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Small-Signal Stability Analysis for Non-Index 1 Hessenberg Form Systems of Delay Differential-Algebraic Equations

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***Abstract*— The paper focuses on the small-signal stability analysis of systems modelled as differential-algebraic equations and with inclusions of delays in both differential equations and algebraic constraints. The paper considers the general case for which the characteristic equation of the system is a series of infinite terms corresponding to an infinite number of delays. The expression of such a series and the conditions for its convergence are first derived analytically. Then, the effect on small-signal stability analysis is evaluated numerically through a Chebyshev discretization of the characteristic equations. Numerical appraisals focus on hybrid control systems recast into delay algebraic-differential equations as well as a benchmark dynamic power system model with inclusion of long transmission lines.**

***Index Terms*— Time delay, delay differential algebraic equations (DDAE), small-signal stability, long transmission line, Chebyshev discretization.**

I. Introduction

Time delays are intrinsic of a variety of electrical, electronic, and communication systems. For example, several applications of electronic circuits include delays, e.g., [1]–[4]. In control applications, there is a huge variety of examples of neutral-type time-delay systems as well as discrete-continuous hybrid systems that can be regarded as Delay Differential-Algebraic Equations (DDAEs). We cite, for example, [5]–[9]. Another relevant example are power systems with long transmission lines, which, under certain assumptions and approximations, can be modelled as DDAEs [10]–[13]. Moreover, recent developments of wide area control schemes, the higher and higher penetration of distributed generations with decentralized controls and the increased number of measurements based on telecommunication systems (e.g., phasor measurement units) lead to an increasing impact of signal delays on power system dynamic response and operation [14]–[21]. The study of the stability of DDAEs is thus relevant for a large number of real-world applications.

Recent works focus on delay stability margin and on the impact of communication delays for load frequency control (LFC) applications, e.g., [22] and [23]. In particular, in [22], the authors consider an approach based on an iterative method that involves the solution of linear matrix inequality (LMI) problems and allows defining the stability of system with both constant and time-varying delays. In [23], the authors provide an exact method to define the stability margin of systems with one or multiple LFC areas. Both methods above are frequency-domain methods that do not involve the solution of an eigenvalue problem. However, such methods are model- and application-dependent and cannot deal with non index-1 Hessenberg form DDAEs, which are the main objective of this paper.

This paper focuses on the evaluation of the small-signal stability of DDAEs. Delays transform the classical problem of finding the roots of the state matrix of the system at the equilibrium point into the solution of a transcendental characteristic equation, with *infinitely many roots*. In this paper, we show that the characteristic equation of the most general form of DDAEs also have *infinitely many delays* which are multiples of the actual delays that appear in the DDAEs.

This paper utilizes a method based on a Chebyshev discretization of a set of partial differential equations (PDEs) that are equivalent to the original DDAEs [24]–[27] to solve the small-signal stability analysis of DDAEs with multiple delays. The Chebyshev discretization has been successfully applied to power systems modelled as index-1 Hessenberg forms with single [28], [29] and multiple delays [30].

The novel contributions of the paper are the following:

- The derivation of the analytical expression of the characteristic equation of general DDAEs. Such an expression consists of a series whose convergence condition is also defined in the paper. To the best of our knowledge, this is the first attempt to define the small-signal stability of DDAEs which are not index-1 Hessenberg form.
- The derivation of the explicit solution of the condition above for power systems models with inclusion of long transmission lines with delays and with and without attenuation.
- A numerical appraisal based on the Chebyshev discretization method of the approximated solution of characteristic equation deduced in the paper for converging and diverging series.

The remainder of the paper is organized as follows. Section II defines the formulation of delay differential-algebraic equations and derives the expression of the characteristic equation for the most general case. Section III briefly recalls the approach based on the Chebyshev discretization to estimate the spectrum of a DDAE with inclusion of multiple delays. Section IV presents several examples based on hybrid control systems and power system models and particularize the structure of the characteristic equation for such specific DDAEs. Section

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V presents numerical results of the small-signal stability analysis using tow simple continuous-discrete hybrid control system examples and the New England 39-bus 10-machine test system. Conclusions are drawn in Section VI.

II. Small-signal Stability of Delayed DAEs

This section recalls definitions of DDAEs and presents the derivation of the characteristic equation of the general case, which leads to an infinite series of matrices. In the following, the case with only a single delay will be considered. The extension to the multiple delay case is straightforward and is considered in Subsection V-C of the case study.

A. Differential-Algebraic Equations with Delays

Let us recall first conventional DAE models without delays. These are described by the following equations:

$$\begin{aligned} \boldsymbol{x}' &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \\ \boldsymbol{0}_{q,1} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) \end{aligned} \tag{1}$$

where $\boldsymbol{f}$ ($\boldsymbol{f} : \mathbb{R}^{p+q} \mapsto \mathbb{R}^p$) are the differential equations; $\boldsymbol{g}$ ($\boldsymbol{g} : \mathbb{R}^{p+q} \mapsto \mathbb{R}^q$) are the algebraic equations; $\boldsymbol{x}$ ($\boldsymbol{x} \in \mathbb{R}^p$) are the state variables; and $\boldsymbol{y}$ ($\boldsymbol{y} \in \mathbb{R}^q$) are the algebraic variables. We also assume that (1) is autonomous, i.e., does not depend explicitly on time t. With $0_{i,j}$ we denote the zero matrix of i rows and j columns.

The DDAE formulation is obtained by introducing time delays in (1). Let

$$\begin{aligned} \boldsymbol{x}_d &= \boldsymbol{x}(t-\tau) \\ \boldsymbol{y}_d &= \boldsymbol{y}(t-\tau) \end{aligned} \tag{2}$$

be the *retarded* or *delayed* state and algebraic variables, respectively, where t is the current simulation time, and τ ($\tau > 0$) is the time delay. In the remainder of this paper, since the main focus is on small-signal stability analysis, time delays are assumed to be constant.

In the most general case, both $\boldsymbol{x}$ and $\boldsymbol{y}$ appear in both the differential and algebraic equations, $\boldsymbol{f}$ and $\boldsymbol{g}$, respectively. This assumption leads to the following system:

$$\begin{aligned} \boldsymbol{x}' &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}_d, \boldsymbol{y}_d) \\ \boldsymbol{0}_{q,1} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}_d, \boldsymbol{y}_d). \end{aligned} \tag{3}$$

Definition 1. The DDAE (3) is index-1 Hessenberg if $\boldsymbol{g}_y$ is nonsingular, i.e. $\det(\boldsymbol{g}_y) \neq 0$ and $y_d = 0$. Using the implicit function Theorem, y is a function of x, t and can be defined uniquely by the algebraic equation and then replaced in the differential equation. The DDAE (3) is non index-1 Hessenberg if $y_d \neq 0$ or $\boldsymbol{g}_y$ is singular, i.e. $\det(\boldsymbol{g}_y) = 0$.

Note that, in (3), $\boldsymbol{g}$ depends on $\boldsymbol{y}_d$. Hence, (3) is not the index-1 Hessenberg considered in most examples of delay power systems, e.g., [28]–[32]. It is not possible to determine a closed form of the characteristic equation of (3). The determination of such a characteristic equation is the main theoretical contribution of this paper and is discussed in the following subsection.

B. Characteristic Equation of General DDAEs

Assume that a stationary solution of (3) is known and has the form:

$$\begin{aligned} \boldsymbol{0}_{p,1} &= \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{y}_0, \boldsymbol{x}_0, \boldsymbol{y}_0) \\ \boldsymbol{0}_{q,1} &= \boldsymbol{g}(\boldsymbol{x}_0, \boldsymbol{y}_0, \boldsymbol{x}_0, \boldsymbol{y}_0) \end{aligned} \tag{4}$$

where it has been used the fact that, in steady-state, $\boldsymbol{x}_{d0} = \boldsymbol{x}_0$ and $\boldsymbol{y}_{d0} = \boldsymbol{y}_0$. Then, differentiating (3) at the stationary solution yields:

$$\Delta \boldsymbol{x}' = \boldsymbol{f}_{\boldsymbol{x}} \Delta \boldsymbol{x} + \boldsymbol{f}_{\boldsymbol{y}} \Delta \boldsymbol{y} + \boldsymbol{f}_{\boldsymbol{x}_d} \Delta \boldsymbol{x}_d + \boldsymbol{f}_{\boldsymbol{y}_d} \Delta \boldsymbol{y}_d \tag{5}$$

$$\boldsymbol{0}_{q,1} = \boldsymbol{g}_{\boldsymbol{x}} \Delta \boldsymbol{x} + \boldsymbol{g}_{\boldsymbol{y}} \Delta \boldsymbol{y} + \boldsymbol{g}_{\boldsymbol{x}_d} \Delta \boldsymbol{x}_d + \boldsymbol{g}_{\boldsymbol{y}_d} \Delta \boldsymbol{y}_d \tag{6}$$

where, neglecting without loss of generality singularity-induced bifurcation points, it can be assumed that $\boldsymbol{g}_{\boldsymbol{y}}$ is non-singular. This assumption holds in the remainder of this paper.

To define the characteristic equation we use the results of the following proposition.

Proposition 1. The linearized system of DDAEs (3) can be written in the following matrix differential equation with multiple delays:

$$\Delta \boldsymbol{x}' = \boldsymbol{A}_0 \Delta \boldsymbol{x} + \boldsymbol{A}_1 \Delta \boldsymbol{x}_d + \sum_{k=2}^{\infty} [\boldsymbol{A}_k \Delta \boldsymbol{x}(t - k\tau)] \,, \tag{7}$$

where

$$\begin{aligned} \boldsymbol{A}_0 &= \boldsymbol{f}_{\boldsymbol{x}} + \boldsymbol{f}_{\boldsymbol{y}} \boldsymbol{A} \,, && (8) \\ \boldsymbol{A}_1 &= \boldsymbol{f}_{\boldsymbol{x}_d} + \boldsymbol{f}_{\boldsymbol{y}_d} \boldsymbol{A} + \boldsymbol{f}_{\boldsymbol{y}} \boldsymbol{D} \,, && (9) \\ \boldsymbol{A}_k &= \boldsymbol{E} \boldsymbol{C}^{k-2} \boldsymbol{D}, \quad k \geq 2, && (10) \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{A} &= -\boldsymbol{g}_{\boldsymbol{y}}^{-1} \boldsymbol{g}_{\boldsymbol{x}} \,, & \boldsymbol{B} &= -\boldsymbol{g}_{\boldsymbol{y}}^{-1} \boldsymbol{g}_{\boldsymbol{x}_d} \,, \\ \boldsymbol{C} &= -\boldsymbol{g}_{\boldsymbol{y}}^{-1} \boldsymbol{g}_{\boldsymbol{y}_d} \,, & \boldsymbol{D} &= \boldsymbol{B} + \boldsymbol{C} \boldsymbol{A} \,, \\ \boldsymbol{E} &= \boldsymbol{f}_{\boldsymbol{y}} \boldsymbol{C} + \boldsymbol{f}_{\boldsymbol{y}_d} \,. \end{aligned} \tag{11}$$

□

The first matrix $\boldsymbol{A}_0$ is the well-known state matrix that is computed for standard DAEs of the form (1). The other matrices are not null only if the system is of retarded type. The matrix $\boldsymbol{A}_1$ is found in any delay differential equations, while matrices $\boldsymbol{A}_k$ appear specifically in DDAEs. Note that the series stops at $k = 2$ if $\boldsymbol{g}_{\boldsymbol{y}_d}$ is null. This is the case considered in most papers on power system models based on DDAEs, e.g., [28].

The series in (7) converges if and only if $\|\boldsymbol{C}\| < 1$, where $\| \cdot \|$ induced norm, or, equivalently, if and only if $\rho(\boldsymbol{C}) < 1$, where $\rho(\cdot)$ spectral radius of the eigenvalues of a matrix. Moreover, if $\rho(\boldsymbol{C}) < 1$, the matrices $\boldsymbol{A}_k$ tend to $\boldsymbol{0}_{p,p}$ as $k \to \infty$. Hence, based on the definition of $\boldsymbol{A}_k$ in (10), the following condition must hold:

$$\rho(\boldsymbol{C}) = \rho(\boldsymbol{g}_{\boldsymbol{y}}^{-1} \boldsymbol{g}_{\boldsymbol{y}_d}) < 1 \,. \tag{12}$$

Note also that, in (10), we assume that $\boldsymbol{C}^0 = \boldsymbol{I}_p$, hence:

$$\boldsymbol{A}_2 = \boldsymbol{E} \boldsymbol{D} \,. \tag{13}$$

For the sake of completeness, we provide below the proof of Proposition 1.

Proof of Proposition 1. From (6), we have

$$\Delta \boldsymbol{y} = -\boldsymbol{g}_{\boldsymbol{y}}^{-1}\boldsymbol{g}_{\boldsymbol{x}}\Delta\boldsymbol{x} - \boldsymbol{g}_{\boldsymbol{y}}^{-1}\boldsymbol{g}_{\boldsymbol{x}_d}\Delta\boldsymbol{x}_d - \boldsymbol{g}_{\boldsymbol{y}}^{-1}\boldsymbol{g}_{\boldsymbol{y}_d}\Delta\boldsymbol{y}_d\,, \tag{14}$$

or, equivalently,

$$\Delta\boldsymbol{y} = \boldsymbol{A}\Delta\boldsymbol{x} + \boldsymbol{B}\Delta\boldsymbol{x}_d + \boldsymbol{C}\Delta\boldsymbol{y}_d\,. \tag{15}$$

Note that $\Delta\boldsymbol{y}$ depends on $\Delta\boldsymbol{y}_d$, which, based on the same (15), can be written as

$$\Delta\boldsymbol{y}_d = \boldsymbol{A}\Delta\boldsymbol{x}_d + \boldsymbol{B}\Delta\boldsymbol{x}_{dd} + \boldsymbol{C}\Delta\boldsymbol{y}_{dd}\,, \tag{16}$$

where $\boldsymbol{x}_{dd} = \boldsymbol{x}(t-2\tau)$ and $\boldsymbol{y}_{dd} = \boldsymbol{y}(t-2\tau)$. In the same vein, $\Delta\boldsymbol{y}_d$ depends on $\Delta\boldsymbol{y}_{dd}$ and so on. Hence, (15) can be rewritten as follows:

$$\Delta\boldsymbol{y} = \boldsymbol{A}\Delta\boldsymbol{x} + \boldsymbol{D}\Delta\boldsymbol{x}_d + \boldsymbol{C}\boldsymbol{B}\Delta\boldsymbol{x}_{dd} + \boldsymbol{C}^2\Delta\boldsymbol{y}_{dd}\,, \tag{17}$$

or, equivalently,

$$\begin{aligned}\Delta\boldsymbol{y} = \boldsymbol{A}\Delta\boldsymbol{x} + \boldsymbol{D}\Delta\boldsymbol{x}_d + \boldsymbol{C}\boldsymbol{D}\Delta\boldsymbol{x}_{dd} + \\ \boldsymbol{C}^2\boldsymbol{B}\Delta\boldsymbol{x}_{ddd} + \boldsymbol{C}^3\Delta\boldsymbol{y}_{ddd}\,,\end{aligned} \tag{18}$$

or, equivalently,

$$\begin{aligned}\Delta\boldsymbol{y} = \boldsymbol{A}\Delta\boldsymbol{x} + \sum_{k=1}^{n}[\,\boldsymbol{C}^{k-1}\boldsymbol{D}\Delta\boldsymbol{x}(t-k\tau)\,] + \\ \boldsymbol{C}^{n}\boldsymbol{B}\Delta\boldsymbol{x}(t-(n+1)\tau) + \\ \boldsymbol{C}^{n+1}\Delta\boldsymbol{y}(t-(n+1)\tau).\end{aligned} \tag{19}$$

If the condition (12) holds, then

$$\Delta\boldsymbol{y} = \boldsymbol{A}\Delta\boldsymbol{x} + \sum_{k=1}^{\infty}[\boldsymbol{C}^{k-1}\boldsymbol{D}\Delta\boldsymbol{x}(t-k\tau)]\,. \tag{20}$$

Substituting the above expression into (5), we obtain

$$\begin{aligned}\Delta\boldsymbol{x}' = (\boldsymbol{f}_{\boldsymbol{x}} + \boldsymbol{f}_{\boldsymbol{y}}\boldsymbol{A})\Delta\boldsymbol{x} + (\boldsymbol{f}_{\boldsymbol{x}_d} + \boldsymbol{f}_{\boldsymbol{y}_d}\boldsymbol{A})\Delta\boldsymbol{x}_d + \\ \boldsymbol{f}_{\boldsymbol{y}}\sum_{k=1}^{\infty}[\boldsymbol{C}^{k-1}\boldsymbol{D}\Delta\boldsymbol{x}(t-k\tau)] + \\ \boldsymbol{f}_{\boldsymbol{y}_d}\sum_{k=1}^{\infty}[\boldsymbol{C}^{k-1}\boldsymbol{D}\boldsymbol{\Delta}\boldsymbol{x}(t-(k+1)\tau)]\,.\end{aligned} \tag{21}$$

By taking into account that

$$\boldsymbol{f}_{\boldsymbol{y}}\textstyle\sum_{k=1}^{\infty}[\boldsymbol{C}^{k-1}\boldsymbol{D}\Delta\boldsymbol{x}(t-k\tau)] =$$

$$\boldsymbol{f}_{\boldsymbol{y}}\boldsymbol{D}\Delta\boldsymbol{x}_d + \boldsymbol{f}_{\boldsymbol{y}}\textstyle\sum_{k=2}^{\infty}[\boldsymbol{C}^{k-1}\boldsymbol{D}\Delta\boldsymbol{x}(t-k\tau)].$$

and

$$\boldsymbol{f}_{\boldsymbol{y}_d}\textstyle\sum_{k=1}^{\infty}[\boldsymbol{C}^{k-1}\boldsymbol{D}\boldsymbol{\Delta}\boldsymbol{x}(t-(k+1)\tau)] =$$

$$\boldsymbol{f}_{\boldsymbol{y}_d}\textstyle\sum_{k=2}^{\infty}[\boldsymbol{C}^{k-2}\boldsymbol{D}\boldsymbol{\Delta}\boldsymbol{x}(t-k\tau)],$$

the equation (21) takes the form

$$\begin{aligned}\Delta\boldsymbol{x}' = (\boldsymbol{f}_{\boldsymbol{x}} + \boldsymbol{f}_{\boldsymbol{y}}\boldsymbol{A})\Delta\boldsymbol{x} + \\ (\boldsymbol{f}_{\boldsymbol{x}_d} + \boldsymbol{f}_{\boldsymbol{y}_d}\boldsymbol{A} + \boldsymbol{f}_{\boldsymbol{y}}\boldsymbol{D})\Delta\boldsymbol{x}_d + \\ \boldsymbol{f}_{\boldsymbol{y}}\sum_{k=2}^{\infty}[\boldsymbol{C}^{k-1}\boldsymbol{D}\Delta x(t-k\tau)] + \\ \boldsymbol{f}_{\boldsymbol{y}_d}\sum_{k=2}^{\infty}[\boldsymbol{C}^{k-2}\boldsymbol{D}\Delta x(t-k\tau)],\end{aligned} \tag{22}$$

or, equivalently,

$$\begin{aligned}\Delta\boldsymbol{x}' = (\boldsymbol{f}_{\boldsymbol{x}} + \boldsymbol{f}_{\boldsymbol{y}}\boldsymbol{A})\Delta\boldsymbol{x} + \\ (\boldsymbol{f}_{\boldsymbol{x}_d} + \boldsymbol{f}_{\boldsymbol{y}_d}\boldsymbol{A} + \boldsymbol{f}_{\boldsymbol{y}}\boldsymbol{D})\Delta\boldsymbol{x}_d + \\ \sum_{k=2}^{\infty}[\boldsymbol{f}_{\boldsymbol{y}}\boldsymbol{C}\boldsymbol{C}^{k-2}\boldsymbol{D}\Delta x(t-k\tau)] + \\ \sum_{k=2}^{\infty}[\boldsymbol{f}_{\boldsymbol{y}_d}\boldsymbol{C}^{k-2}\boldsymbol{D}\Delta x(t-k\tau)],\end{aligned} \tag{23}$$

or, equivalently,

$$\begin{aligned}\Delta\boldsymbol{x}' = (\boldsymbol{f}_{\boldsymbol{x}} + \boldsymbol{f}_{\boldsymbol{y}}\boldsymbol{A})\Delta\boldsymbol{x} + \\ (\boldsymbol{f}_{\boldsymbol{x}_d} + \boldsymbol{f}_{\boldsymbol{y}_d}\boldsymbol{A} + \boldsymbol{f}_{\boldsymbol{y}}\boldsymbol{D})\Delta\boldsymbol{x}_d + \\ \sum_{k=2}^{\infty}[\boldsymbol{E}\boldsymbol{C}^{k-2}\boldsymbol{D}\Delta x(t-k\tau)]\end{aligned} \tag{24}$$

The proof is complete. □

The *characteristic equation* of (7) is given by

$$\det\,\Delta(\lambda) = 0 \tag{25}$$

where

$$\Delta(\lambda) = \lambda\boldsymbol{I}_p - \boldsymbol{A}_0 - \sum_{k=1}^{\infty} e^{-\lambda k\tau}\boldsymbol{A}_k \tag{26}$$

is the *characteristic matrix* [33]. In (26), $\boldsymbol{I}_p$ is the identity matrix of order p. □

The solutions of (25) are called the *characteristic roots* or *spectrum*, similar to the finite-dimensional case (i.e., the case for which $\boldsymbol{A}_k = \boldsymbol{0}$, $\forall k = 1, \dots, \infty$). As for the finite-dimensional case the stability of (3) can be defined based on the sign of the roots of (25), i.e., the stationary point is stable if all roots have negative real part, and unstable if there exists at least one eigenvalue with positive real part.

Equation (25) is transcendental and, hence, shows infinitely many roots. In general, the explicit solution of (25) is not known and only approximated numerical solutions of a subset of the roots of (25) can be found. The Chebyshev discretization method has proved to be accurate for large systems [30]. For this reason, the Chebyshev discretization discussed in the next section is the numerical method utilized for the case studies presented in this paper.

Equation (26) has been determined for the single-delay case. It is straightforward to generalize (26) for the multi-delay case. Assuming that if (3) includes r delays, say τ_i, $i = 1, 2, \dots, r$, (26) can be rewritten as:

$$\Delta(\lambda) = \lambda\boldsymbol{I}_p - \boldsymbol{A}_0 - \sum_{i=1}^{r}\sum_{k=1}^{\infty} e^{-\lambda k\tau_i}\boldsymbol{A}_{i,k} \tag{27}$$

where $\boldsymbol{A}_{i,k}$ is the matrix associated with delay $k\tau_i$.

III. Chebyshev discretization scheme

This approach consists in transforming the original problem of computing the roots of a retarded functional differential equations into a matrix eigenvalue problem of a PDE system of infinite dimensions. No loss of information is involved in this step. Then the dimension of the PDE is made tractable using a discretization based on a finite element method.

In this paper, we consider the Chebyshev discretization, which has led to excellent results for large DDAEs (see, for example, [30]). The discretized matrix is build as follows. Let Ξ_N be the Chebyshev differentiation matrix of order N (see the Appendix I) and define

$$\boldsymbol{M} = \left[\begin{array}{ccccc} \multicolumn{5}{c}{\hat{\boldsymbol{\Psi}} \otimes \boldsymbol{I}_p} \\ \hline \hat{\boldsymbol{A}}_N & \hat{\boldsymbol{A}}_{N-1} & \dots & \hat{\boldsymbol{A}}_1 & \hat{\boldsymbol{A}}_0 \end{array} \right] , \tag{28}$$

where $\otimes$ indicate the *tensor product* or Kronecker product (see Appendix II); $\boldsymbol{I}_p$ is the identity matrix of order p; and $\hat{\boldsymbol{\Psi}}$ is a matrix composed of the first $N-1$ rows of $\boldsymbol{\Psi}$ defined as follows:

$$\boldsymbol{\Psi} = -2\Xi_N/\tau , \tag{29}$$

and matrices $\hat{\boldsymbol{A}}_0, \dots, \hat{\boldsymbol{A}}_N$ are defined as follows.

Let us consider that the characteristic equation (26) is truncated at the first s terms, i.e., $k = 1, 2, \dots, s$. Hence, the matrices $\boldsymbol{A}_k$ in the sum is associated with delay $\tau_k = k\tau$, with $\tau_1 < \tau_2 < \cdots < \tau_{s-1} < \tau_s$. Let us also assume the general case for which $s \neq N$, i.e., the point of the Chebyshev grid are not the same as the number of matrices $\boldsymbol{A}_k$. This assumption will allow generalize the discussion given in this section to the multi-delay case of (27).

The Chebyshev grid corresponds to a delay $\theta_j = (N - j)\Delta\tau$, with $j = 1, 2, \dots, N$ and $\Delta\tau = \tau_s/(N-1)$. Hence, $j = 1$ corresponds to the state matrix $\boldsymbol{A}_s$, which corresponds to the maximum delay $\tau_s = s\,\tau$; and $j = N$ is taken by the non-delayed state matrix $\boldsymbol{A}_0$. If a delay $\tau_k = \theta_j$ for some $j = 2, \dots, N-1$, then the correspondent matrix $\boldsymbol{A}_k$ takes the position j in the grid. However, in general, the delays of the system will not match the points of the grid. In this case, a linear interpolation is considered in this paper, as follows. Let the time delay τ_k, $k \neq j$, satisfy the condition:

$$\theta_j < \tau_k < \theta_{j+1} . \tag{30}$$

Then, the matrices that will be added to the positions j and $j+1$ are, respectively:

$$\hat{\boldsymbol{A}}_{j,k} = \frac{\tau_k - \theta_j}{\Delta\tau} \boldsymbol{A}_k , \quad \hat{\boldsymbol{A}}_{j+1,k} = \frac{\theta_{j+1} - \tau_k}{\Delta\tau} \boldsymbol{A}_k . \tag{31}$$

Then, the resulting matrix of each point j of the grid is computed as the sum of the contributions of each delay that overlaps that point:

$$\hat{\boldsymbol{A}}_j = \sum_{k \in \Omega_k} \hat{\boldsymbol{A}}_{j,k} , \tag{32}$$

where Ω_k is the set of delays τ_k that satisfies (30). Other more sophisticated interpolation schemes can be used. For example, a Lagrange polynomial interpolation is implemented in [34].

The eigenvalues of $\boldsymbol{M}$ are an approximated spectrum of (25). As it can be expected, the number of points N of the grid affects the precision and the computational burden of the method, as it is further discussed in the case study.

The matrix $\boldsymbol{M}$ is the discretization of a set of PDEs where the continuum is represented by the interval $\xi \in [-\tau_{\max}, 0]$. The continuum is discretized along a grid of N points and the position of such points are defined by the Chebyshev polynomial interpolation.

Figure 1 illustrates the Chebyshev discretization approach through a pictorial representation of matrix $\boldsymbol{M}$. Note that, by construction, $\tau_s = \theta_1$ always holds. The other delays, however, might not match exactly the nodes of the grid.

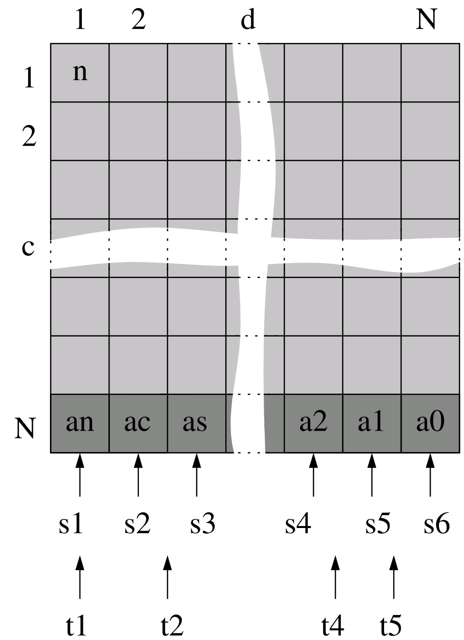

Fig. 1. Representation of the Chebyshev discretization for a system with s delays $\tau_1 < \tau_2 < \cdots < \tau_{s-1} < \tau_s$. In the general case, the delays do not match exactly the grid and, thus, an interpolation between consecutive nodes of the grid is required.

In practice, N cannot be very large, as the size of $\boldsymbol{M}$ would prevent applying any numerical technique to compute the eigenvalues. And, clearly, it is impossible to impose $N \to \infty$. Hence, (26) must be truncated at a certain s, for s sufficiently big to allow properly estimate the spectrum of (26) and sufficiently small to make tractable the solution of the eigenvalue problem. The characteristic matrix that can be evaluated is thus:

$$\Delta(\lambda) \approx \lambda \boldsymbol{I}_p - \boldsymbol{A}_0 - \sum_{k=1}^{s} e^{-\lambda k \tau} \boldsymbol{A}_k \tag{33}$$

for some finite value of s.

The multi-delay case of characteristic equation (27) can be treated just in the same way as the single-delay case above. In this case, the matrices of (27) are $\boldsymbol{A}_0$ (corresponding to the null delay) and r sets, each one composed of s matrices $\boldsymbol{A}_{i,k}$, with $i = 1, 2, \dots, r$ and $k = 1, 2, \dots, s$. The delays associated with these matrices can be arranged in a sequence of $r \cdot s$ terms, corresponding to $r \cdot s$ sorted delays, i.e., $\tau_1 < \tau_2 < \cdots < \tau_{r \cdot s}$. Then the $r \cdot s$ matrices can be accommodated into the Chebyshev grid as discussed above. The resulting truncated characteristic equation of a multi-delay case is as follows

$$\Delta(\lambda) \approx \lambda \boldsymbol{I}_p - \boldsymbol{A}_0 - \sum_{i=1}^{r} \sum_{k=1}^{s} e^{-\lambda k \tau_i} \boldsymbol{A}_{i,k} \tag{34}$$

for some finite value of s.

IV. Applications

This section illustrates applications of the formula deduced in the previous section to neutral-type hybrid systems (Subsection IV-A) as well as to a power system model with inclusion of long transmission lines (Subsection IV-B).

A. Neutral-Type Time-Delay Control Systems

In this section, we illustrate the small-signal stability analysis discussed above through two simple control systems that were originally proposed in [35]. These are examples of hybrid systems that include both continuous and discrete variables. The latter can be reformulated as delayed variables, where the delays are the sampling time of the original discrete variables, as discussed, for example, in [36]. The two systems considered in this subsection are linear and thus allow a straightforward application of (7). These examples are also utilized in the first two case studies included in Section V.

1) DDAE Example 1: Let us assume the following single-delay linear system:

$$(\boldsymbol{y}(t) - \boldsymbol{K}_{22}\boldsymbol{y}(t-\tau))' = \boldsymbol{K}_{11}\boldsymbol{y}(t) + \boldsymbol{K}_{12}\boldsymbol{y}(t-\tau) \,. \tag{35}$$

If we set $\boldsymbol{x}(t) = \boldsymbol{y}(t) - \boldsymbol{K}_{22}\boldsymbol{y}(t-\tau)$, we obtain the following DDAE system:

$$\begin{aligned} \boldsymbol{x}' &= \boldsymbol{K}_{11}\boldsymbol{x} + (\boldsymbol{K}_{11}\boldsymbol{K}_{22} + \boldsymbol{K}_{12})\boldsymbol{y}_d \\ \boldsymbol{0}_{p,1} &= \boldsymbol{x} - \boldsymbol{y} + \boldsymbol{K}_{22}\boldsymbol{y}_d \,. \end{aligned} \tag{36}$$

where:

$$\begin{aligned} \boldsymbol{f_x} &= \boldsymbol{K}_{11} \,, & \boldsymbol{f_{x_d}} &= \boldsymbol{0}_{p,p} \,, \\ \boldsymbol{f_y} &= \boldsymbol{0}_{p,q} \,, & \boldsymbol{f_{y_d}} &= \boldsymbol{K}_{11}\boldsymbol{K}_{22} + \boldsymbol{K}_{12} \,, \\ \boldsymbol{g_x} &= \boldsymbol{I}_p \,, & \boldsymbol{g_{x_d}} &= \boldsymbol{0}_{p,q} \,, \\ \boldsymbol{g_y} &= -\boldsymbol{I}_p \,, & \boldsymbol{g_{y_d}} &= \boldsymbol{K}_{22} \,, \end{aligned}$$

and, according to the notation in (11):

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{I}_p \,, & \boldsymbol{B} &= \boldsymbol{0}_{p,p} \,, \\ \boldsymbol{C} &= \boldsymbol{K}_{22} \,, & \boldsymbol{D} &= \boldsymbol{K}_{22} \,, \\ \boldsymbol{E} &= \boldsymbol{K}_{11}\boldsymbol{K}_{22} + \boldsymbol{K}_{12} \,. \end{aligned} \tag{37}$$

Then $\boldsymbol{A}_0$, $\boldsymbol{A}_1$ and $\boldsymbol{A}_k$, $k \geq 2$, in (8)-(10) take the form

$$\begin{aligned} \boldsymbol{A}_0 &= \boldsymbol{K}_{11} \,, \\ \boldsymbol{A}_1 &= \boldsymbol{K}_{11}\boldsymbol{K}_{22} + \boldsymbol{K}_{12} \,, \\ \boldsymbol{A}_k &= (\boldsymbol{K}_{11}\boldsymbol{K}_{22} + \boldsymbol{K}_{12})(\boldsymbol{K}_{22})^{k-1} \,, \quad k \geq 2 \,. \end{aligned} \tag{38}$$

2) DDAE Example 2: Assume the following single-delay linear system, which is again taken from [35]:

$$\begin{aligned} \boldsymbol{x}'(t) &= \boldsymbol{K}_{11}\boldsymbol{x}(t) + \boldsymbol{K}_{12}\boldsymbol{y}(t) \,, \\ \boldsymbol{y}(t) &= \boldsymbol{K}_{21}\boldsymbol{x}(t) + \boldsymbol{K}_{22}\boldsymbol{y}(t-h) \,, \end{aligned} \tag{39}$$

or, equivalently, using the formulation of (3):

$$\begin{aligned} \boldsymbol{x}' &= \boldsymbol{K}_{11}\boldsymbol{x} + \boldsymbol{K}_{12}\boldsymbol{y} \,, \\ \boldsymbol{0}_{p,1} &= \boldsymbol{K}_{21}\boldsymbol{x} - \boldsymbol{y} + \boldsymbol{K}_{22}\boldsymbol{y}_d \,. \end{aligned} \tag{40}$$

where

$$\begin{aligned} \boldsymbol{f_x} &= \boldsymbol{K}_{11} \,, & \boldsymbol{f_{x_d}} &= \boldsymbol{0}_{p,p} \,, \\ \boldsymbol{f_y} &= \boldsymbol{K}_{12} \,, & \boldsymbol{f_{y_d}} &= \boldsymbol{0}_{p,q} \,, \\ \boldsymbol{g_x} &= \boldsymbol{K}_{21} \,, & \boldsymbol{g_{x_d}} &= \boldsymbol{0}_{q,p} \,, \\ \boldsymbol{g_y} &= -\boldsymbol{I}_p \,, & \boldsymbol{g_{y_d}} &= \boldsymbol{K}_{22} \,, \end{aligned}$$

and according to the notation in (11):

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{K}_{21} \,, & \boldsymbol{B} &= \boldsymbol{0}_{p,p} \,, \\ \boldsymbol{C} &= \boldsymbol{K}_{22} \,, & \boldsymbol{D} &= \boldsymbol{K}_{22}\boldsymbol{K}_{21} \,, \\ \boldsymbol{E} &= \boldsymbol{K}_{12}\boldsymbol{K}_{22} \,. \end{aligned} \tag{41}$$

Then A_0, A_1 and A_k, $k \geq 2$ in (8)-(10) take the form:

$$\begin{aligned} \boldsymbol{A}_0 &= \boldsymbol{K}_{11} + \boldsymbol{K}_{12}\boldsymbol{K}_{21} \,, \\ \boldsymbol{A}_1 &= \boldsymbol{K}_{12}\boldsymbol{K}_{22}\boldsymbol{K}_{21} \,, \\ \boldsymbol{A}_k &= \boldsymbol{K}_{12}(\boldsymbol{K}_{22})^k\boldsymbol{K}_{21} \,, \quad k \geq 2 \,. \end{aligned} \tag{42}$$

According to (12), the series converges if and only if $\rho(\boldsymbol{K}_{22}) < 1$.

B. Power Systems with Long Transmission Lines

In this section, we consider the small-signal stability of a power system model with inclusion of transmission line time delays. Long transmission lines are best modelled through a continuum, which leads to the well-known set of partial differential equations:

$$\begin{aligned} \frac{\partial v(\ell,t)}{\partial \ell} &= Ri(\ell,t) + L\frac{\partial i(\ell,t)}{\partial t} \\ \frac{\partial i(\ell,t)}{\partial \ell} &= Gv(\ell,t) + C\frac{\partial v(\ell,t)}{\partial t} \end{aligned} \tag{43}$$

where R, L, C and G are the resistance, inductance, capacitance and conductance per unit length, respectively.

Equations (43) along with the conditions:

$$\begin{aligned} v(0,t) &= v_i(t), & v(\ell_{ij},t) &= v_j(t) \\ i(0,t) &= i_i(t), & i(\ell_{ij},t) &= i_j(t) = -i_i(t) \end{aligned} \tag{44}$$

where ℓ_{ij} is the total length of the line, define a boundary value problem whose general solution is too complex to be used for systems with hundreds of lines. Thus, some simplifications are required.

The first commonly accepted assumption is to use fast balanced time-varying phasors. The boundary value problem becomes:

$$\begin{aligned} \frac{\partial \bar{v}(\ell,t)}{\partial \ell} &= R\bar{i}(\ell,t) + L\frac{\partial \bar{i}(\ell,t)}{\partial t} + j\omega_0 L\bar{i}(\ell,t) \\ \frac{\partial \bar{i}(\ell,t)}{\partial \ell} &= G\bar{v}(\ell,t) + C\frac{\partial \bar{v}(\ell,t)}{\partial t} + j\omega_0 C\bar{v}(\ell,t) \\ \bar{v}(0,t) &= \bar{v}_i(t), \qquad \bar{v}(\ell_{ij},t) = \bar{v}_j(t) \\ \bar{i}(0,t) &= \bar{i}_i(t), \qquad \bar{i}(\ell_{ij},t) = \bar{i}_j(t) \end{aligned} \tag{45}$$

where ω_0 is the synchronous pulsation.

Assuming $G \approx 0$, the boundary value problem (45) has an explicit solution [12]. Let define the following quantities:

- Characteristic admittance $Y_c = \sqrt{C/L}$.

- Time delay (or *travelling time*) $\tau_{ij} = \ell_{ij}\sqrt{LC}$, i.e., the time required by a wave to pass through the line at the wave speed $1/\sqrt{LC}$.
- Phase shift $\beta_{ij} = \omega_0 \tau_{ij}$ and attenuation factor $\alpha_{ij} = \frac{R\ell_{ij}}{2} Y_c$.

Then, (45) has the solution:

$$\begin{aligned}
0 = &- \bar{i}_i(t) + \bar{i}_i(t - 2\tau_{ij}) e^{-2(\alpha_{ij}+j\beta_{ij})} - Y_c \bar{w}_i(t) \\
&+ 2Y_c \bar{w}_j(t - \tau_{ij}) e^{-(\alpha_{ij}+j\beta_{ij})} \\
&- Y_c \bar{w}_i(t - 2\tau_{ij}) e^{-(\alpha_{ij}+j\beta_{ij})} \\
0 = &- \bar{i}_j(t) + \bar{i}_j(t - 2\tau_{ij}) e^{-2(\alpha_{ij}+j\beta_{ij})} - Y_c \bar{w}_j(t) \\
&+ 2Y_c \bar{w}_i(t - \tau_{ij}) e^{-(\alpha_{ij}+j\beta_{ij})} \\
&- Y_c \bar{w}_j(t - 2\tau_{ij}) e^{-2(\alpha_{ij}+j\beta_{ij})}
\end{aligned} \tag{46}$$

where $\bar{w}_i$ and $\bar{w}_j$ satisfy the following set of complex differential equations:

$$\begin{aligned}
\bar{w}'_i - \bar{v}'_i &= -j\omega_0 \bar{w}_i - \frac{R}{2L}\bar{w}_i + j\omega_0 \bar{v}_i \\
\bar{w}'_j - \bar{v}'_j &= -j\omega_0 \bar{w}_j - \frac{R}{2L}\bar{w}_j + j\omega_0 \bar{v}_j
\end{aligned} \tag{47}$$

If $R \approx 0$ (e.g., loss-less line), equations (46) become:

$$\begin{aligned}
0 = &- \bar{i}_i(t) + \bar{i}_i(t - 2\tau_{ij}) e^{-j2\beta_{ij}} - Y_c \bar{v}_i(t) \\
&+ 2Y_c \bar{v}_j(t - \tau_{ij}) e^{-j\beta_{ij}} - Y_c \bar{v}_i(t - 2\tau_{ij}) e^{-j2\beta_{ij}} \\
0 = &- \bar{i}_j(t) + \bar{i}_j(t - 2\tau_{ij}) e^{-j2\beta_{ij}} - Y_c \bar{v}_j(t) \\
&+ 2Y_{ij} \bar{v}_i(t - \tau_{ij}) e^{-j\beta_{ij}} - Y_c \bar{v}_j(t - 2\tau_{ij}) e^{-j2\beta_{ij}}
\end{aligned} \tag{48}$$

Equations (46)-(47) or (48) are a set of functional differential equations with constant delays. These equations lead to power system models in the form of (3). In particular, assuming that synchronous machines and primary regulators are modelled using conventional delay-free dynamic equations (see, for example, [37]), one obtains:

- Equations (46)-(47) lead to:

$$\begin{aligned}
\boldsymbol{x}' &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \\
\boldsymbol{0}_{q,1} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}_d, \boldsymbol{y}_d).
\end{aligned} \tag{49}$$

  where delayed quantities are transmission line transient voltages $\boldsymbol{x}_d = [\boldsymbol{w}_{\mathrm{re},d}, \boldsymbol{w}_{\mathrm{im},d}]$, with $\bar{\boldsymbol{w}} = \boldsymbol{w}_{\mathrm{re}} + j\boldsymbol{w}_{\mathrm{im}}$ and currents $\boldsymbol{y}_d = [\boldsymbol{i}_{\mathrm{re},d}, \boldsymbol{i}_{\mathrm{im},d}]$, with $\bar{\boldsymbol{i}} = \boldsymbol{i}_{\mathrm{re}} + j\boldsymbol{i}_{\mathrm{im}}$.
- Equations (48) lead to:

$$\begin{aligned}
\boldsymbol{x}' &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \\
\boldsymbol{0}_{q,1} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}_d).
\end{aligned} \tag{50}$$

  where delayed quantities are only transmission line currents $\boldsymbol{y}_d = [\boldsymbol{i}_{\mathrm{re},d}, \boldsymbol{i}_{\mathrm{im},d}]$.

Note that, in (49) and (50) the complex expressions (46)-(47) and (48), respectively, are split into their real and imaginary parts to obtain a set of real equations.

From observing (49), we have, for each delay τ_{ij}:

$$\begin{aligned}
&\boldsymbol{f}_{\boldsymbol{x}} \neq \boldsymbol{0}_{p,p}\,, \quad \boldsymbol{f}_{\boldsymbol{x}_{d,ij}} = \boldsymbol{0}_{p,p}\,, \quad \boldsymbol{f}_{\boldsymbol{y}} \neq \boldsymbol{0}_{p,q}\,, \quad \boldsymbol{f}_{\boldsymbol{y}_{d,ij}} = \boldsymbol{0}_{p,q}\,, \\
&\boldsymbol{g}_{\boldsymbol{x}} \neq \boldsymbol{0}_{q,p}\,, \quad \boldsymbol{g}_{\boldsymbol{x}_{d,ij}} \neq \boldsymbol{0}_{q,p}\,, \quad \boldsymbol{g}_{\boldsymbol{y}} \neq \boldsymbol{0}_{q,q}\,, \quad \boldsymbol{g}_{\boldsymbol{y}_{d,ij}} \neq \boldsymbol{0}_{q,q}\,.
\end{aligned}$$

and, substituting in (10), we obtain:

$$\boldsymbol{A}_{k,ij} = \boldsymbol{f}_{\boldsymbol{y}} \boldsymbol{C}_{ij}^{k-1} \boldsymbol{D}_{ij}, \quad k \geq 1\,, \quad ij \in \Omega_{ij}\,. \tag{51}$$

where Ω_{ij} is the set of transmission lines. Hence, this is a multi-delay system – delays τ_{ij} are as many as the transmission lines – and each delay generates an infinite series of non-null matrices $\boldsymbol{A}_{k,ij}$ associated to delays $k\tau_{ij}$, $k \geq 1$. Such matrices converge to $\boldsymbol{0}_{p,p}$ if and only if $\rho(\boldsymbol{C}_{ij}) < 1$ for $ij \in \Omega_{ij}$.

Loss-less line model (50) shows same Jacobian matrices as (49) but for $\boldsymbol{g}_{\boldsymbol{x}_{d,ij}} = \boldsymbol{0}_{q,p}$, which leads to:

$$\boldsymbol{A}_{k,ij} = \boldsymbol{f}_{\boldsymbol{y}} \boldsymbol{C}_{ij}^{k} \boldsymbol{A}_{ij}, \quad k \geq 1\,, \quad ij \in \Omega_{ij}\,. \tag{52}$$

To complete this section, we provide a proposition and a corollary and their proofs regarding the spectral radii of models (49) and (50), respectively, discussed above.

Proposition 2. The spectral radius of transmission lines modelled with (46)-(47) satisfy the condition

$$\rho(\boldsymbol{C}_{ij}) = e^{-2\alpha_{ij}}\,. \tag{53}$$

Proof of Proposition 2. Let us consider the structure of equations (46)-(47) and their Jacobian matrices $\boldsymbol{g}_{\boldsymbol{y}}$ and $\boldsymbol{g}_{\boldsymbol{y}_d}$. Current injections $\bar{i}_i$ and $\bar{i}_j$ are the only algebraic variables in (46) and appear as instantaneous and as delayed quantities, with delay $2\tau_{ij}$. While instantaneous currents also appear in other equations, at least in the current balances of networks buses, delayed currents do not appear in any other equation of the system. Let us assume that (46) are split into their real and imaginary parts, and write Jacobian matrices $\boldsymbol{g}_{\boldsymbol{y}}$ and $\boldsymbol{g}_{\boldsymbol{y}_d}$ by separating the terms depending on the real and imaginary parts of $\bar{i}_i$ and $\bar{i}_j$, respectively. Then, we obtain:

$$\boldsymbol{g}_{\boldsymbol{y}} = \begin{bmatrix} \boldsymbol{J}_{q-4,q-4} & \boldsymbol{J}_{q-4,4} \\ \boldsymbol{0}_{4,q-4} & -\boldsymbol{I}_4 \end{bmatrix} \tag{54}$$

$$\boldsymbol{g}_{\boldsymbol{y}_d} = \begin{bmatrix} \boldsymbol{0}_{q-4,q-4} & \boldsymbol{0}_{q-4,4} \\ \boldsymbol{0}_{4,q-4} & \boldsymbol{H}_4 \end{bmatrix} \tag{55}$$

where $\boldsymbol{J}_{q-4,q-4}$ and $\boldsymbol{J}_{4,q-4}$ are sparse, non-null matrices and

$$\boldsymbol{H}_4 = \begin{bmatrix} \boldsymbol{H}_2 & \boldsymbol{0}_{2,2} \\ \boldsymbol{0}_{2,2} & \boldsymbol{H}_2 \end{bmatrix} \tag{56}$$

where

$$\boldsymbol{H}_2 = \begin{bmatrix} e^{-2\alpha_{ij}}\cos(2\beta_{ij}) & e^{-2\alpha_{ij}}\sin(2\beta_{ij}) \\ -e^{-2\alpha_{ij}}\sin(2\beta_{ij}) & e^{-2\alpha_{ij}}\cos(2\beta_{ij}) \end{bmatrix} \tag{57}$$

Given the structure of the Jacobian matrices above, we have:

$$\boldsymbol{C}_{ij} = -\boldsymbol{g}_{\boldsymbol{y}}^{-1}\boldsymbol{g}_{\boldsymbol{y}_d} = \begin{bmatrix} \boldsymbol{0}_{q-4,q-4} & \hat{\boldsymbol{J}}_{q-4,4} \\ \boldsymbol{0}_{4,q-4} & \boldsymbol{H}_4 \end{bmatrix} \tag{58}$$

where $\hat{\boldsymbol{J}}_{q-4,4}$ is a sparse, non-null matrix. Hence $\boldsymbol{C}_{ij}$ has $q-4$ zero eigenvalues and two pairs of complex eigenvalues equal to $e^{-2(\alpha_{ij} \pm j\beta_{ij})}$. The proof is complete. □

Corollary 1. The spectral radius of transmission lines modelled with (48) satisfies the condition $\rho(\boldsymbol{C}_{ij}) = 1$.

Proof of Corollary 1. Loss-less transmission lines have $R = 0$, and, by the definition of the attenuation factor, $\alpha_{ij} = 0$. Hence, using Proposition 2, we obtain $\rho(\boldsymbol{C}_{ij}) = e^{0}$. The proof is complete. □

V. Case Studies

In this section, we utilize the DDAE systems discussed above to illustrate, through numerical examples, the impact on small-signal stability of the series given in (7). With this aim, we consider four relevant relevant cases: $\rho(\boldsymbol{C}) \ll 1$; $\rho(\boldsymbol{C}) > 1$; $\rho(\boldsymbol{C}) = 1$; and $\rho(\boldsymbol{C}) < 1$ with $\rho(\boldsymbol{C}) \approx 1$. The first two examples are based on the control systems discussed in Subsections IV-A.1 and IV-A.2, respectively, while the last two examples are based on the power system models with delay line models discussed in Subsection IV-B.

A. *DDAE with* $\rho(\boldsymbol{C}) < 1$

In this subsection, we study the numerical appraisal of the small-signal stability analysis of a DDAE system for which $\rho(\boldsymbol{A}_k) < 1$. With this aim, let us consider the DDAE system (36) described in Subsection IV-A.1. We assume the following parameters:

$$\boldsymbol{K}_{11} = \begin{bmatrix} -5 & 1 & 1 \\ 0 & -5 & 1 \\ 1 & 0 & -5 \end{bmatrix}, \quad \boldsymbol{K}_{12} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix},$$

$$\boldsymbol{K}_{22} = \begin{bmatrix} 0 & 0.5 & 1 \\ 0.1 & 0.1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The equilibrium point $(\boldsymbol{x}, \boldsymbol{y}) = (\mathbf{0}_{p,1}, \mathbf{0}_{q,1})$ is stable for $\tau = 0$. The question is whether the system is stable for $\tau > 0$. Let us assume that $\tau = 0.005$ s. Note that $\rho(\boldsymbol{K}_{22}) = 0.279 < 1$, hence $\boldsymbol{A}_k$ satisfies condition (12) and converges to $\mathbf{0}_{p,p}$ for $k \to \infty$. The rightmost roots of (25) are thus expected to converge to a constant value for sufficiently high values of k. This fact is confirmed by the results shown in Fig. 2 that depicts the three rightmost roots of (25). The eigenvalues have been computed using $N = 80$ for the Chebyshev discretization, which leads to a 120×120 matrix $\boldsymbol{M}$ defined in (28). Moreover, we have used (33) for $s \in [2, 40]$. Figure 2 indicates that, in this case, if $s < 10$, the stability margin of the system, i.e., the distance of the rightmost eigenvalues from the imaginary axis, as well as the dominant oscillation mode can be overestimated. Finally, the rightmost eigenvalues vary within a tolerance of less than 10^{-5} for $s > 35$.

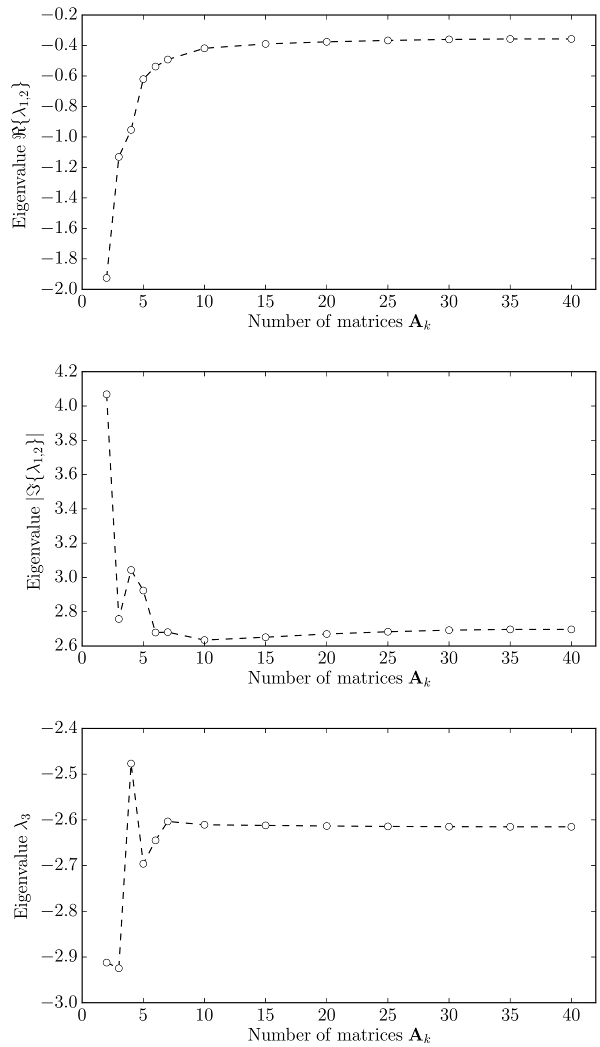

Fig. 2. Three rightmost eigenvalues of (36) for $\tau = 0.005$ s.

B. *DDAE with* $\rho(\boldsymbol{C}) > 1$

In this subsection, we consider the DDAE system described by (40) discussed in Subsection IV-A.2 to illustrate the behaviour of the eigenvalues for an equilibrium point for which $\rho(\boldsymbol{A}_k) > 1$. We assume the following matrices:

$$\boldsymbol{K}_{11} = \begin{bmatrix} -20 & 0 & 10 \\ -10 & -25 & 0 \\ 10 & 1 & -20 \end{bmatrix}, \boldsymbol{K}_{12} = \begin{bmatrix} 0.5 & -1 & -1 \\ 0.5 & 1 & 0 \\ 1 & 0 & 0.5 \end{bmatrix},$$

$$\boldsymbol{K}_{21} = \begin{bmatrix} 0.5 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0.5 & 1 \end{bmatrix}, \boldsymbol{K}_{22} = \begin{bmatrix} 0.9 & 0 & 1 \\ 1 & 0.6 & 0 \\ 0 & -0.1 & 1 \end{bmatrix}.$$

Note that $\rho(\boldsymbol{K}_{22}) = 1.147 > 1$, hence $\boldsymbol{A}_k$ will not converge to $\mathbf{0}_{p,p}$ for $k \to \infty$, so the impact on the system of the delay increases as one increases the value of s in the truncated characteristic matrix (33). For $\tau = 0$, the equilibrium point $(\boldsymbol{x}, \boldsymbol{y}) = (\mathbf{0}_{p,1}, \mathbf{0}_{q,1})$ is stable, but, since the eigenvalues of $\boldsymbol{K}_{22}$ are positive, we expect that the system is actually unstable for $\tau > 0$. The real and imaginary parts of the rightmost eigenvalue are depicted in Fig. 3. Simulations are solved using $\tau = 0.001$ s and $N = 80$ for the Chebyshev discretization grid. The system instability becomes apparent by properly increasing s, in this case for $s \geq 12$. Clearly, the value of s for which the instability of the system is revealed depends on τ.

C. *New England 39-bus System with Long Transmission Lines*

In this subsection, the IEEE 39-bus system is utilized to illustrate the effect of transmission line delays. Base case

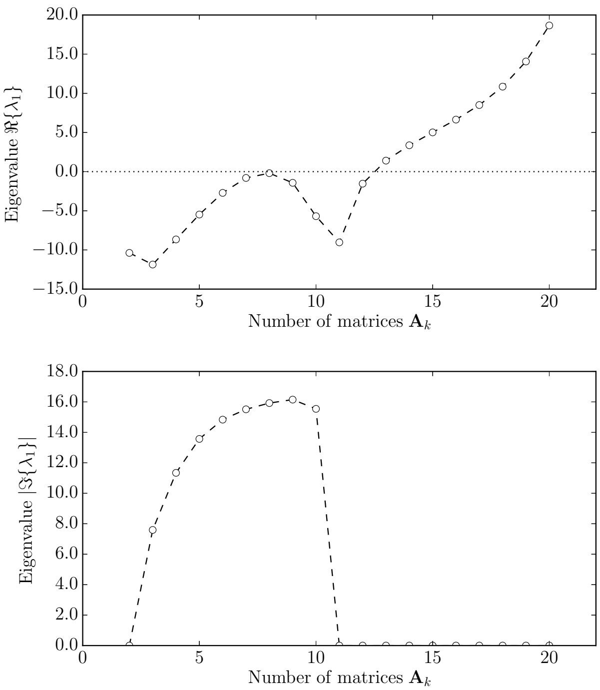

Fig. 3. Real and imaginary parts of the rightmost eigenvalues of (40) for $\tau = 0.001$ s.

TABLE I
LENGTHS, ATTENUATION FACTORS AND DELAYS OF FOUR TRANSMISSION LINES OF THE NEW ENGLAND 39-BUS SYSTEM

Line i-j	ℓ_{ij} [km]	α_{ij} [Np]	$e^{-\alpha_{ij}}$	$e^{-2\alpha_{ij}}$	τ_{ij} [ms]
1-2	276	0.0179	0.9822	0.9648	1.131
23-24	234	0.0152	0.9849	0.9700	0.962
25-26	216	0.0141	0.9860	0.9723	0.887
26-29	418	0.0271	0.9732	0.9472	1.713

dynamic data[1] have been modified as follows: (i) machine inertias are reduced by ten times; (ii) AVR amplifier control gains are reduced by ten times; and (iii) no PSS controller is included.

Table I indicates the four lines that are considered "long" and their lengths. For all these lines, the following per unit-length parameters are assumed: $R = 0.037$ Ω/km, $X = \omega_0 L = 0.367$ Ω/km, and $B = \omega_0 C = 4.518$ μS/km. These are typical data of 345 kV lines as given in [38] and lead to an attenuation factor $\hat{\alpha}_{ij} = 0.00066$ Np/km and phase shift $\hat{\beta}_{ij} = 0.00129$ rad/km. The total attenuation factors $\alpha_{ij} = \hat{\alpha}_{ij} \cdot \ell_{ij}$, coefficients $e^{-\alpha_{ij}}$ and time delays τ_{ij} are indicated in Table I.

We consider three cases:

- Standard lumped models of transmission lines, which lead to a conventional non-delayed DAE as in (1);[2]
- Dynamic delay models of the lines with attenuation, i.e., models (46)-(47), which lead to the DDAE as in (49);
- Delay loss-less line models (48), which lead to the DDAE as in (50).

[1]Power flow and dynamic data of the New England 39-bus system can be easily found on the internet. For example, data in PSS/E format are available at: `http://electrica.uc3m.es/pablole/new_england.html`.

TABLE II
STATISTICS OF THE THREE DYNAMIC MODELS OF THE NEW ENGLAND 39-BUS SYSTEM

	DAE (1)	DDAE (49)	DDAE (50)
# of states, p	70	70	86
# of alg. vars, q	190	222	222
# of eigs., $p \cdot N$	70	700	860
# of delays	-	51	204

Since we assume four long lines, the system includes multiple delays and $r = 4$. Then, to compute the truncated characteristic equation (34), we have assumed $s = 50$. Hence, the total number of terms in (34) are $r \cdot s = 200$. Finally, we have used $N = 10$ to generate the Chebyshev grid. Statistics for the three cases above are given in Table II. Note that the number of total delays considered is not simply a multiple of s, as delays that differs less than a given tolerance are merged into a single delay and delays associated to $\boldsymbol{A}_k$ whose elements have a maximum absolute value below a given threshold (10^{-18} in the simulations solved for this case study) are neglected.

The spectral radii of each matrix $\boldsymbol{C}_{ij}$ associated with each delay are indicated in Table III. As expected, the lines with attenuation show a spectral radius $\rho < 1$. On the other hand, loss-lines have $\rho = 1$. This is thus a special case for which the matrices $\boldsymbol{A}_k$ do not converge to $\mathbf{0}_{p,p}$. Even if the series in (7) does not converge, the values of matrices $\boldsymbol{A}_k$ are sufficiently small not to concern the stability of the system. In the case of line models with delay and attenuation, spectral radii satisfy the condition $\rho < 1$, however, the attenuation is small, and $\rho \approx 1$, so the convergence of matrices $\boldsymbol{A}_k$ is slow. Also in this case, the impact of $\boldsymbol{A}_k$ on the rightmost eigenvalues of the system is negligible well before $\boldsymbol{A}_k$ vanishes. In this case study, for $s = 50$, the error on the rightmost eigenvalues is lower than 10^{-5}. Finally, note that numerical results confirms Proposition 2 given in Subsection IV-B. In fact, the values in the fifth column of Table I and in the second column of Table III are the same.

Table IV shows the 15 rightmost pairs of complex eigenvalues for the three cases above. Note that the impact of line models is not negligible. The three models show the same numbers of poorly damped oscillations modes, however, the models with delays show, in general, a slightly lower damping, in particular the case considering loss-less delay

[2]The lumped model of transmission lines is the standard π-circuit where the series impedance and shunt admittance are evaluated based on per-unit length parameters, as follows:

$$\bar{Z}_{ij} = \frac{1}{Y_c}\sinh(\bar{\gamma}_{ij}\ell_{ij}), \quad \bar{Y}_{ij} = Y_c \tanh(0.5\bar{\gamma}_{ij}\ell_{ij}),$$

where

$$\bar{\gamma} = \sqrt{\frac{R + j\omega_0 L}{G + j\omega_0 C}}.$$

TABLE III

SPECTRAL RADII $\rho(\boldsymbol{C}_{ij})$ OF FOUR TRANSMISSION LINES OF THE NEW ENGLAND 39-BUS SYSTEM

Line i-j	Model (46)-(47)	Model (48)
1-2	0.9648	1.0000
23-24	0.9700	1.0000
25-26	0.9723	1.0000
26-29	0.9472	1.0000

TABLE IV

15 RIGHTMOST COMPLEX PAIRS OF EIGENVALUES FOR THE NEW ENGLAND 39-BUS SYSTEM AND DIFFERENT LINE MODELS

Lumped model	Model (46)-(47)	Model (48)
$-0.1326 \pm j14.250$	$-0.1209 \pm j0.7791$	$-0.1314 \pm j0.7290$
$-0.1505 \pm j0.7320$	$-0.1529 \pm j14.423$	$-0.1454 \pm j15.017$
$-0.1851 \pm j0.2378$	$-0.1765 \pm j0.2383$	$-0.1810 \pm j0.2379$
$-0.1909 \pm j16.385$	$-0.2010 \pm j0.3703$	$-0.1934 \pm j0.3489$
$-0.2067 \pm j0.3473$	$-0.2019 \pm j0.3443$	$-0.1943 \pm j0.5219$
$-0.2119 \pm j0.3662$	$-0.2075 \pm j16.547$	$-0.1971 \pm j0.3724$
$-0.2262 \pm j0.2149$	$-0.2079 \pm j0.3311$	$-0.1990 \pm j0.2169$
$-0.2293 \pm j0.5186$	$-0.2105 \pm j0.2167$	$-0.2050 \pm j0.3264$
$-0.2306 \pm j0.3343$	$-0.2120 \pm j0.5240$	$-0.2127 \pm j0.1967$
$-0.2405 \pm j0.2293$	$-0.2191 \pm j0.2323$	$-0.2130 \pm j0.2307$
$-0.2501 \pm j0.1802$	$-0.2282 \pm j0.1909$	$-0.2277 \pm j16.775$
$-0.2523 \pm j0.4649$	$-0.2510 \pm j0.4674$	$-0.2365 \pm j20.883$
$-0.2810 \pm j24.242$	$-0.2831 \pm j24.396$	$-0.2507 \pm j0.4665$
$-0.2842 \pm j22.897$	$-0.2841 \pm j22.659$	$-0.2910 \pm j24.622$
$-0.3355 \pm j20.643$	$-0.3186 \pm j20.658$	$-0.2921 \pm j22.639$

line models. Note also that, without a proper small-signal stability analysis, the only way to determine the behaviour of the system would be a time domain integration, which, for DDAEs, is particularly involved as it requires computationally demanding implicit integration techniques, e.g., Lobatto IIIC. The interested reader can find more details on the numerical integration of DDAEs in [25].

VI. CONCLUSIONS

The paper provides a derivation of the characteristic equation of DDAEs. This is found to be a series of infinite terms associated with infinite delays, which are multiples of the delay of the DDAE. The condition for the convergence of this series are also provided in the paper. Then the paper discusses the convergence condition of the characteristic equation for power system models with inclusion of long transmission lines and derives an explicit formula. The paper also provides a numerical appraisals based on a Chebyshev discretization methods of the small-signal stability analysis based on truncated version of the characteristic equation previously determined and defines how the convergence of the series impact on the stability of neutral-type hybrid control and power systems.

Future work will focus on the evaluation of the small-signal stability analysis of hybrid systems combining continuous and discrete (e.g., digital) variables and, in particular, on the dynamic interaction of communication and power systems.

APPENDIX I
CHEBYSHEV DIFFERENTIATION MATRIX

Chebyshev's differentiation matrix $\boldsymbol{\Xi}_N$ of dimensions $N+1 \times N+1$ is defined as follows. Firstly, one has to define $N+1$ Chebyshev nodes, i.e., the interpolation points on the normalized interval $[-1, 1]$:

$$x_k = \cos\left(\frac{k\pi}{N}\right), \quad k = 0, \dots, N. \tag{59}$$

Then, the element (i, j) differentiation matrix $\boldsymbol{\Xi}_N$ indexed from 0 to N is defined as [39]:

$$\boldsymbol{\Xi}_{(i,j)} = \begin{cases} \frac{\xi_i(-1)^{i+j}}{\xi_j(x_i - x_j)}, & i \neq j \\ -\frac{1}{2}\frac{x_i}{1-x_i^2}, & i = j \neq 1, N-1 \\ \frac{2N^2+1}{6}, & i = j = 0 \\ -\frac{2N^2+1}{6}, & i = j = N \end{cases} \tag{60}$$

where $\xi_0 = \xi_N = 2$ and $\xi_2 = \cdots = \xi_{N-1} = 1$. For example, $\boldsymbol{\Xi}_1$ and $\boldsymbol{\Xi}_2$ are:

$$\boldsymbol{\Xi}_1 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}, \quad \text{with } x_0 = 1,\ x_1 = -1\ .$$

and

$$\boldsymbol{\Xi}_2 = \begin{bmatrix} \frac{3}{2} & -2 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix}, \text{ with } x_0 = 1,\ x_1 = 0,\ x_2 = -1\ .$$

APPENDIX II
KRONECKER PRODUCT

If $\boldsymbol{A}$ is a $m \times n$ matrix and $\boldsymbol{B}$ is a $p \times q$ matrix, then Kronecker product $\boldsymbol{A} \otimes \boldsymbol{B}$ is an $mp \times nq$ block matrix [40], as follows:

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{bmatrix} a_{11}\boldsymbol{B} & \cdots & a_{1n}\boldsymbol{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\boldsymbol{B} & \cdots & a_{mn}\boldsymbol{B} \end{bmatrix} \tag{61}$$

For example, let $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$. Then:

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{bmatrix} \boldsymbol{B} & 2\boldsymbol{B} & 3\boldsymbol{B} \\ 2\boldsymbol{B} & 2\boldsymbol{B} & \boldsymbol{B} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 & 2 & 6 & 3 \\ 2 & 3 & 4 & 6 & 6 & 9 \\ 6 & 3 & 4 & 2 & 2 & 1 \\ 6 & 9 & 4 & 6 & 2 & 3 \end{bmatrix}$$

Observe that $\boldsymbol{A} \otimes \boldsymbol{B} \neq \boldsymbol{B} \otimes \boldsymbol{A}$.

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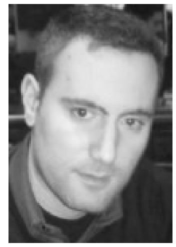

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