Small distortion and volume preserving embedding for Planar and Euclidian metrics

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CSE 254 - Metric Embeddings

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Overview

- Definitions
- Main Results
- Proof
- Further Results
- Open Problems

Main Result

Definition. *Let $\mathcal{G}$ be a class of graphs and let $G \in \mathcal{G}$.*

A graph metric *is the shortest distance metric d on the vertices $V(G)$ of G.*

Definition. *A* planar metric *is a graph metric on the class of all* planar *graphs.*

Theorem. *(Rao's Theorem) Any finite planar metric of cardinality n can be embedded into ℓ_2 with distortion $\mathcal{O}(\sqrt{\log n})$.*

This improves on the general $\mathcal{O}(\log n)$ distortion bound obtained by *Bourgain* for all metrics.

Proof - Outline

We will outline a decomposition method

- which has some nice properties
  (for planar graphs in particular)
- a repeated number of decompositions provide coordinates for embedding
- distant vertices will have independent coordinates

Each decomposition satisfies our purpose with a constant probability

We then estimate the distortion of the composed embedding

Decomposition

Pick $\Delta \in \{1, 2, 4, \dots n\}$
Pick $v_0 \in V(G)$ arbitrarily
Pick $r \in \{0, 1, 2, \dots \Delta - 1\}$ uniformly at random

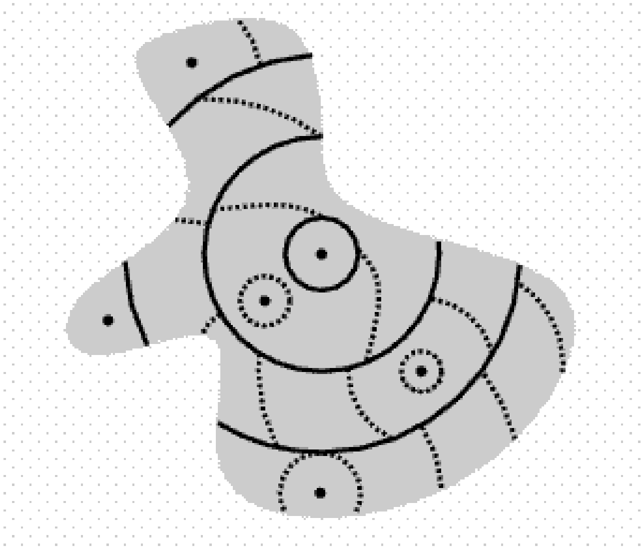
Let

$$S_1 = \{v \in V(G) : d(v_0, v) \equiv r \pmod{\Delta}\}$$

Partition $G \setminus S_1$ into connected components

For each component, repeat this procedure twice (with the same Δ)

Let finally $S = S_1 \cup S_2 \cup S_3$
Decomposition is connected components of $G \setminus S$



Properties

- Each connected component in the decomposition has diameter at most $\mathcal{O}(\Delta)$

This results from a theorem by Klein, Plotkin and Rao
We will sketch the proof shortly

- For each $x \in V(G)$ we have $\mathbb{P}[d(x,S) \geq c_1\Delta] \geq c_2$

Given a Δ,

$$d(v_1, x) \pmod{\Delta}$$

will depend upon the choice of BFS-tree root v_1

Property 1 - Outline of proof

- We do a proof by contradiction
- We assume the existence of a component of diameter greater than $k\Delta$

- We will use the BFS-trees on which we constructed the decomposition to expose a $K_{3,3}$ minor in G

- This implies that the graph can not be planar
- The diameter of each component therefore has to be bounded

Property 1 - Proofsketch

Suppose there is a component C containing u, v such that

$$d(u, v) \geq 34\Delta$$

Let w be the midpoint of the path between them (within C)

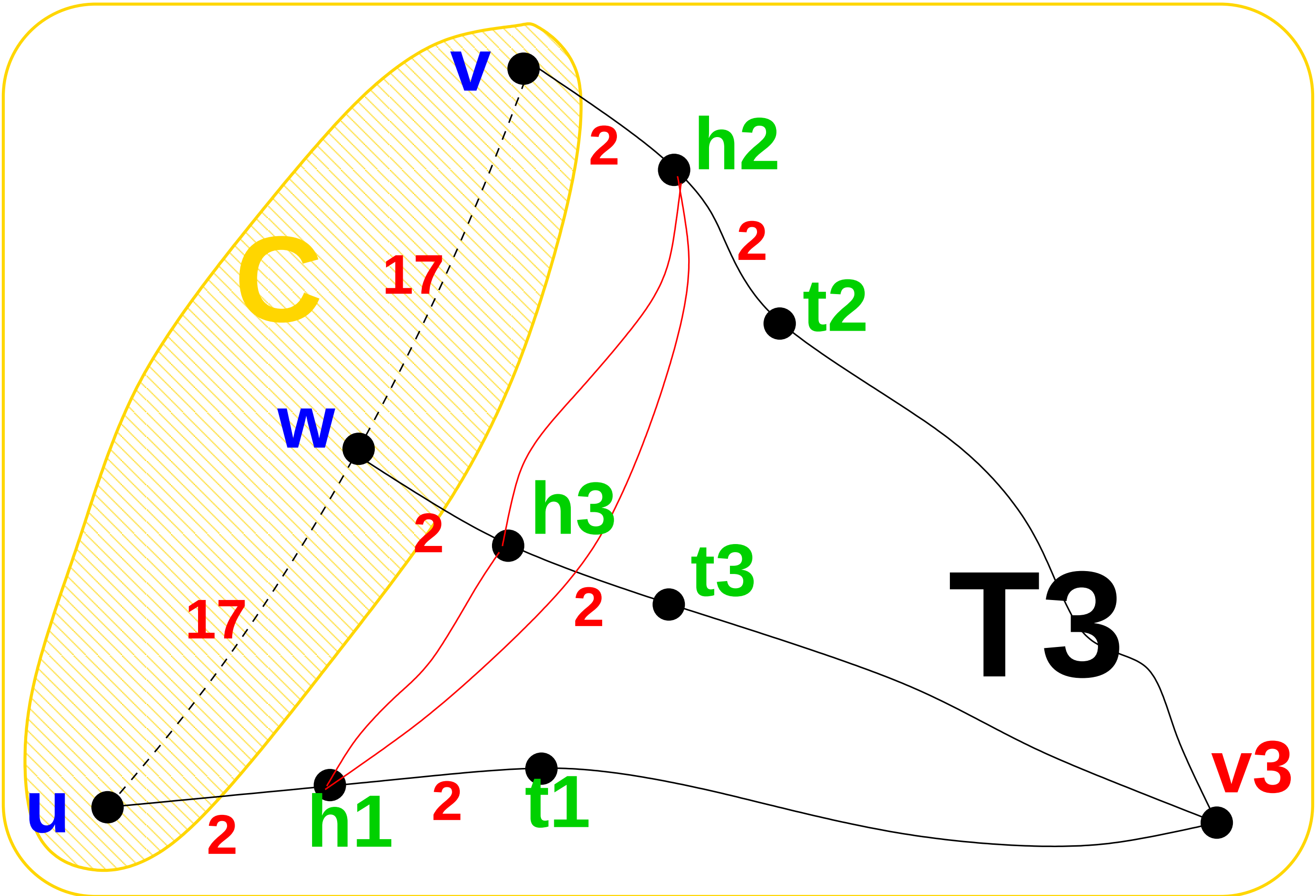
$$d(u, w), d(w, v) \geq 17\Delta$$

Let v_3 be the root of the last BFS-tree used to obtain the component

$\exists$ disjoint paths ut_1, wt_2, vt_3 of length 4Δ in the tree
Let h_1,h_2,h_3 be their midpoints, i.e. $d(u, h_1) = 2\Delta$, etc
We then have that

$$d(h_i, h_j) > 12\Delta \quad \text{for all} \quad i \neq j$$

Property 1 - Diagram 1



Property 1 - Proofsketch

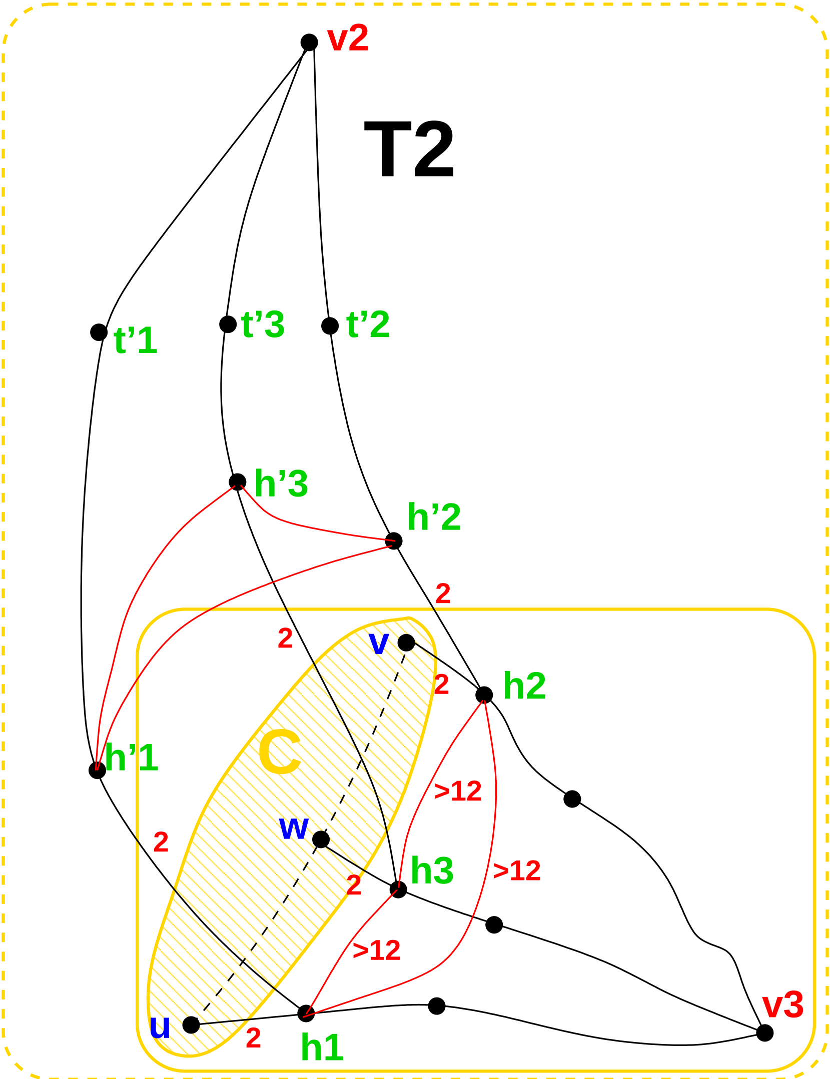
Now, let v_2 be the root of the BFS-tree of the previous level
It has disjoint paths $h_1 t_1^{'}$, $h_2 t_2^{'}$, $h_3 t_3^{'}$ of length 4Δ and if we let $h_1^{'}$,$h_2^{'}$,$h_3^{'}$ be their midpoints

$$d(h_i^{'}, h_j^{'}) > 8\Delta \quad \text{for all} \quad i \neq j$$

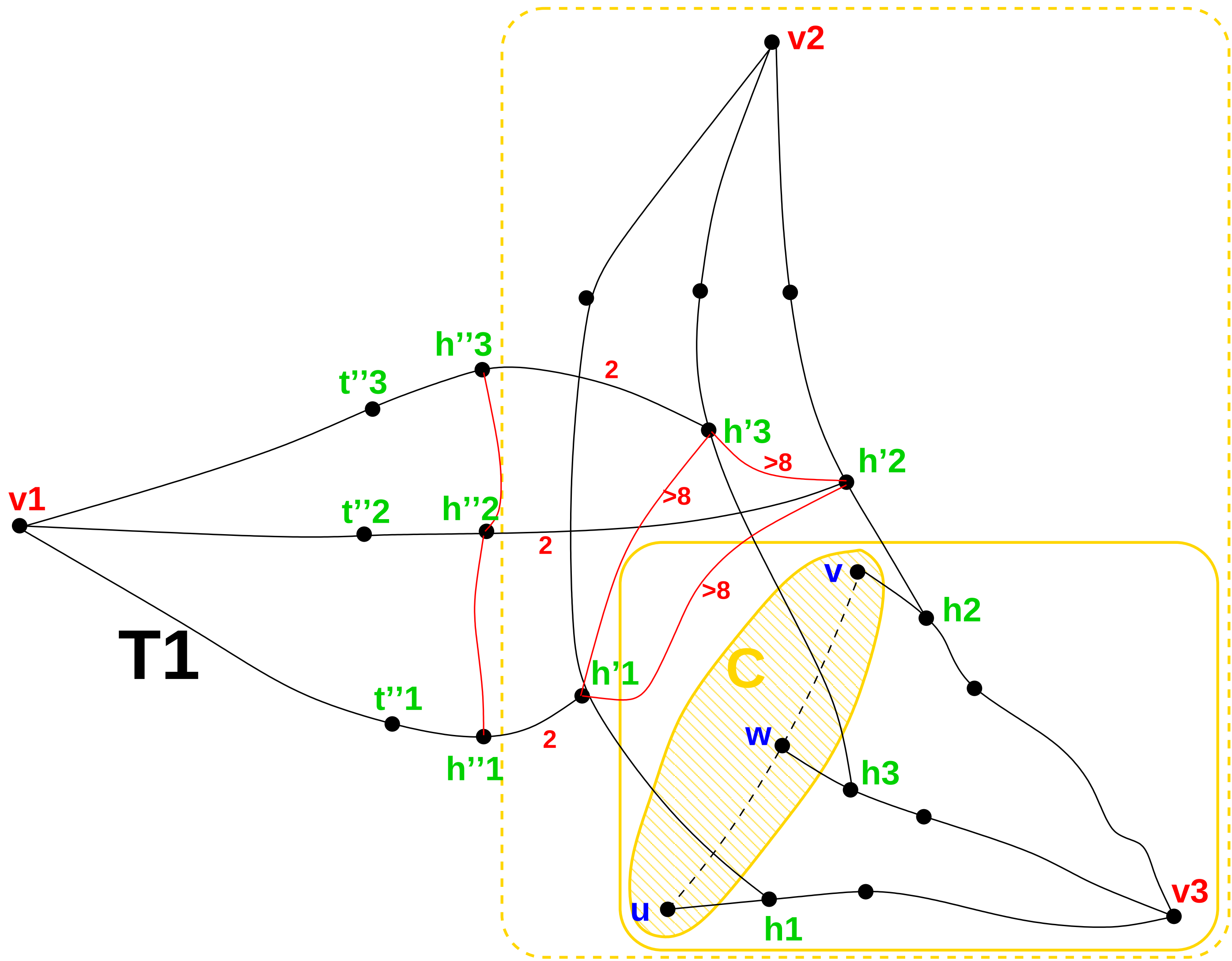
Similarly, for the first BFS-tree of the decomposition

Look at disjoint paths in the tree
Define $h_1^{''}$,$h_2^{''}$,$h_3^{''}$ as before
$h_1^{''}$,$h_2^{''}$,$h_3^{''}$ are pairwise more than 4Δ apart

Property 1 - Diagram 2



Property 1 - Diagram 3



Red super nodes

Definition. *A* super node *of a graph G is a connected subgraph*

Let us define the following *red* super nodes:

$$A(v_1), A(v_2), A(v_3)$$

$A(v_3)$ is the union of three paths of the tree T_3

- from the root v_3 to vertices t_1, t_2, t_3
- (but not including t_1, t_2, t_3)
- each of t_1, t_2, t_3 is at distance 4Δ from one of u, v, w

Similarly, define $A(v_2), A(v_1)$ with respect to the t_i''s and t_i'''s

Blue super nodes

Let us define the following *blue* super nodes:

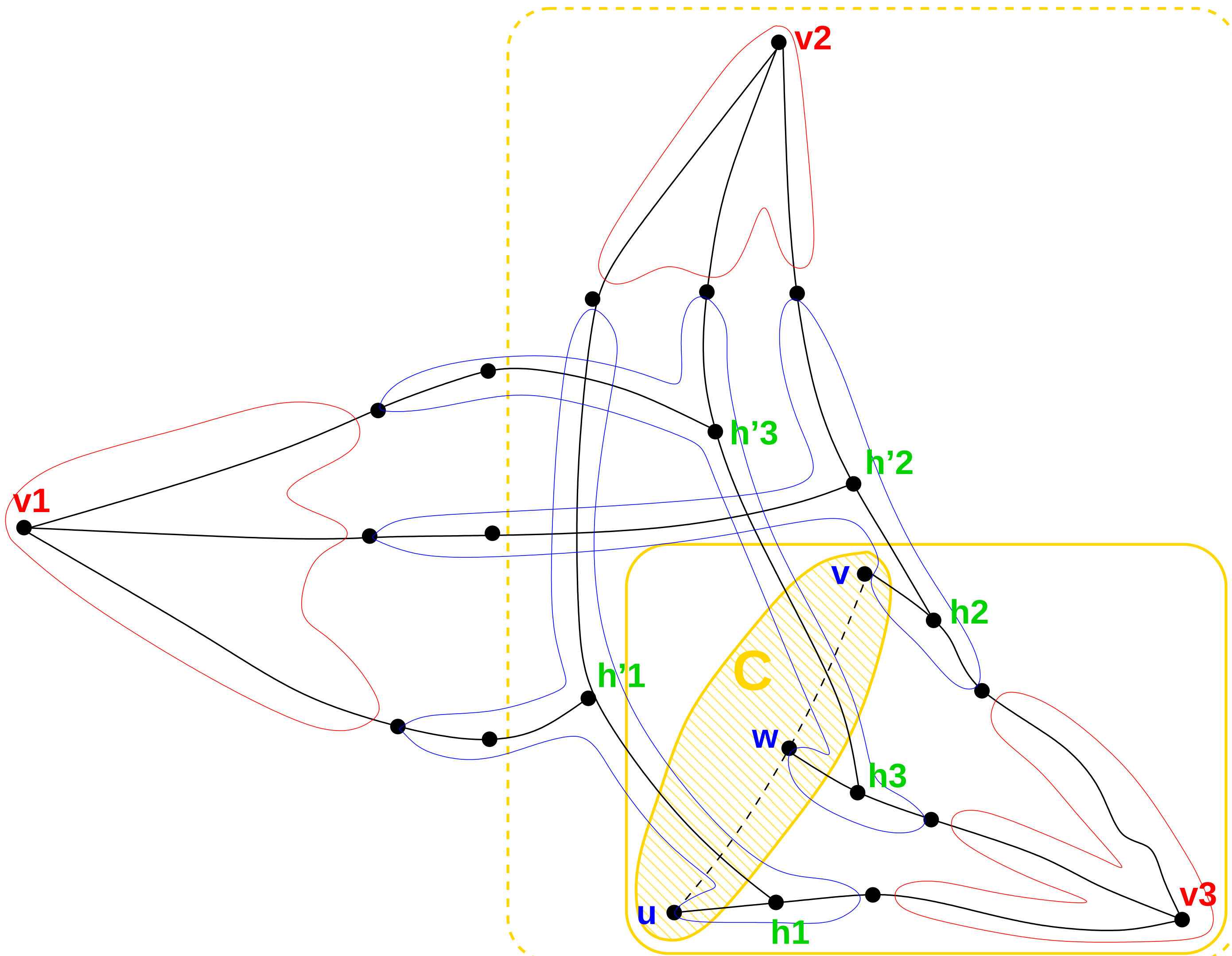
$$A(u), A(v), A(w)$$

$A(u)$ is the union of

- the path on tree T_3 joining u and t_1
- the path on tree T_2 joining h_1 and t_1'
- the path on tree T_1 joining h_1' and t_1''

$A(v)$ and $A(w)$ are defined similarly

Super nodes



Super nodes are disjoint

Claim. $A(u), A(v), A(w)$ *(Blue super nodes) are pairwise disjoint*

Proof. Each blue node is only 8Δ in diameter (see diagram)

Yet, they each contain one of u, v, w, any two of which are $> 16\Delta$ apart □

Claim. $A(v_1), A(v_2), A(v_3)$ *(Red super nodes) are pairwise disjoint*

Proof. $A(v_1), A(v_2), A(v_3)$ are separated by the decomposition

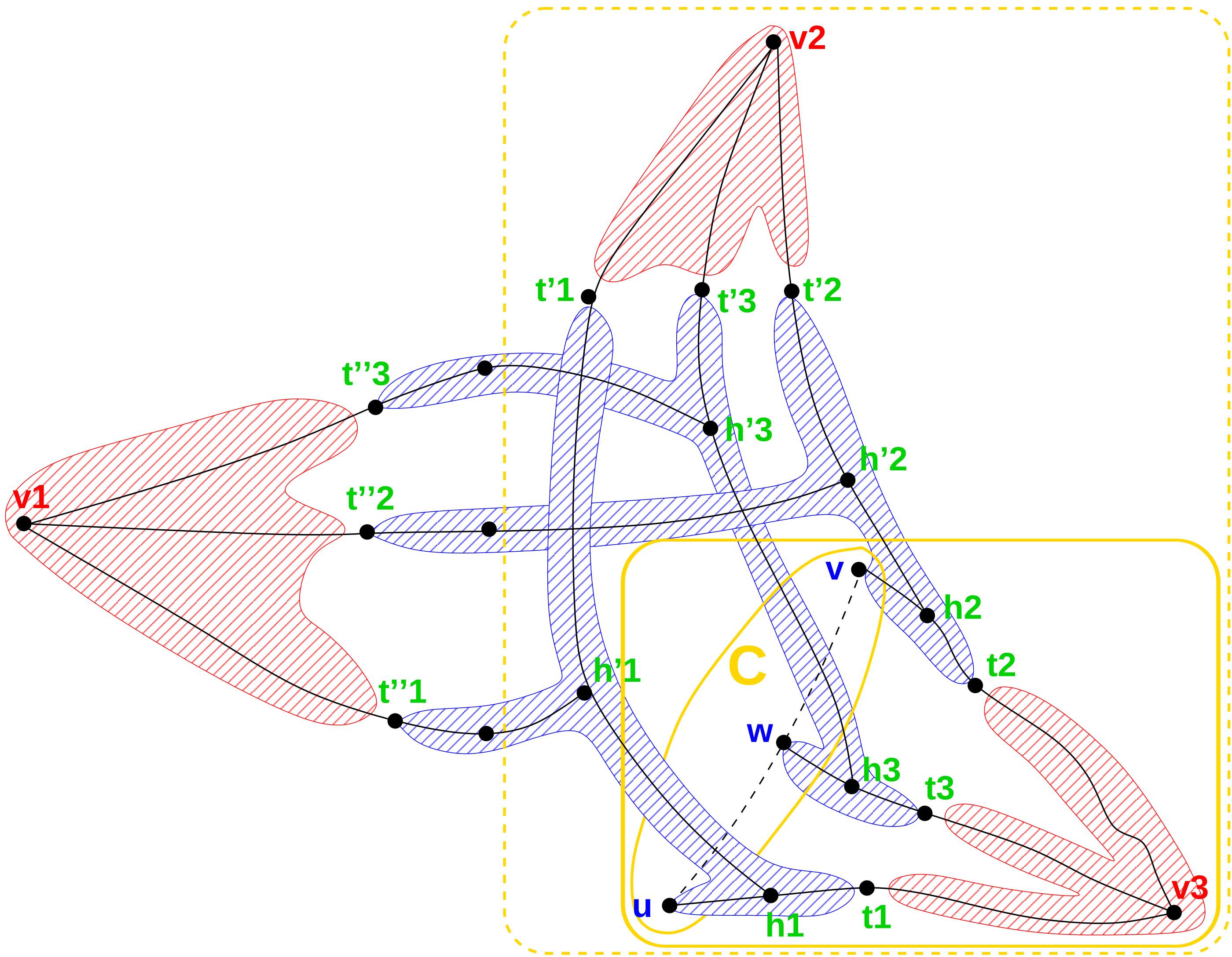
Each of h_1, h_2, h_3 is $> 4\Delta$ away from $A(v_2)$

Thus $A(v_2) \cap T_3 = \emptyset$ and a fortiori $A(v_2) \cap A(v_3) = \emptyset$

Same argument applies to $A(v_1)$ with respect to either of $A(v_2)$ and $A(v_3)$ □

Finally, similar arguments will show that any red super node is disjoint from any blue super node

Super nodes



Red nodes, blue nodes

Claim. $A(v_1)$ *is disjoint from blue nodes* $A(u), A(v), A(w)$

Proof. Visibly, the parts that belong to trees T_2 and T_3 can not intersect with $A(v_1)$ (because they are all within the same $\Delta - 1$ levels of vertices 4Δ apart from $A(v_1)$)

Question is:
could a vertex x of $A(v_1)$ belong to one of $t_1''h_1'$, $t_2''h_2'$, $t_2''h_2'$, say $t_1''h_1'$?
Then, $d(x, h_1') \leq 4\Delta$, thus $d(x, u) \leq 8\Delta$

Without loss of generality, let's say x is on path $v_1 h_2'$

Now $d(x, h_2') \leq 5\Delta - 1$
(because h_1' and h_2' are within consecutive $\Delta - 1$ levels)
So, $d(x, v) \leq 9\Delta - 1$ and thus $d(u, v) \leq 17\Delta - 1$, a contradiction $\square$

Red nodes, blue nodes

Claim. $A(v_2)$ *is disjoint from blue nodes* $A(u), A(v), A(w)$

Proof. Repeating the arguments for $A(v_1)$,

we only need to worry about paths $t_1''h_1'$, $t_2''h_2'$, $t_2''h_2'$ intersecting $A(v_2)$

Pick an x in T_2 on $t_1''h_1'$ and y in $A(v_2)$

Without loss of generality, let's say y is on path v_2h_2

Then, restricting our distance metric to T_2 we get:

$d(v_2, x) \geq d(v_2, h_1') - (\Delta - 1)$ (consecutive $\Delta - 1$ levels)

So, $d(v_2, x) \geq d(v_2, h_1) - 2\Delta - (\Delta - 1) = d(v_2, h_1) - 3\Delta + 1$

But then $d(v_2, x) \geq d(v_2, h_2) - (\Delta - 1) - 3\Delta + 1 = d(v_2, h_2) - 4\Delta + 2$

But, y being in $A(v_2)$, $d(y, h_2) \geq 4\Delta$, so $d(v_2, h_2) - 4\Delta + 2 \geq d(v_2, y) + 2$

Thus, $d(v_2, x) \geq d(v_2, y) + 2$, x and y are therefore distinct $\square$

Red nodes, blue nodes

Claim. $A(v_3)$ *is disjoint from blue nodes* $A(u), A(v), A(w)$

Proof. The exact same technique as in the previous proof covers all the cases we need to consider ... □

Property 1 - End of Proofsketch

By contracting the super nodes, we observe a $K_{3,3}$
This violates the assumption of the graph being planar

(Kuratowski)

$\therefore$ For each component C, we have $Diam(C) < 34\Delta$

By induction, Klein, Plotkin and Rao in their paper actually proof the following stronger statement:

Theorem. *If G excludes $K_{r,r}$ as a minor, any connected component obtained through r iterations of the described decomposition method has diameter* $\mathcal{O}(r^3\Delta)$

Properties

Given a random decomposition, with parameter Δ

- Each component in the decomposition has diameter at most $\mathcal{O}(\Delta)$
- For each $x \in V(G)$ we have $\mathbb{P}[d(x,S) \geq c_1\Delta] \geq c_2$

We now furthermore have: For any $x, y \in V(G)$, with $d(x,y) \geq 34\Delta$

- $x, y \notin S$ with constant probability
- x, y are in different connected components C_i, C_j
- $d(x,S), d(y,S) \geq c_1\Delta$ with constant probability

Now, for r_i, r_j random numbers chosen uniformly from $[1,2]$
$|r_i d(x,S) - r_j d(y,S)| \geq c_1\Delta$ with constant probability

Embedding

We will now define the embedding:

For each $\Delta \in \{2^j | 1 \le 2^j \le Diam(G)\}$

- perform $4 \log n$ random decompositions

For each component C_k in a decomposition

- uniformly pick a random r_k from $[1, 2]$

For $x \in C_k$ define its coordinate as $r_k \cdot d(x, S)$

This defines a mapping

$$f_{\Delta,i} : x \mapsto r_k \cdot d(x, S)$$

for all Δ and all $i \in \{2^j | 1 \le 2^j \le \log n\}$

Finally, let

$$f : x \mapsto \left(\frac{1}{2 \log n} f_{\Delta,i}(x) : \Delta, i \right)$$

Embedding

The embedding is a contraction

Let x, y in $V(G)$, then

$$\|f(x) - f(y)\|^2 = \sum_{\Delta,i} \frac{1}{(2\log n)^2}(f_{\Delta,i}(x) - f_{\Delta,i}(y))^2$$

$$\leq \frac{1}{4\log^2 n} \sum_{\Delta,i} (2d(x,y))^2$$

$$\leq \frac{1}{4\log^2 n} 4\log^2 n(4d(x,y)^2) = d(x,y)^2$$

Embedding

The embedding has distortion $\mathcal{O}(\sqrt{\log n})$

Let x, y in $V(G)$, and pick a Δ such that

$$34\Delta < d(x,y) < 68\Delta$$

then

$$\|f(x) - f(y)\|^2 \geq \sum_i \frac{1}{(2\log n)^2} (f_{\Delta,i}(x) - f_{\Delta,i}(y))^2$$

$$\geq \sum_i \frac{1}{(2\log n)^2} (\Omega(1) d(x,y))^2$$

$$\geq \frac{1}{\Omega(1)\log n} (d(x,y))^2$$

Applications

Using this result, we can obtain
a $\mathcal{O}(\sqrt{\log n})$-approximative max flow min cut theorem
for multicommodity flow problems in planar graphs

Further results

Definition. *For a set of k points S in $\mathbb{R}^L$* the volume $Evol(S)$ *is the $k-1$ dimensional ℓ_2 - volume of the convex hull of S*

Definition. *The* volume *of a k-point metric space (S,d) is*

$$Vol(S) = \sup_{f:S\to\ell_2} Evol(f(S))$$

(the maximum being taken over all contractions f)

Further results

Definition. *A* (k,c)-volume preserving embedding *of a metric space (S,d) is a contraction* $f : X \mapsto \ell_2$ *where for all* $P \subset S$ *with* $|P| = k$*,*

(1) $$\left(\frac{Vol(S)}{Evol(f(S))}\right)^{1/(k-1)} \leq c$$

The k-distortion *of* f *is*

(2) $$\sup_{P \subseteq S, |P|=k} \left(\frac{Vol(S)}{Evol(f(S))}\right)^{1/(k-1)}$$

With the help of some results from Feige, we can prove the following:

Theorem. *Rao's Theorem For every finite planar metric of cardinality* n *there exists a (k,c)-volume preserving embedding of k-distortion* $\mathcal{O}(\sqrt{\log n})$

References

S. Rao. Small distortion and volume preserving embeddings for planar and euclidean metrics. In *Proceedings of the 15th Annual Symposium on Computational Geometry*, ACM Press, 1999

P. Klein, S. Rao and S. Plotkin. Excluded minors, network decompositions, and multicommodity flow. In *Proceedings of the 25th Annual ACM Symposium on Theory of Computing*, 1993

U. Feige. Approximating the bandwidth via volume respecting embeddings. In *Proceedings of the 30th Annual ACM Symposium on Theory of Computing*, 1998

J. Matousek. *Lectures on Discrete Geometry*. Springer-Verlag, New York, 2002