Letters to the Editor

On the Flow of a Perfect Fluid between Two Plates from a Circular Inlet Channel

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In a Letter to the Editor in the November 1948 Journal of Applied Physics, p. 1092, E. T. Benedikt gives a definite integral formula for the velocity potential ϕ of an ideal incompressible, non-viscous fluid flowing from the region between two thick parallel plates a distance h apart into a circular cylindrical hole of radius r_0 in one of them. For some purposes it may be more convenient to express his result as a series. This could be done by contour integration, but it is shorter to start over. The velocity potential $d\phi$ of a ring source of linear strength $2qdr'$ at $z=0$, $r=r'$ between two plates at $z=h$ and $z=-h$ is found by the usual method[1] to be, when $r<r'$,

$$d\phi_1=(qr'dr'/h)[\ln r'+2+\sum_{n=1}^{\infty} I_0(Nr)K_0(Nr')\cos Nz], \quad (1)$$

where $I_0(x)$ and $K_0(x)$ are modified Bessel Functions, and N is $n\pi/h$. An interchange of r and r' inside the bracket gives the potential $d\phi_2$ when $r>r'$. For the uniform disk source of radius r_0 and strength q per unit area assumed by Benedikt the potential is, according as $0<r<r_0$ or $r_0<r$,

$$\phi_1=\int_{r=0}^{r=r} d\phi_2+\int_{r=r}^{r=r_0} d\phi_1 \quad \text{or} \quad \phi_2=\int_0^{r_0} d\phi_2. \quad (2)$$

The velocities v_r and v_z are the quantities of chief interest. Integration for ϕ_1 and ϕ_2, use of the relation $v^{-1}=I_1(v)K_0(v)+I_0(v)K_1(v)$, and differentiation with respect to r give v_r when $r<r_0$ to be

$$(qr_0/h)[\tfrac{1}{2}(r/r_0)-2\sum_{n=1}^{\infty} I_1(Nr)K_1(Nr_0)\cos Nz].$$

When $r>r_0$, interchange r and r_0 inside the bracket. Differentiation with respect to z gives for v_z

$$r<r_0, \quad (q/h)[h-z-2r_0\sum_{n=1}^{\infty} I_0(Nr)K_1(Nr_0)\sin Nz],$$

$$r>r_0, \quad (2qr_0/h)\sum_{n=1}^{\infty} I_1(Nr_0)K_0(Nr)\sin Nz. \quad (3)$$

Except when $r=r_0$, the convergence is very rapid because the $I_n(v)K_n(v)$ product becomes proportional to $\exp(-n\pi|r-r_0|/h)$ as $n\to\infty$.

As Benedikt points out, the assumption of a uniform source gets rapidly worse as h decreases. If $h\ll r_0$, however, another method gives increasingly accurate results. The distribution of flow across the gap between the plates then approaches that of the two-dimensional flow into a crack of width $2h$ in an infinite plane faced block. At $r=r_0$ the radial velocity v_r is then given[2] to one part in 5000 by the formula

$$v_0\{A+B\cos(\pi z'/h)+C(z'/h)^2+D[\sec(\tfrac{1}{2}\pi z'/h)]^{0.365}\}, \quad (4)$$

where z' is measured from the unperforated plate $0<z'<h$ and A, B, C, and D are numerical constants.[2] The mean radial velocity at $r=r_0$ is v_0. When $r>r_0$, v_r may be written as the series

$$v_0\{1+\sum_{n=1}^{\infty} A_n[K_1(Nr)/K_1(Nr_0)]\cos(Nz)\}. \quad (5)$$

Where the coefficients A_n have been calculated[2] for $0<n<21$. When $r<r_0$, the boundary conditions are $v_z=0$ at $z'=0$, $v_r=0$ when $h<z'<\infty$, and v_r is given by (4) when $0<z'<h$. If the velocity is assumed uniform in the tube at some value of z' such as $z'=5h$, the velocity potential when $0<z'<5h$ can be expressed as a Fourier series somewhat similar to (5) whose coefficients involve gamma-functions. Otherwise a Fourier integral is required.

[1] W. R. Smythe, *Static and Dynamic Electricity* (McGraw-Hill Book Company, Inc., New York, 1939), Art. 5.324.
[2] W. R. Smythe, Rev. Mod. Phys. **20**, 176 (1948).

Small Spherical Particles of Exceptionally Uniform Size

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During the course of some quantitative work with the electron microscope we have chanced upon some small, spherical particles of remarkable uniformity of size. The particles are contained in a polystyrene latex, made by the Dow Chemical Company, and are described as Dow latex 580-G, Lot 3584. Their dimensional uniformity immediately suggests a number of applications in electron microscopy, and three of the uses will be discussed here.

1. *Magnification Calibration*:—Inasmuch as the particles are readily suspended in distilled water, they can be applied directly with a small pipette to an electron microscope specimen, and can serve as a standard of size for any pictures taken of that specimen. An example of the appearance of the latex, diluted 1000 to 1 with water, deposited upon a specimen, is shown in Fig. 1. In this case the speci-

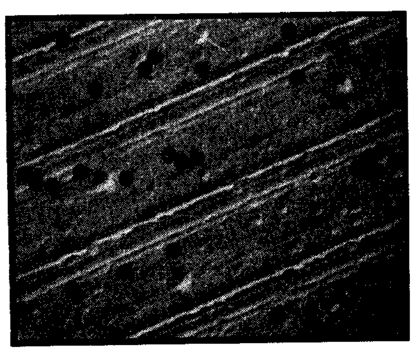

Fig. 1. Electron micrograph of Dow latex polystyrene particles supported upon a pre-shadowed replica of a 15,000-line/inch diffraction grating.

men is a shadow-cast collodian replica of a glass diffraction grating of 15,000 lines per inch.

A determination of the mean diameter of the individual particles has been made, as well as a determination of the dispersion in diameter from particle to particle. The method first tried for a determination of size was to prepare shadow-cast, organic replicas of a diffraction grating of known spacing, and then to make a direct comparison with the particle diameter, as could be done by the use of Fig. 1. After a few measurements, however, it became apparent that the replicas were more dimensionally variable than were the particles. We then decided to make a determination of particle size through the intermediate use of a light microscope whose eyepiece reticle had been calibrated against a standard etched-glass scale. The first step in the measurement was to allow the objects to adhere to the surface of fine glass fibers, following immersion of the fibers in a concentrated suspension of the particles. The partially coated fibers were then spread at random across a microscope specimen screen. Upon examination with the light microscope certain unique clusters of particles could be identified, some 40 to 50 microns apart, and their separation could be measured with the use of the calibrated reticle. The same fiber was then found in the field of the electron microscope, and by the use of several overlapping exposures, the distance along the fiber between the clusters could be measured on the micrographs. These measurements, of course, allowed a determination to be made of the diameters of the numerous individual particles found between the clusters. It was thought that a correction for distortion in the electron micrographs would be necessary, but no measurable difference in magnification exists across the field of the micrographs. The instrument used is an RCA, type EMB, of an early make, and the pictures were taken at a magnification of about 10,000.

The result of the measurements is a value of 2590 ± 25A for the mean diameter of the particles.* The probable error is thought to reside largely in the errors of measurement, rather than in a dispersion of diameter of the objects. Observations of the appearance of the shadow-cast particles, as well as of the appearance of the particles when seen in silhouette along the glass fibers, lead us to believe that the objects are truly spherical.

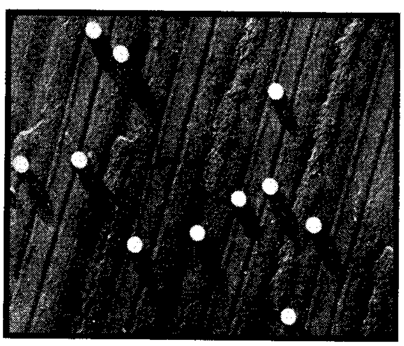

FIG. 2. Electron micrograph of uranium-shadowed Dow latex particles supported upon a replica of a diffraction grating.

A determination has also been made of the dispersion in diameter of about 500 of the small spheres, photographed on 20 pictures of identical magnification. The values of the diameters yield an internal probable error of ± 10A, indicating a very high degree of dimensional uniformity. In measuring this large number of particles, we found there are some (about 2 percent of the total) which are clearly "sports" in that their diameters are different from the ordinary ones by almost a factor of two.

2. *Contours of Replica Surfaces*:—A second use of the latex spheres is illustrated by Fig. 2, which is a micrograph of the particles as placed upon the specimen film followed by uranium shadowing of particles and substrate. The ratio of length of shadow to diameter of sphere gives the value of the local angle of shadowing, and since this is frequently not constant over the area of a specimen, its evaluation at any one point is desirable. By comparing the lengths and shapes of the shadows of the particles over an extended region of the specimen, one may obtain a fair quantitative notion of the contours of the surface. The shadow-casting technique itself, of course, gives one a qualitative impression of specimen contours, but the shadows of the spherical particles allow a more quantitative estimate to be made. In Fig. 2, for example, one may note the considerable difference in shadow lengths, depending upon whether the particle is on an "up-hill" or "down-hill" portion of the replica surface.

3. *Thickness of Film Used in Shadowing*:—If some latex particles are placed on the specimen prior to shadowing, and some subsequent to shadowing, evidence can be obtained of the thickness of the evaporated film used for shadowing. It will be noticed in Fig. 2 that the spheres are slightly elongated in the direction of the length of the shadows. The elongation is due to the accretion of the evaporated film on the side of the particle nearest the filament during the shadow-casting; in Fig. 2 the elongation is about 35A. The thickness of the evaporated film on the surface of the substrate is calculated by measuring the difference between the mean diameter of the unshadowed particles and the mean major diameter of the shadowed ones, and dividing the difference in diameters by the ratio of the shadow length to the particle diameter.

* This is 50A greater than the preliminary value reported in the abstracts of the E. F. Burton Memorial meeting of the Electron Microscope Society of America.

The Coefficient of Anomalous Viscosity

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IN a recent paper on "Anomalous flow as an order-disorder transition" the writer gave the following expression for the coefficient of viscosity for anomalous flow:[1]

$$\eta = (n_\beta - n_\alpha) b\sigma / v, \tag{1}$$