)^{-1/p}. \tag{3.2}$$

Indeed, for any $\lambda > 0$ and $x \in \mathbb{R}^n$ we have

$$\|\lambda x\|_{K_p(f)}^{-p} = \frac{p}{f(0)} \int_0^\infty f(r\lambda x) r^{p-1} dr = \frac{p}{\lambda^p f(0)} \int_0^\infty f(rx) r^{p-1} dr = \frac{1}{\lambda^p} \|x\|_{K_p(f)}^{-p}.$$

Note that if f is even, then $K_p(f)$ is symmetric for all $p > 0$.

Integrating in polar coordinates we see that, for any $\theta \in S^{n-1}$,

$$\begin{aligned} \int_{K_{n+1}(f)} \langle x, \theta\rangle dx &= n\omega_n \int_{S^{n-1}} \langle \phi, \theta\rangle \int_0^{1/\|\phi\|_{K_{n+1}(f)}} r^n dr d\sigma(\phi) \\ &= \frac{n\omega_n}{f(0)} \int_{S^{n-1}} \langle \phi, \theta\rangle \int_0^\infty r^n f(r\theta) dr d\sigma(\phi) \\ &= \frac{1}{f(0)} \int_{\mathbb{R}^n} \langle x, \theta\rangle f(x) dx. \end{aligned}$$

So, if f has center of mass at the origin then $K_{n+1}(f)$ has also center of mass at the origin.

The same argument shows that, for every $p > 0$ and $\theta \in S^{n-1}$,

$$\begin{aligned}\int_{K_{n+p}(f)} |\langle x, \theta\rangle|^p dx &= n\omega_n \int_{S^{n-1}} |\langle \phi, \theta\rangle|^p \int_0^{1/\|\phi\|_{K_{n+p}(f)}} r^{n+p-1} dr d\sigma(\phi) \\ &= \frac{n\omega_n}{f(0)} \int_{S^{n-1}} |\langle \phi, \theta\rangle|^p \int_0^{\infty} r^{n+p-1} f(r\theta) dr d\sigma(\phi) \\ &= \frac{1}{f(0)} \int_{\mathbb{R}^n} |\langle x, \theta\rangle|^p f(x) dx.\end{aligned}$$

Throughout the rest of the paper we will assume that the function f satisfies

$$0 < |K_p(f)| < \infty$$

for every $p > 1$. This yields that for every $p \geq 1$ we have

$$\int_{K_{n+p}(f)} |\langle x, \theta\rangle|^p dx = |K_{n+p}(f)|^{1+\frac{p}{n}} \int_{\widetilde{K_{n+p}(f)}} |\langle x, \theta\rangle|^p dx.$$

So, we conclude that for $p \geq 1$,

$$Z_p(\widetilde{K_{n+p}(f)}) |K_{n+p}(f)|^{\frac{1}{p}+\frac{1}{n}} f(0)^{1/p} = Z_p(f). \tag{3.3}$$

Let V be a star-shaped body in $\mathbb{R}^n$ and let $\|x\|_V$ be the gauge function of V. Working in the same manner we see that for $-n < p \leq \infty$, $p \neq 0$,

$$\begin{aligned}\int_{K_{n+p}(f)} \|x\|_V^p dx &= n\omega_n \int_{S^{n-1}} \|\phi\|_V^p \int_0^{1/\|\phi\|_{K_{n+p}(f)}} r^{n+p-1} dr d\sigma(\phi) \\ &= \frac{n\omega_n}{f(0)} \int_{S^{n-1}} \|\phi\|_V^p \int_0^{\infty} r^{n+p-1} f(r\theta) dr d\sigma(\phi) \\ &= \frac{1}{f(0)} \int_{\mathbb{R}^n} \|x\|_V^p f(x) dx.\end{aligned}$$

Setting $V = B_2^n$ we get

$$I_p(\widetilde{K_{n+p}(f)}) |K_{n+p}(f)|^{\frac{1}{p}+\frac{1}{n}} f(0)^{1/p} = I_p(f). \tag{3.4}$$

The family of bodies K_p was introduced by K. Ball in [1], where the following theorem was proved:

Theorem 3.1. *If f is a* log-*concave function then $K_p(f)$ is a convex set for all $p > 0$.*

We will use the following standard lemma:

Lemma 3.2. *Let $f : [0, \infty) \to [0, \infty)$ be a* log-*concave function. Then, for all $1 < p \leq q$ we have*

$$\left(\frac{p}{\|f\|_\infty} \int_0^\infty t^{p-1} f(t) dt\right)^{1/p} \leq \left(\frac{q}{\|f\|_\infty} \int_0^\infty t^{q-1} f(t) dt\right)^{1/q} \tag{3.5}$$

and

$$\left(\frac{q}{\Gamma(q+1) f(0)} \int_0^\infty t^{q-1} f(t) dt\right)^{1/q} \leq \left(\frac{p}{\Gamma(p+1) f(0)} \int_0^\infty t^{p-1} f(t) dt\right)^{1/p}. \tag{3.6}$$

Comment. The proof of both facts is well-known to specialists and can be found in [21]. The first claim can be derived from Lemma 2.1 in [21, page 76], whereas the second claim can be derived from Corollary 2.7 in [21, page 81]. Both facts are also corollaries of a result of Borell (see [4]).

If f is log-concave and even, then $\|f\|_\infty = f(0)$. If f is log-concave and has center of mass at the origin then the quantities $\|f\|_\infty$ and $f(0)$ are comparable. More precisely, we have the following theorem of M. Fradelizi (see [8]).

Theorem 3.3. *Let $f : \mathbb{R}^n \to [0,\infty]$ be a* log*-concave function with center of mass at the origin. Then,*

$$\|f\|_\infty \le e^n f(0). \tag{3.7}$$

Proposition 3.4. *Let $f : \mathbb{R}^n \to [0,\infty]$ be a* log*-concave function with center of mass at the origin and $\int_{\mathbb{R}^n} f(x)dx = 1$. Then, for $p \ge 1$ one has*

$$\frac{1}{e} Z_p(\widetilde{K_{n+p}(f)}) \subseteq f(0)^{1/n} Z_p(f) \subseteq e\frac{n+p}{n} Z_p(\widetilde{K_{n+p}(f)}). \tag{3.8}$$

Moreover, for $-(n-1) < p \le \infty$,

$$\frac{1}{e} I_p(\widetilde{K_{n+p}(f)}) \le f(0)^{1/n} I_p(f) \le e\frac{n+p}{n} I_p(\widetilde{K_{n+p}(f)}). \tag{3.9}$$

Moreover, if f is even then the constant on the left hand side in the previous two inclusions can be chosen to be 1 instead of $\frac{1}{e}$.

Proof. Using (3.5), (3.6) and (6) we see that if f is log-concave then

$$\|x\|_{K_p(f)} \le \frac{\Gamma(q+1)^{1/q}}{\Gamma(p+1)^{1/p}} \|x\|_{K_q(f)} \tag{3.10}$$

and

$$\|x\|_{K_q(f)} \le \left(\frac{\|f\|_\infty}{f(0)}\right)^{\frac{1}{p}-\frac{1}{q}} \|x\|_{K_p(f)}. \tag{3.11}$$

Moreover, if f has center of mass at the origin, then (3.11) becomes

$$\|x\|_{K_q(f)} \le e^{\frac{n}{p}-\frac{n}{q}} \|x\|_{K_p(f)}. \tag{3.12}$$

So, if f is log-concave and has center of mass at the origin, we get the following volumetric estimates for $1 < p \le q$:

$$e^{\frac{n^2}{q}-\frac{n^2}{p}} |K_p(f)| \le |K_q(f)| \le \left(\frac{\Gamma(q+1)^{1/q}}{\Gamma(p+1)^{1/p}}\right)^n |K_p(f)|. \tag{3.13}$$

Once again, integrating in polar coordinates we get

$$|K_n(f)| = \frac{1}{f(0)} \int_{\mathbb{R}^n} f(x)dx. \tag{3.14}$$

So, if f is a log-concave function with center of mass at the origin and $\int_{\mathbb{R}^n} f(x)dx = 1$ then, combining (3.14) and (3.13) we get that, for $p > 0$,

$$\frac{e^{-\frac{np}{n+p}}}{f(0)} \le |K_{n+p}(f)| \le ((n+p)!)^{\frac{n}{n+p}} \frac{1}{n! f(0)},$$

and hence,

$$\frac{1}{e} \le f(0)^{\frac{1}{n}+\frac{1}{p}} |K_{n+p}(f)|^{\frac{1}{n}+\frac{1}{p}} \le \frac{((n+p)!)^{1/p}}{(n!)^{\frac{n+p}{np}}}.$$

Using the bounds

$$\frac{((n+p)!)^{1/p}}{(n!)^{\frac{n+p}{np}}} \leq (n+p)\frac{(n!)^{1/p}}{(n!)^{\frac{n+p}{np}}} = \frac{n+p}{(n!)^{1/n}} \leq e\frac{n+p}{n},$$

we conclude that

$$\frac{1}{e} \leq f(0)^{\frac{1}{n}+\frac{1}{p}}|K_{n+p}(f)|^{\frac{1}{n}+\frac{1}{p}} \leq e\frac{n+p}{n}. \tag{3.15}$$

Working in the same way for $0 \leq p \leq n-1$, we get

$$e^{-\frac{np}{n-p}}|K_{n-p}(f)| \leq \frac{1}{f(0)} \leq \frac{n!}{((n-p)!)^{\frac{n}{n-p}}}|K_{n-p}(f)|,$$

and hence,

$$\frac{1}{e} \leq f(0)^{\frac{1}{n}-\frac{1}{p}}|K_{n-p}(f)|^{\frac{1}{n}-\frac{1}{p}} \leq \frac{((n!))^{\frac{n-p}{np}}}{((n-p)!)^{1/p}}.$$

Using the bounds

$$\frac{((n!))^{\frac{n-p}{np}}}{((n-p)!)^{1/p}} \leq n^{\frac{n-p}{n}}\frac{((n-p)!)^{\frac{n-p}{np}}}{((n-p)!)^{1/p}} \leq e\frac{n}{n-p},$$

we conclude that

$$\frac{1}{e} \leq f(0)^{\frac{1}{n}-\frac{1}{p}}|K_{n-p}(f)|^{\frac{1}{n}-\frac{1}{p}} \leq e\frac{n}{n-p}. \tag{3.16}$$

Combining (3.15), (3.16) and (3.3), (3.4) we complete the proof. □

Working in the same spirit we can also compare the symmetric convex bodies $Z_q(\widetilde{K_{n+r_1}(f)})$ and $Z_q(\widetilde{K_{n+r_2}(f)})$ for $-(n-1) < r_1 \leq r_2 \leq \infty$ and $q \geq 1$. We have the following

Proposition 3.5. *Let $f : \mathbb{R}^n \to [0,\infty)$ be a* log*-concave function with center of mass at the origin and $\int_{\mathbb{R}^n} f(x)dx = 1$. Then, for every $1 \leq q \leq n$, one has*

$$c_1 f(0)^{1/n} Z_q(f) \subseteq Z_q(\widetilde{K_{n+1}(f)}) \subseteq c_2 f(0)^{1/n} Z_q(f) \tag{3.17}$$

and

$$c_3 f(0)^{1/n} Z_q(f) \subseteq Z_q(\widetilde{K_{n+2}(f)}) \subseteq c_4 f(0)^{1/n} Z_q(f), \tag{3.18}$$

where $c_1, c_2, c_3, c_4 > 0$ are universal constants.

Proof. Let $\theta \in S^{n-1}$. We have that

$$\begin{aligned}
\frac{h^q_{Z_q(\widetilde{K_{n+r_1}(f)})}(\theta)}{h^q_{Z_q(\widetilde{K_{n+r_2}(f)})}(\theta)} &= \left(\frac{|K_{n+r_2}|}{|K_{n+r_1}|}\right)^{1+\frac{q}{n}} \frac{\int_{K_{n+r_1}} |\langle x,\theta\rangle|^q dx}{\int_{K_{n+r_2}} |\langle x,\theta\rangle|^q dx} \\
&= \left(\frac{|K_{n+r_2}|}{|K_{n+r_1}|}\right)^{1+\frac{q}{n}} \frac{\frac{n\omega_n}{n+q}\int_{S^{n-1}} |\langle \phi,\theta\rangle|^q \|\phi\|^{-(n+q)}_{K_{n+r_1}} d\sigma(\phi)}{\frac{n\omega_n}{n+q}\int_{S^{n-1}} |\langle \phi,\theta\rangle|^q \|\phi\|^{-(n+q)}_{K_{n+r_2}} d\sigma(\phi)}.
\end{aligned}$$

Using (3.11) and (3.12) we get

$$\frac{(\Gamma(n+r_1))^{\frac{n+q}{n+r_1}}}{(\Gamma(n+r_2))^{\frac{n+q}{n+r_2}}} \leq \frac{\|\phi\|^{-(n+q)}_{K_{n+r_1}(f)}}{\|\phi\|^{-(n+q)}_{K_{n+r_2}(f)}} \leq e^{n\frac{(r_2-r_1)(n+q)}{(n+r_1)(n+r_2)}}.$$

Also, (3.13) implies that

$$e^{-n^2\frac{r_2-r_1}{(n+r_1)(n+r_2)}} \leq \frac{|K_{n+r_2}|}{|K_{n+r_1}|} \leq \frac{(\Gamma(n+r_2))^{\frac{n}{n+r_2}}}{(\Gamma(n+r_1))^{\frac{n}{n+r_1}}}.$$

So,

$$\begin{aligned} e^{-\frac{n(r_2-r_1)(n+q)}{q(n+r_1)(n+r_2)}}\frac{(\Gamma(n+r_1))^{\frac{n+q}{q(n+r_1)}}}{(\Gamma(n+r_2))^{\frac{n+q}{q(n+r_2)}}} &\leq \frac{h_{Z_q(\widetilde{K_{n+r_1}(f)})}(\theta)}{h_{Z_q(\widetilde{K_{n+r_2}(f)})}(\theta)} \\ &\leq e^{\frac{n(r_2-r_1)(n+q)}{q(n+r_1)(n+r_2)}}\frac{(\Gamma(n+r_2))^{\frac{n+q}{q(n+r_2)}}}{(\Gamma(n+r_1))^{\frac{n+q}{q(n+r_1)}}}. \end{aligned}$$

For $n \in \mathbb{N}$, $q > 0$ and $-n < r_1 \leq r_2 \leq \infty$ we define

$$A_{n,q,r_1,r_2} := e^{\frac{n(r_2-r_1)(n+q)}{q(n+r_1)(n+r_2)}}\frac{(\Gamma(n+r_2))^{\frac{n+q}{q(n+r_2)}}}{(\Gamma(n+r_1))^{\frac{n+q}{q(n+r_1)}}}. \tag{3.19}$$

So, we have shown that if f is a log-concave function in $\mathbb{R}^n$ with center of mass at the origin, then for every $q \geq 1$, for every $-(n-1) < r_1 \leq r_2 \leq \infty$ and for all $\theta \in S^{n-1}$, one has

$$A^{-1}_{q,r_1,r_2,n} \leq \frac{h_{Z_q(\widetilde{K_{n+r_1}(f)})}(\theta)}{h_{Z_q(\widetilde{K_{n+r_2}(f)})}(\theta)} \leq A_{q,r_1,r_2,n}, \tag{3.20}$$

or equivalently

$$A^{-1}_{q,r_1,r_2,n} Z_q(\widetilde{K_{n+r_2}(f)}) \subseteq Z_q(\widetilde{K_{n+r_1}(f)}) \subseteq A_{q,r_1,r_2,n} Z_q(\widetilde{K_{n+r_2}(f)}). \tag{3.21}$$

We are interested in the case where $r_2 = q$ and $r_2 = 1$ or $r_2 = 2$. We have that

$$\begin{aligned} A_{n,q,1,q} &= e^{\frac{n(q-1)}{q(n+1)}}\frac{(\Gamma(n+q))^{\frac{1}{q}}}{(\Gamma(n+1))^{\frac{n+q}{q(n+1)}}} \\ &= \left(\frac{e^{\frac{n(q-1)}{n+1}}(n+1)\dots(n+q-1)}{(n!)^{\frac{q-1}{n+1}}}\right)^{\frac{1}{q}} \\ &\leq \left(e^{2\frac{n(q-1)}{n+1}}\frac{(n+q-1)^{q-1}}{n^{q-1}}\right)^{\frac{1}{q}} \\ &\leq e^2\frac{n+q}{n}. \end{aligned}$$

A similar computation shows that $A_{n,q,2,q} \leq e^2\frac{n+q}{n}$. So, we get that that for $r = 1$ or $r = 2$,

$$\frac{n}{e^2(n+q)} Z_q(\widetilde{K_{n+q}(f)}) \subseteq Z_q(\widetilde{K_{n+r}(f)}) \subseteq e^2\frac{n+q}{n} Z_q(\widetilde{K_{n+q}(f)}). \tag{3.22}$$

Then, for $q \leq n$, using (3.8), (3.21) we get (3.17), (3.18). □

We will also use the following:

Lemma 3.6. *Let K be a convex body in $\mathbb{R}^n$ with volume one and center of mass at the origin. Then, for every $p \geq n$,*

$$Z_p(K) \supseteq c_1\,\mathrm{co}\{K, -K\} \tag{3.23}$$

and

$$c_1 \leq |Z_p(K)|^{1/n} \leq c_2 \tag{3.24}$$

where $c_1, c_2 > 0$ are universal constants.

Proof. Under our assumptions, one can prove that for every $\theta \in S^{n-1}$,

$$h_{Z_p(K)}(\theta) \geq \left(\frac{\Gamma(p+1)\Gamma(n)}{2e\Gamma(n+p+1)}\right)^{1/p} \max\{h_K(\theta), h_K(-\theta)\}.$$

For a proof of this well-known fact see [25]. It follows that if $p \geq n$ then $h_{Z_p(K)} \geq c_1 \max\{h_K(\theta), h_K(-\theta)\}$, which proves (27).

This in turn means that $|Z_p(K)|^{1/n} \geq c_1|\mathrm{co}\{K,-K\}|^{1/n} \geq c_1|K| \geq c_1$. Taking into account the fact that $Z_p(K) \subseteq \mathrm{co}\{K,-K\}$ and using an inequality due to Rogers and Shephard (see [30]) we readily see that $|\mathrm{co}\{K,-K\}| \leq 2^n|K|$. This proves (3.24). □

Recall that if f has center of mass at the origin then $K_{n+1}(f)$ has also its center of mass at the origin. So, combining the previous Lemma with (3.17) we get the following:

Proposition 3.7. *Let $f : \mathbb{R}^n \to [0,\infty)$ be a* log*-concave function with center of mass at the origin and $\int_{\mathbb{R}^n} f(x)dx = 1$. Then,*

$$\frac{c_1}{f(0)^{1/n}} \leq |Z_n(f)|^{1/n} \leq \frac{c_2}{f(0)^{1/n}}, \tag{3.25}$$

where $c_1, c_2 > 0$ are universal constants.

4. Marginals and Projections

Let $f : \mathbb{R}^n \to [0,\infty)$ be an integrable function. Let $1 \leq k < n$ be an integer and let $F \in G_{n,k}$. We define the marginal $\pi_F(f) : F \to \mathbb{R}_+$ of f with respect to F by

$$\pi_F(f)(x) := \int_{x+F^\perp} f(y)dy. \tag{4.1}$$

Note that, by Fubini's theorem,

$$\int_F \pi_F(f)(x)dx = \int_{\mathbb{R}^n} f(x)dx \tag{4.2}$$

and, for every $\theta \in S_F$,

$$\int_F \langle x,\theta\rangle \pi_F(f)(x)dx = \int_{\mathbb{R}^n} \langle x,\theta\rangle f(x)dx. \tag{4.3}$$

In particular, if f has center of mass at the origin then for every $F \in G_{n,k}$, $\pi_F(f)$ has the same property.

The same argument gives that, for every $p > 0$ and $\theta \in S_F$,

$$\int_{\mathbb{R}^n} |\langle x,\theta\rangle|^p f(x)dx = \int_F |\langle x,\theta\rangle|^p \pi_F(f)(x)dx$$

and, for every $-k < p \leq \infty$,

$$\int_{\mathbb{R}^n} \|P_F x\|_2^p f(x)dx = \int_F \|x\|_2^p \pi_F(f)(x)dx.$$

We will use the notation

$$I_p(f,F) := \left(\int_{\mathbb{R}^n} \|P_F x\|_2^p f(x)dx\right)^{1/p}.$$

So, we have the following:

Proposition 4.1. *Let $f : \mathbb{R}^n \to [0,\infty)$ be an integrable function with $\int_{\mathbb{R}^n} f(x)dx = 1$. Then, for $1 \leq k \leq n$, $F \in G_{n,k}$ and $p > 0$, one has*

$$P_F(Z_p(f)) = Z_p(\pi_F(f)). \tag{4.4}$$

Also, for any $-k < p \leq \infty$,

$$I_p(f,F) = I_p(\pi_F(f)). \tag{4.5}$$

Let f be a log-concave function with center of mass at the origin and $\int_{\mathbb{R}^n} f(x)dx = 1$. Then, for every $F \in G_{n,k}$, the same holds true for $\pi_F(f)$. So, we may apply Proposition 3.7 to get

$$\frac{c_1}{\pi_F(f)(0)^{1/k}} \leq |Z_k(\pi_F(f))|^{1/k} \leq \frac{c_2}{\pi_F(f)(0)^{1/k}}.$$

This last fact, combined with (4.4), proves the following.

Proposition 4.2. *Let f be a* log*-concave function with center of mass at the origin and $\int_{\mathbb{R}^n} f(x)dx = 1$. Then, for any $1 \leq k < n$ and $F \in G_{n,k}$, one has*

$$c_1 \leq \pi_F(f)(0)^{1/k} |P_F Z_p(f)|^{1/k} \leq c_2, \tag{4.6}$$

where $c_1, c_2 > 0$ are universal constants.

Consider the special case where K is a convex body of volume 1 and has center of mass at the origin and $f := \mathbf{1}_K$. Observe that $\pi_F(f)(0) = |K \cap F^{\perp}|$. Then, the previous proposition can be viewed as an "L_q-version" of the following inequality due to Rogers-Shephard [29] (see [32] or [22] for the lower bound).

Theorem 4.3. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. Let $1 \leq k \leq n$ and let $F \in G_{n,k}$. Then,*

$$|P_F(K)||K \cap F^{\perp}| \leq \binom{n}{k}.$$

If K has center of mass at the origin, then

$$1 \leq |P_F(K)||K \cap F^{\perp}|.$$

The term "Rogers-Shephard inequality" is usually used for the upper bound. A more general inequality can be easily obtained by the following formula for mixed volumes, which is due to Fedotov (see [3] or [31]): Let $F \in G_{n,k}$, let $K_1, \dots K_k$ be convex bodies in $\mathbb{R}^n$ and let $L_i, \dots, L_{n-k}$ be compact convex subsets of $F^{\perp}$. Then,

$$\binom{n}{k} V(K_1, \dots K_k, L_1, \dots L_{n-k}) = V(P_F K_1, \dots, P_F K_k) V(L_1, \dots, L_{n-k}). \tag{4.7}$$

In the special case where $L_1 = L_{n-k} = K \cap F^{\perp}$, (4.7) implies that

$$\binom{n}{k} V(K_1, \dots K_k, K \cap F^{\perp} \dots K \cap F^{\perp}) = V(P_F K_1, \dots, P_F K_k)|K \cap F^{\perp}|.$$

The Rogers-Shephard inequality follows if we take $K_1 = K_k = K$ and use the monotonicity property of mixed volumes.

Note that one can rewrite the inequality in the following form:

$$1 \leq \left(|P_F(K)||K \cap F^{\perp}|\right)^{1/k} \leq e\frac{n}{k}.$$

In the special case where K is an ellipsoid of volume 1 one actually has

$$c_1\sqrt{\frac{n}{k}} \leq \left(|P_F(K)||K \cap F^{\perp}|\right)^{1/k} \leq c_2\sqrt{\frac{n}{k}},$$

where $c_1, c_2 > 0$ are universal constants.

The following direct consequence of Proposition 4.2 can be viewed as an "L_q version of the Rogers-Shephard inequality":

Theorem 4.4. *Let K be a convex body in $\mathbb{R}^n$ with center of mass at the origin and volume 1. Then, for every $F \in G_{n,k}$ one has*

$$c_1 \leq |K \cap F^{\perp}|^{1/k}|P_F(Z_k(K)|^{1/k} \leq c_2, \tag{4.8}$$

where $c_1, c_2 > 0$ are universal constants.

The inequality of Theorem 4.4 is sharp up to a universal constant. A disadvantage is that the constants are not optimal (in contrast, the equality cases in the classical Rogers-Shephard inequality are known).

The L_q-version of the Rogers-Shephard inequality played an important role in [27]. In that paper, our approach was based on the bodies $B_p(K,F)$ which had appeared already in the classical paper of Milman and Pajor [21]. Our approach in the present paper is a little more general. We will recall the definition in order to provide a unified setting for our results.

Let us first recall the definition of isotropicity for convex bodies: Let K be a convex body in $\mathbb{R}^n$ with center of mass at the origin and volume 1. We define the isotropic constant of K as follows:

$$L_K := \left(\frac{|Z_2(K)|}{|D_n|}\right)^{1/n}.$$

We will say that K is isotropic if $Z_2(K) = L_K D_n$.

Next, let K be a convex body of volume 1 in $\mathbb{R}^n$, let $1 \leq k < n$, $F \in G_{n,k}$ and $p > 0$. We define a convex body $B_p(K,F)$ in F by

$$B_p(K,F) := K_{p+1}(\pi_F\left((\mathbf{1}_K)\right).$$

Then, we have the following:

Theorem 4.5. *Let K be a convex body of volume 1 in $\mathbb{R}^n$ and let $1 \leq k < n$, $F \in G_{n,k}$ and $p > 0$. Then,*

(i) *If K has center of mass at the origin, then $B_k(K,F)$ has also center of mass at the origin.*

(ii) *If K is symmetric, then $B_p(K,F)$ is also symmetric. Moreover, if K is symmetric and isotropic, then $\widetilde{B_{k+1}}(K,F)$ is also isotropic.*

(iii) *If K has center of mass at the origin then, for any $q \leq k$ we have*

$$Z_q(\widetilde{B_{k+1}}(K,F)) \simeq |K \cap F^{\perp}|^{\frac{1}{k}} P_F(Z_q(K)) \simeq Z_q(\widetilde{B_k}(K,F)).$$

(iv) *If K is isotropic, then*

$$|K \cap F^{\perp}| \simeq \frac{L_{\widetilde{B_{k+1}(K,F)}}}{L_K}.$$

Moreover, if K has center of mass at the origin, then $L_{\widetilde{B_{k+1}}(K,F)} \simeq L_{\widetilde{B_k}(K,F)}$.

Proof. (i) Recall that if $\mathbf{1}_K$ has center of mass at the origin, then $\pi_F(\mathbf{1}_K)$ has center of mass at the origin. This implies that $K_{k+1}(\pi_F(\mathbf{1}_K))$ has center of mass at the origin.

(ii) Since $\mathbf{1}_K$ is an even function, the same is true for $\pi_F(\mathbf{1}_K)$. This implies that $K_p(\pi_F(\mathbf{1}_K))$ is symmetric. We also have that if K is isotropic then, for every $F \in G_{n,k}$, $L_K B_F = P_F Z_2(\mathbf{1}_K) = Z_2(\pi_F(\mathbf{1}_K))$, where we have also used (4.4). Moreover, (3.3) implies that if $Z_2(\pi_F(\mathbf{1}_K))$ is homothetic to B_2^n then the same holds true for $Z_2(K_{k+2}(\pi_F(\mathbf{1}_K))$. So, if K is also symmetric, then $\widetilde{B_{k+1}}(K,F)$ is isotropic.

(iii) Note that $|K \cap F^{\perp}| = \pi_F(\mathbf{1}_K)(0)$. Using (3.17) and (4.4) we get

$$Z_q\left(\widetilde{(K_{k+1}}(\pi_F((\mathbf{1}_K)))\right) \simeq \pi_F(\mathbf{1}_K)(0)^{1/k} Z_q(\pi_F(\mathbf{1}_K)) \simeq |K \cap F^{\perp}|^{1/k} P_F(Z_q(\mathbf{1}_K)).$$

We work similarly for $\widetilde{K_{k+2}}(\pi_F(\mathbf{1}_K))$, this time using (3.18) instead of (3.17).

(iv) It follows immediately from (ii) and (iii). □

Statement (iv) in the previous theorem can be found explicitly in [21]. Note that the body $B_p(K,F)$ that we have defined here is homothetic to the one defined in [21] or [27]. On the other hand, the assertions of the previous theorem are independent of scaling.

The following identity will play a crucial role in the proof of the main theorem:

Proposition 4.6. *Let f be an integrable function on $\mathbb{R}^n$ and let $k < n$ be a positive integer. Then,*

$$I_{-k}(f) = c_{n,k}\left(\int_{G_{n,k}} \pi_F(f)(0)\, d\mu(F)\right)^{-1/k}, \tag{4.9}$$

where $c_{n,k} = \left(\frac{(n-k)\omega_{n-k}}{n\omega_n}\right)^{1/k} \simeq \sqrt{n}$. *Moreover, if f is also* log*-concave and has center of mass at the origin then, for every $y \in \mathbb{R}^n$,*

$$I_{-k}(f_y) \geq \frac{1}{e} I_{-k}(f). \tag{4.10}$$

Proof. Let $f : \mathbb{R}^n \to R_+$ and $y \in \mathbb{R}^n$. Then, for every $F \in G_{n,k}$,

$$\begin{aligned} \pi_F(f_y)(x) &= \int_{x+F^{\perp}} f_y(z)dz = \int_{x+F^{\perp}} f(z + P_F y + P_{F^{\perp}} y)dz \\ &= \int_{x+F^{\perp}} f(z + P_F y)dz = \int_{x+P_F y + F^{\perp}} f(z)dz = \pi_F(f)(x + P_F y). \end{aligned}$$

In particular, if $f : \mathbb{R}^n \to R_+$ is a log-concave function with center of mass at the origin and if $y \in \mathbb{R}^n$, using Theorem 3.3 we get

$$\pi_F(f_y)(0) \leq \|\pi_F(f)\|_{\infty} \leq e^k \pi_F(f)(0). \tag{4.11}$$

Also, for any integrable f and $1 \leq m < n$ we have that

$$\begin{aligned}\int_{G_{n,n-m}} \pi_F(f)(0)d\mu(F) &= \int_{G_{n,m}} \pi_{F^\perp}(f)(0)d\mu(F) = \int_{G_{n,m}} \int_F f(z)dzd\mu(F) \\ &= \int_{G_{n,m}} m\omega_m \int_{S_F} \int_0^\infty r^{m-1} f(r\theta)drd\sigma_F(\theta)d\mu(F) \\ &= \frac{m\omega_m}{n\omega_n} n\omega_n \int_{S^{n-1}} \int_0^\infty r^{m-1} f(r\theta)drd\sigma(\theta) \\ &= \frac{m\omega_m}{n\omega_n} \int_{\mathbb{R}^n} \frac{f(x)}{\|x\|_2^{n-m}} dx = \frac{m\omega_m}{n\omega_n} I_{-(n-m)}^{-(n-m)}(f).\end{aligned}$$

Equivalently, we may write that, for every integer $k < n$,

$$I_{-k}(f) = c_{n,k} \left(\int_{G_{n,k}} \pi_F(f)(0)d\mu(F) \right)^{-1/k}, \tag{4.12}$$

where $c_{n,k} = \left(\frac{(n-k)\omega_{n-k}}{n\omega_n} \right)^{1/k} \simeq \sqrt{n}$.

Let $f : \mathbb{R}^n \to R_+$ be a log-concave function with center of mass at the origin and let $y \in \mathbb{R}^n$. Using (4.11) we get that, for every integer $k < n$,

$$\begin{aligned}I_{-k}(f_y) &= c_{n,k} \left(\int_{G_{n,k}} \pi_F(f_y)(0)d\mu(F) \right)^{-1/k} \geq \frac{c_{n,k}}{e} \left(\int_{G_{n,k}} \pi_F(f)(0)d\mu(F) \right)^{-1/k} \\ &= \frac{1}{e} I_{-k}(f).\end{aligned}$$

The proof is complete. □

The following argument is a variation of an argument of Milman and Pajor (see [21]).

Proposition 4.7. *Let K be a compact set of volume 1 in $\mathbb{R}^n$. Then, if $-(n-1) \leq p \leq \infty$, $p \neq 0$,*

$$I_p(K) \geq I_p(\widetilde{B_2^n}) \simeq \sqrt{n}. \tag{4.13}$$

Proof. Let V be a star shaped body and write $\|x\|_V$ for the gauge function of V. Then, for every $-n \leq p \leq \infty$, $p \neq 0$, one has

$$\begin{aligned}\left(\int_K \|x\|_V^p dx \right)^{1/p} &= \left(\int_{K \cap \widetilde{V}} \|x\|_V^p dx + \int_{K \setminus \widetilde{V}} \|x\|_V^p dx \right)^{1/p} \\ &\geq \left(\int_{K \cap \widetilde{V}} \|x\|_V^p dx + \int_{\widetilde{V} \setminus K} \|x\|_V^p dx \right)^{1/p} \\ &= \left(\int_{\widetilde{V}} \|x\|_V^p dx \right)^{1/p} \\ &= \left(\frac{n}{n+p} \right)^{\frac{1}{p}} |V|^{-\frac{1}{n}}.\end{aligned}$$

If we choose $V = B_2^n$ we get the left hand side inequality in (4.13). To complete the proof observe that for $-(n-1) \leq p \leq \infty$, $p \neq 0$, we have $I_p(\widetilde{B_2^n}) \simeq \sqrt{n}$. □

Proposition 4.8. *Let $f : \mathbb{R}^n \to R_+$ be a* log-*concave function with center of mass at the origin and $\int_{\mathbb{R}^n} f(x)dx = 1$. Let $1 \leq k \leq \frac{n}{2}$. For every $y \in \mathbb{R}^n$,*

$$I_{-k}(f_y) \geq \frac{c\sqrt{n}}{f(0)^{1/n}}, \tag{4.14}$$

where $c > 0$ is a universal constant.

Proof. Using (3.9), (4.10) and (4.13) we see that

$$f(0)^{1/n} I_p(f_y) \geq f(0)^{1/n} I_p(f) \geq \frac{1}{e} I_{-k}(\widetilde{K_{n-k}(f)}) \geq \frac{1}{e} I_{-k}(\widetilde{B_2^n}) \geq c\sqrt{n},$$

where $c > 0$ is a universal constant. □

5. Constant behavior of moments

Let C be a symmetric convex body in $\mathbb{R}^n$ and let $-\infty \leq p \leq \infty$, $p \neq 0$. We define

$$W_p(C) := \left(\int_{S^{n-1}} h_C^p(\theta) d\sigma(\theta) \right)^{1/p}. \tag{5.1}$$

Also, we denote by $k_*(C)$ the "Dvoretzky number" of C: roughly speaking, this is the maximal dimension such that a random projection of C is 4-Euclidean, i.e.

$$\frac{1}{2} W(C) B_F \subseteq P_F C \subseteq 2W(C) B_F.$$

A remarkable formula due to V. D. Milman (see [20]) states that the Dvoretzky number of C is determined from "global" parameters of C (see also [24]):

$$k_*(C) \simeq n \left(\frac{W(C)}{R(C)} \right)^2. \tag{5.2}$$

The following theorem was proved in [17]:

Theorem 5.1. *Let C be a symmetric convex body in $\mathbb{R}^n$. Then,*
(i) *If $1 \leq q \leq k_*(C)$ then $W(C) \leq W_q(C) \leq c_1 W(C)$.*
(ii) *If $k_*(C) \leq q \leq n$ then $c_2 \sqrt{q/n}\, R(C) \leq W_q(C) \leq c_3 \sqrt{q/n}\, R(C)$.*
(iii) *If $k_*(C) \geq n$ then $c_2\, R(C) \leq W_q(C) \leq R(C)$.*
In the statements above, $c_1, c_2 > 0$ are universal constants.

In particular, we see that we have almost constant behavior of the moments $W_q(C)$ until q becomes of the order of $k_*(C)$. The same phenomenon occurs also for negative moments: we have the following theorem (see [15] and [14]):

Theorem 5.2. *Let C be a symmetric convex body. Then, for $p \leq c_1 k_*(C)$,*

$$W_{-p}(C) \geq c_2 W(C),$$

where $c_1, c_2 > 0$ are universal constants.

Combining Theorems 5.1 and 5.2, and adjusting the constants, we get:

Proposition 5.3. *Let C be a symmetric convex body in $\mathbb{R}^n$. Then, $W_p(C) \simeq W_{-p}(C)$ if and only if $p \leq k \simeq k_*(C)$.*

Remark. To be more precise, Theorem 5.1 implies that if for some $\delta \geq 1$ one has that $W_{-p}(C) \geq \frac{1}{\delta} W_p(C)$ then $p \leq c\delta^2 k_*(C)$, where $c > 0$ is an universal constant.

The Santaló inequality asserts that, for every symmetric convex body K in $\mathbb{R}^n$,

$$|K||K^\circ| \leq \omega_n^2.$$

The reverse Santaló inequality proved by Bourgain and Milman (see [2]) asserts that

$$|K||K^\circ| \geq c^n \omega_n^2,$$

where $c > 0$ is a universal constant. Combining the two results we may write

$$c \leq \left(\frac{|K||K^\circ|}{|B_2^n|^2} \right)^{1/n} \leq 1, \tag{5.3}$$

where $c > 0$ is a universal constant.

Using (5.3) we can express negative moments of the support function of a convex body as an average of volumes of projections:

Proposition 5.4. *Let $f : \mathbb{R}^n \to R_+$ be a* log-*concave function with center of mass at the origin and $\int_{\mathbb{R}^n} f(x)dx = 1$. For every integer $k < n$,*

$$W_{-k}(Z_k(f)) \simeq \sqrt{k} \left(\int_{G_{n,k}} \pi_F(f)(0) d\mu(F) \right)^{-\frac{1}{k}} \tag{5.4}$$

and

$$I_{-k}(f) \simeq \sqrt{\frac{n}{k}} W_{-k}(Z_k(f)). \tag{5.5}$$

Proof. For $1 \leq k \leq n$ and any symmetric convex body C in $\mathbb{R}^n$,

$$\begin{aligned} W_{-k}^{-1}(C) &= \left(\int_{S^{n-1}} \frac{1}{h_C^k(\theta)} d\sigma(\theta) \right)^{1/k} \\ &= \left(\frac{1}{\omega_k} \int_{G_{n,k}} \omega_k \int_{S_F} \frac{1}{\|\theta\|_{(P_F C)^\circ}^k} d\sigma(\theta) d\mu(F) \right)^{1/k} \\ &= \left(\int_{G_{n,k}} \frac{|(P_F(C))^\circ|}{|B_2^k|} d\mu(F) \right)^{1/k} \\ &\simeq \left(\int_{G_{n,k}} \frac{|B_2^k|}{|P_F(C)|} d\mu(F) \right)^{1/k}, \end{aligned}$$

and hence,

$$W_{-k}(C) \simeq \sqrt{k} \left(\int_{G_{n,k}} |P_F C|^{-1} d\mu(F) \right)^{-\frac{1}{k}}. \tag{5.6}$$

Now, let $f : \mathbb{R}^n \to R_+$ be a log-concave function with center of mass at the origin and $\int_{\mathbb{R}^n} f(x)dx = 1$. Consider an integer $k < n$ and let $F \in G_{n,k}$. Recall that (from 4.6)

$$\frac{1}{|P_F Z_k(f)|^{1/k}} \simeq \pi_F(f)(0)^{1/k}.$$

Then, by (5.6) and (4.12) we complete the proof. □

Proposition 5.5. *Let f be an integrable function on $\mathbb{R}^n$ and let $p \geq 1$. Then,*

$$W_p(Z_p(f)) \simeq \sqrt{\frac{p}{n+p}} I_p(f). \tag{5.7}$$

Proof. We will use the following simple fact (see e.g. [25]): For any $x \in \mathbb{R}^n$ and any $p \geq 1$ one has

$$\left(\int_{S^{n-1}} |\langle x, \theta\rangle|^p d\sigma(\theta)\right)^{1/p} \simeq \sqrt{\frac{p}{n+p}} \|x\|_2. \tag{5.8}$$

So, if f is an integrable function in $\mathbb{R}^n$, by Fubini's theorem we have that for every $p \geq 1$,

$$\begin{aligned} W_p(Z_p(f)) &= \left(\int_{S^{n-1}} \int_{\mathbb{R}^n} |\langle x, \theta\rangle|^p f(x) dx d\sigma(\theta)\right)^{1/p} \\ &= \left(\int_{\mathbb{R}^n} \int_{S^{n-1}} |\langle x, \theta\rangle|^p d\sigma(\theta) f(x) dx\right)^{1/p} \\ &\simeq \sqrt{\frac{p}{n+p}} \left(\int_{\mathbb{R}^n} \|x\|_2^p f(x) dx\right)^{1/p} \\ &\simeq \sqrt{\frac{p}{n+p}} I_p(f). \end{aligned}$$

This proves the proposition. □

The formulae (5.7) and (5.5) lead us to the following definition (in the case of convex bodies it first appeared in [26]): Let f be an integrable function with $\int_{\mathbb{R}^n} f(x)dx = 1$ and $\delta > 0$. We define

$$q_*(f) := \max\{k \leq n : k_*(Z_k(f)) \geq k\}, \tag{5.9}$$

and

$$q_*(f, \delta) := \max\{k \leq n : k_*(Z_k(f)) \geq \frac{k}{\delta^2}\}.$$

Combining (5.5) and (5.7) with Proposition 5.3 we get:

Theorem 5.6. *Let $f : \mathbb{R}^n \to R_+$ be a* log-*concave function with center of mass at the origin and $\int_{\mathbb{R}^n} f(x)dx = 1$. For any integer $k < n$ we have $I_{-k}(f) \simeq I_k(f)$ if and only if $k \leq cq_*(f)$, where $c > 0$ is an absolute constant.*

In particular, from the previous theorem we see that for all $k \leq q_*(f)$ one has $I_k(f) \leq CI_2(f)$, where $C > 0$ is a universal constant. This was the main result of [27]. Moreover note that Theorem 5.6 implies Theorem 1.4.

Remark. To be more precise, if for some $\delta \geq 1$ and some integer k one has that $I_{-k}(f) \geq \frac{1}{\delta} I_k(f)$, then $k \leq q_*(f, c\delta)$, where $c > 0$ is an universal constant.

The following bound for the quantity $q_*(f)$ was proved in [27]:

Proposition 5.7. *Let f be an integrable function on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} f(x)dx = 1$. Assume that f is ψ_α with constant b_α for some $\alpha \geq 1$. Then,*

$$q_*(f) \geq \frac{c}{b_\alpha^\alpha} \left(k_*(Z_2(f))\right)^{\frac{\alpha}{2}}, \tag{5.10}$$

where $c > 0$ is a universal constant.

It is well known that there exists a universal constant $C > 0$ such that every log-concave function f is ψ_1 with constant C. Note that (4.4) implies that if f is a ψ_α function with constant b_α for some $\alpha \geq 1$, then the same is true for $\pi_F(f)$, for every $F \in G_{n,k}$.

We conclude this section with the following fact.

Proposition 5.8. *Let f be an integrable function on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} f(x)dx = 1$. Assume that f is ψ_α with constant b_α for some $\alpha \geq 1$. Then, for every $F \in G_{n,k}$,*

$$q_*(\pi_F(f)) \geq \frac{c}{b_\alpha^\alpha} \left(k_*(Z_2(\pi_F(f))\right)^{\frac{\alpha}{2}}, \tag{5.11}$$

where $c > 0$ is a universal constant.

6. Small ball probability

Proposition 5.8 suggests that one has the best bounds for the quantity q_* if the ellipsoid $Z_2(f)$ is a multiple of the Euclidean ball. We have the following:

Definition 6.1. Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be an integrable function with $\int_{\mathbb{R}^n} f(x)dx = 1$. We say that f is isotropic if f has center of mass at the origin and $Z_2(f) = B_2^n$. Equivalently if, for every $\theta \in S^{n-1}$,

$$\int_{\mathbb{R}^n} |\langle x, \theta\rangle|^2 f(x)dx = 1. \tag{6.1}$$

Note that if f is isotropic then $I_2(f) = \sqrt{n}$.

It is known that given any f one can find $T \in SL_n$ such that $f \circ T^{-1}$ is isotropic. Also, the isotropic condition (6.1) is known to be equivalent with the following:

$$\int_{\mathbb{R}^n} \langle x, Ax\rangle f(x)dx = \mathrm{tr}(A) \tag{6.2}$$

for every $n \times n$ matrix A. In particular, one has that, if f is isotropic then

$$\int_{\mathbb{R}^n} \|Ax\|_2^2 f(x)dx = \|A\|_{\mathrm{HS}}^2. \tag{6.3}$$

Let f be isotropic and let $T \in SL_n$. Then,

$$Z_2(f \circ T^{-1}) = T(Z_2(f)) = T(B_2^n).$$

Note that $W(T(B_2^n)) = \frac{\|T\|_{\mathrm{HS}}}{\sqrt{n}}$ and $R(T(B_2^n)) = \|T\|_{\mathrm{op}}$. So, using (5.2), we have that

$$k_*(Z_2(f \circ T^{-1}) \simeq \left(\frac{\|T\|_{\mathrm{HS}}}{\|T\|_{\mathrm{op}}}\right)^2. \tag{6.4}$$

Also, if $F \in G_{n,k}$ then $Z_2(\pi_F(f \circ T^{-1})) = P_F(Z_2(f \circ T^{-1})) = P_F(T(B_2^n))$. Therefore,

$$k_\star(Z_2(\pi_F(f \circ T^{-1}))) \simeq \left(\frac{\|P_F T\|_{\mathrm{HS}}}{\|P_F T\|_{\mathrm{op}}}\right)^2. \tag{6.5}$$

We are now ready to give a proof of the Theorem 1.3. Actually we will prove the following more general statement (Theorem 1.3. corresponds to the case $\alpha = 2$):

Theorem 6.2. *Let X be an isotropic* log*-concave random vector in $\mathbb{R}^n$ which is ψ_α with constant b_α for some $\alpha \geq 1$. Let A be a non-zero $n\times n$ matrix, let $y \in \mathbb{R}^n$ and $\varepsilon \in (0, c_1)$. Then, one has*

$$\mathbb{P}\left(\|AX-y\|_2 \leq \varepsilon\|A\|_{\mathrm{HS}}\right) \leq \varepsilon^{\frac{c_2}{b_\alpha^\alpha}\left(\frac{\|A\|_{\mathrm{HS}}}{\|A\|_{\mathrm{op}}}\right)^\alpha}, \tag{6.6}$$

where $c_1, c_2 > 0$ are absolute constants.

For the convenience of the reader the proof is divided in 2 parts: First, we will deal with the case where the operator is invertible.

Proposition 6.3. *Let $\alpha \geq 1$ and let f be an isotropic* log*-concave function on $\mathbb{R}^n$, which is ψ_α with constant b_α. Let $S \in GL_n$ and $y \in \mathbb{R}^n$. Then,*

$$\mathbb{P}\left(\|Sx-y\|_2 \leq \varepsilon c_1\|S\|_{\mathrm{HS}}\right) \leq \varepsilon^{\frac{c}{b_\alpha^\alpha}\left(\frac{\|S\|_{\mathrm{HS}}}{\|S\|_{\mathrm{op}}}\right)^\alpha}. \tag{6.7}$$

Proof. We set

$$m := \frac{c}{b_\alpha^\alpha}\left(\frac{\|T\|_{\mathrm{HS}}}{\|T\|_{\mathrm{op}}}\right)^\alpha.$$

We have chosen $c > 0$ such that $m \in \mathbb{N}$ and (see (6.4) and (5.11)),

$$m \leq \frac{C}{b_\alpha^\alpha}\left(\frac{\|T\|_{\mathrm{HS}}}{\|T\|_{\mathrm{op}}}\right)^\alpha \leq \frac{C_1}{b_\alpha^\alpha}\left(k_*(Z_2(f\circ T^{-1}))\right)^{\frac{\alpha}{2}} \leq C' q_*(f\circ T^{-1}).$$

Note that $m < n$. Then, Theorem 5.6 implies that

$$I_{-m}(f\circ T^{-1}) \geq c I_2(f\circ T^{-1}).$$

Using (4.10), (6.3) we get

$$\begin{aligned} \int_{\mathbb{R}^n}\|Tx-y\|_2^{-m} f(x)dx &\leq e^m\int_{\mathbb{R}^n}\|x\|_2^{-m} f(T^{-1}x)dx \\ &= I_{-m}^{-m}(f\circ T^{-1}) \\ &\leq c_1^m I_2^{-m}(f\circ T^{-1}) \\ &= c_1^m\left(\int_{\mathbb{R}^n}\|Tx\|_2^2 f(x)dx\right)^{-\frac{m}{2}} \\ &= \left(c_1\|T\|_{\mathrm{HS}}\right)^{-m}. \end{aligned}$$

Then, from Markov's inequality we get that for every $\varepsilon \in (0,1)$,

$$\mathbb{P}\left(\|Tx-y\|_2 \leq \varepsilon c_1\|T\|_{\mathrm{HS}}\right) \leq \varepsilon^m = \varepsilon^{\frac{c}{b_\alpha^\alpha}\left(\frac{\|T\|_{\mathrm{HS}}}{\|T\|_{\mathrm{op}}}\right)^\alpha}. \tag{6.8}$$

Given $S \in GL_n$, let $T := |\det S|^{-1/n}S$; then, $T \in SL_n$. Observe that (6.8) holds for every $y \in \mathbb{R}^n$ and is homogeneous in T. The proof is complete. □

Proof of Theorem 6.2: Let $1 \leq k := \mathrm{rank}(A) < n$. There exist $F \in G_{n,k}$ ($F := \mathrm{Im}(A)$) and $B_1 \in GL_n$ such that $A = (\det B_1)P_F(B)$, where $B = (\det B_1)^{-1}B_1 \in SL_n$.

Let $m := \frac{c}{b_\alpha^\alpha}\left(\frac{\|A\|_{\mathrm{HS}}}{\|A\|_{\mathrm{op}}}\right)^\alpha$. We have chosen $c > 0$ so that $m \in \mathbb{N}$ and (see (6.5) and (5.11)),

$$m \leq \frac{C}{b_\alpha^\alpha}\left(\frac{\|P_F B\|_{\mathrm{HS}}}{\|P_F B\|_{\mathrm{op}}}\right)^\alpha \leq \frac{C_1}{b_\alpha^\alpha}\left(k_*(Z_2(\pi_F(f\circ B^{-1})))\right)^{\frac{\alpha}{2}} \leq q_*(\pi_F(f\circ B^{-1})).$$

Note that $m < k$. Then, Proposition 4.6 implies that

$$I_{-m}(\pi_F(f\circ B^{-1})) \geq c I_2(\pi_F(f\circ B^{-1})).$$

Then, for every $y \in \mathbb{R}^n$ we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \|Ax-y\|_2^{-m} f(x)dx &\leq e^m \int_{\mathbb{R}^n} \|Ax\|_2^{-m} f(x)dx \\
&= \left(\frac{e}{\det B_1}\right)^m \int_{\mathbb{R}^n} \|P_F x\|_2^{-m} f(B^{-1}x)dx \\
&= \left(\frac{e}{\det B_1}\right)^m I_{-m}^{-m}(f\circ B^{-1}, F) \\
&= \left(\frac{e}{\det B_1}\right)^m I_{-m}^{-m}(\pi_F(f\circ B^{-1})) \\
&\leq \left(\frac{c_1}{\det B_1}\right)^m \big(I_2((\pi_F(f\circ B^{-1}))\big)^{-m} \\
&= \left(\frac{c_1}{\det B_1}\right)^m \big(I_2(F\circ B^{-1}, F)\big)^{-m} \\
&= \left(\frac{c_1}{\det B_1}\right)^m \left(\int_{\mathbb{R}^n} \|P_F Bx\|_2^2 f(x)dx\right)^{-\frac{m}{2}} \\
&= c_1^m \left(\int_{\mathbb{R}^n} \|Ax\|_2^2 f(x)dx\right)^{-\frac{m}{2}} \\
&= c_1^m \|A\|_{\mathrm{HS}}^{-m},
\end{aligned}$$

where we have also used (4.5) and (6.3).

So, from Markov's inequality again, we get that for every $\varepsilon \in (0,1)$,

$$\mathbb{P}\left(\|Ax-y\|_2 \leq \varepsilon c_1 \|A\|_{\mathrm{HS}}\right) \leq \varepsilon^m = \varepsilon^{\frac{c}{b_\alpha^\alpha}\left(\frac{\|A\|_{\mathrm{HS}}}{\|A\|_{\mathrm{op}}}\right)^\alpha}. \tag{6.9}$$

This finishes the proof of Theorem 6.2 and Theorem 1.3. □

Remark. Note that the dependence in Theorem 1.3 is better than the one in Theorem 1.1, although it is not clear if it is the right one. The best dependence is related to a major open question in Convex Geometry known as the Hyperplane Conjecture: Let K be a convex body of volume 1, with center of mass at the origin. Then, there exists $\theta \in S^{n-1}$ such that

$$|K \cap \theta^\perp| \geq c,$$

where $c > 0$ is an universal constant.

An equivalent formulation of the problem is the following: There exists a universal constant $C > 0$ such that $L_K \leq C$ for every convex body K with center of mass at the origin.

It is well known (it also follows from Proposition 3.5) that the previous statement is equivalent to the following:

Hyperplane Conjecture: *There exists a universal constant $C > 0$ such that, for every isotropic* log-*concave function f on $\mathbb{R}^n$,*

$$f(0)^{1/n} \leq C. \tag{6.10}$$

The best known bound is due to B. Klartag: $f(0)^{1/n} \leq Cn^{1/4}$ (see [11]). For more informations on isotropicity and the Hyperplane Conjecture we refer to [21] or [9].

Let A be a projection matrix and let $F := \mathrm{Im}(A)$ and $k = \mathrm{rank}(A) = \dim(F)$. Note that $\|A\|_{\mathrm{HS}} = \sqrt{k}$ and $\|A\|_{\mathrm{op}} = 1$. Assume that the Hyperplane Conjecture is true. Then, by Proposition 4.8 we have that, for every $y \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \|Ax-y\|_2^{-(k-1)} f(x)dx \le \left(c\frac{\pi_F(f)(0)^{1/k}}{\sqrt{k}} \right)^{k-1} \le \left(\frac{c_1}{\sqrt{k}} \right)^{k-1}.$$

So, from Markov's inequality, we get that for every $\varepsilon \in (0,1)$,

$$\mathbb{P}\left(\|Ax-y\|_2 \le \varepsilon C \|A\|_{\mathrm{HS}} \right) \le \varepsilon^{k-1}.$$

This means that in this case we have no dependence on the ψ_α constant! In fact, the Hyperplane Conjecture is closely related to the question of the dependence in the ψ_α constant in Theorem 6.3. To fully reveal this connection we need different tools; we will present this connection elsewhere.

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