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SMALL SCALE LIMIT THEOREMS FOR THE INTERSECTION LOCAL TIMES OF BROWNIAN MOTION

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Abstract: In this paper we contribute to the investigation of the fractal nature of the intersection local time measure on the intersection of independent Brownian paths. We particularly point out the difference in the small scale behaviour of the intersection local times in three-dimensional space and in the plane by studying almost sure limit theorems motivated by the notion of average densities introduced by Bedford and Fisher. We show that in $\mathbb{R}^3$ the intersection local time measure μ of two paths has an average density of order two with respect to the gauge function $\varphi(r) = r$, but in the plane, for the intersection local time measure μ_p of p Brownian paths, the average density of order two fails to converge. The average density of order three, however, exists for the gauge function $\varphi_p(r) = r^2[\log(1/r)]^p$. We also prove refined versions of the above results, which describe more precisely the fluctuations of the volume of small balls around these gauge functions by identifying the density distributions, or lacunarity distributions, of the intersection local times.

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1 Introduction and statement of results

This paper is a contribution to the study of the fractal nature of the intersection local time measure μ, the natural Hausdorff measure on the intersection of independent Brownian paths in 3-space and in the plane. We investigate the notions of average densities and density distributions of μ and particularly point out the striking difference between the spatial and the planar case. In this section we motivate these notions in a general context and embed our results in this context, leaving the precise definition and properties of intersection local time to the next section.

An important role for the fine geometry of fractal measures μ is played by the behaviour, as $r \downarrow 0$, of the functions

$$d_\varphi : r \mapsto \frac{\mu(B(x,r))}{\varphi(r)} \, ,$$

where $B(x,r)$ is the open ball centred in x of radius r and $\varphi : (0,\varepsilon) \to (0,\infty)$ is a suitably chosen gauge function. For a smooth measure, for example a measure μ absolutely continuous with respect to the surface measure on an m-submanifold, this function converges for the gauge function $\varphi(r) = r^m$, as $r \to 0$, for μ-almost every x to a nonzero limit. Conversely, a measure μ where we encounter such a convergence has strong regularity properties, see [PM95]. Hence the fluctuations of this function are a means to describe the irregularities of a measure μ.

For the random measures appearing in the study of nonsmooth stochastic processes, like for example occupation measures and local times, typically, there is *no* gauge function φ such that the function $d_\varphi(r)$ converges to a nonzero limit as $r \downarrow 0$ for all x on a set of positive measure. It is, however, of interest to find a gauge function ψ such that $\limsup_{r\downarrow 0} d_\psi(r)$ is positive and finite, as this allows to compare μ to the ψ-Hausdorff measure. Similarly, a gauge function θ such that $\liminf_{r\downarrow 0} d_\theta(r)$ is positive and finite allows a comparison of μ and the θ-packing measure. See [JT86] for a survey of such results and methods for measures μ arising in the context of stochastic processes. These results refer to the behaviour of $r \mapsto \mu(B(x,r))$ along certain extreme sequences $r_n \downarrow 0$, which asymptotically describe its lower and upper hull. It is natural to try and describe the oscillation between the lower and upper hull and also find a suitable average value for $\mu(B(x,r))$. A first step in this direction is the investigation of the *average densities* introduced by Bedford and Fisher in [BF92], see also [KF97] for an introduction.

For certain fractal measures Bedford and Fisher observed that, although $d_\varphi(r)$ does not converge to a nonnegative limit, it is possible to define a generalized limit using classical summation techniques of Hardy and Riesz. This generalized limit defines an interesting parameter, which is closely related to Mandelbrot's concept of fractal lacunarity (see e.g. [BM95]). This parameter may be used to compare the lacunarity (or mass density) of different fractals with the same dimension gauge, see [LL94] or [KF97] for explicit calculations. Bedford and Fisher used *logarithmic averaging of order two* to define the *average densities of order two* of μ at x as

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\log(1/\varepsilon)} \int_\varepsilon^1 \frac{\mu(B(x,r))}{\varphi(r)} \, \frac{dr}{r} \, .$$

For many fractal measures this limit was shown to exist for gauge functions of the type $\varphi(r) = r^\alpha$. Examples include Hausdorff measures on deterministic and random self–similar sets, mixing repellers or occupation measures of stable processes, see [BF92], [PZ94], [KF92] and [FX95]. We remark that average densities were also used to characterize geometric regularity of sets, see [FS95], [JM96], [PM97], or symmetry properties of measures, see [M98a], [MP98].

Our first result shows that for the intersection local time measure on the intersection of two Brownian paths in 3-space an average density of order two may be defined using such a gauge function.

Theorem 1.1 *Suppose μ is the intersection local time of two independent Brownian paths in $\mathbb{R}^3$ started at arbitrary points and running for unit time. Define the gauge function $\varphi(r) = r$. Then, with probability one, the average density of order two with respect to φ exists at μ-almost every x and we have*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\log(1/\varepsilon)} \int_\varepsilon^1 \frac{\mu(B(x,r))}{\varphi(r)} \frac{dr}{r} = \frac{4}{\pi} . \tag{1}$$

In the next theorem we show that for the intersection local time measure μ_p of p independent Brownian motions in the plane the behaviour of the average densities is different from the behaviour observed in the cases above, namely the average density of order two fails to exist for any gauge function. In this case it is natural to use logarithmic averaging of higher order. Following [BF92] we define the *average density of order three* at x by

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \frac{\mu_p(B(x,r))}{\varphi_p(r)} \frac{dr}{r \log(1/r)} .$$

There is a hierarchy in the notions of average densities: The existence of average densities of order two implies the existence of average densities of order three with the same value, see e.g. [BF92]. With the choice of a gauge function $\varphi_p(r)$ involving a logarithmic correction we get a positive convergence result for the average densities of order three.

Theorem 1.2 *Suppose μ_p is the intersection local time of p independent Brownian paths in $\mathbb{R}^2$ started at arbitrary points and running for unit time. Then, with probability one,*

(a) *for every gauge function at μ_p-almost every x the average density of order two fails to exist,*

(b) *for the gauge function $\varphi_p(r) = r^2\pi[\log(1/r)/\pi]^p$ the average density of order three with respect to φ_p exists at μ_p-almost every x and we have*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \frac{\mu_p(B(x,r))}{\varphi_p(r)} \frac{dr}{r \log(1/r)} = 2^p . \tag{2}$$

Remarks:

- It is not hard to see that both our theorems hold irrespective of the choice of the finite (and in the first case even infinite) running times of the Brownian motions.
- In the case of occupation measure of a Brownian path similar results hold, in the case of dimensions larger than three this was proved in [FX95] and in the planar case in [M98b].
- A heuristic explanation for the non-existence of the order-two densities for μ_p is that the spectrum of the oscillations of $\mu_p(B(x,r))$ contains smaller frequencies than in the case of 3-space, a fact which is due to the longer range of dependence of the random variables $\mu_p(B(x,r))$ in the planar case.

In order to get a finer picture of the oscillation of $\mu(B(x,r))$ around r and $\mu_p(B(x,r))$ around $r^2\pi[\log(1/r)/\pi]^p$ we study, for fixed Brownian paths, the distributions of $\mu(B(x,r))/\varphi(r)$ for a natural random choice of r. This leads us to the notion of density distributions, or lacunarity distributions, due also to [BF92]. For a fixed measure μ the *density distribution of order n of μ at x* is the asymptotic distribution as $T \to \infty$ of

$$\frac{\mu(B(x, 1/\exp^{(n-1)}(X)))}{\varphi(1/\exp^{(n-1)}(X))},$$

where X is uniformly distributed on $(0,T)$ and $\exp^{(n)}$ denotes the nth iterate of the exponential function. A simple substitution confirms that the density distributions of order two are the limit distributions as $\varepsilon \downarrow 0$ of

$$\frac{1}{\log(1/\varepsilon)} \int_\varepsilon^1 \delta_{\{d_\varphi(r)\}} \frac{dr}{r},$$

and the density distributions of order three are the limits of

$$\frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \delta_{\{d_\varphi(r)\}} \frac{dr}{r\,\log(1/r)},$$

where $\delta_{\{a\}}$ stands for the point mass in a. A straightforward modification of the proof of Theorem 1.1 shows that for the intersection local time measure in 3-space, with probability one, the density distribution of order two with respect to $\varphi(r) = r$ exists at μ-almost every x and equals the distribution of the total intersection local time of two independent two-sided Brownian motions in the unit ball. In the planar case we get an interesting almost-sure limit theorem.

Theorem 1.3 *Suppose μ_p is the intersection local time of p independent Brownian paths in the plane started at arbitrary points and running for unit time. For the gauge function $\varphi_p(r) = r^2\pi[\log(1/r)/\pi]^p$ the density distribution of order three exists at μ_p-almost every x and equals the distribution of the product of p independently with parameter two gamma–distributed random variables. More explicitly,*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\log(\log(1/\varepsilon))} \int_\varepsilon^{1/e} \delta_{\left\{\frac{\mu_p(B(x,r))}{\varphi_p(r)}\right\}} \frac{dr}{r\,\log(1/r)} = \int_0^\infty \cdots \int_0^\infty \delta_{\{a_1\cdots a_p\}} \prod_{i=1}^p a_i\, e^{-a_i}\, da_i\,. \quad (3)$$

Remarks:

- The corresponding result for the case of occupation measure was obtained in [M98b].
- As in Theorem 1.2(a) it may be shown that for the intersection local time measure in the plane, with probability one, the density distribution of order two fails to exist.
- Our theorem shows that, for almost every x, the function $r \mapsto \mu_p(B(x,r))$ oscillates around the gauge functions $\varphi_p(r)$ in such a way that for "most" scales the ratio $\mu_p(B(x,r))/\varphi_p(r)$ is bounded away from 0 and ∞ and hence this gauge function describes the typical behaviour of $\mu_p(B(x,r))$. To make this more explicit recall the definition of logarithmic densities.

The *logarithmic density of order two*, resp. three, of a set $N \subset (0,\infty)$ is the value of the limit

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\log(1/\varepsilon)} \int_\varepsilon^1 1_N(r)\, \frac{dr}{r} \quad \text{, resp. } \lim_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} 1_N(r)\, \frac{dr}{r\log(1/r)}\,,$$

if it exists. For the intersection local time of two independent Brownian paths in $\mathbb{R}^3$, with probability one, for every $\varepsilon > 0$ there are $0 < c < C < \infty$ such that, for μ-almost every x, we have

$$c \cdot r < \mu(B(x,r)) < C \cdot r$$

for all r outside a set N of logarithmic density of order two smaller than ε. For the intersection local time of p independent Brownian paths in the plane, with probability one, for every $\varepsilon > 0$ there are $0 < c < C < \infty$ such that, for μ-almost every x, we have

$$c \cdot r^2(\log(1/r))^p < \mu_p(B(x,r)) < C \cdot r^2(\log(1/r))^p$$

for all r outside a set N of logarithmic density of order three smaller than ε. These statements are immediate from the existence of the density distributions upon recalling Prohorov's Theorem: Weak compactness of a family of probability distributions implies uniform tightness of the family.

- The gauge functions $\varphi(r) = r$ and $\varphi_p(r) = r^2[\log(1/r)]^p$ in the previous remark should be compared to the gauge functions governing the limsup-behaviour of the density functions (and thus the dimension gauge) obtained by Le Gall [LG87]. These are in the case of two Brownian motions in space

$$\psi(r) = r \cdot [\log\log(1/r)]^2$$

and in the case of p Brownian motions in the plane

$$\psi_p(r) = r^2 \cdot [\log(1/r)\log\log\log(1/r)]^p\,.$$

The gauge functions for the liminf-behaviour seem to be unknown for $p > 1$, see Section 6.

The idea common to the proofs of our theorems is to reduce the problem first to the study of the intersection local time of independent Brownian paths at a common starting point, say the origin. To do this we introduce a Palm distribution associated with the intersection local time (Theorem 3.1) and then derive a 0-1 law (Theorem 3.2) — a technique suitable for the study of intersection local time in all dimensions. The problem at the origin is then dealt with by means of the ergodic scaling flow (Section 4), in the case of Brownian paths in space, and by means of an approximation of the intersection local times by crossing numbers, in the case of planar paths (Section 5). Some of these methods have been used in [M98b] in the case of occupation measures, but we believe that the full strength of these methods, in particular the Palm distribution technique, becomes only apparent in the study of the more complicated intersection local times.

The paper is organized as follows: In the next section we give a precise definition and collect some facts about intersection local times. In Section 3 we introduce the Palm distribution associated with the intersection local time. The following section contains the proofs of our theorems in the case of Brownian paths in space and in Section 5 we treat the case of Brownian paths in the plane. Section 6 contains some open questions.

2 Intersection local time as canonical measure on the intersection of Brownian paths

We consider a family of $p \geq 2$ independent two-sided Brownian motions $B^1, \ldots, B^p$ in $\mathbb{R}^d$ with $B_0^1 = x_1, \ldots, B_0^p = x_p$. Let

$$C_d = \{f : \mathbb{R} \to \mathbb{R}^d, \quad f \text{ is continuous and } f(0) = 0\},$$

equipped with the standard Wiener measure W on the σ–algebra $\mathcal{B}(C_d)$ generated by the cylinder subsets of C_d. We conveniently assume the motions to be the coordinate processes on the space $\Omega = C_d^{\otimes p}$ with $\mathcal{F} = \mathcal{B}(C_d)^{\otimes p}$ and $\mathbb{P}_0 = W^{\otimes p}$, so that for every vector $x = (x_1, \ldots, x_p)$ of initial points $x_i \in \mathbb{R}^d$ and $\omega = (\omega_1, \ldots, \omega_p) \in \Omega$ the p independent Brownian motions in $\mathbb{R}^d$ with initial points x_i are represented by $x + \omega$ or, more precisely, $B_s^i = x_i + \omega_i(s)$.

For all time vectors $S = (S_1, \ldots, S_p)$ and $T = (T_1, \ldots, T_p)$ with $-\infty < S_i < T_i < \infty$ we study the set

$$B^1[S_1, T_1] \cap \cdots \cap B^p[S_p, T_p] = \Big\{z \in \mathbb{R}^d \, : \, z = B_{t_1}^1 = \cdots = B_{t_p}^p \text{ for some } t_i \in [S_i, T_i]\Big\}$$

of intersections of the Brownian paths. By classical results of Dvoretzky, Erdös, Kakutani and Taylor these intersections are nonempty with positive probability if and only if either $d = 2$ and p is arbitrary or $d = 3$ and $p = 2$. In these cases Dynkin [ED81] and Geman, Horowitz and Rosen [GH84] have constructed canonical random measures μ_S^T on this set. We follow the construction of [GH84], see also [LG86]. There is a Borel set $\Omega_0 \subset \Omega$ with $\mathbb{P}_0(\Omega_0) = 1$, such that, for every $\omega \in \Omega_0$, every initial vector $x = (x_1, \ldots, x_d)$, and all S and T, there is a family $\{\lambda_y \, : \, y \in (\mathbb{R}^d)^{p-1}\}$ of finite measures $\lambda_y = \lambda_y[x + \omega]$ on $\prod_{i=1}^p [S_i, T_i]$ with the following two properties:

(i) the mapping $y \mapsto \lambda_y$ is continuous with respect to the vague topology on the space $\mathcal{M}(\mathbb{R}^p)$ of locally finite measures on $\mathbb{R}^p$,

(ii) for all Borel functions $g : (\mathbb{R}^d)^{p-1} \to [0, \infty]$ and $f : \prod_{i=1}^p [S_i, T_i] \to [0, \infty]$,

$$\int_{S_1}^{T_1} \cdots \int_{S_p}^{T_p} f(s_1, \ldots, s_p) \, g(B_{s_1}^1 - B_{s_2}^2, \ldots, B_{s_{p-1}}^{p-1} - B_{s_p}^p) \, ds_p \ldots ds_1 = \int_{\mathbb{R}^{d(p-1)}} g(y) \Big[\int f d\lambda_y \Big] \, dy.$$

The above properties imply that

(iii) λ_y is supported by the set

$$\Lambda_y = \Big\{(s_1, \ldots, s_p) \in \prod_{i=1}^p [S_i, T_i] \, : \, B_{s_1}^1 - B_{s_2}^2 = y_1, \ldots, B_{s_{p-1}}^{p-1} - B_{s_p}^p = y_{p-1}\Big\},$$

(iv) for all Borel functions $f : \prod_{i=1}^p [S_i, T_i] \to [0, \infty]$,

$$\int_{S_1}^{T_1} \cdots \int_{S_p}^{T_p} f(s_1, \ldots, s_p) \, ds_p \ldots ds_1 = \int_{\mathbb{R}^{d(p-1)}} \Big[\int f d\lambda_y \Big] \, dy \, .$$

Note that (iii) follows from (ii) by choosing g_ε to be a nonnegative function supported by $B(y,\varepsilon)$ with $\int g_\varepsilon(x)\,dx = 1$. For every continuous function f with support disjoint from Λ_y the integral vanishes as $\varepsilon \downarrow 0$ and hence λ_y is supported by Λ_y. (iv) follows from (ii) by letting $g \equiv 1$.

By these properties the image measure $\mu_S^T = \mu_S^T[x+\omega]$ of λ_0 under the mapping $(t_1,\dots,t_p) \mapsto B^1_{t_1}$ is a finite measure supported by the intersections of the Brownian paths, which we call the *intersection local time* of the p Brownian paths. We remark that many authors reserve the term intersection local time for the family $\{\lambda_y\}$ itself.

Properties (i) and (ii) imply that, for all $y \in (\mathbb{R}^d)^{p-1}$ and f nonnegative and continuous, the mapping $\omega \to \int f d\lambda_y[\omega+x]$ may be defined as a limit of measurable mappings and hence the mapping

$$\Lambda : (\Omega_0, \mathcal{F} \cap \Omega_0) \longrightarrow \mathcal{M}(\mathbb{R}^p)\,,\ \omega \mapsto \lambda_y[\omega+x]$$

into the space $\mathcal{M}(\mathbb{R}^p)$ of locally finite measures, with the Borel structure induced by the vague topology, is measurable. This also implies measurability of the mapping

$$M : (\Omega_0, \mathcal{F} \cap \Omega_0) \longrightarrow \mathcal{M}(\mathbb{R}^d)\,,\ \omega \mapsto \mu_S^T[\omega+x]\,.$$

Alternative characterizations show that the intersection local time is indeed a canonical measure on the intersection of the paths. For example, Le Gall has given a description in terms of the volume of Wiener sausages. Fix time vectors S and T and define the *Wiener sausage* S^i_ε as

$$S^i_\varepsilon = S^i_\varepsilon(S,T)[x+\omega] = \Big\{ y \in \mathbb{R}^d \,:\, \inf\{|B^i_s - y| \,:\, S_i \le s \le T_i\} \le \varepsilon \Big\}\,. \tag{4}$$

Define a finite measure μ^ε on $\mathbb{R}^d$, in the case $d=3$ and $p=2$ by

$$\mu^\varepsilon(A) = \frac{1}{(2\pi\varepsilon)^2}\,\ell^3\Big(S^1_\varepsilon \cap S^2_\varepsilon \cap A\Big)\,, \tag{5}$$

and in the case $d=2$ by

$$\mu^\varepsilon(A) = \left(\frac{\log(1/\varepsilon)}{\pi}\right)^p \ell^2\Big(S^1_\varepsilon \cap \cdots \cap S^p_\varepsilon \cap A\Big)\,. \tag{6}$$

Le Gall has shown in [LG86] that μ_S^T can be characterized by

$$\int f\, d\mu_S^T = \lim_{\varepsilon\downarrow 0} \int f\, d\mu^\varepsilon \tag{7}$$

for all $f : \mathbb{R}^d \to \mathbb{R}$ continuous and bounded, where convergence holds in probability and in the L^q-sense for any $1 \le q < \infty$. This implies that there is a sequence $\varepsilon_n \downarrow 0$ such that μ^{ε_n} converges almost surely to μ_S^T on the space $\mathcal{M}(\mathbb{R}^d)$.

The most interesting characterization given by Le Gall in [LG87] shows that μ_S^T may be defined intrinsically as a constant multiple of the ψ-Hausdorff measure on the random set $B^1[S_1,T_1] \cap \cdots \cap B^p[S_p,T_p]$, in the case of two spatial Brownian motions for the gauge function

$$\psi(r) = r[\log\log(1/r)]^2$$

and in the case of p planar Brownian motions for

$$\psi_p(r) = r^2[\log(1/r)\log\log\log(1/r)]^p.$$

Although we are not explicitly using this characterization in our proofs, it is our main motivation for studying intersection local times.

3 Palm distributions associated with intersection local times

In this section we suppose that either $d = 2$ and $p \geq 2$ is an arbitrary integer or $d = 3$ and $p = 2$. Here we refer to $\mu = \mu[x + \omega] = \mu_0^1[x + \omega]$ as the intersection local time measure of p independent Brownian motions started at time 0 in arbitrary points $x_1, \ldots, x_p$ and running for unit time.

We now address the problem of reducing the investigation of the local geometry of the intersection local time measure at almost every point to an investigation of the intersection local time measure at a single typical point. The main difficulty in this reduction lies in the fact that the typical tuples $(t_1, \ldots, t_p)$ with $B^1_{t_1} = \cdots = B^p_{t_p}$ cannot be realized as stopping times and therefore the strong Markov property cannot be applied. We use the idea of Palm distributions to overcome this difficulty. Palm distributions are also a common tool in other branches of probability such as queuing theory or point processes, see [OK83] for a general reference and [UZ88], [PZ94], [MP98] for applications in fractal geometry.

Definition: Denote by $\mathcal{M}(\mathbb{R}^d)$ the Polish space of all locally finite Borel measures on $\mathbb{R}^d$ equipped with the vague topology and by ℓ^d the Lebesgue measure on $\mathbb{R}^d$. A *stationary quasi-distribution* is a σ-finite measure Q on $\mathcal{M}(\mathbb{R}^d)$ that is invariant with respect to the mapping T^u given by $T^u\nu(A) = \nu(u+A)$. The *intensity of* Q is the number $\lambda = \int \nu(B)\, Q(d\nu)/\ell^d(B)$, which, by stationarity, is independent of the choice of a Borel set B of positive and finite Lebesgue measure. With every stationary quasi-distribution Q of finite intensity λ we associate the *Palm distribution* P, which is the probability distribution defined by

$$P(M) = \frac{1}{\lambda \cdot \ell^d(B)} \int\!\!\int_B 1_M(T^u\nu)\, \nu(du)\, Q(d\nu)\,, \tag{8}$$

for all Borel sets $M \subset \mathcal{M}(\mathbb{R}^d)$. Note that, by stationarity, this definition does not depend on the choice of a Borel set $B \subset \mathbb{R}^d$ of positive and finite Lebesgue measure. It is easy to see that P is the unique probability distribution such that

$$\lambda \cdot \int\!\!\int G(\nu, u)\, du\, P(d\nu) = \int\!\!\int G(T^u\nu, u)\, \nu(du)\, Q(d\nu)\,, \tag{9}$$

for every measurable $G : \mathcal{M}(\mathbb{R}^d) \times \mathbb{R}^d \to [0, \infty]$.

Theorem 3.1 *Suppose that either $d = 2$ and $p \geq 2$ is an arbitrary integer or $d = 3$ and $p = 2$. Denote,*

- *for every $x = (x_1, \ldots, x_p)$ with $x_i \in \mathbb{R}^d$, by P_x the probability distribution on $\mathcal{M}(\mathbb{R}^d)$ defined by $P_x(M) = \mathbb{P}_0\{\mu[x + \omega] \in M\}$ for $M \subset \mathcal{M}(\mathbb{R}^d)$ Borel, i.e. the distribution of the intersection local times μ of p independent Brownian motions in $\mathbb{R}^d$ started in x and running for unit time,*

- *by Q the σ-finite measure on $\mathcal{M}(\mathbb{R}^d)$ given by $Q(M) = \int P_x(M \setminus \{\phi\})\, dx$ for all Borel sets $M \subset \mathcal{M}(\mathbb{R}^d)$, where ϕ denotes the zero measure,*

- *by P the probability distribution on $\mathcal{M}(\mathbb{R}^d)$ defined by*

$$P(M) = \int_0^1 \cdots \int_0^1 \mathbb{P}_0\Big\{\mu_{S(y)}^{T(y)}[\omega] \in M\Big\}\, dy_1 \cdots dy_p \text{ for } M \subset \mathcal{M}(\mathbb{R}^d) \text{ Borel,}$$

  where $S(y) = (y_1 - 1, \dots, y_p - 1)$ and $T(y) = (y_1, \dots, y_p)$. In other words, P is the distribution of the intersection local times $\mu_{S(Y)}^{T(Y)}$ for an independent family $Y_1, \dots, Y_p$ of uniformly distributed random variables on $[0,1]$, which are independent of the Brownian motions.

Then P is the Palm distribution associated with Q.

Proof: Let us first check that Q is indeed σ-finite. For this purpose let $M_n = \{\nu \in \mathcal{M}(\mathbb{R}^d) : \nu(B(0,n)) > 0\}$. Observe that $\mathcal{M}(\mathbb{R}^d) = \bigcup_{n=1}^\infty M_n \cup \{\phi\}$ and recall the definition of the Wiener sausage from (4) to obtain

$$\begin{aligned} Q(M_n) &= \int_{\mathbb{R}^{pd}} \mathbb{P}_0\{\mu[x+\omega](B(0,n)) > 0\}\, dx \\ &\le \int \ell^d(S_n^1) \cdots \ell^d(S_n^p)\, \mathbb{P}_0(d\omega) \\ &\le \Big(\ell^d(B(0,1))\Big)^p \mathbb{E}\Big\{\prod_{i=1}^p \Big(\max_{0 \le s \le 1} |\omega_i(s)| + n\Big)^d\Big\} < \infty\,. \end{aligned}$$

Hence Q is σ-finite. To show the Palm property of P fix a function $G : \mathcal{M}(\mathbb{R}^d) \times \mathbb{R}^d \to [0,\infty]$. For $u \in \mathbb{R}^d$ and $x = (x_1, \dots, x_p) \in \mathbb{R}^p d$ we simply write $x + u$ for the vector $(x_1 + u, \dots, x_p + u)$. Observe that $T^u \mu[x+\omega] = \mu[x - u + \omega]$. Hence, recalling the notation $t = (t_1, \dots, t_p)$,

$$\begin{aligned} &\iint G\Big(T^u\nu, u\Big)\, \nu(du)\, Q(d\nu) \\ &= \int_{\mathbb{R}^{pd}} \int \Big[\int G\Big(\mu[x - u + \omega], u\Big)\, \mu[x+\omega](du)\Big]\, \mathbb{P}_0(d\omega)\, dx \\ &= \int_{\mathbb{R}^{pd}} \int \Big[\int G\Big(\mu[x - x_1 - \omega_1(t_1) + \omega], x_1 + \omega_1(t_1)\Big)\, \lambda_0[x+\omega](dt)\Big]\, \mathbb{P}_0(d\omega)\, dx\,. \end{aligned}$$

For $t = (t_1, \dots, t_p) \in [S_i, T_i]^p$ we write $\omega(t) = (\omega_1(t_1), \dots, \omega_p(t_p))$. By property (iii) of the family λ_y, for every $1 \le i \le p$ and $\lambda_0[x+\omega]$-almost every t,

$$x_1 + \omega_1(t_1) = x_i + \omega_i(t_i)\,,$$

and hence the last expression equals

$$\int_{\mathbb{R}^{pd}} \int \Big[\int G\Big(\mu[\omega - \omega(t)], x_1 + \omega_1(t_1)\Big)\, \lambda_0[x+\omega](dt)\Big]\, \mathbb{P}_0(d\omega)\, dx\,.$$

Observe that $\lambda_y[x+\omega] = \lambda_{y(x)}[\omega]$, where

$$y(x) = (y_1 + x_2 - x_1, \dots, y_{p-1} + x_p - x_{p-1})\,.$$

Hence we may substitute $y = (x_2 - x_1, \dots, x_p - x_{p-1})$ for $(x_2, \dots, x_p)$ and obtain, using Fubini's Theorem and property (iv) of λ_y,

$$\begin{aligned}
&\int_{\mathbb{R}^{pd}} \int \Big[\int G\Big(\mu[\omega - \omega(t)], x_1 + \omega_1(t_1)\Big)\, \lambda_0[x+\omega](dt) \Big]\, \mathbb{P}_0(d\omega)\, dx \\
&\quad = \iint_{\mathbb{R}^d} \int_{\mathbb{R}^{d(p-1)}} \Big[\int G\Big(\mu[\omega - \omega(t)], x_1 + \omega_1(t_1)\Big)\, \lambda_y[\omega](dt) \Big]\, dy\, dx_1\, \mathbb{P}_0(d\omega) \\
&\quad = \iint_{\mathbb{R}^d} \Big[\int_0^1 \cdots \int_0^1 G\Big(\mu[\omega - \omega(s)], x_1 + \omega_1(s_1)\Big)\, ds_p \dots ds_1 \Big]\, dx_1\, \mathbb{P}_0(d\omega)\,.
\end{aligned}$$

Observe now that, for $s_1, \dots, s_p$ fixed, the distribution of the process $\omega_i(t) - \omega_i(s_i)$ and $\omega_i(t - s_i)$ under $\mathbb{P}_0$ coincide. Hence the distribution of $\mu[\omega - \omega(s)]$ and $\mu_{S(y)}^{T(y)}$ coincide for $y_i = 1 - s_i$. Using again Fubini's Theorem and substitutions $u = x_1 + \omega_1(s_1)$ and $y_i = 1 - s_i$,

$$\begin{aligned}
&\iint_{\mathbb{R}^d} \Big[\int_0^1 \cdots \int_0^1 G\Big(\mu[\omega - \omega(s)], x_1 + \omega_1(s_1)\Big)\, ds_p \dots ds_1 \Big]\, dx_1\, \mathbb{P}_0(d\omega) \\
&\quad = \int_0^1 \cdots \int_0^1 \int \Big[\int_{\mathbb{R}^d} G\Big(\mu[\omega - \omega(s)], u\Big)\, du \Big]\, \mathbb{P}_0(d\omega)\, ds_p \dots ds_1 \\
&\quad = \int_0^1 \cdots \int_0^1 \int \Big[\int G\Big(\mu_{S(y)}^{T(y)}[\omega], u\Big)\, du \Big]\, \mathbb{P}_0(d\omega)\, dy_p \dots dy_1 \\
&\quad = \iint_{\mathbb{R}^d} G(\nu, u)\, du\, P(d\nu)\,.
\end{aligned}$$

Altogether, we have shown that

$$\iint G\Big(T^u \nu, u\Big)\, \nu(du)\, Q(d\nu) = \iint_{\mathbb{R}^d} G(\nu, u)\, du\, P(d\nu)\,.$$

Plugging $G(\nu, u) = 1_B(u)$ into this formula also gives

$$\lambda = \frac{1}{\ell^d(B)} \iint 1_B(u)\, \nu(du)\, Q(d\nu) = \frac{1}{\ell^d(B)} \iint_{\mathbb{R}^d} 1_B(u)\, du\, P(d\nu) = 1\,.$$

Hence we have verified formula (9), identifying P as the Palm distribution of Q. $\square$

The Palm distribution P is the principal tool in the proof of the following theorem, which includes a 0-1 law.

Theorem 3.2 *Consider a Borel set $M \subset \mathcal{M}(\mathbb{R}^d) \times \mathbb{R}^d$ with the properties*

- *if $(\nu_1, x) \in M$ and there are $\nu_2 \in \mathcal{M}(\mathbb{R}^d)$ and $\varepsilon > 0$ with $\nu_1 = \nu_2$ on $B(x, \varepsilon)$, then $(\nu_2, x) \in M$,*
- *if $(\nu, x) \in M$, then $(T^u \nu, x - u) \in M$ for all $u \in \mathbb{R}^d$.*

Suppose $X = (X_1, \dots, X_p)$ is an arbitrary random vector with $X_i[\omega] < 0$ and $Y = (Y_1, \dots, Y_p)$ is an arbitrary random vector with $Y_i[\omega] > 0$. Denote by $\tilde{\mu}[\omega]$ the intersection local time with respect to the time domain $\prod_{i=1}^p [X_i, Y_i]$. Then the condition

$$\mathbb{P}_0\Big\{\omega : (\tilde{\mu}[\omega], 0) \in M\Big\} > 0$$

implies that, for every choice x of initial points,

$$P_x\Big\{\mu\,:\,(\mu,y)\in M \text{ for } \mu\text{-almost every point } y\Big\}=1\,.$$

We prepare the proof of this proposition by verifying a formula of Mecke [JM67], see also [UZ88], which characterizes every Palm distribution P.

Lemma 3.3 *Let P be the Palm distribution associated with a stationary random measure Q of finite intensity. Then, for every Borel function $G:\mathcal{M}(\mathbb{R}^d)\times\mathbb{R}^d\to[0,\infty]$, we have*

$$\iint G(\nu,u)\,\nu(du)\,P(d\nu)=\iint G(T^u\nu,-u)\,\nu(du)\,P(d\nu)\,. \tag{10}$$

Proof: Using first (8) and then (9) we infer that, for every $G:\mathcal{M}(\mathbb{R}^d)\times\mathbb{R}^d\to[0,\infty]$ Borel,

$$\begin{aligned}
&\iint G(T^u\nu,-u)\,\nu(du)\,P(d\nu)\\
&=\frac{1}{\ell^d(B)}\iint_B\int G(T^{v+u}\nu,-u)\,T^v\nu(du)\,\nu(dv)\,Q(d\nu)\\
&=\frac{1}{\ell^d(B)}\iiint_B G(T^u\nu,v-u)\,\nu(dv)\,\nu(du)\,Q(d\nu)\\
&=\frac{1}{\ell^d(B)}\iiint_{B-u} G(T^u\nu,v)\,T^u\nu(dv)\,\nu(du)\,Q(d\nu)\\
&=\frac{1}{\ell^d(B)}\iint_{\mathbb{R}^d}\int_{B-u} G(\nu,v)\,\nu(dv)\,du\,P(d\nu)\\
&=\iint G(\nu,u)\,\nu(du)\,P(d\nu)\,.
\end{aligned}$$

□

Proof of Theorem 3.2: In the proof we consider the product space $\Omega_1=\Omega\times[0,1]^p$ endowed with the product measure $\mathbb{P}_1=\mathbb{P}_0\otimes\ell^p$, where ℓ^p is the uniform distribution on $[0,1]^p$ and $\mathbb{P}_0$ is as before. We denote the elements of Ω_1 by (ω,y) and define a family of p independent Brownian motions on our space by $B^i_s(\omega)=\omega_i(s)$. A random measure μ, which is distributed according to our Palm distribution P, and a random measure $\tilde\mu$, as in the theorem, are realized on our space Ω_1 as

$$\mu[\omega,y]=\mu^{T(y)}_{S(y)}[\omega] \text{ and } \tilde\mu[\omega]=\mu^{Y[\omega]}_{X[\omega]}[\omega]\,.$$

For the first step, assume that $\mathbb{P}_0\big\{\omega\,:\,(\tilde\mu[\omega],0)\in M\big\}>0$. We note that, as the set $\{0\}$ is a polar set for each of our p independent Brownian paths, for every $\delta>0$, there exists a (random) $\varepsilon>0$ such that none of the ω_i intersects $B(0,\varepsilon)$ in the time interval $(\delta,Y_i[\omega])$ and $(X_i[\omega],-\delta)$. As the condition $(\tilde\mu[\omega],0)\in M$ depends only on the behaviour of the intersection local time in an arbitrarily small neighbourhood of the origin, we infer from Blumenthal's 0-1-law that

$$\mathbb{P}_0\Big\{\omega\,:\,(\tilde\mu[\omega],0)\in M\Big\}=1\,.$$

By the same argument as above there is a random $\varepsilon > 0$ such that the random measures $\mu[\omega, y]$ and $\tilde{\mu}[\omega]$ coincide on the ball $B(0, \varepsilon)$ and we infer that

$$\mathbb{P}_1\Big\{(\omega, y) \,:\, (\mu[\omega, y], 0) \in M\Big\} = 1\,.$$

As $\mu[\omega, y]$ is Palm distributed we may apply (10) to the function $G(\nu, u) = 1 - 1_M(\nu, u)$. From the second property of M we know that $G(\nu, 0) = G(T^u\nu, -u) = 0$, for every u, and hence

$$\int\!\!\int G(T^u\nu, -u)\,\nu(du)P(d\nu) = \int\!\!\int G(\nu, 0)\,\nu(du)P(d\nu) = 0\,.$$

(10) implies that $\int\!\int G(\nu, u)\,\nu(du)P(d\nu) = 0$, i.e.

$$P\Big\{\mu \,:(\mu, u) \in M \text{ for } \mu\text{-almost every } u\Big\} = 1. \tag{11}$$

We now distinguish two cases: In the first case the given initial points $x_1, \ldots, x_p$ coincide. Then we can obviously assume that this point is the origin. Let $\delta > 0$. We may choose $\varepsilon > 0$ so small that, with probability exceeding $1-\delta$, the paths $\{B_t^2 \,:\, -\varepsilon \le t \le 0\}$ and $\{B_t^2 \,:\, 1-\varepsilon \le t \le 1\}$ do not hit the path $E_\delta := \{B_t^1 \,:\, \delta \le t \le 1-\delta\}$. This event implies that around every $u \in E_\delta$ there is a small neighbourhood on which the measures $\mu[\omega, y]$ coincide for every value of $y \in [1-\varepsilon, 1]^p$. Recall that the event $(\mu, u) \in M$ depends only on the behaviour of μ in an arbitrarily small neighbourhood of u. From (11) and the independence of y and the Brownian motions ω we thus infer that for the intersection local time μ on the time interval $[0, 1]^p$ we have

$$P_x\Big\{\mu \,:\, (\mu, u) \in M \text{ for } \mu\text{-almost all } u \in E_\delta\Big\} \ge 1 - \delta\,.$$

Letting $\delta \downarrow 0$ implies the statement in the first case.

In the case that not all initial points are identical we may assume that $x_1 \ne x_2$. We apply (8) and infer from (11) that

$$Q\Big(\Big\{\mu \,:\, (\mu, u) \in M \text{ for } \mu\text{-almost all } u\Big\}^c\Big) = 0\,.$$

Hence the conclusion of the proposition holds for all initial vectors $x = (x_1, \ldots, x_p)$ outside a set $N \subset \mathbb{R}^{dp}$ of Lebesgue measure zero. We now find, for every $\delta > 0$, some $\varepsilon > 0$ such that, with probability larger than $1 - \delta$, the paths $\{B_t^1 \,:\, 0 \le t \le \varepsilon\}$ and $\{B_t^1 \,:\, 1 \le t \le 1 + \varepsilon\}$ do not intersect the path $\{B_t^2 \,:\, 0 \le t \le 1 + \varepsilon\}$. As P_x-almost surely $(B_\varepsilon^1, \ldots, B_\varepsilon^p) \notin N$, our conclusion holds for the intersection local time measure of the Brownian motions with respect to the interval $[\varepsilon, 1 + \varepsilon]^p$, which coincides with probability at least $1 - \delta$ with the intersection local time measure with respect to the interval $[0, 1]^p$. As $\delta > 0$ was arbitrary, we infer that

$$P_x\Big\{\mu \,:\, (\mu, y) \in M \text{ for } \mu\text{-almost every point } y\Big\} = 1\,,$$

as required to complete the proof. $\square$

Remarks:

- In the remainder of this paper we shall apply Theorem 3.2 to the following Borel subsets of $\mathcal{M}(\mathbb{R}^d) \times \mathbb{R}^d$.

$$
\begin{aligned}
M_2(a) &= \Big\{(\mu,x) : \lim_{\varepsilon\downarrow 0} \frac{1}{\log(1/\varepsilon)} \int_\varepsilon^1 \frac{\mu(B(x,r))}{\varphi(r)}\,\frac{dr}{r} = a\Big\}, \\
M_2^c &= \Big\{(\mu,x) : \lim_{\varepsilon\downarrow 0} \frac{1}{\log(1/\varepsilon)} \int_\varepsilon^1 \frac{\mu(B(x,r))}{\varphi(r)}\,\frac{dr}{r} \text{ fails to exist}\Big\}, \\
M_3(a) &= \Big\{(\mu,x) : \lim_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \frac{\mu(B(x,r))}{\varphi(r)}\,\frac{dr}{r\log(1/r)} = a\Big\}, \\
L_2(\gamma) &= \Big\{(\mu,x) : \lim_{\varepsilon\downarrow 0} \frac{1}{\log(1/\varepsilon)} \int_\varepsilon^1 \delta_{\Big\{\frac{\mu(B(x,r))}{\varphi(r)}\Big\}}\,\frac{dr}{r} = \gamma\Big\}, \\
L_3(\gamma) &= \Big\{(\mu,x) : \lim_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \delta_{\Big\{\frac{\mu(B(x,r))}{\varphi(r)}\Big\}}\,\frac{dr}{r\log(1/r)} = \gamma\Big\}.
\end{aligned}
$$

- In the case $d = 3$, $p = 2$ and $x_1 = x_2$ a more direct approach to the reduction problem, which is inspired by the technique of [LG92], is possible. We believe that this approach is also related to the concept of Palm distributions, yet the precise nature of this relation is unclear. The interested reader may contact N.-R. Shieh for a manuscript on this approach.

4 Proofs for intersections of Brownian paths in space

In this section we complete the proof of Theorem 1.1. Throughout the proof we will rely on the transience of Brownian motion in $\mathbb{R}^3$. We may define the last exit times

$$X_i(r)[\omega] = \inf\{t \le 0 \,:\, \omega_i(t) \in B(0,r)\} \text{ and } Y_i(r)[\omega] = \sup\{t \ge 0 \,:\, \omega_i(t) \in B(0,r)\}\,.$$

We define $X(r) = (X_1(r), X_2(r))$ and $Y(r) = (Y_1(r), Y_2(r))$ with associated random measures $\tilde\mu(r)[\omega]$ as in Theorem 3.2. By Theorem 3.2 it suffices to show that, for $M = M_2(4/\pi)$,

$$\mathbb{P}_0\Big\{\omega \in \Omega \,:\, (\tilde\mu(1)[\omega], 0) \in M\Big\} = 1\,. \tag{12}$$

For this purpose we introduce a group of scaling operators as follows. For every $a > 0$ and $\omega \in \Omega$ or $\omega \in C_3$, we set

$$(\Delta_a\omega)(t) = \frac{\omega(at)}{\sqrt{a}}, \quad t \ge 0. \tag{13}$$

Recall the definition of Ω_0 from Section 2. Let $\omega \in \Omega_0$ and $a > 0$. We claim that, for every pair $S = (S_1, S_2)$, $T = (T_1, T_2)$ of time vectors and every initial vector $x = (x_1, x_2)$ there is a family $\{\lambda_y[x + \Delta_a\omega]\}$ satisfying the conditions (i) and (ii). Indeed, we pick the measures $\lambda_y[\omega]$ with respect to the time vectors $aS = (aS_1, aS_2)$ and $aT = (aT_1, aT_2)$, and we choose

$$\lambda_y[x + \Delta_a\omega](M) = \frac{1}{\sqrt{a}}\,\lambda_{\sqrt{a}y}[\sqrt{a}x + \omega](aM) \text{ for every Borel set } M \subset [S_1, T_1] \times [S_2, T_2]\,.$$

The continuity (i) is clearly satisfied and (ii) follows from the following scaling argument. For all Borel functions $g:\mathbb{R}^3\to[0,\infty]$ and $f:[S_1,T_1]\times[S_2,T_2]\to[0,\infty]$,

$$\begin{aligned}
&\int_{\mathbb{R}^3} g(y)\int f(t_1,t_2)\,\lambda_y[x+\Delta_a\omega](dt_1,dt_2)\,dy\\
&= \frac{1}{\sqrt{a}}\int_{\mathbb{R}^3} g(y)\int f(t_1/a,t_2/a)\,\lambda_{\sqrt{a}y}[\sqrt{a}x+\omega](dt_1,dt_2)\,dy\\
&= \frac{1}{a^2}\int_{\mathbb{R}^3} g(y/\sqrt{a})\int f(t_1/a,t_2/a)\,\lambda_y[\sqrt{a}x+\omega](dt_1,dt_2)\,dy\\
&= \frac{1}{a^2}\int_{aS_1}^{aT_1}\int_{aS_2}^{aT_2} f(s_1/a,s_2/a)\,g\Big(\frac{(\omega_1(s_1)+\sqrt{a}x_1)-(\omega_2(s_2)+\sqrt{a}x_2)}{\sqrt{a}}\Big)\,ds_2\,ds_1\\
&= \int_{S_1}^{T_1}\int_{S_2}^{T_2} f(s_1,s_2)\,g\Big((\Delta_a\omega_1(s_1)+x_1)-(\Delta_a\omega_2(s_2)+x_2)\Big)\,ds_2\,ds_1\,.
\end{aligned}$$

We can therefore define the intersection local times $\tilde{\mu}(r)[\Delta_a\omega]$ for all $\omega\in\Omega_0$ and $a>0$. Observing that the last exit times satisfy

$$X_i(r)[\Delta_a\omega]=\inf\Big\{s\le 0:\big|\omega_i(as)\big|=\sqrt{a}r\Big\}=a^{-1}X_i(\sqrt{a}r)[\omega]$$

and

$$Y_i(r)[\Delta_a\omega]=\sup\Big\{s\ge 0:\big|\omega_i(as)\big|=\sqrt{a}r\Big\}=a^{-1}Y_i(\sqrt{a}r)[\omega]\,,$$

we get the following scaling property

$$\begin{aligned}
\tilde{\mu}(r)[\Delta_a\omega](B(0,r)) &= \frac{1}{\sqrt{a}}\,\lambda_0[\omega]\Big(\{(at_1,at_2)\,:\,\Delta_a\omega_1(t_1)\in B(0,r)\}\Big)\\
&= \frac{1}{\sqrt{a}}\,\lambda_0[\omega]\Big(\{(s_1,s_2)\,:\,\omega_1(s_1)\in B(0,\sqrt{a}r)\}\Big)\\
&= \frac{1}{\sqrt{a}}\,\tilde{\mu}(\sqrt{a}r)[\omega](B(0,\sqrt{a}r))\,. \qquad (14)
\end{aligned}$$

We define

$$\Omega'=\big\{f\,:\,[0,\infty)\to\mathbb{R},\quad f \text{ is monotonically increasing and } f(0)=0\big\}$$

and denote by $\mathcal{F}'$ the σ-algebra generated by the cylinder sets. We now let

$$\Omega_1=\Big\{\omega\in\Omega\,:\ \text{there is } a>0 \text{ such that } \Delta_a\omega\in\Omega_0\Big\}\supset\Omega_0\,.$$

This set is obviously a Δ-invariant set of full measure. Recall that Δ-invariance means that $\omega\in\Omega_1$ implies $\Delta_a\omega\in\Omega_1$ for every $a>0$. It is clear that $r\mapsto\tilde{\mu}(r)[\omega](B(0,r))$ is monotonically increasing for all $\omega\in\Omega_1$ and hence, for every $\omega\in\Omega_1$, the function $H[\omega]:r\mapsto\tilde{\mu}(r)[\omega](B(0,r))$ defines an element of Ω'. Moreover, the mapping $H:(\Omega_1,\mathcal{F}_1)\to(\Omega',\mathcal{F}')$ is measurable, where $\mathcal{F}_1$ denotes the restriction of $\mathcal{F}$ to Ω_1. Define a probability distribution $\mathbb{P}'$ on $(\Omega',\mathcal{F}')$ as the distribution of the random function H, or more precisely, let

$$\mathbb{P}'(A)=\mathbb{P}_0(\{\omega\in\Omega_0\,:\,H[\omega]\in A\})\ ,\text{ for } A\in\mathcal{F}'.$$

We now introduce a second group of scaling operators. For every $a > 0$ and $f \in \Omega'$, we set

$$(\tilde{\Delta}_a f)(r) = \frac{f(\sqrt{a}r)}{\sqrt{a}}, \quad r \geq 0. \tag{15}$$

We also set

$$\tau_s = \Delta_{\exp(s)}, \quad \tilde{\tau}_s = \tilde{\Delta}_{\exp(s)}, \quad -\infty < s < \infty.$$

(14) implies that

$$\tilde{\Delta}_a(H\omega) = H(\Delta_a \omega)\,.$$

This is the flow–homomorphism property, as it has been termed in [BF92, p119]. By definition, $\tau_s = \Delta_{\exp(s)}$ is a measure–preserving flow on $(\Omega_1, \mathcal{F}_1, \mathbb{P}_0)$. It is well known that this flow is ergodic (in fact, this is the ergodicity of the Ornstein-Uhlenbeck stationary process). Hence, by the above flow–homomorphism, $\tilde{\tau}_s = \tilde{\Delta}_{\exp(s)}$ is also an ergodic flow on $(\Omega', \mathcal{F}', \mathbb{P}')$. By Birkhoff's Ergodic Theorem, for $\mathbb{P}'$-almost all f,

$$\lim_{R\to\infty} \frac{1}{R} \int_0^R F(\tilde{\tau}_{-s} f)\, ds = \mathbb{E}' F, \tag{16}$$

whenever $F \in L^1(\mathbb{P}')$, where $\mathbb{E}'$ denotes expectation with respect to $\mathbb{P}'$. We define $F(f) = f(1),\ f \in \Omega'$. Then

$$\mathbb{E}' F = \int_{\Omega'} F(f)\, \mathbb{P}'(df) = \int_{\Omega_0} F(H[\omega])\, \mathbb{P}_0(d\omega) = \mathbb{E} H(1).$$

This value may be explicitly calculated using the formula for the total intersection local time in the unit ball for two one-sided Brownian motions starting at the origin and running till infinity, see [LG87, (2.c)]. Observe that in our case we have to add the contributions of the intersection local times of 4 pairs of one-sided Brownian motions.

$$\mathbb{E} H(1) = 4 \int_{B(0,1)} \big[G(0,y)\big]^2\, dy = \frac{1}{\pi^2} \int_{B(0,1)} \frac{1}{|y|^2}\, dy = \frac{4}{\pi}\,,$$

where $G(x,y) = \frac{1}{2\pi|x-y|}$ is the potential kernel. Altogether we get, $\mathbb{P}_0$-almost surely,

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\log(1/\varepsilon)} \int_\varepsilon^1 \frac{\tilde{\mu}(1)(B(0,r))}{r}\, \frac{dr}{r} = \lim_{R\to\infty} \frac{1}{R} \int_0^R \frac{H(e^{-s/2})}{e^{-s/2}}\, ds = \frac{4}{\pi}\,,$$

which is (12) and hence we obtain Theorem 1.1 by applying Theorem 3.2 to the set $M_2(4/\pi)$.

To obtain the statement about the density distributions it suffices, by Theorem 3.2 applied to the set $L_2(\gamma)$ with the appropriate choice of γ, to consider the limit

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\log(1/\varepsilon)} \int_\varepsilon^1 \delta_{\left\{\frac{\tilde{\mu}(1)[\omega](B(0,r))}{r}\right\}}\, \frac{dr}{r}\,.$$

We choose $F_\lambda(f) = \exp(-\lambda f(1))$ in (16) and get, $\mathbb{P}_0$-almost surely, for all rational $\lambda > 0$,

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\log(1/\varepsilon)} \int_\varepsilon^1 \exp\Big\{-\lambda \tilde{\mu}(B(0,r))/r\Big\}\, \frac{dr}{r} = \lim_{R\to\infty} \frac{1}{R} \int_0^R F_\lambda(\tilde{\tau}_{-s})\, ds = \mathbb{E}' F\,,$$

and by monotonicity this follows for all positive λ. The continuity theorem for Laplace transforms now finishes the proof of the convergence of the density distributions of order two in the case of 3-space.

Remark: The method in this section has also been used "dually" to prove a certain growth condition of Brownian intersection points in [NS97].

5 Proofs for intersections of Brownian paths in the plane

The arguments used in this section are quite natural extensions of the arguments used in [M98b] in the case of occupation measure. We let $X = (X_1, \dots, X_p)$ and $Y = (Y_1, \dots, Y_p)$ be given by the hitting times

$$X_i[\omega] = \sup \Big\{ t < 0 \,:\, |\omega_i(t)| = 1 \Big\} \text{ and } Y_i[\omega] = \inf \Big\{ t > 0 \,:\, |\omega_i(t)| = 1 \Big\} .$$

In order to prove Theorem 1.2 it remains to verify the condition of Theorem 3.2 for the measures $\tilde{\mu}$ coming from this choice of X and Y and the set $M = M_3(2^p)$.

For the moment fix a number $b > 0$ and define $a_n = e^{-bn}$. We define the *crossing numbers* $N_1^i(n)$ as the number of downward crossings of the interval (a_n, a_{n-1}) performed by the process $X_t = |B_t^i|$ for $t \geq 0$ before it first reaches the level 1. Analogously define the crossing numbers $N_2^i(n)$ as the number of downcrossings of (a_n, a_{n-1}) for the process $X_t = |B_{-t}^i|$ for $t \geq 0$. The next lemma collects the necessary facts about the behaviour of the crossing numbers.

Lemma 5.1 *(i) $\mathbb{P}_0$-almost surely, for all $(k_1, \dots, k_p) \in \{1, 2\}^p$,*

$$\lim_{n \to \infty} \frac{1}{\log n} \sum_{m=1}^{n} \frac{1}{m} \, \frac{N_{k_1}^1(m) \cdots N_{k_p}^p(m)}{m^p} = 1 \, .$$

(ii) $\mathbb{P}_0$-almost surely,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \sum \frac{N_{k_1}^1(m) \cdots N_{k_p}^p(m)}{m^p} \textit{ fails to exist,}$$

where the (second) summation is with respect to all $(k_1, \dots, k_p) \in \{1, 2\}^p$.

(iii) $\mathbb{P}_0$-almost surely,

$$\begin{aligned} &\lim_{n \to \infty} \frac{1}{\log n} \sum_{m=1}^{n} \frac{1}{m} \delta_{\{(N_j^i(m)/m : 1 \leq i \leq p \,,\, 1 \leq j \leq 2)\}} \\ &= \int_0^\infty \cdots \int_0^\infty \delta_{\{(a_j^i : 1 \leq i \leq p \,,\, 1 \leq j \leq 2)\}} \prod_{i=1}^{p} \prod_{j=1}^{2} e^{-a_j^i} \, da_j^i \, . \end{aligned}$$

Proof: This may be proved using the arguments in [M98b]. There it is shown that, for all

$(k_1,\ldots,k_p)\in\{1,2\}^p$, $1\le i\le p$ and $\kappa>0$,

$$\lim_{m\to\infty}\mathbb{E}\Big\{\frac{N^i_{k_i}(m)}{m}\Big\}=1 \text{ and } \lim_{m\to\infty}\mathbb{E}\Big\{\exp\Big(-\kappa N^i_{k_i}(m)/m\Big)\Big\}=\frac{1}{1+\kappa}\,.$$

By independence this shows that (i) holds in expectation. By Lemma 3.5 in [M98b], for all $l\ge m>0$, we have $l(m-1)\le\mathbb{E}\{N^i_{k_i}(m)N^i_{k_i}(l)\}\le 2ml$ and

$$1-\frac{m}{2l}\ge\frac{\mathbb{E}\Big\{N^i_{k_i}(m)\Big\}\mathbb{E}\Big\{N^i_{k_i}(l)\Big\}}{\mathbb{E}\Big\{N^i_{k_i}(m)N^i_{k_i}(l)\Big\}}\ge 1-\frac{m}{l}\,.$$

Using independence and taking p-th powers we get, for some constant $C>0$,

$$1-\frac{m}{2l}\ge\frac{\mathbb{E}\Big\{\prod_{i=1}^p N^i_{k_i}(m)\Big\}\mathbb{E}\Big\{\prod_{i=1}^p N^i_{k_i}(l)\Big\}}{\mathbb{E}\Big\{\prod_{i=1}^p N^i_{k_i}(m)N^i_{k_i}(l)\Big\}}\ge 1-C\cdot\frac{m}{l}\,.$$

From this we infer that, for some constants $C_p>0$ and $D_p>0$,

$$\mathrm{Var}\Big\{\frac{1}{\log n}\sum_{m=1}^n\frac{1}{m}\,\frac{N^1_{k_1}(m)\cdots N^p_{k_p}(m)}{m^p}\Big\}\le\frac{2^{p+1}C}{(\log n)^2}\sum_{m=1}^n\sum_{l=m}^n\frac{1}{ml}\,\frac{m}{l}\le\frac{C_p}{\log n}\ \text{ and } \tag{17}$$

$$\mathrm{Var}\Big\{\frac{1}{n}\sum_{m=1}^n\sum\ \frac{N^1_{k_1}(m)\cdots N^p_{k_p}(m)}{m^p}\Big\}\ge\frac{2}{n^2}\sum_{m=2}^n\sum_{l=m}^n\frac{m}{2l}\ge D_p\,. \tag{18}$$

Now we argue as in [M98b]. From (17), Chebyshev's Inequality and the Borel-Cantelli Lemma we infer that (i) holds for the subsequence $n_k=\exp(k^2)$. The monotonicity of the sum, together with the fact that $\lim_{k\to\infty}\log(n_{k+1})/\log(n_k)=1$, then yield (i) for any sequence. For (ii) we observe that, if the expression converged, the limit would necessarily be equal to 2^p by (i) and the consistency of the averaging procedures. As we know from [M98b, Lemma 3.5] that the third moments of the expression in (ii) are bounded, this would imply that the variance (18) converges to 0, a contradiction to (18). Finally, to prove (iii), we recall from Lemma 4.1 in [M98b] that, for all fixed $\kappa_{i,j}>0$,

$$\frac{\mathbb{E}\Big\{\exp(-\kappa_{i,k_i}N^i_{k_i}(m)/m)\Big\}\mathbb{E}\Big\{\exp(-\kappa_{i,k_i}N^i_{k_i}(l)/l)\Big\}}{\mathbb{E}\Big\{\exp(-\kappa_{i,k_i}N^i_{k_i}(m)/m)\exp(-\kappa_{i,k_i}N^i_{k_i}(l)/l)\Big\}}\ge 1-C\cdot\frac{m}{l}\,,$$

and we may argue as in the proof of (i) to get, for all rational $\kappa_{i,j}>0$,

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{m=1}^n\frac{1}{m}\exp\Big\{-\sum_{i=1}^p\sum_{j=1}^2\kappa_{i,j}N^i_j(m)/m\Big\}=\prod_{i=1}^p\prod_{j=1}^2\frac{1}{1+\kappa_{i,j}}\,.$$

The continuity theorem for Laplace transforms now implies (iii). □

In the next lemma we describe the approximation of the intersection local time of small balls by means of the crossing numbers, which follows from the results of [LG87]. The idea of approximating occupation measure of planar Brownian motion in small balls by crossing numbers appeared already in [DR63]. It was first used for intersection local times in [LG87].

For every $0 < r \le 1$ we denote by $A(r, e^b)$ the expected mass of $B(0, r)$ induced by the intersection local time of p independent Brownian paths started at independent uniformly distributed points on the unit sphere and stopped at their first hitting time of the sphere of radius e^b.

Lemma 5.2 $\mathbb{P}_1$*-almost surely,*

$$\tilde{\mu}\Big(B(0, a_n)\Big) = A(1, e^b) \cdot a_n^2 \cdot \sum \prod_{i=1}^{p} N_{k_i}^i(n) + o(n^p a_n^2) \ , \ \textit{as } n \to \infty, \tag{19}$$

where summation is with respect to all $(k_1, \dots, k_p) \in \{1, 2\}^p$.

Proof: Observe that we are dealing with altogether $2p$ Brownian motions $\{B_t^i\}_{t \ge 0}$ and $\{B_{-t}^i\}_{t \ge 0}$ for $i = 1, \dots, p$. Contributions to the intersection local time $\tilde{\mu}[\omega]$ come from the (altogether 2^p) p-tuples of paths with differing superscripts and $\tilde{\mu}[\omega]$ is the sum of the contributions of these p-tuples. Hence it suffices for our proof to consider a single such tuple, say $\{B_t^i\}_{t \ge 0}$ for $i = 1, \dots, p$, and let $\hat{\mu}[\omega] = \mu_0^{Y[\omega]}[\omega]$.

The following inequality is proved in [LG87, Lemma 7] for the case $b = \log 2$, it can be generalized to arbitrary $b > 0$ without further effort: For some $C > 0$ we have

$$\mathbb{E}\Big\{ \Big(\frac{\hat{\mu}(B(0, a_n))}{a_n^2} - A(1, e^b) \prod_{i=1}^{p} N_1^i(n) \Big)^4 \Big\} \le C \cdot n^{4p-2} \,.$$

It follows that

$$\sum_{n=1}^{\infty} \mathbb{E}\Big\{ \Big(\frac{\hat{\mu}(B(0, a_n))}{a_n^2 n^p} - A(1, e^b) \prod_{i=1}^{p} \frac{N_1^i(n)}{n} \Big)^4 \Big\} < \infty \,,$$

from which we infer that, $\mathbb{P}_0$-almost surely,

$$\lim_{n \to \infty} \frac{\hat{\mu}(B(0, a_n))}{a_n^2 n^p} - A(1, e^b) \prod_{i=1}^{p} \frac{N_1^i(n)}{n} = 0 \,,$$

and the desired result follows, as explained in the beginning, by summing the 2^p contributions of this form. □

Lemma 5.3 *For all* $0 < r \le 1$ *we have* $A(r, e^b) = r^2 \pi \cdot (b/\pi)^p$.

Proof: We use the Wiener sausage approximation. By S we denote the unit sphere and by σ

the uniform distribution on S. In this proof we denote by $Z_i[x+\omega]$ the first hitting times of the sphere of radius e^b by the Brownian motions $x+\omega$ and denote the Wiener sausages by

$$S^i_\varepsilon[\omega,x] = \Big\{ y \in \mathbb{R}^2 \,:\, \inf\{|B^i_s - y| \,:\, 0 \le s \le Z_i[x+\omega]\} \le \varepsilon \Big\}\,.$$

For all $0<r<1$, the formulae on p.115 of [LG87] show that

$$\begin{aligned} A(r,e^b) &= \int_{S^p} \lim_{\varepsilon\downarrow 0} \Big(\frac{\log(1/\varepsilon)}{\pi}\Big)^p \mathbb{E}\Big\{ \ell^2\Big(S^1_\varepsilon[\omega,x] \cap \ldots \cap S^p_\varepsilon[\omega,x] \cap B(0,r)\Big)\Big\}\, d\sigma^p(x_1,\ldots,x_p) \\ &= \int_{S^p} \lim_{\varepsilon\downarrow 0} \Big(\frac{\log(1/\varepsilon)}{\pi}\Big)^p \Big[\int_{B(0,r)} \prod_{i=1}^p \mathbb{P}_0(y \in S^i_\varepsilon[\omega,x])\, dy\Big]\, d\sigma^p(x_1,\ldots,x_p) \\ &= \int_{S^p} \int_{B(0,r)} \prod_{i=1}^p G_b(x_i,y)\, dy\, d\sigma^p(x_1,\ldots,x_p)\,, \end{aligned}$$

where

$$G_b(x,y) = \frac{1}{\pi} \log\Big(\frac{|e^b x - e^{-b} y|}{|x-y|}\Big)$$

denotes the Green function for the Laplace equation with boundary value zero on the circle of radius e^b. To evaluate the integral, we differentiate

$$\frac{\partial}{\partial b} \int_S G_b(x_i,y)\, d\sigma(x_i) = \frac{1}{\pi} \int_S \frac{\partial}{\partial b} \log\Big(|e^b x_i - e^{-b} y|\Big)\, d\sigma(x_i) = \frac{1}{\pi} \int_S \frac{1-|e^{-2b}y|^2}{|e^{-2b}y - x_i|^2}\, d\sigma(x_i) = \frac{1}{\pi}\,,$$

using the Poisson integral formula in the last step. Hence $\int_S G_b(x_i,y)\, d\sigma(x_i) = b/\pi$ for every $y \in B(0,r)$ and we conclude that

$$A(r,e^b) = \int_{B(0,r)} \prod_{i=1}^p \int_S G_b(x_i,y)\, d\sigma(x_i)\, dy = r^2\pi \cdot (b/\pi)^p\,,$$

for all $0<r<1$ and, letting $r\to 1$, also for $r=1$. □

Putting together the previous lemmas we achieve the following approximation.

Lemma 5.4 *Let $\delta>0$. Then we can find an arbitrarily small $b>0$ such that for $a_n = e^{-bn}$ and the corresponding crossing numbers $\{N^i_j(n)\}_{n\ge 1}$ there is, $\mathbb{P}_1$-almost surely, an integer N such that, for all $a_{n+1} \le r \le a_n$ and $n > N$,*

$$(1-\delta)\Big[\frac{1}{(n+1)^p} \cdot \sum \prod_{i=1}^p N^i_{k_i}(n+1) - \delta\Big] \le \frac{\tilde\mu(B(0,r))}{r^2\pi(\log(1/r)/\pi)^p} \le (1+\delta)\Big[\frac{1}{n^p} \cdot \sum \prod_{i=1}^p N^i_{k_i}(n) + \delta\Big]\,,$$

where summation is with respect to all $(k_1,\ldots,k_p) \in \{1,2\}^p$.

Proof: Let $\delta>0$ and fix $b>0$ such that $e^{2b} < 1+\delta$ and $e^{-2b} > 1-\delta$. By Lemmas 5.2 and 5.3

we find, $\mathbb{P}_0$-almost surely, an integer N such that, for all $n > N$,

$$\Big|\frac{\tilde\mu(B(0,a_n))}{(bn/\pi)^p a_n^2\pi} - \frac{1}{n^p}\cdot\sum\prod_{i=1}^p N^i_{k_i}(n)\Big| < \delta\,.$$

We conclude that, whenever $a_{n+1} \le r \le a_n$, then

$$\frac{\tilde\mu(B(0,r))}{r^2\pi(\log(1/r)/\pi)^p} \le e^{2b}\cdot\frac{\tilde\mu(B(0,a_n))}{(bn/\pi)^p a_n^2\pi} \le (1+\delta)\cdot\Big(\frac{1}{n^p}\cdot\sum\prod_{i=1}^p N^i_{k_i}(n)+\delta\Big)\,.$$

Analogously, we can prove the reverse estimate and we are done. □

To finish the proof of Theorem 1.2, we choose, for given $\delta > 0$, b as in Lemma 5.4 and pick for small $\varepsilon > 0$ the index n such that $a_{n+1} \le \varepsilon \le a_n$. Using Lemma 5.1(i) we get, $\mathbb{P}_0$-almost surely,

$$\begin{aligned}
&\limsup_{\varepsilon\downarrow 0}\frac{1}{\log\log(1/\varepsilon)}\int_\varepsilon^{1/e}\frac{\tilde\mu(B(0,r))}{r^2\pi(\log(1/r)/\pi)^p}\,\frac{dr}{r\log(1/r)}\\
\le\;& \limsup_{n\to\infty}\frac{1}{\log\log(1/a_n)}\sum_{k=1}^n\int_{a_{k+1}}^{a_k}\frac{\tilde\mu(B(0,r))}{r^2\pi(\log(1/r)/\pi)^p}\,\frac{dr}{r\log(1/r)}\\
\le\;& \lim_{n\to\infty}\frac{1+\delta}{\log n}\Big(\sum_{k=1}^n\frac{1}{k}\sum\prod_{i=1}^p\frac{N^i_{k_i}(k)}{k}\Big)+\delta \le 2^p\cdot(1+\delta)+\delta\,.
\end{aligned}$$

Analogously, we get, $\mathbb{P}_0$-almost surely,

$$\liminf_{\varepsilon\downarrow 0}\frac{1}{\log\log(1/\varepsilon)}\int_\varepsilon^{1/e}\frac{\tilde\mu(B(0,r))}{r^2\pi(\log(1/r)/\pi)^p}\,\frac{dr}{r\log(1/r)} \ge 2^p\cdot(1-\delta)-\delta\,.$$

As $\delta > 0$ was arbitrary, the hypothesis of Theorem 3.2 and hence the result of Theorem 1.2(b) follows.

To show that the average densities of order two fail to exist, we argue by contradiction. Assuming that, for some vector x of initial points, the probability is positive that there is a set of positive measure such that the average density of order two exists for all points from this set, we may infer from Theorem 3.2 for the set $M = M_2^c$ that

$$\mathbb{P}_0\Big\{\omega\,:\,(\tilde\mu[\omega],0)\in M\Big\} = 0\,.$$

In other words, the average density of order two of $\tilde\mu[\omega]$ at 0 exists almost surely. By our previous result and the consistency of averaging procedures this average density must be equal to 2^p and in particular it must be deterministic. We use the approximation of Lemma 5.4 and the fact that, due to scaling, the distribution of the crossing numbers is independent of the choice of $b > 0$ to infer that, $\mathbb{P}_0$-almost surely,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^n\sum\prod_{i=1}^p\frac{N^i_{k_i}(m)}{m} = 2^p\,,$$

where summation is with respect to all $(k_1,\dots,k_p) \in \{1,2\}^p$. This is a contradiction to Lemma 5.1(ii) and hence the proof of Theorem 1.2 is complete.

Finally, in order to prove Theorem 1.3, it suffices, by applying Theorem 3.2 to the set $L_3(\gamma)$, to consider the limit

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \delta_{\left\{\frac{\tilde\mu[\omega](B(0,r))}{r^2\log(1/r)}\right\}} \frac{dr}{r\log(1/r)}\,.$$

Using essentially the same line of argument as above, we apply the approximation given in Lemma 5.4 and infer from the third part of Lemma 5.1(ii) that, P-almost surely,

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \delta_{\left\{\frac{\tilde\mu(B(0,r))}{r^2\pi[\log(1/r)/\pi]^p}\right\}} \frac{dr}{r\,\log(1/r)} = \int_0^\infty\cdots\int_0^\infty \delta_{\{\sum a^1_{k_1}\cdots a^p_{k_p}\}} \prod_{i=1}^p\prod_{j=1}^2 e^{-a^i_j}\,da^i_j\,.$$

We finally observe, from a substitution $a_i = a^i_1 + a^i_2$, that

$$\int_0^\infty\cdots\int_0^\infty \delta_{\{\sum a^1_{k_1}\cdots a^p_{k_p}\}} \prod_{i=1}^p\prod_{j=1}^2 e^{-a^i_j}\,da^i_j = \int_0^\infty\cdots\int_0^\infty \delta_{\{a_1\cdots a_p\}} \prod_{i=1}^p a_i\, e^{-a_i}\,da_i\,,$$

which is the distribution of the product of p independent gamma(2)-distributed random variables. This finishes the proof of Theorem 1.3.

6 Open problems and remarks

- As mentioned in the introduction, the lower density behaviour of the density functions of the intersection local time measures seems to be unknown. It would be interesting to compare a gauge function θ_p such that

$$0 < \liminf_{r\downarrow 0} \frac{\mu_p(B(x,r))}{\theta_p(r)} < \infty \text{ for } \mu_p\text{-almost every } x,$$

  with the gauge functions ψ_p for the upper hull behaviour and the gauge functions φ_p for the average behaviour of $r \mapsto \mu_p(B(x,r))$. However, results of Le Gall and Taylor [LT87] on the occupation measure case $p = 1$ strongly suggest that for the intersection local time of p independent Brownian paths in the plane no such gauge function θ_p exists. In this case it would be interesting to see an integral test as in [LT87].

- In this paper we concentrate on the fractal geometry of the intersections of independent paths. Perhaps the more interesting object is the set of multiple points of a *single* Brownian path. The questions raised in this paper, however, do not make sense in this context, as the canonical measure on this set, the self-intersection local time, is not a locally finite measure. It is unclear how the lacunarity of such a set can be studied.

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