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SMALL EIGENVALUES OF CLOSED SURFACES

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Abstract

Generalizing recent work of Otal and Rosas, we show that the Laplacian of a Riemannian metric on a closed surface S with Euler characteristic $\chi(S) < 0$ has at most $-\chi(S)$ small eigenvalues.

1. Introduction

Relations between the spectrum of the Laplacian and the geometry and topology of the underlying Riemannian manifold are a fascinating topic at the crossroads of a number of mathematical fields. We are concerned with the case of closed Riemannian surfaces S. Then the spectrum (of the Laplacian Δ) of S is discrete and consists of eigenvalues with finite multiplicity. We enumerate these in increasing order,

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$$

where an eigenvalue is counted as often as its multiplicity requires.

The most intriguing Riemannian metrics on the closed surface $S = S_g$ of genus $g \geq 2$ are the hyperbolic ones, that is, Riemannian metrics of constant curvature -1. For this reason, the spectrum of hyperbolic metrics on S_g attracted special attention. In [**5**], Buser showed that, for any $n \in \mathbb{N}$ and $\varepsilon > 0$, S_g admits hyperbolic metrics with $\lambda_{2g-3} < \varepsilon$ and $\lambda_n < 1/4 + \varepsilon$. He also showed that $\lambda_{4g-2} > 1/4$ for any hyperbolic metric on S_g. Refining the arguments of [**5**], Schmutz improved the latter result to $\lambda_{4g-4} > 1/4$ and showed that any hyperbolic metric on S_2 satisfies $\lambda_2 \geq 1/4$ [**13**, **14**]. These results show that the optimal result in this direction would be that $\lambda_{2g-2} > 1/4$ for any hyperbolic metric on S_g, and this was in fact conjectured by Buser and Schmutz.

The distinguished role of the eigenvalue λ_{2g-2} is also emphasized by an estimate of Schoen-Wolpert-Yau, who showed that there is a constant $c > 0$, which depends only on g, such that

$$\lambda_{2g-2} > c \tag{1.1}$$

for any Riemannian metric on S_g, whose curvature satisfies $K \leq -1$ [**15**]. In [**6**, Theorem 8.1.4], Buser proved that, for hyperbolic metrics, the constant c in (1.1) can in fact be chosen to be absolute, that is,

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independent of g. Finally, the more recent work [**12**] of Otal and Rosas implies an extended version of the conjecture of Buser and Schmutz, namely that

$$\lambda_{2g-2} > 1/4 \tag{1.2}$$

for any analytic Riemannian metric on S_g with curvature $K \le -1$. In (1.4) below, we state their corresponding result in full generality.

In their work, Buser, Schmutz and Schoen-Wolpert-Yau rely on decompositions of the surface into appropriate pieces and monotonicity properties of eigenvalues. In contrast, the method of Otal and Rosas involves a careful examination of topological properties of the nodal lines and domains of finite linear combinations of eigenfunctions.

For a domain Ω in a Riemannian surface, the *bottom of the L^2-spectrum* of the Laplacian on Ω is given by

$$\lambda_0(\Omega) = \inf R(\phi), \tag{1.3}$$

where ϕ runs over all non-zero smooth functions on Ω with compact support in the interior of Ω and $R(\phi) = \int \|\nabla\phi\|^2 / \int \phi^2$ denotes the *Rayleigh quotient* of ϕ. For the hyperbolic plane H, we have $\lambda_0(H) = 1/4$, and this is intimately related to the $1/4$ in the above results. In fact, Theorem 1 of [**12**] asserts that

$$\lambda_{2g-2} > \lambda_0(\tilde{S}) \tag{1.4}$$

for any analytic Riemannian metric on $S = S_g$ with curvature $K \le -1$, where $\tilde{S}$ denotes the universal covering surface of S with the lifted Riemannian metric. This is sharper than (1.2) since the isoperimetric inequality in dimension two gives $\lambda_0(\tilde{S}) \ge 1/4$.

In his thesis [**10**] (see also [**11**]), the third named author showed that, for any hyperbolic metric on S_g, there is a constant $\delta > 0$, which only depends on g and the systole of the metric, such that $\lambda_{2g-2} > 1/4 + \delta$. In his Bachelor thesis [**9**], the second named author showed that the assumption of negative curvature in the estimate (1.4) of Otal and Rosas can be omitted. Since smooth Riemannian metrics can be approximated by real analytic metrics, the latter result implies that $\lambda_{2g-2} \ge \lambda_0(\tilde{S})$ for any smooth Riemannian metric on S_g.

Suppose now that S is a closed surface with a Riemannian metric and let

$$\Lambda_D(S) = \inf \lambda_0(\Omega),\ \Lambda_A(S) = \inf \lambda_0(\Omega),\ \Lambda_C(S) = \inf \lambda_0(\Omega), \tag{1.5}$$

where Ω runs over all embedded discs respectively annuli respectively cross caps in S with piecewise smooth boundary, and set

$$\Lambda(S) = \min\{\Lambda_D(S), \Lambda_A(S), \Lambda_C(S)\}. \tag{1.6}$$

Observe that any embedded disc, annulus or cross cap with piecewise smooth boundary in S can be lifted isometrically to $\tilde{S}$ or a cyclic quotient of $\tilde{S}$. Hence, by Theorem 1 in [**4**] and (1.3), we have $\Lambda(S) \geq \lambda_0(\tilde{S})$. Our main result is the following

Theorem 1.7. *A closed Riemannian surface S with Euler characteristic $\chi(S) < 0$ has at most $-\chi(S)$ eigenvalues which are $\leq \Lambda(S)$; in other words,*

$$\lambda_{-\chi(S)} > \Lambda(S).$$

Since $\Lambda(S) \geq \lambda_0(\tilde{S})$, Theorem 1.7 implies the strict inequality $\lambda_{-\chi(S)} > \lambda_0(\tilde{S})$ as opposed to the weak inequality, which would follow from [**9**] (at least in the orientable case, as explained further up). We also emphasize that the statement of Theorem 1.7 is curvature free.

The main lines of the proof of Theorem 1.7 stem from [**12**] and [**16**]. We show that the space spanned by eigenfunctions with eigenvalue $\leq \Lambda(S)$ has dimension $\leq -\chi(S)$. However, in the general situation we consider, nodal lines and domains of finite linear combinations of eigenfunctions might be quite complicated. In particular, we cannot rely on the regularity theory of analytic functions as in [**12**] and need to pass to approximate nodal lines and domains. As a result, the analytical and topological arguments become more involved.

If Ω is a compact domain with piecewise smooth boundary, then $\lambda_0(\Omega)$ is the first Dirichlet eigenvalue of Ω, that is, $\lambda_0(\Omega)$ is the smallest $\lambda \in \mathbb{R}$ such that the problem

$$\Delta\phi = \lambda\phi \quad \text{on } \mathring{\Omega}, \quad \phi = 0 \quad \text{on } \partial\Omega, \tag{1.8}$$

admits a non-zero solution ϕ which is smooth on $\mathring{\Omega}$ and continuous on $\bar{\Omega}$. From this characterization and (1.3) it follows easily that, for any two compact domains Ω_1 and Ω_2 with piecewise smooth boundary,

$$\lambda_0(\Omega_1) > \lambda_0(\Omega_2) \;\text{ whenever }\; \Omega_1 \subsetneq \Omega_2. \tag{1.9}$$

In view of this, we suspect that always $\Lambda(S) > \lambda_0(\tilde{S})$. Indeed, for orientable closed surfaces with hyperbolic metrics, we have

$$\Lambda(S) \geq 1/4 + \delta(S) \tag{1.10}$$

by Theorem 1.1 in [**10**] (or, respectively, Theorem 2.1.4 in [**11**]), where

$$\delta(S) = \min\{\pi, s(S)^2/|S|\} \cdot \frac{1}{|S|} > 0. \tag{1.11}$$

Here $s(S)$ and $|S|$ denote the systole and the area of S, respectively.

With respect to estimating $\Lambda(S) - \lambda_0(\tilde{S})$ in general, it would be desirable to get the existence of minimizers, of one kind or another, of $\lambda_0(\Omega)$ in a given isotopy class of domains $\Omega \subseteq S$. In our case, this may be viewed as an optimal design problem for graphs in S. For example,

$\Lambda_D(S)$ is the infimum over all $\lambda_0(S \setminus G)$, where G runs over all finite graphs in S such that $\Omega = S \setminus G$ is diffeomorphic to an open disc.

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2. Approximate nodal sets and domains

In what follows, S is a closed Riemannian surface with negative Euler characteristic. We denote by Δ the positive Laplacian of S. For any $\lambda \geq 0$, we let $\mathbb{E}_\lambda = \{\phi \in L^2(M) \mid \Delta\phi = \lambda\phi\}$ be the λ-eigenspace of Δ in $L^2(M)$ (where we allow for $\mathbb{E}_\lambda = \{0\}$). We let

$$\mathbb{E} = \oplus_{\lambda \leq \Lambda(S)} \mathbb{E}_\lambda \tag{2.1}$$

and $\mathbb{S}$ be the unit sphere in $\mathbb{E}$ with respect to the L^2-norm. The assertion of Theorem 1.7 is that $\dim \mathbb{E} \leq -\chi(S)$.

Recall that any eigenfunction of Δ is smooth (elliptic regularity). Hence each function in $\mathbb{S}$ is smooth. For any $\phi \in \mathbb{S}$,

$$Z_\phi := \{x \in S \mid \phi(x) = 0\} \tag{2.2}$$

is called the *nodal set* of ϕ. The connected components of the complement $S \setminus Z_\phi$ are called *nodal domains* of ϕ.

Lemma 2.3. *With respect to the area element of S, we have $\nabla\phi(x) = 0$ for almost any $x \in Z_\phi$.*

Proof. The set of points of density of Z_ϕ has full measure in Z_ϕ, and, clearly, $\nabla\phi(x) = 0$ at any such point x. q.e.d.

We say that $\epsilon > 0$ is *regular* or, more precisely, *ϕ-regular*, if ϵ and $-\epsilon$ are regular values of ϕ. By Sard's theorem, almost any $\epsilon > 0$ is regular.

For any $\epsilon > 0$, we call

$$Z_\phi(\epsilon) := \{x \in S \mid |\phi(x)| \leq \epsilon\} \tag{2.4}$$

the *ϵ-nodal set* of ϕ. We are only interested in the case where ϵ is regular. Then $Z_\phi(\epsilon)$ is a subsurface of S with smooth boundary, may be empty or may consist of more than one component, and the boundary components of $Z_\phi(\epsilon)$ are embedded smooth circles along which ϕ is constant $\pm\epsilon$.

Let $\epsilon > 0$ be regular. Delete from $Z_\phi(\epsilon)$ all the components which are contained in the interior of an embedded closed disc in S and obtain the *derived ϵ-nodal set* $Z'_\phi(\epsilon)$. By definition, no component of $Z'_\phi(\epsilon)$ is contained in the interior of an embedded closed disc in S. Since $Z'_\phi(\epsilon)$ is important in our discussion, we view its definition also from a different angle: If $D \subseteq S$ is an embedded closed disc, then the components of

$Z_\phi(\epsilon)$ contained in the interior of D are compact and bounded by smooth circles. Each such circle is the boundary of an embedded closed disc D' in D, by the Schoenfliess theorem. By definition, the boundary circle $\partial D'$ of any such disc D' is also a boundary circle of a component C of $Z_\phi(\epsilon)$. There are two possible types for $\partial D'$: Either C is in the outer part or in the inner part of $\partial D'$ with respect to the interior of D. We say that D' is *essential* if C is in the inner part of $\partial D'$. In other words, D' is essential if a neighborhood of $\partial D'$ in D' is contained in $Z_\phi(\epsilon)$. Essential discs in S are either disjoint or one is contained in the other; they are partially ordered by inclusion. Therefore each essential disc is contained in a unique maximal essential disc.

For any regular $\epsilon > 0$, $Y_\phi(\epsilon) := S \setminus \mathring{Z}'_\phi(\epsilon)$ is a smooth subsurface of S.

Lemma 2.5. *For any regular $\epsilon > 0$, we have:*
1) $Y_\phi(\epsilon)$ is a smooth and incompressible subsurface of S.
2) Each component C of $Y_\phi(\epsilon)$ is the union of some component C_0 of $\{\phi \geq \epsilon\}$ or of $\{\phi \leq -\epsilon\}$ with a finite number (≥ 0) of maximal essential discs which are attached to C' along $\partial C'$. In particular, $\phi|_{\partial C} = +\epsilon$ or $\phi|_{\partial C} = -\epsilon$.
3) The function ϕ_ϵ on S, defined by

$$\phi_\epsilon(x) = \begin{cases} \phi(x) - \epsilon & \text{if } \phi(x) \geq \epsilon, \\ \phi(x) + \epsilon & \text{if } \phi(x) \leq -\epsilon, \\ 0 & \text{otherwise}, \end{cases}$$

belongs to $H^1(S)$. Moreover, $\lim_{\epsilon \to 0} \phi_\epsilon = \phi$ in $H^1(S)$.

Proof. 1) Since $Z'_\phi(\epsilon)$ is a union of components of the smooth subsurface $Z_\phi(\epsilon)$ of S, it is a smooth subsurface of S. Hence the complement $Y_\phi(\epsilon)$ of its interior is also a smooth subsurface of S.

It remains to show that there is no loop c in $Y_\phi(\epsilon)$ which is not homotopic to zero in $Y_\phi(\epsilon)$, but is homotopic to zero in S. We suppose the contrary and assume without loss of generality that c is simple and contained in the interior of $Y_\phi(\epsilon)$. Since c is homotopic to zero in S, it bounds an embedded closed disc D in S, by Lemma A.1. Now D is not contained in $Y_\phi(\epsilon)$ since c is not homotopic to zero in $Y_\phi(\epsilon)$. Hence D contains components of $Z'_\phi(\epsilon)$. These are in the interior of D since c lies in the interior of $Y_\phi(\epsilon)$. But this is in contradiction to the definition of $Z'_\phi(\epsilon)$.

2) For each component of $Z_\phi(\epsilon)$ which is contained in the interior of an embedded closed disc, choose an essential disc as explained in our discussion of the definition of $Z'_\phi(\epsilon)$ further up. Since each essential disc is contained in a unique maximal essential disc, it follows that $Y_\phi(\epsilon)$ is equal to the (possibly non-disjoint) union of $S \setminus \mathring{Z}_\phi(\epsilon) = \{|\phi| \geq \epsilon\}$ with maximal essential discs. Hence each of the components of $Y_\phi(\epsilon)$ consists of some component C_0 of $\{\phi \geq \epsilon\}$ or of $\{\phi \leq -\epsilon\}$ together with a finite

number (≥ 0) of maximal essential discs which are attached to C' along $\partial C'$.

3) For all $x \in S$, we have $|\phi_\epsilon(x)| \leq |\phi(x)|$. Hence ϕ_ϵ is in $L^2(M)$. Moreover, $\phi_\epsilon(x) \to \phi(x)$ for all $x \in S$, hence $\lim_{\epsilon\to 0} \phi_\epsilon = \phi$ in $L^2(M)$. Furthermore, ϕ_ϵ has weak gradient

$$\nabla\phi_\epsilon(x) = \begin{cases} \nabla\phi(x) & \text{if } |\phi(x)| \geq \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that ϕ_ϵ is in $H^1(S)$. Furthermore, $\lim_{\epsilon\to 0} \nabla\phi_\epsilon = \nabla\phi$ in $H^1(S)$, by Lemma 2.3. q.e.d.

We let $Y'_\phi(\epsilon)$ be the union of the components C of $Y_\phi(\epsilon)$ with Euler characteristic $\chi(C) < 0$. That is, $Y'_\phi(\epsilon)$ is the union of the components of $Y_\phi(\epsilon)$ which are not diffeomorphic to a disc, an annulus, or a cross cap.

Lemma 2.6. *For all sufficiently small regular $\epsilon > 0$, we have $\chi(Y'_\phi(\epsilon)) < 0$. In other words, $Y'_\phi(\epsilon)$ is non-empty for all sufficiently small $\epsilon > 0$.*

Proof. Case 1: Assume first that the Rayleigh quotient $R(\phi) < \Lambda(S)$ and choose a $\delta > 0$ such that

$$R(\phi) \leq \Lambda(S) - 2\delta.$$

By Lemma 2.5.3, we have, for any sufficiently small regular $\epsilon > 0$,

$$\frac{\sum_C \int_C |\nabla\phi_\epsilon|^2}{\sum_C \int_C \phi_\epsilon^2} \leq \frac{\int_S |\nabla\phi|^2 dv}{\int_S \phi^2 dv} + \delta = R(\phi) + \delta \leq \Lambda(S) - \delta,$$

where the sums run over the components C of $Y_\phi(\epsilon)$. We conclude that there is a component C such that

$$R(\phi_\epsilon|_C) = \frac{\int_C |\nabla\phi_\epsilon|^2}{\int_C \phi_\epsilon^2} \leq \Lambda(S) - \delta.$$

Since ϕ_ϵ vanishes along ∂C, it follows from the definition of $\Lambda(S)$ that C is neither a disc, nor an annulus, nor a cross cap. Hence the Euler characteristic of C is negative.

Case 2: Assume now that $\mathcal{R}(\phi) = \Lambda(S)$. This is the only part of the proof which requires the regularity theory of the nodal sets of eigenfunctions, and it is needed to establish that the inequality in Theorem 1.7 is strict.

Since $\mathbb{E}$ is the sum of the eigenspaces of Δ with eigenvalues $\leq \Lambda(S)$, the equality $\mathcal{R}(\phi) = \Lambda(S)$ implies that ϕ is an eigenfunction of Δ with eigenvalue $\Lambda(S)$. Now it is a classical result that non-zero eigenfunctions of the Laplacian cannot vanish of infinite order at any point; see e.g. [**1**]. Therefore, by the main result of [**2**], at any critical point $z \in Z_\phi$ of ϕ, there are Riemannian normal coordinates (x, y) about z, a spherical harmonic $p = p(x, y) \neq 0$ of some order $n \geq 2$, and a constant $\alpha \in (0, 1)$ such that

$$\phi(x, y) = p(x, y) + O(r^{n+\alpha}),$$

where we write $(x, y) = (r \cos\theta, r \sin\theta)$. By Lemma 2.4 of [**7**], there is a local C^1-diffeomorphism Φ about $0 \in \mathbb{R}^2$ fixing 0 such that

$$\phi = p \circ \Phi.$$

Note that, up to a rotation of the (x, y)-plane, we have

$$p = p(x, y) = cr^n \cos n\theta$$

for some constant $c \neq 0$. It follows that the nodal set Z_ϕ of ϕ is a finite graph with critical points of ϕ as vertices ([**7**, Theorem 2.5]). It also follows that, for any sufficiently small $\epsilon > 0$, the only critical points of ϕ in $\{|\phi| \leq \epsilon\}$ are already contained in Z_ϕ. In particular, the gradient flow of ϕ can be used to obtain a deformation retraction of $S \setminus Z_\phi$ onto $\{|\phi| \geq \epsilon\}$.

For any component C of $S \setminus Z_\phi$, the restriction of ϕ to C vanishes nowhere on C, and hence ϕ is the eigenfunction for the first Dirichlet eigenvalue of C. It follows that $\lambda_0(C) = \Lambda(S)$.

Since ϕ is perpendicular to the constant functions, the interior of the complement of a component C as above is non-empty. Hence C can be strictly enlarged within S, keeping the topological type of C, while strictly decreasing $\lambda_0(C)$; see (1.9). It follows that no component C of $S \setminus Z_\phi$ is diffeomorphic to a disc or an annulus or a cross cap (with piecewise smooth boundary). Thus each component of $S \setminus Z_\phi$ has negative Euler characteristic.

It follows also that the graph Z_ϕ does not contain non-trivial loops which are homotopic to zero in S since otherwise $S \setminus Z_\phi$ would contain a component which is a disc. Hence, for all sufficiently small regular $\epsilon > 0$, no component of $Z_\epsilon(\phi)$ is contained in a disc and each component of $Y_\phi(\epsilon)$ has negative Euler characteristic. Thus $Y'_\phi(\epsilon) = Y_\phi(\epsilon)$, for all sufficiently small $\epsilon > 0$. q.e.d.

Lemma 2.7. *For all regular $\epsilon > 0$, we have $\chi(S) \leq \chi(Y'_\phi(\epsilon))$.*

Proof. By definition, $Y'_\phi(\epsilon)$ and $S \setminus \mathring{Y}'_\phi(\epsilon)$ are smooth subsurfaces of S which intersect along their common boundary, a finite number of circles. Hence

$$\chi(S) = \chi(Y'_\phi(\epsilon)) + \chi(S \setminus \mathring{Y}'_\phi(\epsilon)),$$

by the Mayer-Vietoris sequence. No component of $S \setminus \mathring{Y}'_\phi(\epsilon)$ is a disc since otherwise the boundary of the disc would be a loop in $Y'_\phi(\epsilon)$ which is not homotopic to zero in $Y'_\phi(\epsilon)$, but homotopic to zero in S. This would be in contradiction to Lemma 2.5.1. It follows that $\chi(S \setminus \mathring{Y}'_\phi(\epsilon)) \leq 0$. q.e.d.

For later purposes, we want to attach signs to the components C of $Y'_\phi(\epsilon)$: We say that C is *positive* or *negative* if C is the union of maximal essential discs with a component of $\{\phi \geq \epsilon\}$ or a component of $\{\phi \leq -\epsilon\}$, respectively. We denote by $Y'^+_\phi(\epsilon)$ and $Y'^-_\phi(\epsilon)$ the subsets of positive and negative components of $Y'_\phi(\epsilon)$, respectively.

Lemma 2.8. *Let $\epsilon_1 > \epsilon_2 > 0$ be regular. Then*

$$Y'_\phi(\epsilon_1) \subseteq Y'_\phi(\epsilon_2) \quad \textit{and} \quad \chi(Y'_\phi(\epsilon_2)) \leq \chi(Y'_\phi(\epsilon_1)).$$

Moreover, if $\chi(Y'_\phi(\epsilon_2)) = \chi(Y'_\phi(\epsilon_1))$, then $Y'_\phi(\epsilon_2)$ arises from $Y'_\phi(\epsilon_1)$ by attaching annuli and cross caps along boundary curves of $Y'_\phi(\epsilon_1)$. The analogous statements hold for $Y'^\pm_\phi(\epsilon_1)$ and $Y'^\pm_\phi(\epsilon_2)$ in place of $Y'_\phi(\epsilon_1)$ and $Y'_\phi(\epsilon_1)$, respectively.

Proof. By definition, $Z_\phi(\epsilon_2) \subseteq Z_\phi(\epsilon_1)$. If a component of $Z_\phi(\epsilon_1)$ is contained in the interior of an embedded closed disc, then also all the components of $Z_\phi(\epsilon_2)$ it contains. It follows that $Z'_\phi(\epsilon_2) \subseteq Z'_\phi(\epsilon_1)$ and hence that $Y_\phi(\epsilon_1) \subseteq Y_\phi(\epsilon_2)$.

Let C_1 be a component of $Y'_\phi(\epsilon_1)$ and C be the component of $Y_\phi(\epsilon_2)$ which contains it. Let B be the union of the components of $Y'_\phi(\epsilon_1)$ which are contained in C. Since $\epsilon_1 \neq \epsilon_2$, the boundaries of B and C are disjoint, by Lemma 2.5.2. Since the Euler characteristics of the components of B are negative, boundary curves of B are not homotopic to zero in B.

Assume that $\chi(C) > \chi(B)$. Then one of the components of $C \setminus \mathring{B}$ is a (closed) disc. Then a boundary curve of B would be homotopic to zero in S in contradiction to the incompressibilty of B; see Lemma 2.5.1. We conclude that $\chi(C) \leq \chi(B)$. Since $\chi(B) < 0$, we also conclude that $C \subseteq Y'_\phi(\epsilon_2)$. Therefore $Y'_\phi(\epsilon_1) \subseteq Y'_\phi(\epsilon_2)$ and $\chi(Y'_\phi(\epsilon_2)) \leq \chi(Y'_\phi(\epsilon_1))$. Equality implies that the differences $C \setminus \mathring{B}$ as above consists of annuli and cross caps.

By what we just said, the last assertion follows if $Y'^\pm_\phi(\epsilon_1) \subseteq Y'^\pm_\phi(\epsilon_2)$. To show this, let C_1 be a positive component of $Y'_\phi(\epsilon_1)$ and C be the component of $Y'_\phi(\epsilon_2)$ containing it. Assume first that $C_1 \neq S$, that is, that $\partial C_1 \neq \emptyset$. Now C is the union of a number of maximal essential discs (with respect to ϵ_2) with a component C_0 of $\{\phi \geq \epsilon_2\}$ or $\{\phi \leq -\epsilon_2\}$. Since C_1 is incompressible in S and the boundary curves of C_1 are not homotopic to zero in C_1, ∂C_1 is not contained in any of the maximal discs. Therefore ∂C_1 intersects C_0 non-trivially. Since $\phi|_{\partial C_1} = \epsilon_1$, we conclude that C_0 is a component of $\{\phi \geq \epsilon_2\}$. Hence C is positive and therefore $Y'^+_\phi(\epsilon_1) \subseteq Y'^+_\phi(\epsilon_2)$.

The case $C_1 = S$ follows from the Schoenfliess theorem. The proof of the inclusion $Y'^-_\phi(\epsilon_1) \subseteq Y'^-_\phi(\epsilon_2)$ is similar. q.e.d.

We want fo modify the subsurfaces $Y_\phi'^{\pm}(\epsilon)$ so that their isotopy type in S becomes independent of ϵ as $\epsilon \to 0$: For any regular $\epsilon > 0$, we let $X_\phi^+(\epsilon)$ be the union of $Y_\phi'^+(\epsilon)$ with the components of the complement of the interior of $Y_\phi'^+(\epsilon)$ in S which are annuli and cross caps. Note that $\phi = \epsilon$ on the boundary of such annuli and cross caps. We define $X_\phi^-(\epsilon)$ accordingly and set $X_\phi(\epsilon) = X_\phi^+(\epsilon) \cup X_\phi^-(\epsilon)$. Note that

$$\chi(X_\phi^{\pm}(\epsilon)) = \chi(Y_\phi'^{\pm}(\epsilon)) \quad \text{and} \quad \chi(X_\phi(\epsilon)) = \chi(Y_\phi'(\epsilon)). \tag{2.9}$$

By construction and Lemma 2.5.2, $\phi|_{\partial C} = \pm\epsilon$ for any component C of $X_\phi^{\pm}(\epsilon)$. Observe that $X_{-\phi}^+(\epsilon) = X_\phi^-(\epsilon)$, and accordingly for $Y_\phi'^{\pm}(\epsilon)$.

Lemma 2.10. *Let $\epsilon_1 > \epsilon_2 > 0$ be regular and suppose that $\chi(X_\phi(\epsilon_1)) = \chi(X_\phi(\epsilon_2))$. Then $(S, X_\phi^+(\epsilon_1), X_\phi^-(\epsilon_1))$ is isotopic to $(S, X_\phi^+(\epsilon_2), X_\phi^-(\epsilon_2))$; that is, there is a diffeomorphism of S which is isotopic to the identity and which restricts to a diffeomorphism between $X_\phi^+(\epsilon_1)$ and $X_\phi^+(\epsilon_2)$ respectively between $X_\phi^-(\epsilon_1)$ and $X_\phi^-(\epsilon_2)$.*

Proof. By (2.9), we have $\chi(Y_\phi'(\epsilon_1)) = \chi(Y_\phi'(\epsilon_2))$. Hence $Y_\phi'^{\pm}(\epsilon_2)$ arises from $Y_\phi'^{\pm}(\epsilon_1)$ by attaching annuli and cross caps, by Lemma 2.8. The point of the argument below is that all boundary curves of $Y_\phi'^{\pm}(\epsilon_2)$ arise by attaching an annulus A to a boundary curve of $Y_\phi'^{\pm}(\epsilon_1)$. Then ϕ is equal to ϵ_1 on one of the boundary curves of A and equal to ϵ_2 on the other.

Without loss of generality, we only consider the X^+-spaces. It suffices to show that $X_\phi^+(\epsilon_2)$ arises from $X_\phi^+(\epsilon_1)$ by attaching annuli A such that ϕ is equal to ϵ_1 on one of the boundary curves of A and equal to ϵ_2 on the other.

There are several cases in the passage from the Y'-spaces to the X-spaces. We exemplify the argument in one of the cases.

Suppose that, in the passage from $Y_\phi'^+(\epsilon_1)$ to $X_\phi^+(\epsilon_1)$, an annulus A is attached to $Y_\phi'^+(\epsilon_1)$ such that ϕ is equal to ϵ_1 on the boundary curves of A. Then $\phi = \epsilon_1$ on ∂A and either $\phi > \epsilon_2$ on A or else, by Lemma 2.8, there are disjoint annuli $A', A'' \subseteq A$, each of them sharing a boundary curve with A, such that ϕ is equal to ϵ_2 on the other boundary curve and such that A' and A'' belong to $Y_\phi'^+(\epsilon_2)$. By Lemma A.2, we get an annulus A''' in A between A' and A'' and sharing one of its boundary curves with A' and the other with A''. In particular, ϕ is equal to ϵ_2 on both boundary curves of A'''. We conclude that $A = A' \cup A''' \cup A''$ belongs to $X_\phi(\epsilon_2)$. q.e.d.

3. Concluding Theorem 1.7

By the above Lemmas 2.6 – 2.10, we obtain a partition of the unit sphere $\mathbb{S}$ in $\mathbb{E}$ into the subsets

$$\mathcal{C}_i := \{\phi \in \mathbb{S} \mid \chi(X_\phi(\epsilon)) = i \text{ for all sufficiently small } \epsilon > 0\}, \tag{3.1}$$

where $\chi(S) \le i < 0$. By definition, $\phi \in \mathcal{C}_i$ if and only if $-\phi \in \mathcal{C}_i$. Hence $\mathcal{C}_i$ is the preimage of the subset $\mathcal{B}_i = \pi(\mathcal{C}_i)$ in the projective space $\mathbb{P} = \mathbb{S}/\pm\,\mathrm{id}$ under the canonical projection $\pi : \mathbb{S} \to \mathbb{P}$.

Lemma 3.2. *Let $\epsilon > 0$ and $U \subseteq \mathbb{S}$ be the subset of ϕ such that ϵ is ϕ-regular. Then U is open and the isotopy types of $(S, X_\phi^+(\epsilon), X_\phi^-(\epsilon))$ are locally constant as functions of $\phi \in U$.*

Proof. Note that U is open since S is compact. Consider the map

$$F\colon U \times S \to \mathbb{R}, \quad F(\phi, z) = \phi(z).$$

Since $\mathbb{E}$ is finite dimensional, any two norms on $\mathbb{E}$ are equivalent. In particular, F is continuously differentiable. If $\phi \in U$ and $z \in S$ satisfy $\phi(z) = \epsilon$, then $d\phi_z \neq 0$, and hence $dF_{(\phi,z)} \neq 0$. Choose a vector $v \in T_zS$ with $dF_z(v) \neq 0$ and coordinates (x, y) of S about z such that $z = (0, 0)$ and $v = \partial/\partial y$. Then, by the implicit function theorem, there are open intervals $I \ni 0$ and $J \ni \epsilon$, an open neighborhood W of ϕ in $\mathbb{E}$, and a smooth function $y\colon W \times I \times J \to \mathbb{R}$ such that $F(\psi, x, y(\psi, x, \tau)) = \tau$ for all $(\psi, x, \tau) \in W \times I \times J$. It follows that the families of curves $\{\phi = \epsilon\}$ depend smoothly on $\phi \in U$; and similarly for $-\epsilon$. The claim of Lemma 3.2 now follows easily from the construction of the $X^\pm$-spaces.
q.e.d.

Now we are ready for the final steps of the proof of Theorem 1.7.

Proof of Theorem 1.7. For $\mathcal{B}_i = \pi(\mathcal{C}_i)$ as above, we show that $\pi\colon \mathbb{S} \to \mathbb{P}$ is trivial over $\mathcal{B}_i$. To that end, we note that, for $\phi \in \mathcal{C}_i$, we have $-\phi \in \mathcal{C}_i$ and that

$$(S, X_{-\phi}^+(\epsilon), X_{-\phi}^-(\epsilon)) = (S, X_\phi^-(\epsilon), X_\phi^+(\epsilon)).$$

Now $(S, X^+\phi(\epsilon), X_\phi^-(\epsilon))$ is not isotopic to $(S, X_\phi^-(\epsilon), X_\phi^+(\epsilon))$, by Theorem A.5. Hence the partition of $\mathcal{C}_i$ into the open subsets $U_{i,j}$ with the same isotopy type has the property that $U_{i,j} \cap -U_{i,j} = \emptyset$. Note also that $U_{i,j} \cup -U_{i,j}$ is the preimage of a subset $V_{i,j}$ in $\mathbb{P}$. It follows that $\pi|_{\mathcal{B}_i}$ is trivial. We conclude that $-\chi(S) > \dim \mathbb{P} = \dim \mathbb{E} - 1$, by Lemma 8 in [**16**] (see also the final paragraph in the proof of Lemma 5 in [**12**]).
q.e.d.

Appendix A. On the topology of surfaces

For the convenience of the reader (and the authors), we collect some facts from the topology of surfaces.

In what follows, let S be a surface of finite type, that is, S is diffeomorphic to the interior of a compact surface with boundary, with Euler number $\chi(S) \leq 0$. In other words, S is of finite type, but is not diffeomorphic to the sphere or the real projective plane. In the orientable case, the first two assertions are Corollary A.7 and Proposition A.11 in [**6**].

Lemma A.1. *Any homotopically trivial simple closed curve in S bounds an embedded disc.*

Lemma A.2. *Let c_0 and c_1 be smooth two-sided simple closed curves in S which are freely homotopic (up to their orientation) and which do not intersect. Then $c_0 \cup c_1$ bounds an embedded annulus in S.*

Lemma A.3. *Let $C \subseteq S$ be a connected subsurface with smooth boundary which is closed as a subset of S. Assume that C contains a closed curve which is homotopic to zero in S, but not in C. Then $S \setminus \mathring{C}$ contains a connected component which is diffeomorphic to a closed disc.*

Proof. Without loss of generality we may assume that $\mathring{C}$ contains a smoothly immersed simple closed curve c which is homotopic to zero in S, but not in C. Now c bounds a smooth disc D in S, by Lemma A.1, and $B = C \cap D$ is a smooth and closed subsurface in D. Furthermore, B is connected since $c = \partial D \subseteq B$ and $B \neq D$ since otherwise c would be homotopic to zero in C. Hence the interior of D contains boundary circles c' of B, and the interior D' of any such c' in D is disjoint from C. By the Schoenfliess theorem, any such D' is diffeomorphic to a disc. q.e.d.

A subsurface $C \subseteq S$ is called *incompressible* in S if any closed curve in C, which is homotopic to zero in S, is already homotopic to zero in C.

Corollary A.4. *Let $C \subseteq S$ be a connected subsurface with boundary which is closed as a subset of S. Assume that no component of $S \setminus \mathring{C}$ is diffeomorphic to a closed disc. Then C is incompressible in S.*

For a proof of the following result, we refer to Chapter 1 of [**8**].

Theorem A.5. *Let S be a compact and connected surface with $\chi(S) < 0$ and $L \subseteq S$ be a closed one-dimensional submanifold. Let $F\colon S \to S$ be a diffeomorphism which is isotopic to the identity and such that $F(L) = L$. Then F leaves all components of L and $S \setminus L$ invariant.*

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