Small-time asymptotics of stopped Lévy bridges and simulation schemes with controlled bias

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Abstract

We characterize the small-time asymptotic behavior of the exit probability of a Lévy process out of a two-sided interval and of the law of its overshoot, conditionally on the terminal value of the process. The asymptotic expansions are given in the form of a first order term and a precise computable error bound. As an important application of these formulas, we develop a novel adaptive discretization scheme for the Monte Carlo computation of functionals of killed Lévy processes with controlled bias. The considered functionals appear in several domains of mathematical finance (e.g. structural credit risk models, pricing of barrier options, and contingent convertible bonds) as well as in natural sciences. The proposed algorithm works by adding discretization points sampled from the Lévy bridge density to the skeleton of the process until the overall error for a given trajectory becomes smaller than the maximum tolerance given by the user.

Keywords and phrases: Small-time asymptotics, Lévy bridge, killed Lévy process, exit probability, bridge Monte Carlo methods, adaptive discretization, barrier options.

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1 Introduction

Small-time asymptotics for the distributions of Lévy processes and related Markov processes have a long history going back to the seminal work of Léandre [30], who obtained the leading order term of the transition density of a Markov process solving a stochastic differential equation with jumps. In the case of a Lévy process, the main result of [30] reads

$$\lim_{t\to 0}\frac{1}{t}f_t(x)=s(x), \qquad (x\neq 0), \tag{1.1}$$

where $f_t(x) := \frac{d}{dx}\mathbb{P}(X_t \leq x)$ is the marginal density of the Lévy process X and s is the Lévy density of X, whose existence and smoothness need to be assumed. Léandre's approach was to consider separately the small jumps (say, those with sizes smaller than an $\varepsilon > 0$) and the large jumps of the underlying Lévy process, and to condition on the number of large jumps by time t. A similar approach has been applied during the last decade to obtain high-order asymptotic expansions for the transition distributions and densities of Lévy processes and other Markov processes with jumps (see [38], [19], [20], and [21]). These small-time asymptotic results have found a wide scope of applications ranging from estimation methods based on high-frequency sampling observations of the process (see, e.g., [17], [11], [37], and references therein) to asymptotic results for option prices and Black-Scholes volatilities in short-time (c.f. [43], [18], [19]).

In the present paper, we adopt Leandre's approach to study the asymptotic behavior of the generalized moments of the Lévy process stopped at the time it exits a two-sided interval (a, b), conditionally on the terminal value of the process. Specifically, for a Lévy process $(X_t)_{t\geq 0}$ with Lévy density s that is smooth outside any neighborhood of the

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origin and for a bounded Lipschitz function φ, we prove that

$$\mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\tau\leq t}|\, X_t = y\right) = \frac{t}{2}\int_{(a,b)^c}\varphi(v)\,\frac{s(v)s(y-v)}{s(y)}\,dv + o(t), \qquad (t\to 0, y\in(a,b)\backslash\{0\}), \tag{1.2}$$

where $\tau := \inf\{u \geq 0 : X_u \notin (a,b)\}$ with $-\infty \leq a < 0 < b \leq \infty$. In the case $\varphi \equiv 1$, (1.2) can be written as follows:

$$\mathbb{P}\left(\exists u\in[0,t]:\; X_u \notin (a,b)|\, X_t = y\right) = \frac{t}{2}\int_{(a,b)^c}\frac{s(v)s(y-v)}{s(y)}\,dv + o(t), \qquad (t\to 0), \tag{1.3}$$

for $y \in (a,b)\backslash\{0\}$. As in the case of the small-time asymptotics for the marginal distributions of the process, the main intuition can be drawn from considering the pure-jump case with finite jump activity. Intuitively, formulas (1.2)-(1.3) tell us that if, within a small time period, a Lévy process goes out of the interval (a,b) and then comes back to the point $y \in (a,b)$, this essentially happens with two large jumps: the first jump takes the process out of (a,b), while the second jump brings it back to y.

Our study of the short-time behavior of (1.2) and (1.3) is motivated by applications in the Monte Carlo evaluation of functionals of the form

$$\mathbb{E}[F(X_T)\mathbf{1}_{\tau>T}], \qquad \tau = \inf\{t\geq 0 : X_t \notin (a,b)\}. \tag{1.4}$$

In financial mathematics, such functionals arise in structural credit risk models based on Lévy processes [16] and in the pricing of barrier options (cf. [27], [7]), which is one of the most popular classes of exotic options. Very recently, a renewed interest to these problems has emerged in relation to the so-called contingent convertible bonds, where the conversion is triggered by a passage across a level and which exhibit a high sensitivity to jump risk [13]. In natural sciences, Lévy processes (under the name of Lévy flights) are used as models for certain diffusion-like phenomena in physics and chemistry (so-called anomalous or super-diffusion) [32, 41, 3] as well as to describe movement patterns of foraging animals [44, 5], and there is considerable interest towards the study of Lévy flights in bounded domains and related first passage problems giving rise to functionals of type (1.4) [10, 8, 22]. In all these settings, closed-form expressions are rarely available and Monte Carlo is often the method of choice.

The simplest procedure to evaluate the functional (1.4) by Monte Carlo consists in simulating the process $(X_t)_{t\geq 0}$ at evenly spaced times $t_k^n := kh_n$, with $h_n := T/n$ and $k = 0,\dots,n$, over the interval $[0,T]$, and approximating the exit time τ by

$$\tilde{\tau}_n := \inf\{t_k^n : X_{t_k^n} \notin (a,b)\}.$$

This simple method introduces two types of errors: the statistical error and the discretization error. The latter is known to be quite significant (cf. [2] and Example 2 in Section 5 below); [31] reports errors of up to 10% in the context of barrier options for a time discretization of one point per day.

In the context of continuous diffusions, short-time asymptotics have been successfully employed to alleviate the bias due to the discretization error. One of the earliest procedures of this type, due to Baldi [2], is based on an approximation of the probability, $p(x,y,t)$, that the process X has gone out of a domain (a,b) during the small time interval $[s,s+t]$ conditioning on $X_s = x$ and $X_{s+t} = y$; i.e.,

$$p(x,y,t) := \mathbb{P}\left(\exists u\in[s,s+t]:\; X_u \notin (a,b)|\, X_s = x, X_{s+t} = y\right). \tag{1.5}$$

Given such an approximation $\tilde{p}(x,y,t)$ of the functional $p(x,y,t)$, the procedure simulates iteratively $X_{t_{k+1}^n}$ at each step $k = 0,\dots,n-1$, and if $X_{t_{k+1}^n} \in (a,b)$, it proceeds to kill the process with probability $\tilde{p}(X_{t_k^n}, X_{t_{k+1}^n}, h_n)$ and choose $t_{k+1}^n = (k+1)h_n$ as an approximation of the exit time τ. A similar idea was used in [33] to price barrier options with payoff $\varphi(S_\tau,\tau)$ by Monte Carlo.

In the context of Lévy processes, an attempt to apply a similar methodology has been made in [45, 36]. The authors remarked that the discretization bias can be reduced by using the identity

$$\mathbb{E}\left(F(X_T)\mathbf{1}_{\{\tau<T\}}\right) = \mathbb{E}\left(F(X_T)\left(1 - \prod_{k=0}^{n-1}\left\{1 - p(X_{t_k^n}, X_{t_{k+1}^n}, h_n)\right\}\right)\right) \tag{1.6}$$

and replacing the exact exit probability $p(x, y, t)$ with a suitable small-time approximation $\tilde{p}(x, y, t)$. However, these papers propose no general formula for $\tilde{p}(x, y, t)$ and, as shown in [4], the Monte Carlo method proposed in [45, 36] could lead to a large discretization bias. On the other hand, in the specific case of the parametric variance gamma model, there exist discretization algorithms (cf. [1]) allowing to simulate the running minimum and maximum with error bounds. Let us also remark the recent work of [28] where a method for the joint simulation of the running maximum and the position of a Lévy process is introduced based on the Wiener-Hopf decomposition of the process.

Our short-time asymptotic result (1.3) provides an approximation of the exit probability (1.5) via the formula

$$\tilde{p}(x, y, t) := \frac{t}{2} \int_{(a-x, b-x)^c} \frac{s(v)s(y-x-v)}{s(y-x)} \, dv = \frac{t}{2} \int_{(a,b)^c} \frac{s(u-x)s(y-u)}{s(y-x)} \, du, \qquad (x \neq y), \tag{1.7}$$

which is valid under mild regularity conditions on the Lévy process X (see Section 2 for details). The first order approximation (1.7), together with an appropriate error bound for it, enable us to develop a general adaptive Monte Carlo method for evaluating the functional (1.4) with a given precision. Given a target error level γ, the idea is to generate a "random skeleton" $\{(T_k, X_{T_k})\}_{k=1}^{N}$ of the process X such that the error in each subinterval $[T_k, T_{k+1}]$, i.e.

$$e := p(X_{T_k}, X_{T_{k+1}}, T_{k+1} - T_k) - \tilde{p}(X_{T_k}, X_{T_{k+1}}, T_{k+1} - T_k), \tag{1.8}$$

satisfies $|e| \leq \frac{T_{k+1}-T_k}{T}\gamma$. The functional (1.4) is then approximated as follows:

$$\mathbb{E}[F(X_T)\mathbf{1}_{\tau > T}] \approx \mathbb{E}\left(F(X_T) \prod_{k=0}^{N-1} \left\{1 - \tilde{p}(X_{T_k}, X_{T_{k+1}}, T_{k+1} - T_k)\right\} \right), \tag{1.9}$$

and it is shown that the total bias of this computation will be less then γ. As a result of this adaptiveness, the algorithm generates more frequent points when the process X is close to the boundary, and takes large time steps (thus saving computational time) when the process is far from the boundary. Let us remark that, unlike the formula (1.6), where the sampling times $\{t_k^n\}$ are deterministic and fixed, the decomposition (1.9) for random skeletons $\mathcal{X} := \{(T_k, X_{T_k})\}_{k=0}^{N}$ requires precise (and also novel to the best of our knowledge) conditions under which this formula holds (see Section 4 for the details).

The proposed adaptive algorithm works as follows. First, the endpoint X_T is generated and added to the skeleton. Next, if the error (1.8) is too large for a given subinterval $[T_k, T_{k+1}]$, the procedure splits the interval into two and generates the midpoint $X_{\bar{T}_k}$ with $\bar{T}_k := (T_k + T_{k+1})/2$ from the bridge distribution. This is repeated iteratively until the desired error bound is satisfied for every subinterval $[T_k, T_{k+1}]$ of the sampling times $0 = T_0 < \cdots < T_N = T$. Such retrospective sampling (starting from the endpoint) has a number of advantages over the classical uniform discretization, especially in the context of rare event simulation, where it enables one to easily implement variance reduction by importance sampling. Indeed, the process can be directed to the region of interest by modifying the distribution of the terminal value, while keeping unchanged the rest of the algorithm. On the other hand, this method requires fast simulation from the bridge distribution of $X_{t/2}$ conditioned to $X_t = y$. To this end, as another contribution of particular interest on its own, we also propose a new method to simulate from this Lévy bridge distribution based on the classical rejection method.

As previously explained, in order to implement the above adaptive algorithm, precise computable bounds for the approximation errors in (1.2)-(1.3) are also needed. We obtain such bounds by developing explicit inequalities for the tail probabilities and transition densities of a Lévy process whose Lévy density has a small compact support. This type of concentration inequalities in turn allows us to estimate the different components of the error, which, as explained above, originate from conditioning the desired functional on the number of big jumps by time t (see Section 3 for the details). The resulting error bounds are given in terms of the Lipschitz and L_∞ norms of φ as well as several computable quantities related to the Lévy density s such as $\sup_{|x|\geq\varepsilon} s(x)$, $\sup_{|x|\geq\varepsilon} |s'(x)|$, $\int_{|x|\geq\varepsilon} s(x)dx$, and $\int_{|x|\leq\varepsilon} x^2 s(x)dx$.

Let us also remark that an adaptive simulation method similar to the one introduced in the present paper was proposed in [15] to compute a functional of the form $\mathbb{E}\varphi(X_\tau, \tau)$ for a homogeneous diffusion process X without jumps. Adaptive numerical methods for finding weak approximation of diffusions without jumps and with finite intensity jumps (but with the adaptiveness only concerning the diffusion part) have also been proposed in [42] and [34], respectively. As in our paper, the idea therein is to sample from inside of a subinterval $[t_k^n, t_{k+1}^n]$ whenever the approximation error in that subinterval has not reached a desired low level, specified by the user.

The paper is organized as follows. In Section 2, we obtain the leading term of the functional $\mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\tau\leq t} \middle| X_t = y\right)$ when $t \to 0$. The explicit estimate of the approximation error is given in Section 3. The development of the adaptive discretization schemes for the Monte Carlo computation of the functional $\mathbb{E}[F(X_T)\mathbf{1}_{\tau>T}]$ as well as the algorithm to simulate random observations from the Lévy bridge distribution are given in Section 4. Our methods are illustrated numerically in Section 5 for Cauchy process. Finally, the proofs of the technical results are deferred to the appendix.

2 Small-time asymptotics for Lévy bridges

Let X be a real-valued Lévy process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Lévy triplet (σ^2, ν, μ) with respect to truncation function $h(x) = 1_{|x|\leq 1}$. Throughout, $(\mathcal{F}_t)_{t\geq 0}$ denotes the natural filtration generated by the process X and augmented by the null sets of $\mathcal{F}$ so that it satisfies the usual conditions (see, e.g., Chapter I.4 in [35]). The following standing assumptions are imposed throughout the paper:

- The Lévy measure ν admits a continuously differentiable density $s : \mathbb{R}\backslash\{0\} \to (0, \infty)$, with respect to the Lebesgue measure (hereafter denoted by $\mathcal{L}$), which satisfies

$$\sup_{|x|\geq\varepsilon} s(x) < \infty, \qquad \sup_{|x|\geq\varepsilon} |s'(x)| < \infty, \tag{2.1}$$

  for any $\varepsilon > 0$.
- The distribution of X_t admits a density f_t for all $t > 0$. Since ν is already assumed to admit a density, for this assumption to hold, it suffices to additionally require that $\nu(\mathbb{R}) = \infty$ or $\sigma > 0$ (see Theorem 27.7 in Sato [40]).
- The density of X_t satisfies $f_t(x) > 0$ for all $x \in \mathbb{R}$ and $t > 0$ (see Theorem 24.10 in Sato [40] for mild sufficient conditions for this property to hold).

As it is usually done with Lévy processes, we shall decompose X into a compound Poisson process and a process with bounded jumps. More specifically, for any $\varepsilon \in (0, 1)$, we select a function $c_\varepsilon \in C^\infty(\mathbb{R})$, which is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$ and such that $\mathbf{1}_{|x|\geq\varepsilon} \leq c_\varepsilon(x) \leq \mathbf{1}_{|x|\geq\varepsilon/2}$. Next, we define the truncated Lévy densities

$$s_\varepsilon(x) := c_\varepsilon(x)s(x) \quad \text{ and } \quad \bar{s}_\varepsilon(x) := \bar{c}_\varepsilon(x)s(x),$$

with $\bar{c}_\varepsilon(x) := 1 - c_\varepsilon(x)$. Let Z^ε be a compound Poisson process with Lévy measure $s_\varepsilon(x)dx$ and X^ε be a Lévy process, independent from Z^ε, with characteristic triplet $(\sigma^2, \bar{s}_\varepsilon(x)dx, \mu_\varepsilon)$, where

$$\mu_\varepsilon := \mu - \int_{|x|\leq 1} xc_\varepsilon(x)s(x)dx. \tag{2.2}$$

It is clear that $X^\varepsilon + Z^\varepsilon$ has the same law as X and that the intensity and probability density of the jumps of Z^ε are $\lambda_\varepsilon := \int s_\varepsilon(x)dx$ and $s_\varepsilon(x)/\lambda_\varepsilon$, respectively. Throughout the paper, we let $(N^\varepsilon_t)_{t\geq 0}$ be the jump counting process of Z^ε and $(Y^\varepsilon_k)_{k\geq 1}$ be the jump sizes of Z^ε. Thus, $Z^\varepsilon_t = \sum_{k=1}^{N^\varepsilon_t} Y^\varepsilon_k$. Note that the distribution of X^ε_t is also absolutely continuous since $\sigma > 0$ or $\int \bar{s}_\varepsilon(x)dx = \infty$, for any $\varepsilon > 0$. For future reference, let us remark that

$$\mathbb{E}\left(X^\varepsilon_t\right) = t\left(\mu_\varepsilon + \int_{|x|\geq 1} x\bar{s}_\varepsilon(x)dx\right) = t\mu_\varepsilon, \qquad \mathrm{Var}\left(X^\varepsilon_t\right) = t\left(\sigma^2 + \int x^2\bar{s}_\varepsilon(x)dx\right) =: t\sigma^2_\varepsilon, \tag{2.3}$$

since $\varepsilon \in (0, 1)$ (see, e.g., Example 25.12 in [40] for the mean and variance formulas of a Lévy process).

The following Lemma will be needed in what follows (c.f. Propositions I.4 and III.2 in [30]). See also Sections 3.1-3.2 below for explicit expressions for the constants $C_p(\eta, \varepsilon)$ and $c_p(\eta, \varepsilon)$.

Lemma 2.1. *Let f^ε_t be the transition density of the small-jump component process $(X^\varepsilon_t)_{t\geq 0}$. Then, for any fixed positive real η and positive integer p, there exist an $\varepsilon_0(\eta, p) > 0$ and positive constants $t_0(\eta, \varepsilon)$, $c_p(\eta, \varepsilon)$, and $C_p(\eta, \varepsilon) < \infty$ for any $\varepsilon < \varepsilon_0$ such that*

$$\text{(i) } \mathbb{P}\left(\sup_{0\leq s\leq t} |X^\varepsilon_s| \geq \eta\right) < C_p(\eta, \varepsilon)t^p, \qquad \text{(ii) } \sup_{|x|\geq\eta} f^\varepsilon_t(x) < c_p(\eta, \varepsilon)t^p, \tag{2.4}$$

for all $0 < t \leq t_0$ and $0 < \varepsilon \leq \varepsilon_0$.

The following result provides the key tool for establishing the small-time asymptotics of the moments of the Lévy bridge "stopped" at the exit time from an interval (a,b). Its proof is presented in Appendix A.

Theorem 2.2. *For fixed constants $a \in [-\infty, 0)$ and $b \in (0, \infty]$, define*

$$\tau := \inf \{u \geq 0 : X_u \notin (a,b)\}.$$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be bounded and Lipschitz on $\mathbb{R}$ and let $\delta_0 \in \left(0, \frac{b-a}{2}\right)$. Then, for any $y \in (a+\delta_0, b-\delta_0)$ and $0 < \delta < \delta_0$,

$$\mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\{\tau \leq t, X_t \in (y-\delta, y+\delta)\}}\right) = \int_{y-\delta}^{y+\delta} \left(\frac{t^2}{2}\int_{(a,b)^c} \varphi(v)s(v)s(u-v)dv + \mathcal{R}_t(u)t^2\right) du, \tag{2.5}$$

where the remainder term $\mathcal{R}_t(u)$ is such that

$$\lim_{t\to 0} \operatorname*{ess\,sup}_{u \in (a+\delta_0, b-\delta_0)} |\mathcal{R}_t(u)| = 0. \tag{2.6}$$

Remark 2.3. *By the definition of conditional expectation,*

$$\begin{aligned}\mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\{\tau \leq t, X_t \in (y-\delta, y+\delta)\}}\right) &= \mathbb{E}\left(\mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\{\tau\leq t\}}\middle| X_t\right)\mathbf{1}_{\{X_t\in(y-\delta,y+\delta)\}}\right)\\ &= \int_{y-\delta}^{y+\delta} \mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\{\tau\leq t\}}\middle| X_t = u\right) f_t(u)du, \end{aligned}\tag{2.7}$$

where $f_t(u)$ is the density of X_t and, as usual, $\Phi(u) := \mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\{\tau\leq t\}}\middle| X_t = u\right)$ is such that $\Phi(X_t)$ is a version of $\mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\{\tau\leq t\}}\middle| X_t\right)$. Comparing (2.7) and (2.5), it then follows that, for $\mathcal{L}$-a.e. $y \in (a,b)$,

$$\mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\{\tau\leq t\}}\middle| X_t = y\right) = \frac{\frac{t^2}{2}\int_{(a,b)^c}\varphi(v)s(v)s(y-v)dv}{f_t(y)} + \frac{\mathcal{R}_t(y)t^2}{f_t(y)}. \tag{2.8}$$

If, in addition, the transition density f_t satisfies the asymptotic formula (1.1)[1] *then, for $\mathcal{L}$-a.e. $y \in (a,b)\backslash\{0\}$,*

$$\mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\{\tau\leq t\}}\middle| X_t = y\right) = t\,\frac{\int_{(a,b)^c}\varphi(v)s(v)s(y-v)dv}{2s(y)} + o(t). \tag{2.9}$$

Formulas (2.5) *and* (2.8) *can be interpreted as large deviation results for the trajectories of Lévy processes in small time. When $\varphi(x) \equiv 1$, (2.9) gives the following small-time approximation for the exit probability of the Lévy bridge:*

$$\mathbb{P}\left(\tau \leq t \middle| X_t = y\right) = t\,\frac{\int_{(a,b)^c} s(v)s(y-v)dv}{2s(y)} + o(t). \tag{2.10}$$

We conclude this section with a simpler result for the case when X_t is outside the interval. Its proof is outlined in Appendix A.

Proposition 2.4. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be bounded and Lipschitz on $\mathbb{R}$, and let $\delta_0 > 0$. Then, under the same notation and conditions as in Theorem 2.2, for any $y \in (a-\delta_0, b+\delta_0)^c$ and $\delta < \delta_0$,*

$$\mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\{X_t\in(y-\delta,y+\delta)\}}\right) = \int_{y-\delta}^{y+\delta} \left(t\varphi(u)s(u) + \mathcal{R}_t(u)t\right) du, \tag{2.11}$$

where the remainder term $\mathcal{R}_t(u)$ is such that

$$\lim_{t\to 0} \operatorname*{ess\,sup}_{u \in (a-\delta_0, b+\delta_0)^c} |\mathcal{R}_t(u)| = 0. \tag{2.12}$$

Remark 2.5. *Analogously to Remark 2.3, (2.11) enables us to establish the following natural asymptotic formula:*

$$\mathbb{E}\left(\varphi(X_\tau)\middle| X_t = y\right) = t\,\frac{\varphi(y)s(y)}{f_t(y)} + o(1) = \varphi(y) + o(1), \qquad (t \to 0),$$

for $\mathcal{L}$-a.e. $y \in [a,b]^c$. The second equality above holds whenever $f_t(y)$ satisfies (1.1).

[1] As stated in the introduction, (1.1) holds for a large class of Markov processes with jumps as proved by [30]. For Lévy processes, [38] provided a more elementary proof using the same conditions and similar approach as in [30]. Higher order short-time expansions for the transition densities were obtained in [19].

3 On a precise bound for the remainder term

In the previous section, we developed the necessary results for finding estimates of the functional

$$f(0,y,t) := \mathbb{E}\left[\varphi(X_\tau)\mathbf{1}_{\tau\leq t}|\, X_t = y\right] \tag{3.1}$$

in short-time. Indeed, as explained in Remark 2.3, Theorem 2.2 yields the following natural estimate for $f(0,y,t)$:

$$\tilde{f}(0,y,t) = \frac{\frac{t^2}{2}\int_{(a,b)^c}\varphi(v)s(v)s(y-v)dv}{f_t(y)}. \tag{3.2}$$

The estimate (3.2) will be used below to develop adaptive discretization schemes for the Monte Carlo computation of functionals of the killed Lévy process (see Section 4). To this end, we first need to find an explicit estimate for the remainder $\mathcal{R}_t(y)$ appearing in (2.5). Such an estimate can be expressed in terms of bounds for the tail probability and transition densities of the small-jump component $(X_t^\varepsilon)_{t\geq 0}$. Hence, we start by providing explicit expressions for the upper bounds appearing in (2.4) and then proceed to give a precise error bound for $|f(0,y,t) - \tilde{f}(0,y,t)|$.

3.1 Bounding the tail probability of the supremum

The following exponential inequality for Lévy processes with bounded jumps will be important to estimate the supremum of the small-jump component (X_t^ε) defined in Section 2. Its proof, which is provided in Appendix B for completeness, is a variation of the bound obtained in [38] (which in turn is based on Lemma 26.4 in [40]).

Lemma 3.1. *Let (M_t) be a martingale Lévy process with $|\Delta M_t| \leq \varepsilon$ and $\langle M, M\rangle_t = \sigma_\varepsilon^2 t$. Then,*

$$\mathbb{P}\left(\sup_{s\leq t}(M_s + \mu s) \geq \eta\right) \leq t^{\frac{\eta}{\varepsilon}}\bar{C}_\ell(\eta,\varepsilon;\mu) \qquad (\ell = 0,1), \tag{3.3}$$

with the following constants $\bar{C}_\ell(\eta,\varepsilon;\mu)$ and corresponding conditions:

(1) $\bar{C}_0(\eta,\varepsilon;\mu) = e^{\frac{\mu\vee 0}{\varepsilon}e^{-1}}e^{\sigma_\varepsilon^2/\varepsilon^2}$ for all $\eta > 0$ and $0 < t < \eta/(\mu\vee 0)$ (with the convention here and below that the fraction is $+\infty$ if the denominator is zero);

(2) $\bar{C}_1(\eta,\varepsilon;\mu) = e^{\frac{\mu\vee 0}{\varepsilon}e^{-1}}\left(\frac{e\sigma_\varepsilon^2}{\varepsilon\eta}\right)^{\eta/\varepsilon}$ for all $\eta > 0$ and $0 < t < \eta/(\mu\vee 0)$ if either (i) $\mu \leq 0$ or (ii) $\mu > 0$ and $\eta \leq \sigma_\varepsilon^2/\varepsilon$;

In order to apply Lemma 3.1 for $(X_t^\varepsilon)_{t\geq 0}$, we recall that $0 < \varepsilon < 1$ so that $\mathbb{E}X_t^\varepsilon = \mu_\varepsilon t$. Then, the martingale part $M_t^\varepsilon := X^\varepsilon - \mu_\varepsilon t$ of X^ε is such that

$$\langle M^\varepsilon, M^\varepsilon\rangle_t = \left(\sigma^2 + \int \bar{c}_\varepsilon(x)x^2\nu(dx)\right)t = \sigma_\varepsilon^2 t.$$

Thus, fixing

$$t_0(\varepsilon,\eta) := \frac{\eta}{2(\mu_\varepsilon\vee 0)}, \tag{3.4}$$

it follows that, for all $0 < t < t_0$,

$$\mathbb{P}\left(\sup_{s\leq t}X_s^\varepsilon \geq \eta\right) \leq \mathbb{P}\left(\sup_{s\leq t}M_s^\varepsilon + |\mu_\varepsilon|t \geq \eta\right) \leq \mathbb{P}\left(\sup_{s\leq t}M_s^\varepsilon \geq \frac{\eta}{2}\right) \leq t^{\frac{\eta}{2\varepsilon}}C\left(\frac{\eta}{2},\varepsilon\right), \tag{3.5}$$

with $C(\eta,\varepsilon)$ is defined by

$$C(\eta,\varepsilon) := \left(\frac{e\sigma_\varepsilon^2}{\varepsilon\eta}\right)^{\frac{\eta}{\varepsilon}}. \tag{3.6}$$

Similarly, we have $\mathbb{P}(\sup_{s\leq t}|X_s^\varepsilon| \geq \eta) \leq 2t^{\frac{\eta}{2\varepsilon}}C(\eta/2,\varepsilon)$.

3.2 Bounding the transition density of the small-jump component

To obtain explicit expressions for the constants appearing in the bounds for the density f_t^ε in Lemma 2.1, we shall assume that the process X is such that X_t^ε has a unimodal distribution for all $t>0$ and $\varepsilon>0$. By Yamazato's theorem (see Theorem 53.1 in [40]), a sufficient condition for this is that the process X is self-decomposable, which is the case if and only if the Lévy density s is of the form $s(x)=\frac{k(x)}{|x|}$ for a function k which is increasing on $(-\infty,0)$ and decreasing on $(0,\infty)$ (see Corollary 15.11 in [40]). In particular, most of the parametric models used in the literature (such as stable, tempered stable, variance gamma, and normal inverse Gaussian processes) are self-decomposable and so these processes as well as their truncated versions have unimodal densities at all times.

Let m_t^ε be the mode of X_t^ε. If $m_t^\varepsilon\in[-\underline{\eta},\underline{\eta}]$ and $\eta>\underline{\eta}$, then the density can be estimated by

$$\sup_{|x|\geq\eta} f_t^\varepsilon(x)\leq\frac{2}{\eta-\underline{\eta}}P[|X_t^\varepsilon|\geq\underline{\eta}], \tag{3.7}$$

simply because the density is decreasing in $(\underline{\eta},\infty)$ and increasing in $(-\infty,-\underline{\eta})$. The relation (3.7) in turn leads to a bound of the form (2.4-ii) by applying the tail bound (2.4-i). It remains to find conditions for $m_t^\varepsilon\in[-\underline{\eta},\underline{\eta}]$. Since obviously X^ε has finite second moment, the following bound due to Johnson and Rogers [26] can be applied

$$|m_t^\varepsilon-\mathbb{E}X_t^\varepsilon|^2\leq 3\mathrm{Var}(X_t^\varepsilon). \tag{3.8}$$

Thus, recalling the mean and variance formulas given in (2.3), $m_t^\varepsilon\in[-\underline{\eta},\underline{\eta}]$ whenever $0<t<t_1$, where t_1 is such that

$$t_1|\mu_\varepsilon|+\sqrt{3}\,t_1^{1/2}\left(\sigma^2+\int_{|x|\leq\varepsilon}|x|^2\nu(dx)\right)^{1/2}=\underline{\eta}. \tag{3.9}$$

By taking $\underline{\eta}=\eta/2$, we will have

$$\sup_{|x|\geq\eta} f_t^\varepsilon(x)\leq\frac{4}{\eta}\mathbb{P}\left[|X_t^\varepsilon|\geq\frac{\eta}{2}\right]\leq\frac{8C(\eta/4,\varepsilon)}{\eta}t^{\frac{\eta}{2\varepsilon}}, \tag{3.10}$$

for any $0<t<t_1\wedge t_0$ with t_0 defined as in (3.4).

3.3 Precise bound for the remainder

We are now ready to give an explicit bound for the reminder term $\mathcal{R}_t(y)$ appearing in (2.5), which in turn will produce an error bound for $|f(0,y,t)-\tilde{f}(0,y,t)|$. Throughout, we shall use the following notation:

(i) $a_\varepsilon:=\sup_x s_\varepsilon(x)$ and $a'_\varepsilon:=\sup_x|s'_\varepsilon(x)|$, where, as before, $s_\varepsilon(x):=c_\varepsilon(x)s(x)$ is the Lévy density s, truncated in a neighborhood of the origin;

(ii) $\lambda_\varepsilon:=\int s(x)c_\varepsilon(x)dx$, $\mu_\varepsilon:=\mu-\int_{|x|\leq 1}xc_\varepsilon(x)s(x)dx$, and $\sigma_\varepsilon^2:=\sigma^2+\int\bar{c}_\varepsilon(x)x^2s(x)dx$;

(iii) $C(\eta,\varepsilon)$ is defined as in (3.6), $t_0(\varepsilon,\eta)$ is defined as in (3.4), and $t_1(\varepsilon,\underline{\eta})$ is defined as in (3.9).

The following result, whose proof is given in Appendix B, gives an estimate for $\mathcal{R}_t(y)$ in terms of the previously defined notation and the L_∞- and Lipschitz norms of φ denoted hereafter by

$$\|\varphi\|_\infty:=\operatorname*{ess\,sup}_x|\varphi(x)|,\qquad\|\varphi\|_{\mathrm{Lip}}:=\inf\left\{K\geq 0:|\varphi(x)-\varphi(y)|\leq K|x-y|,\quad\forall\,x,y\in\mathbb{R}\right\}.$$

Theorem 3.2. *Using the notation of Theorem 2.2, assume that the process X is such that X_t^ε has a unimodal distribution for all $t>0$ and $\varepsilon>0$. Let $c:=b\wedge|a|$ and $\Delta_y:=(b-y)\wedge(y-a)>0$. Then,*

$$|\mathcal{R}_t(y)|\leq\frac{1}{t^2}\,e_{\mathcal{R}}(0,y,t),$$

for all $0 < t < t_0(\varepsilon, (\Delta_y/2) \wedge c) \wedge t_1(\varepsilon, \Delta_y/2)$*, where*

$$\begin{aligned} e_{\mathcal{R}}(0,y,t) := {} & e^{-\lambda_\varepsilon t}\|\varphi\|_\infty C(\Delta_y/4,\varepsilon) t^{\frac{\Delta_y}{4\varepsilon}} \left\{ \frac{8}{\Delta_y} + 2a_\varepsilon t + a_\varepsilon \lambda_\varepsilon t^2 \right\} + 2e^{-\lambda_\varepsilon t}\|\varphi\|_\infty a_\varepsilon C(c/2,\varepsilon) t^{1+\frac{c}{2\varepsilon}} \left\{1 + t\lambda_\varepsilon\right\} \\ & + \frac{\|\varphi\|_\infty \lambda_\varepsilon^2 a_\varepsilon}{2} t^3 + \|\varphi\|_\infty a_\varepsilon \lambda_\varepsilon^{-1} \left(1 - e^{-\lambda_\varepsilon t}[1 + \lambda_\varepsilon t + (\lambda_\varepsilon t)^2/2]\right) \\ & + e^{-\lambda_\varepsilon t} t^2 \left[a_\varepsilon \lambda_\varepsilon \|\varphi\|_{\mathrm{Lip}} + 2\|\varphi\|_\infty a_\varepsilon^2 + \|\varphi\|_\infty \lambda_\varepsilon a'_\varepsilon\right] (\sigma_\varepsilon t^{1/2} + \frac{|\mu_\varepsilon|}{2} t). \end{aligned} \tag{3.11}$$

Two immediate conclusions can be drawn. First, note that, by taking $\varepsilon < \frac{\Delta_y}{8} \wedge \frac{c}{2}$, we obtain a bound for the remainder satisfying condition (2.6). Second, as seen in Remark 2.3, the previous bound implies the following error bound

$$|f(0,y,t) - \tilde{f}(0,y,t)| \le \frac{e_{\mathcal{R}}(0,y,t)}{f_t(y)} =: e_f(0,y,t),$$

with f and $\tilde{f}$ defined as in (3.1)-(3.2).

Remark 3.3. *The approximation for the conditional exit probability* $p(0,y,t) := \mathbb{P}[\tau \le t | X_t = y]$ *is obtained by substituting* $\varphi \equiv 1$ *into* (2.8)*:*

$$\tilde{p}(0,y,t) = \frac{\frac{t^2}{2}\int_{(a,b)^c} s(v)s(y-v)dv}{f_t(y)}.$$

Making this substitution in the previous bound, it follows that $|p(0,y,t) - \tilde{p}(0,y,t)| \le e_p(0,y,t)$ *with* $e_p(0,y,t)$ *given by*

$$\begin{aligned} e_p(0,y,t) := \frac{1}{f_t(y)} \Bigg(& e^{-\lambda_\varepsilon t} C(\Delta_y/4,\varepsilon) t^{\frac{\Delta_y}{4\varepsilon}} \left\{ \frac{8}{\Delta_y} + 2a_\varepsilon t + a_\varepsilon \lambda_\varepsilon t^2 \right\} + 2e^{-\lambda_\varepsilon t} a_\varepsilon C(c/2,\varepsilon) t^{1+\frac{c}{2\varepsilon}} \left\{1 + t\lambda_\varepsilon\right\} + \frac{\lambda_\varepsilon^2 a_\varepsilon}{2} t^3 \\ & + a_\varepsilon \lambda_\varepsilon^{-1} \left(1 - e^{-\lambda_\varepsilon t}[1 + \lambda_\varepsilon t + (\lambda_\varepsilon t)^2/2]\right) + e^{-\lambda_\varepsilon t} t^2 \left[2a_\varepsilon^2 + \lambda_\varepsilon a'_\varepsilon\right] (\sigma_\varepsilon t^{1/2} + \frac{|\mu_\varepsilon|}{2} t) \Bigg), \end{aligned}$$

valid for all $t < t_0(\varepsilon, (\Delta_y/2) \wedge c) \wedge t_1(\varepsilon, \Delta_y/2)$*. The one-sided case* ($a = -\infty$) *can similarly be obtained.*

4 Adaptive simulation of killed Lévy processes

Our goal in this section is to design a type of adaptive Monte Carlo estimators for functionals of the form

$$\mathbb{E}[F(X_T)\mathbf{1}_{\tau > T}], \tag{4.1}$$

where F is a Borel measurable function and $\tau := \inf\{t \ge 0 : X_t \notin D\}$ with $D := (a,b)$, for some $a \in [-\infty, 0)$ and $b \in (0, \infty]$. From now on, to simplify notation and with no loss of generality, we shall take $T = 1$.

For $0 < s < t$, $x \in \mathbb{R}$, and $y \in \mathbb{R}$, we denote by $\mathbb{P}^{BR}_{(s,t,x,y)}[\cdot]$ the bridge law of the Lévy process X on the time interval $[s,t]$ with starting value x and terminal value y; that is, this is a version of the regular conditional distribution of $\{x + X_{u-s}\}_{u \in [s,t]}$ given $X_{t-s} = y - x$. Since X_t has a strictly positive density on $\mathbb{R}$ for every $t > 0$, the bridge law is uniquely defined for $\mathcal{L}$-almost every $y \in \mathbb{R}$ (recall that $\mathcal{L}$ stands for the Lebesgue measure), which is sufficient for our purposes. We also let $p(x,y,t)$ denote the exit probability from the domain D before time t for the Lévy bridge:

$$p(x,y,t) := \mathbb{P}^{BR}_{(0,t,x,y)}[\tau \le t] = \mathbb{P}\left[\exists u \in [0,t] : x + X_u \notin (a,b) | X_t = y\right]. \tag{4.2}$$

Our approach is based on the following decomposition:

$$\mathbb{E}[F(X_1)\mathbf{1}_{\tau > 1}] = \mathbb{E}\left[F(X_1) \prod_{i=0}^{N-1} \left(1 - p(X_{T_i}, X_{T_{i+1}}, T_{i+1} - T_i)\right)\right], \tag{4.3}$$

where $0 = T_0 \le \cdots \le T_N = 1$ are suitable sampling times. Formula (4.3) directly follows from the Markov property when the sampling points are deterministic. In that case, the set of points $\mathcal{X} := \{(T_i, X_{T_i})\}_{i=0}^N$ is called a deterministic skeleton. In our setting both the number of points N and the sampling times $0 = T_0 \le T_1 \le \cdots \le T_N = 1$ are random and we need to formalize under what conditions on $\mathcal{X}$ (4.3) still holds. The following result will suffice for our purposes.

Lemma 4.1. *Let N be a random variable with support $\mathcal{N} \subseteq \mathbb{N}$, such that $N > 0$, and let $0 = T_0 \leq \cdots \leq T_N = 1$ be random points such that*

(1) Each T_i takes values in a countable set $\mathcal{K} \subset [0,1]$;

(2) For each $n \in \mathcal{N}$ and $(s_0, \ldots, s_n) \in \mathcal{K}^{n+1}$ with $0 = s_0 \leq \cdots \leq s_n = 1$, the event $\{N = n, (T_0, \ldots, T_n) = (s_0, \ldots, s_n)\}$ is $\sigma(X_{s_i} : i = 0, \ldots, n)$-measurable.

Then, (4.3) is satisfied for any measurable function F with $\mathbb{E}[|F(X_1)|] < \infty$ and, furthermore, for every $t \in (0,1)$, $n \in \mathcal{N}$, and $A \in \mathcal{B}(\mathbb{R})$,

$$\mathbb{P}\left[X_t \in A | N = n, T_0, \ldots, T_N, X_{T_0}, \ldots, X_{T_N}\right] = \mathbb{P}^{BR}_{T_{i^*}, T_{i^*+1}, X_{T_{i^*}}, X_{T_{i^*+1}}}\left[X_t \in A\right], \quad \text{where} \quad i^* = \max\{i : T_i \leq t\}. \tag{4.4}$$

***Proof*.** Throughout, we let $\bar{p}(x,y,t) := 1 - p(x,y,t)$, $\vec{\mathcal{K}}^n := \{(s_0, \ldots, s_n) \in \mathcal{K}^{n+1} : 0 = s_0 \leq \cdots \leq s_n = 1\}$, $U_0 := [s_0, s_1]$, and $U_i := (s_i, s_{i+1}]$, with $i = 1, \ldots, n-1$. We also use the notation

$$\mathcal{I}_U := \mathbf{1}_{\{X_u \in (a,b) : \forall\, u \in U\}}, \text{ for a domain } U \subset \mathbb{R}_+, \quad \text{and} \quad \mathcal{I}_\emptyset = 1. \tag{4.5}$$

Then, by Markov property

$$\begin{aligned}
\mathbb{E}[F(X_1)\mathbf{1}_{\tau > 1}] &= \sum_{n \in \mathcal{N}} \sum_{(s_0, \ldots, s_n) \in \vec{\mathcal{K}}^n} \mathbb{E}\left[F(X_1)\mathcal{I}_{[0,1]}\mathbf{1}_{\{N=n,(T_0,\ldots,T_n)=(s_0,\ldots,s_n)\}}\right] \\
&= \sum_{n \in \mathcal{N}} \sum_{(s_1, \ldots, s_n) \in \vec{\mathcal{K}}^n} \mathbb{E}\left[F(X_1)\mathbf{1}_{\{N=n,(T_0,\ldots,T_n)=(s_0,\ldots,s_n)\}}\mathbb{E}\left[\prod_{i=0}^{n-1} \mathcal{I}_{U_i} \Big| X_{s_j} : j = 0, \ldots, n\right]\right] \\
&= \sum_{n \in \mathcal{N}} \sum_{(s_1, \ldots, s_n) \in \vec{\mathcal{K}}^n} \mathbb{E}\left[F(X_1)\mathbf{1}_{\{N=n,(T_0,\ldots,T_n)=(s_0,\ldots,s_n)\}} \prod_{i=0}^{n-1} \bar{p}\left(X_{T_i}, X_{T_{i+1}}, T_{i+1} - T_i\right)\right] \\
&= \mathbb{E}\left[F(X_1) \prod_{i=0}^{N-1} \bar{p}\left(X_{T_i}, X_{T_{i+1}}, T_{i+1} - T_i\right)\right],
\end{aligned}$$

which proves (4.3). Similarly,

$$\begin{aligned}
&\mathbb{P}[X_t \in A | N = n, T_0, \ldots, T_N, X_{T_0}, \ldots, X_{T_N}] \\
&= \sum_{(s_0, \ldots, s_n) \in \vec{\mathcal{K}}^n} \mathbb{P}[X_t \in A | N = n, (T_0, \ldots, T_n) = (s_0, \ldots, s_n), X_{T_0}, \ldots, X_{T_N}]\mathbf{1}_{(T_0,\ldots,T_n)=(s_0,\ldots,s_n)} \\
&= \sum_{(s_0, \ldots, s_n) \in \vec{\mathcal{K}}^n} \mathbb{P}^{BR}_{s_{i^*}, s_{i^*+1}, X_{s_{i^*}}, X_{s_{i^*+1}}}[X_t \in A]\mathbf{1}_{(T_0,\ldots,T_n)=(s_0,\ldots,s_n)} = \mathbb{P}^{BR}_{T_{i^*}, T_{i^*+1}, X_{T_{i^*}}, X_{T_{i^*+1}}}[X_t \in A].
\end{aligned}$$

□

From (4.3), it is now evident that, for the computation of (4.1) by Monte Carlo, it suffices to simulate independent replicas of the random variable $\mathcal{Y} := F(X_1)N(\mathcal{X})$, where hereafter we denote

$$N(\mathcal{X}) := \prod_{i=0}^{N-1} \left(1 - p(X_{T_i}, X_{T_{i+1}}, T_{i+1} - T_i)\right).$$

The exit probability $p(x,y,t)$ does not typically admit a closed form expression and some type of approximation must be applied for its evaluation. The short-time asymptotics (2.8) yields the following natural estimate for $p(x,y,t)$ when $x, y \in D$:

$$\tilde{p}(x,y,t) := (\breve{p}(x,y,t) \vee 0) \wedge 1, \quad \text{with} \quad \breve{p}(x,y,t) := \frac{t^2}{2} \int_{(a,b)^c} \frac{s(u-x)s(y-u)}{f_t(y-x)}\, du. \tag{4.6}$$

We also set $\tilde{p}(x, y, t) = 1$ if $x \notin D$ or $y \notin D$. This approximation satisfies

$$\left|\tilde{p}(x, y, t) - p(x, y, t)\right| \leq e_p(x, y, t), \tag{4.7}$$

where the error bound $e_p(x, y, t)$ is defined as in Remark 3.3 for $x, y \in D$ and by $e_p(x, y, t) = 0$ if $x \notin D$ or $y \notin D$. We can then approximate $N(\mathcal{X})$ by

$$\tilde{N}(\mathcal{X}) := \prod_{i=0}^{N-1} \left(1 - \tilde{p}(X_{T_i}, X_{T_{i+1}}, T_{i+1} - T_i)\right). \tag{4.8}$$

Replacing the true exit probability $p(x, y, t)$ with its approximation $\tilde{p}(x, y, t)$ introduces a bias into the evaluation of $N(\mathcal{X})$, which is hard to quantify if the process X is discretized using the uniformly spaced grid $T_i = i/N$. For this reason, we now propose an adaptive algorithm for the determination of the sampling times, which starts by simulating the terminal value X_1 and then refines the sampling grid, using more discretization points when the estimate of the approximation error is "large". The algorithm is parameterized by a real number $\gamma > 0$, which represents the error tolerance and ensures that under suitable conditions on e_p, the global discretization error for approximating the quantity of interest (4.1) will be bounded by γ (see Proposition 4.2 below). The algorithm also requires simulation from the marginal distribution f_1 of X_1 and the bridge distribution of $X_{t/2}$ conditioned to $X_t = y$ $(t > 0)$. Hereafter, we denote the density of this bridge distribution by $f^{br}_{t/2}(x, y)$ and recall the following well-known formula:

$$f^{br}_{t/2}(x, y) := \frac{f_{t/2}(x) f_{t/2}(y - x)}{f_t(y)}. \tag{4.9}$$

At the end of this section, we introduce a new method to simulate variates from the density (4.9).

The procedure to generate the skeleton of X is outlined in pseudo-code in Algorithm 1 below. Assume that this algorithm terminates in finite time a.s. (see Proposition 4.2 for sufficient conditions for this to hold). The algorithm then defines a pair N and $\mathcal{T} := (T_0, \ldots, T_N)$, which satisfies the conditions of Lemma 4.1. Indeed, by construction, each T_i takes values in the dyadic grid $\{i2^{-m}, i = 0, \ldots, 2^m, m = 0, 1, \ldots\}$, which is a countable set. To check the second condition of the lemma, we fix n and a partition $\pi := \{s_0, \ldots, s_n\}$ of $[0, 1]$, and proceed as follows to write the event $E := \{N = n, T_0 = s_0, T_1 = s_1, \ldots, T_n = s_n\}$ in terms of $\{X_{s_i}\}_{i=0}^n$:

- We can and will assume with no loss of generality that π is a *recursive dyadic partition*, meaning that $\{0, 1\} \subset \pi$ and, for every $t \in (0, 1) \cap \pi$, there exists $k \in \mathbb{N}$ with $2^k t \in \mathbb{N}$, and if we take the smallest such k then also $t + \frac{1}{2^k} \in \pi$ and $t - \frac{1}{2^k} \in \pi$. By construction, if π does not have this property, the event E has zero probability.
- We shall assume that $n \geq 2$ because if $n = 1$ then necessarily $s_0 = 0$ and $s_1 = 1$ and, therefore,

$$E = \{X_1 \notin D\} \cup \{X_1 \in D, e_p(X_0, X_1, 1) \leq \gamma\} \in \sigma(X_0, X_1).$$

- For each $\ell \in \{0, \ldots, n-1\}$, define $\pi_\ell := \{s_i \in \pi : 2^{n-\ell} s_i \text{ is an even integer}\}$. The number of elements of π_ℓ is denoted n_ℓ and the sorted elements of π_ℓ are denoted $s_1^\ell < \cdots < s_{n_\ell}^\ell$. Clearly, $\pi_0 = \pi$ and $\pi_{n-1} \neq \pi$ since $1/2 \in \pi$ whenever $n \geq 2$; we let $\ell^* = \max\{l \geq 0 : \pi_l = \pi\}$ and $\pi^* = \pi \setminus \pi_{\ell^*+1}$.
- For each $i = 1, \ldots, n_\ell - 1$, define the event

$$E_i^\ell := \begin{cases} \{\omega : e_p\left(X_{s_i^\ell}(\omega), X_{s_{i+1}^\ell}(\omega), s_{i+1}^\ell - s_i^\ell\right) \leq \gamma(s_{i+1}^\ell - s_i^\ell)\}, & \text{if} \quad \pi \cap (s_i^\ell, s_{i+1}^\ell) = \emptyset, \\ \{\omega : e_p\left(X_{s_i^\ell}(\omega), X_{s_{i+1}^\ell}(\omega), s_{i+1}^\ell - s_i^\ell\right) > \gamma(s_{i+1}^\ell - s_i^\ell)\}, & \text{if} \quad \pi \cap (s_i^\ell, s_{i+1}^\ell) \neq \emptyset. \end{cases}$$

  Then, it follows that

$$E = \left\{\bigcap_{i=0}^{n} \{X_{s_i} \in D\} \cap \bigcap_{\ell=\ell^*}^{n-1} \bigcap_{i=1}^{n_\ell - 1} E_i^\ell \right\} \cup \left\{\bigcup_{s \in \pi^*} \{X_s \notin D\} \cap \bigcap_{s \in \pi_{\ell^*+1}} \{X_s \in D\} \cap \bigcap_{\ell=\ell^*+1}^{n-1} \bigcap_{i=1}^{n_\ell - 1} E_i^\ell \right\}$$

  which clearly belongs to $\sigma(X_{s_i} : i = 0, \ldots, n)$.

Algorithm 1 $[\mathcal{X}]$ = GenerateSkeleton(γ)

$N_0 = 0$, $N_1 = 1$, $m = 1$
$T_0^1 = 0$, $T_1^1 = 1$, $X_0 = 0$
Generate an observation X_1 from the density f_1
while $N_m \neq N_{m-1}$ and $\{X_{T_i^m} \in D, \text{ for } i = 1, ..., N_m\}$ **do**
  $n = 0$, $T_0^{m+1} = 0$
  for $i = 0 \to N_m - 1$ **do**
    $\Delta T = T_{i+1}^m - T_i^m$
    if $e_p(X_{T_i^m}, X_{T_{i+1}^m}, \Delta T) > \gamma \Delta T$ **then**
      $T_{n+1}^{m+1} = (T_i^m + T_{i+1}^m)/2, \quad T_{n+2}^{m+1} := T_{i+1}^m$
      Generate an observation $X_{T_{n+1}^{m+1}}$ from the bridge density $f_{\Delta T/2}^{br}(\cdot, X_{T_{i+1}^m} - X_{T_i^m})$
      $n = n + 2$
    else
      $T_{n+1}^{m+1} := T_{i+1}^m$
      $n = n + 1$
    end if
  end for
  $N_{m+1} = n$
  $m = m + 1$
end while
RETURN $\mathcal{X} = \{(T_i^m, X_{T_i^m})\}_{i=0}^{N_m}$.

To see that $X_{T_{n+1}^{m+1}}$ can be sampled from the bridge density $f_{\Delta T/2}^{br}(\cdot, X_{T_{i+1}^m} - X_{T_i^m})$ in Algorithm 1, we can apply the second part of Lemma 4.1 to the couple $(k, \mathcal{T}_k)$, where $\mathcal{T}_k = \{T_0, \dots, T_k\}$ contains the first $k+1$ sampling times which have been added to the grid by the algorithm, in increasing order.

Algorithm 1 terminates when at least one of the sampling observations X_{T_i} is out of the domain D or the error over each subinterval of the sampling times $0 = T_0 < \cdots < T_N = 1$ is small enough in the following sense:

$$e_p(X_{T_i}, X_{T_{i+1}}, T_{i+1} - T_i) \leq \gamma(T_{i+1} - T_i), \qquad i = 0, \dots, N-1. \tag{4.10}$$

At first glance, it is not obvious that the algorithm will actually terminate in finite time. The following result gives conditions under which this is the case and shows that the global error of the estimate is of order γ.

Proposition 4.2. *The following assertions hold:*

(i) Let X be a Lévy process satisfying one of the following two (non mutually exclusive) conditions:

1. *X does not hit points; that is, $\mathbb{P}(\tau^{\{x\}} < \infty) = 0$ for all x, where $\tau^{\{x\}} := \inf\{s > 0 : X_s = x\}$ or, equivalently,*
$$\int_{\mathbb{R}} \Re\left(\frac{1}{1 + \psi(u)}\right) du = \infty,$$
where $\psi(u) = \log \mathbb{E}[e^{iuX_1}]$ (see [29, Theorem 7.12]);
2. *X is a finite variation process.*

Additionally, assume that the upper bound of the approximation error $e_p(x, y, t)$ satisfies

$$\lim_{t \downarrow 0} \frac{1}{t} \sup_{x, y \in (a', b')} e_p(x, y, t) = 0, \qquad \forall a', b' \in (a, b). \tag{4.11}$$

Then, Algorithm 1 terminates in finite time a.s.

(ii) Assume that $\mathbb{E}|F(X_1)| < \infty$. Let $\mathcal{X} = \{(T_i, X_{T_i})\}_{i=0}^N$ be a skeleton of X on $[0, 1]$ satisfying (4.10) and $\tilde{N}(\mathcal{X})$ be given by (4.8). Then,

$$|\mathbb{E}[F(X_1)\mathbf{1}_{\tau > 1}] - \mathbb{E}[F(X_1)\, \tilde{N}(\mathcal{X})]| \leq \gamma \mathbb{E}[|F(X_1)|]. \tag{4.12}$$

Remark 4.3. *In view of Proposition 4.2,* $\mathbb{E}[F(X_1)\mathbf{1}_{\tau>1}]$ *can be approximated by the Monte Carlo estimator*

$$\frac{1}{M}\sum_{k=1}^{M} F\left(X_1^{(k)}\right)\tilde{N}\left(\mathcal{X}^{(k)}\right),$$

where $X^{(k)}$ *are independent copies of the process* X *and* $\tilde{N}(\mathcal{X}^{(k)})$ *are corresponding values computed with formula (4.8). This estimator has a statistical error which can be estimated in the usual way, and a discretization bias, which is bounded from above by* $\gamma\mathbb{E}[|F(X_1)|]$*. In view of* (4.13) *below, a more precise a posteriori estimate of the bias is*

$$\frac{1}{M}\sum_{k=1}^{M}|F(X_1^{(k)})|\,\mathbf{1}_{S_N^{(k)}}\sum_{i=1}^{N} e_p\left(X^{(k)}_{T_i^{(k)}}, X^{(k)}_{T_{i+1}^{(k)}}, T_{i+1}^{(k)}-T_i^{(k)}\right), \quad \textit{with} \quad S_N := \{(X_{T_0},\dots,X_{T_N})\in D^{N+1}\}.$$

Lemma 4.4. *Let* X *be a Lévy process such that for all* $t>0$*, the law of* X_t *has no atom. Then, for all* $x\in\mathbb{R}$*,*

$$\mathbb{P}[\{t\in[0,1]:\Delta X_t\neq 0, X_{t-}=x\}=\emptyset]=1; \quad \mathbb{P}[\{t\in[0,1]:\Delta X_t\neq 0, X_t=x\}=\emptyset]=1.$$

Proof. We only prove the first identity, the second one follows by similar arguments (or alternatively by time reversal). Let $N_1^\varepsilon = \#\{t\in[0,1]:|\Delta X_t|>\varepsilon, X_{t-}=x\}$. Then

$$\mathbb{P}[\{t\in[0,1]:\Delta X_t\neq 0, X_{t-}=x\}\neq\emptyset]\le \mathbb{E}[N_1^0]\le\sum_{n=1}^{\infty}\mathbb{E}[N_1^{\frac{1}{n}}].$$

But by the compensation formula (see [6, section 0.5]),

$$\mathbb{E}[N_1^{\frac{1}{n}}]=\mathbb{E}\left[\int_0^1\int_{|y|>\varepsilon}1_{X_s=x}\nu(dy)ds\right]=\int_{|y|>\varepsilon}\nu(dy)\int_0^1\mathbb{P}[X_s=x]ds=0.$$

□

Proof of Proposition 4.2*. Part (i).* With the aim of obtaining a contradiction, assume that the statement of the proposition is not true, and the algorithm does not terminate. Let $\{\tilde{T}_i\}_{i\ge1}$ be the infinite sequence of different sampling times produced by the algorithm (in the order in which they were generated, that is, not necessarily ordered in time). Let $\tilde{X}_i := X_{\tilde{T}_i}$ be the corresponding sampling observations. Since the sequence $\{\tilde{T}_i\}$ is bounded, we can find indices $\{i_k\}_{k\ge1}$ such that $\tilde{T}_{i_k}\to T^*$. Moreover, since every point $\tilde{T}_i$ (for $i\ge2$) is obtained as a midpoint of a certain interval, we can find two sequences $\{T_i^-\}$ and $\{T_i^+\}$ such that $T_i^-\uparrow T^*$, $T_i^+\downarrow T^*$, $T^*\in[T_i^-,T_i^+]$ for all i and and $e_p(X_{T_i^-},X_{T_i^+},T_i^+-T_i^-)>\gamma(T_i^+-T_i^-)$ for all i. In addition, since the process X has right and left limits, both $\lim X_{T_i^+}=X^+$ and $\lim X_{T_i^-}=X^-$ exist. There are three possibilities.

If $X^-\in(-\infty,a)\cup(b,\infty)$ or $X^+\in(-\infty,a)\cup(b,\infty)$ then for some i, $\tilde{X}_i\notin D$, so that the algorithm must have stopped in finite time and we have a contradiction.

If $X^-\in(a,b)$ and $X^+\in(a,b)$ then, using the property (4.11), we can find a contradiction with $e_p(X_{T_i^-},X_{T_i^+},T_i^+-T_i^-)>\gamma(T_i^+-T_i^-)$.

It remains to treat the case when X^- or X^+, or both, are at the boundary of D. Then, either $X^-=X^+=X_{T^*}$ or $\Delta X_{T^*}\neq0$. The latter case is ruled out by Lemma 4.4 and in the case when X cannot hit points, the former case is ruled out as well.

We may therefore assume that X is a finite variation process with nonzero drift μ (cf. [29, Theorem 7.12]) and, to fix the notation, that $X^-=X^+=X_{T^*}=b$. We may also assume that T^* is irrational, since for every $t\in\mathbb{Q}\cap[0,1]$, $\mathbb{P}[X_t=b]=0$. The fact that $T^*\notin\mathbb{Q}$ implies that $T_i^-<T^*<T_i^+$ for every i, and we can also assume that $X_{T_i^+}$ and $X_{T_i^-}$ belong to D for each i, because otherwise the algorithm would have stopped in finite time.

Introduce two sequences of stopping times:

$$\tau_0=\inf\{t>0:X_t\ge b\}\wedge1,\quad \sigma_n=\inf\{t>\tau_n:X_t\le b\}\wedge1,\quad \tau_{n+1}=\inf\{t>\sigma_n:X_t\ge b\}\wedge1,\quad n\ge0.$$

The sequences $\{\tau_n\}$ and $\{\sigma_n\}$ do not have an accumulation point except $t = 1$ and for each $n \geq 0$, $\sigma_n > \tau_n$ if $\tau_n < 1$ and $\tau_{n+1} > \sigma_n$ if $\sigma_n < 1$. This holds because for a finite variation process X with drift $\mu \neq 0$, $\{0\}$ is irregular for $[0, \infty)$ if $\mu < 0$ and for $(-\infty, 0]$ if $\mu > 0$ [40, Theorem 43.20], and X may only creep in the direction opposite to the drift [29, Theorem 7.11]. Then clearly, for every $\tau \in [0, 1]$ such that $X_\tau = b$, either there is $n \geq 0$ with $\sigma_n = \tau$, which means that for some $\varepsilon > 0$, $X_t \notin D$ for $t \in (\tau - \varepsilon, \tau)$, or there is $n \geq 0$ with $\tau_n = \tau$, which means that for some $\varepsilon > 0$, $X_t \notin D$ for $t \in (\tau, \tau + \varepsilon)$. In both cases, there is a contradiction with the fact that $X_{T_i^+}$ and $X_{T_i^-}$ belong to D for each i.

Part (ii). Below, we denote $\bar{p}(x, y, t) := 1 - p(x, y, t)$, $\bar{\tilde{p}}(x, y, t) = 1 - \tilde{p}(x, y, t)$, and $S_N := \{(X_{T_0}, \ldots, X_{T_N}) \in D^{N+1}\}$. Then, since

$$N(\mathcal{X}) - \tilde{N}(\mathcal{X}) = \prod_{i=0}^{N-1} \bar{p}\left(X_{T_i}, X_{T_{i+1}}, T_{i+1} - T_i\right) - \prod_{i=0}^{N-1} \bar{\tilde{p}}\left(X_{T_i}, X_{T_{i+1}}, T_{i+1} - T_i\right),$$

we get

$$|\mathbb{E}[F(X_1)\mathbf{1}_{\tau>1}] - \mathbb{E}[F(X_1)\tilde{N}(\mathcal{X})]| \leq \mathbb{E}\Bigg[|F(X_1)|\mathbf{1}_{S_N} \sum_{i=0}^{N-1} e_p(X_{T_i}, X_{T_{i+1}}, T_{i+1} - T_i)\Bigg], \tag{4.13}$$

which can be bounded by $\gamma E\left[|F(X_1)|\right]$. □

Simulation of Lévy bridges The adaptive method presented in this section requires fast simulation from the bridge distribution of $X_{t/2}$ conditioned to $X_t = y$ (with $t > 0$), whose density is given by (4.9). We now propose a simple yet efficient method for simulating from the bridge distribution, valid for Lévy processes with unimodal density at all times. As remarked in Section 3, a sufficient condition for a Lévy process to have a unimodal density for all $t > 0$ is that it belongs to the class of self-decomposable processes which includes most of the parametric models used in the literature. The algorithm is based on the following simple estimate.

Proposition 4.5. *Let X be a Lévy process such that the density f_t of X_t is unimodal for all $t > 0$. Then,*

$$f^{br}_{t/2}(x, y) \leq \frac{f_{t/2}(y/2)}{f_t(y)} \max\{f_{t/2}(x), f_{t/2}(y - x)\}. \tag{4.14}$$

Proof. For all x and y,

$$f_{t/2}(x)f_{t/2}(y - x) = \max\{f_{t/2}(x), f_{t/2}(y - x)\} \min\{f_{t/2}(x), f_{t/2}(y - x)\}$$

By the assumption of unimodality, the density f_t may not have a local minimum, hence, for all a, b, $\min(f_{t/2}(a), f_{t/2}(b)) \leq f_{t/2}\left(\frac{a+b}{2}\right)$ and the result follows. □

As a consequence of the previous result, random variates with density $f^{br}_{t/2}(x, y)$ can be simulated using the classical rejection method [14], with the proposal density given by $\bar{f}(x) = \frac{1}{2}(f_{t/2}(x) + f_{t/2}(y - x))$, provided that the following two requirements are met:

(a) random variates with density $f_t(x)$ can be simulated in bounded time;

(b) the density $f_t(x)$ is known explicitly or can be evaluated in bounded time,

Assumptions (a) and (b) are satisfied, e.g., for the variance gamma process, normal inverse gaussian process, or for stable processes. Simulating a random variable X with density $\bar{f}(x) = \frac{1}{2}(f_{t/2}(x) + f_{t/2}(y - x))$ is straightforward: simulate a random variate Z with density $f_{t/2}$ and an independent Bernoulli random variate U; then, take $X = Z$ if $U = 0$ and $X = y - Z$ otherwise.

The expected number of iterations needed until the acceptance for a given value of y is equal to $C = \frac{2f_{t/2}(y/2)}{f_t(y)}$. This number is bounded for Lévy processes with Pareto tails such as stable. For processes with lighter tails it may be unbounded for large y, but the probability of having a large value of y in an adaptive simulation is very small. For example, if we want to simulate $X_{t/2}$ and X_t by first simulating X_t and then $X_{t/2}$ from the bridge law using formula (4.14), we find that the conditional expectation of the number of iterations given X_t equals $\frac{2f_{t/2}(X_t/2)}{f_t(X_t)}$, and the unconditional expectation is

$$\mathbb{E}\left[\frac{2f_{t/2}(X_t/2)}{f_t(X_t)}\right] = 2\int_{\mathbb{R}} f_{t/2}(x/2)dx = 4.$$

5 Numerical illustrations

In this section, to simplify the discussion, we assume that the interval D is of the form $D = (-\infty, b)$. For the numerical implementation of Algorithm 1 given in Section 4, one needs to be able to perform the following computations efficiently:

- Simulation of the increments of X_t for arbitrary t;
- Evaluation of the density f_t of X_t for arbitrary t;
- Evaluation of the "incomplete convolution" of the Lévy density: $\mathcal{C}(b,y) := \int_b^\infty s(v)s(y-v)dv$;
- Evaluation of the error bound $e_p(x,y,t)$, appearing in Algorithm 1.

These computations can be performed relatively easily, for example, for α-stable Lévy processes with Lévy density $s(x) = |x|^{-\alpha-1}(c_-\mathbf{1}_{x<0} + c_+\mathbf{1}_{x>0})$ and for the variance gamma process with Lévy density $s(x) = |x|^{-1}(ce^{-\lambda_-|x|}\mathbf{1}_{x<0} + ce^{-\lambda_+|x|}\mathbf{1}_{x>0})$. For α-stable processes, the increments can be simulated with an explicit algorithm (cf. [9]), the density can be computed using a rapidly convergent series [39] or expressed via special functions (cf. [23]), tabulated for $t = 1$ and computed by the scaling property for other values of t. The incomplete convolution is given by

$$\mathcal{C}(b,y) = c_+c_-b^{-1-2\alpha}B(1+2\alpha,1)F\left(1+\alpha, 1+2\alpha, 2+2\alpha, \frac{y}{b}\right), \tag{5.1}$$

where B is the beta function and F is the hypergeometric function, for which a rapidly converging series is available [24] and which can also be tabulated prior to the Monte Carlo computation. For the variance gamma process, the density is explicit and the increments are straightforward to simulate [12]. The incomplete convolution is given by

$$\mathcal{C}(b,y) = \frac{c^2}{y}\left\{e^{-y\lambda_+}\mathrm{Ei}(\lambda(b-y)) - e^{y\lambda_-}\mathrm{Ei}(\lambda b)\right\}$$

where $\mathrm{Ei}(x) := \int_x^\infty \frac{e^{-z}}{z}dz$, which can also be tabulated, and $\lambda := \lambda_- + \lambda_+$. The error bound e_p for the α-stable or the variance gamma process can be obtained along the lines of the general computation of Section 3 or the specific computation for the Cauchy process in the Appendix C.

For the numerical simulations in this section we shall concentrate on the Cauchy process, which is an α-stable process with $c_+ = c_- := c$ and $\alpha = 1$. For this process, formula (5.1) simplifies to

$$\mathcal{C}(b,y) = \frac{c^2}{3b^3}\left\{1 + 3\sum_{n=1}^{\infty}\frac{n+1}{n+3}\left(\frac{y}{b}\right)^n\right\} = \frac{c^2}{b^3}\left\{1 + \frac{b}{y} + \frac{2b^2}{y^2} + \frac{y}{b-y} + \frac{2b^3}{y^3}\log\left(1-\frac{y}{b}\right)\right\}.$$

Note that for small y, the series representation has more stable behavior than the exact formula. The error estimate e_p is computed as explained in section C of the Appendix. In both examples below, we take $c = 1$.

Example 1. In our first example, we evaluate the probability $\mathbb{P}[\sup_{0\le s\le 1} X_s \le 1] = \mathbb{P}(\tau > 1)$, which can be expressed in terms of the function (4.1) by taking $T = 1$, $F(X_1) = 1$, and the domain $(a,b) = (-\infty, 1)$. Note that in this case, the starting value of the process is relatively far from the boundary, and hence the advantage of using the adaptive algorithm is less important. The process will typically cross the boundary by a large jump with a large overshoot, which makes the exit easy to detect, even with a uniform discretization.

We study the performance of our adaptive algorithm for various values of γ, and compare it to the standard uniform discretization. When interpreting the results of simulations, one needs to distinguish between the actual error (that is, the difference between the computed value and the true value), and the theoretical value of the bias (computed as

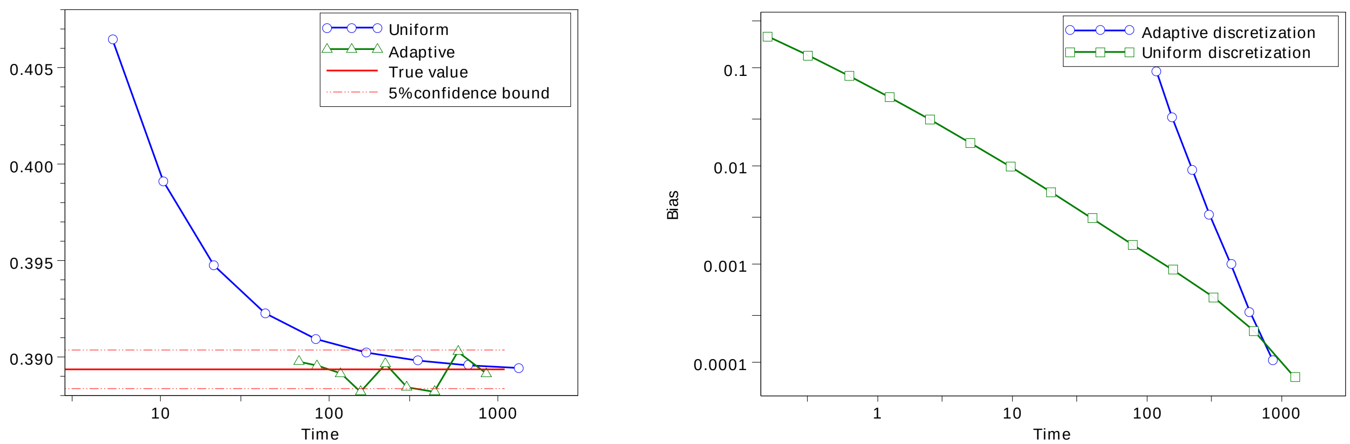

Figure 1: Illustration for Example 1. Left: values returned by the uniform discretization algorithm and the adaptive algorithm, as function of the computational time for 10^6 paths, measured in seconds. Different points on the graph correspond to different numbers of discretization dates for the uniform discretization (ranging from 32 to 8192) and different values of the tolerance parameter γ for the adaptive algorithm (ranging from 7 to 7×10^{-4}). The curve for the uniform discretization is smooth because all the points have been generated using the same trajectories, while for the adaptive discretization different paths have been used. Right: comparison of the theoretical bias of the adaptive algorithm with the actual discretization bias of the uniform discretization.

explained in Remark 4.3 above), which does not require the knowledge of the true value. As an estimate of the true value, we use the value computed in an independent simulation by uniform discretization with 16384 points and 10^7 trajectories, which is approximately equal to 0.38935 with a standard deviation of 10^{-4}. The difference between the values for 8192 and 16384 points (on the same trajectories) is smaller than 10^{-4}, hence one can presume that, for all practical purposes, convergence up to this precision has been achieved.

Figure 1 shows the dependence of the values computed by the two algorithms on the computational time required for 10^6 MC trajectories, for different numbers of discretization points (for the uniform discretization) and different values of the tolerance parameter γ (for the adaptive algorithm). While the uniform discretization algorithm exibits a clear bias which decreases as the number of discretization dates increases, the adaptive algorithm removes the bias completely; all values returned by this algorithm are within confidence bounds of the true value.

The theoretical bias, computed as explained in Remark 4.3, is greater than the actual error, because the error estimates of Appendix C are upper bounds, and because it does not take into account the possible cancellation of errors on different intervals. Figure 1, right graph, compares the theoretical estimate of the bias of the adaptive algorithm with the actual bias of the uniform discretization. One can see that for small computational times, the theoretical bias for the adaptive algorithm is greater than the error of the uniform discretization, however, the theoretical bias converges to zero much faster, and for relatively large computational times is actually smaller than the error of the uniform discretization. The empirical convergence rate (estimated from the slope of the straight lines) is $T^{-0.81}$ for the uniform discretization and $T^{-3.4}$ for the theoretical bias of the adaptive algorithm.

Example 2. In our second example, we evaluate the probability $\mathbb{P}[\sup_{0\leq s\leq 1} X_s \leq 10^{-2}]$, which again can be expressed in terms of the function (4.1) by taking $T = 1$, $F(X_1) = 1$, and the domain $(a, b) = (-\infty, 10^{-2})$. In contrast to Example 1, here we consider a situation where the starting point is close to the boundary. In this case, as we shall see below, the advantage of the adaptive algorithm is more striking, since the process can cross the boundary and come back while it is still close to the starting point and, hence, a very fine discretization will be necessary to detect this event with uniformly spaced observations. As a result, for the uniform discretization we do not observe convergence to a sufficient precision even with 16384 points, and therefore the true value cannot be estimated as in the previous example. Instead, we shall use as the true value the value produced by the adaptive algorithm with 10^7 Monte Carlo paths and equal to

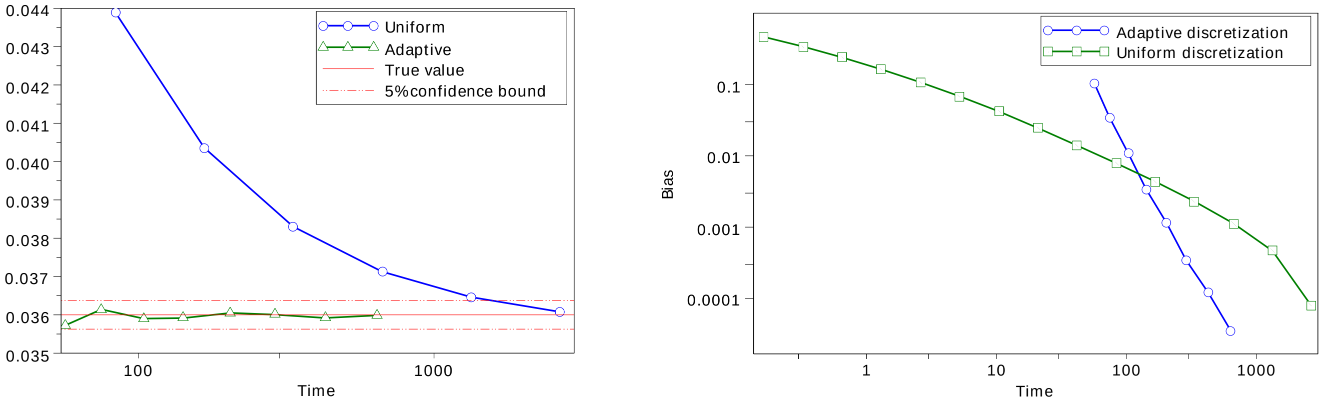

Figure 2: Illustration for Example 2. Left: values returned by the uniform discretization algorithm and the adaptive algorithm, as function of the computational time for 10^6 paths, measured in seconds. Different points on the graph correspond to different numbers of discretization dates for the uniform discretization (ranging from 256 to 16384) and different values of the tolerance parameter γ for the adaptive algorithm (ranging from 9 to 9×10^{-3}). Right: comparison of the theoretical bias of the adaptive algorithm with the actual discretization bias of the uniform discretization.

0.0360, with standard deviation of 6×10^{-5} and theoretical bias of 3×10^{-5}.

Similarly to the previous example, Figure 2 shows the dependence of the values computed by the two algorithms on the computational time required for 10^6 MC trajectories. Here, the adaptive algorithm exibits the same kind of behavior as in the Example 1 above: all the points generated by the algorithm are within the confidence bounds of the true value. However, for the uniform discretization, the convergence is much slower than before and only the last value obtained with 16384 discretization points falls within the confidence bounds. Figure 2, right graph, compares the theoretical estimate of the bias of the adaptive algorithm with the actual bias of the uniform discretization. Once again, the behavior of the adaptive algorithm is roughly the same as in the previous example, showing that the method is robust with respect to the parameters on the problem. On the other hand, as expected, the uniform discretization presents a significant bias in this case (the convergence rates are similar to those obtained in the previous example, but the constant for the uniform discretization is much bigger).

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A Proofs of Section 2

A.1 Proof of Theorem 2.2

Throughout the proof, we shall use the notation

$$\bar{Y}_t := \sup_{0\le s\le t} Y_s \qquad \text{and} \qquad \underline{Y}_t := \inf_{0\le s\le t} Y_s, \tag{A.1}$$

for a given cádlág process $(Y_t)_{t\ge0}$. Without loss of generality (by considering separately the positive and the negative part), we can and will assume that φ is nonnegative. Additionally, assume that $a \in (-\infty, 0)$ and $b \in (0, \infty)$. The cases $a = -\infty$ and $b = \infty$ will be evident from the proof below. We also let $\|\varphi\|_\infty :=$ ess $\sup_x \varphi(x)$, $\|\varphi\|_{\mathrm{Lip}}$ be the Lipschitz norm of φ, $I_\delta(y) := (y-\delta, y+\delta)$, $\eta := \delta_0/2$, $c = b \wedge |a|$, $B := \{\tau \le t\} = \{\bar{X}_t \ge b \text{ or } \underline{X}_t \le a\}$, $U_t^\varepsilon := \sup_{s\le t} |X_s^\varepsilon|$, and $a_\varepsilon := \sup_x s_\varepsilon(x)$, which are finite in light of (2.1). In what follows, $\mathcal{F}_t^\varepsilon := \sigma\left(X_s^\varepsilon : s \le t\right) \vee \mathcal{N}$ where $\mathcal{N}$ denotes the null sets of $\mathcal{F}$. To lighten the notation below, whenever the ess sup of a function g, defined $\mathcal{L}$-a.e. in some region, is considered, we shall simply write $\sup_u g(u)$ instead of ess $\sup_u g(u)$.

The idea is to condition on the number of jumps of the compound Poisson component Z^ε. To this end, let us denote

$$A_k(t) = \mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\{\tau\le t, X_t\in I_\delta(y), N_t^\varepsilon = k\}}\right), \quad \text{for } k=0,1,2, \quad \text{and} \quad A_3(t) = \mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\{\tau\le t, X_t\in I_\delta(y), N_t^\varepsilon \ge 3\}}\right),$$

so that clearly

$$\mathbb{E}(\varphi(X_\tau)\mathbf{1}_{\{\tau\le t, X_t\in I_\delta(y)\}}) = A_0(t)+\cdots+A_3(t). \tag{A.2}$$

Note that each of the terms on the right-hand side of the previous equation can be expressed as

$$A_k(t) = \int_{y-\delta}^{y+\delta} P_t^k(u)du, \quad (k=0,\dots,3), \tag{A.3}$$

for some nonnegative functions $P_t^k(u)$. Indeed, for $k=0,1,2$, by the standard definition of conditional expectation,

$$\begin{aligned} A_k(t) &= \mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\{\tau\le t, X_t\in I_\delta(y), N_t^\varepsilon = k\}}\right) = \mathbb{E}\left(\mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\{\tau\le t, N_t^\varepsilon=k\}}\middle|\, X_t\right)\mathbf{1}_{\{X_t\in I_\delta(y)\}}\right) \\ &= \int_{y-\delta}^{y+\delta} \mathbb{E}(\varphi(X_\tau)\mathbf{1}_{\{\tau\le t, N_t^\varepsilon=k\}}\,|\, X_t=u) f_t(u)du =: \int_{y-\delta}^{y+\delta} P_t^k(u)du. \end{aligned} \tag{A.4}$$

The case $k=3$ is treated in the same way. Let us analyze each of the four terms in the right-hand side of (A.2).

(1) No big jump. Note that, on the event $N_t^\varepsilon = 0$, $X_s = X_s^\varepsilon$ for all $s\le t$ and, thus, $\{\tau\le t\} = \{\tau^\varepsilon \le t\}$, where $\tau^\varepsilon := \inf\{u\ge 0 : X_u^\varepsilon \notin (a,b)\}$. Therefore,

$$A_0(t) = \mathbb{E}\left(\varphi(X_{\tau^\varepsilon}^\varepsilon)\mathbf{1}_{\{\tau^\varepsilon\le t, X_t^\varepsilon\in I_\delta(y), N_t^\varepsilon=0\}}\right) = \mathbb{E}\left(\varphi(X_{\tau^\varepsilon}^\varepsilon)\mathbf{1}_{\{\tau^\varepsilon\le t, X_t^\varepsilon\in I_\delta(y)\}}\right)\mathbb{P}\left(N_t^\varepsilon = 0\right),$$

where in the last equality we used the independence of X^ε and N^ε. Next, conditioning on $\mathcal{F}_{\tau^\varepsilon}^\varepsilon$, it follows that

$$A_0(t) = e^{-\lambda_\varepsilon t}\mathbb{E}(\varphi(X_{\tau^\varepsilon}^\varepsilon)\mathbf{1}_{\{\tau^\varepsilon\le t, X_t^\varepsilon\in I_\delta(y)\}}) = e^{-\lambda_\varepsilon t}\mathbb{E}\left(\mathbb{E}(\mathbf{1}_{\{X_t^\varepsilon\in I_\delta(y)\}}\,|\,\mathcal{F}_{\tau^\varepsilon}^\varepsilon)\,\varphi(X_{\tau^\varepsilon}^\varepsilon)\mathbf{1}_{\{\tau^\varepsilon\le t\}}\right).$$

By Markov's property,

$$A_0(t) = e^{-\lambda_\varepsilon t}\mathbb{E}\left(\mathbb{E}\left(\mathbf{1}_{\{X_t^\varepsilon - X_{\tau^\varepsilon}^\varepsilon + X_{\tau^\varepsilon}^\varepsilon\in I_\delta(y)\}}\middle|\,\mathcal{F}_{\tau^\varepsilon}^\varepsilon\right)\varphi(X_{\tau^\varepsilon}^\varepsilon)\mathbf{1}_{\{\tau^\varepsilon\le t\}}\right) = e^{-\lambda_\varepsilon t}\mathbb{E}\left(F\left(X_{\tau^\varepsilon}^\varepsilon, t-\tau^\varepsilon\right)\varphi(X_{\tau^\varepsilon}^\varepsilon)\mathbf{1}_{\{\tau^\varepsilon\le t\}}\right),$$

where $F(z,s) = \mathbb{P}\left(z + X_s^\varepsilon \in I_\delta(y)\right)$. Note that if $\tau^\varepsilon = t$, then $F\left(X_{\tau^\varepsilon}^\varepsilon, t-\tau^\varepsilon\right) = 0$ since $X_{\tau^\varepsilon}^\varepsilon \in (a,b)^c$ and $I_\delta(y)\subset(a,b)$. On the other hand, on the event $\tau^\varepsilon < t$,

$$F\left(X_{\tau^\varepsilon}^\varepsilon, t-\tau^\varepsilon\right) = \int_{y-\delta}^{y+\delta} f_{t-\tau^\varepsilon}^\varepsilon(u - X_{\tau^\varepsilon}^\varepsilon)du \le \int_{y-\delta}^{y+\delta} \sup_{0<s\le t}\sup_{x\in(a,b)^c} f_s^\varepsilon(u-x)du,$$

since again $X_{\tau^\varepsilon}^\varepsilon\in(a,b)^c$. Putting the two previous cases together and recalling (A.3), we have

$$A_0(t) = \int_{y-\delta}^{y+\delta} P_t^0(u)du \le \int_{y-\delta}^{y+\delta}\left(e^{-\lambda_\varepsilon t}\|\varphi\|_\infty \sup_{0<s\le t}\sup_{x\in(a,b)^c} f_s^\varepsilon(u-x)\right)du =: \int_{y-\delta}^{y+\delta}\bar{P}_t^0(u)du, \tag{A.5}$$

implying that $P_t^0(u)\le \bar{P}_t^0(u)$, for $\mathcal{L}$-a.e. $u\in(a+\delta_0, b-\delta_0)$. Furthermore, using (2.4-ii),

$$\sup_{a+\delta_0<u<b-\delta_0} P_t^0(u) \le \sup_{a+\delta_0<u<b-\delta_0}\bar{P}_t^0(u) \le \|\varphi\|_\infty c_3(\delta_0,\varepsilon)t^3, \qquad (t<t_0).$$

(2) One big jump. Let τ_i and Y_i be the time and size of the i^{th} jump of Z^ε. Clearly, on the event $\{N_t^\varepsilon = 1\}$,

$$\begin{aligned} \varphi(X_\tau)\mathbf{1}_{\{\tau\le t, X_t\in I_\delta(y), N_t^\varepsilon=1\}} &= \varphi(X_\tau^\varepsilon)\mathbf{1}_{\{\tau<\tau_1, X_t^\varepsilon + Y_1\in I_\delta(y), N_t^\varepsilon=1\}} + \varphi(X_\tau^\varepsilon + Y_1)\mathbf{1}_{\{\tau_1\le\tau\le t, X_t^\varepsilon+Y_1\in I_\delta(y), N_t^\varepsilon=1\}} \\ &\le \|\varphi\|_\infty \mathbf{1}_{\{X_t^\varepsilon+Y_1\in I_\delta(y), N_t^\varepsilon=1\}}\left(\mathbf{1}_{\{\bar{X}_t^\varepsilon\ge b \text{ or } \underline{X}_t^\varepsilon\le a\}} + \mathbf{1}_{\{\bar{X}_t^\varepsilon+Y_1\ge b \text{ or } \underline{X}_t^\varepsilon+Y_1\le a\}}\right). \end{aligned}$$

It follows that

$$
\begin{aligned}
0 \le A_1(t) &\le \|\varphi\|_\infty \mathbb{E}\left(\mathbf{1}_{\{U_t^\varepsilon \ge c, X_t^\varepsilon + Y_1 \in I_\delta(y), N_t^\varepsilon = 1\}}\right) + \|\varphi\|_\infty \mathbb{E}\left(\mathbf{1}_{\left\{Y_1 \ge b - \bar{X}_t^\varepsilon \text{ or } Y_1 \le a - \underline{X}_t^\varepsilon\right\}} \mathbf{1}_{\{X_t^\varepsilon + Y_1 \in I_\delta(y), N_t^\varepsilon = 1\}}\right) \\
&= \underbrace{e^{-\lambda_\varepsilon t}\|\varphi\|_\infty \lambda_\varepsilon t\, \mathbb{E}\left(\mathbf{1}_{\{U_t^\varepsilon \ge c, X_t^\varepsilon + Y_1 \in I_\delta(y)\}}\right)}_{A_{1,1}(t)} + \underbrace{e^{-\lambda_\varepsilon t}\|\varphi\|_\infty \lambda_\varepsilon t\, \mathbb{E}\left(\mathbf{1}_{\left\{Y_1 \ge b - \bar{X}_t^\varepsilon \text{ or } Y_1 \le a - \underline{X}_t^\varepsilon\right\}} \mathbf{1}_{\{X_t^\varepsilon + Y_1 \in I_\delta(y)\}}\right)}_{A_{1,2}(t)},
\end{aligned}
$$

where in the last equality we use the joint independence of N^ε, Y_1, and X^ε. Conditioning on $\sigma(X_s^\varepsilon : s \ge 0)$ and applying Fubini,

$$
A_{1,1}(t) = e^{-\lambda_\varepsilon t}\|\varphi\|_\infty t \mathbb{E}\left(\mathbf{1}_{\{U_t^\varepsilon \ge c\}} \int_{y-\delta-X_t^\varepsilon}^{y+\delta-X_t^\varepsilon} s_\varepsilon(v) dv\right) = \int_{y-\delta}^{y+\delta} \underbrace{e^{-\lambda_\varepsilon t}\|\varphi\|_\infty t \mathbb{E}\left(\mathbf{1}_{\{U_t^\varepsilon \ge c\}} s_\varepsilon(u - X_t^\varepsilon)\right)}_{\bar{P}_t^{1,1}(u)} du. \tag{A.6}
$$

Using (2.1) and Lemma 2.1, $\sup_u \bar{P}_t^{1,1}(u) \le e^{-\lambda_\varepsilon t} t \|\varphi\|_\infty a_\varepsilon \mathbb{P}\left(U_t^\varepsilon \ge c\right) \le e^{-\lambda_\varepsilon t} a_\varepsilon \|\varphi\|_\infty C_2(c,\varepsilon) t^3$, where $\varepsilon > 0$ is chosen small enough. Similarly, conditioning on $\sigma(X_s^\varepsilon : s \ge 0)$, making the substitution $u = X_t^\varepsilon + v$, and applying Fubini,

$$
\begin{aligned}
A_{1,2}(t) &= e^{-\lambda_\varepsilon t}\|\varphi\|_\infty t \mathbb{E}\left(\int \mathbf{1}_{\{v \le a - \underline{X}_t^\varepsilon \text{ or } v \ge b - \bar{X}_t^\varepsilon\}} \mathbf{1}_{\{y - \delta < X_t^\varepsilon + v \le y + \delta\}} s_\varepsilon(v) dv\right) \\
&= \int_{y-\delta}^{y+\delta} \underbrace{e^{-\lambda_\varepsilon t}\|\varphi\|_\infty t \mathbb{E}\left(\mathbf{1}_{\{u \le a + X_t^\varepsilon - \underline{X}_t^\varepsilon \text{ or } u \ge b + X_t^\varepsilon - \bar{X}_t^\varepsilon\}} s_\varepsilon(u - X_t^\varepsilon)\right)}_{\bar{P}_t^{1,2}(u)} du.
\end{aligned} \tag{A.7}
$$

Using again Lemma 2.1,

$$
\begin{aligned}
\sup_{u \in (a+\delta_0, b-\delta_0)} \bar{P}_t^{1,2}(u) &\le e^{-\lambda_\varepsilon t}\|\varphi\|_\infty t a_\varepsilon \mathbb{P}\left(X_t^\varepsilon - \underline{X}_t^\varepsilon \ge \delta_0 \text{ or } \bar{X}_t^\varepsilon - X_t^\varepsilon \ge \delta_0\right) \le e^{-\lambda_\varepsilon t}\|\varphi\|_\infty t a_\varepsilon \mathbb{P}\left(\bar{X}_t^\varepsilon - \underline{X}_t^\varepsilon \ge \delta_0\right) \\
&\le e^{-\lambda_\varepsilon t}\|\varphi\|_\infty t a_\varepsilon \mathbb{P}\left(\sup_{s \le t} |X_s^\varepsilon| \ge \delta_0/2\right) \le e^{-\lambda_\varepsilon t}\|\varphi\|_\infty a_\varepsilon C_2(\delta_0/2, \varepsilon) t^3.
\end{aligned} \tag{A.8}
$$

Therefore, recalling from (A.3), the nonnegative function $P_t^1(u)$ is such that,for $\mathcal{L}$-a.e. $u \in (a+\delta_0, b-\delta_0)$, $0 \le P_t^1(u) \le \sum_{\ell=1}^2 \bar{P}_t^{1,\ell}(u) \le \|\varphi\|_\infty a_\varepsilon t^3 \left(C_2(c,\varepsilon) + C_2(\eta,\varepsilon)\right)$.

(3) Two big jumps. As before, let τ_i and Y_i be the time and size of the i^{th} jump of Z^ε. Clearly,

$$
\begin{aligned}
\varphi(X_\tau)\mathbf{1}_{\{\tau \le t, X_t \in I_\delta(y), N_t^\varepsilon = 2\}} &= \varphi(X_\tau^\varepsilon)\mathbf{1}_{\{\tau < \tau_1, X_t^\varepsilon + Y_1 + Y_2 \in I_\delta(y), N_t^\varepsilon = 2\}} \\
&\quad + \varphi(X_\tau^\varepsilon + Y_1)\mathbf{1}_{\{\tau_1 \le \tau < \tau_2, X_t^\varepsilon + Y_1 + Y_2 \in I_\delta(y), N_t^\varepsilon = 2\}} \\
&\quad + \varphi(X_\tau^\varepsilon + Y_1 + Y_2)\mathbf{1}_{\{\tau_2 \le \tau \le t, X_t^\varepsilon + Y_1 + Y_2 \in I_\delta(y), N_t^\varepsilon = 2\}} \\
&\le \|\varphi\|_\infty \mathbf{1}_{\{\exists s < \tau_1 : X_s^\varepsilon \notin (a,b); X_t^\varepsilon + Y_1 + Y_2 \in I_\delta(y); N_t^\varepsilon = 2\}} \\
&\quad + \varphi(X_\tau^\varepsilon + Y_1)\mathbf{1}_{\{\exists s \in [\tau_1, \tau_2) : X_s^\varepsilon + Y_1 \notin (a,b); X_t^\varepsilon + Y_1 + Y_2 \in I_\delta(y); N_t^\varepsilon = 2\}}, \\
&\quad + \|\varphi\|_\infty \mathbf{1}_{\{\exists s \in [\tau_2, t] : X_s^\varepsilon + Y_1 + Y_2 \notin (a,b); X_t^\varepsilon + Y_1 + Y_2 \in I_\delta(y); N_t^\varepsilon = 2\}}.
\end{aligned}
$$

Then, using the independence of N^ε, the Y_i's, and X^ε in the first and last terms, we have the inequality:

$$
\begin{aligned}
A_2(t) &\le e^{-\lambda_\varepsilon t}(t^2/2)\lambda_\varepsilon^2 \|\varphi\|_\infty \mathbb{E}\left(\mathbf{1}_{\{U_t^\varepsilon \ge c, X_t^\varepsilon + Y_1 + Y_2 \in I_\delta(y)\}}\right) \\
&\quad + \mathbb{E}\left(\varphi(X_\tau^\varepsilon + Y_1)\mathbf{1}_{\{\bar{X}_t^\varepsilon + Y_1 \ge b \text{ or } \underline{X}_t^\varepsilon + Y_1 \le a; X_t^\varepsilon + Y_1 + Y_2 \in I_\delta(y); N_t^\varepsilon = 2\}}\right) \\
&\quad + e^{-\lambda_\varepsilon t}(t^2/2)\lambda_\varepsilon^2 \|\varphi\|_\infty \mathbb{E}\left(\mathbf{1}_{\{\bar{X}_t^\varepsilon + Y_1 + Y_2 \ge b \text{ or } \underline{X}_t^\varepsilon + Y_1 + Y_2 \le a; X_t^\varepsilon + Y_1 + Y_2 \in I_\delta(y)\}}\right), \\
&=: A_{2,1}(t) + A_{2,2}(t) + A_{2,3}(t).
\end{aligned} \tag{A.9}
$$

As before, conditioning on $\sigma(X_s^\varepsilon : s \ge 0)$, changing variable from w to $u = X_t^\varepsilon + v + w$, and applying Fubini,

$$
\begin{aligned}
A_{2,1}(t) &= e^{-\lambda_\varepsilon t} 2^{-1} \|\varphi\|_\infty t^2 \mathbb{E}\left(\iint \mathbf{1}_{\{U_t^\varepsilon \ge c\}} \mathbf{1}_{\{y - \delta < X_t^\varepsilon + w + v < y + \delta\}} s_\varepsilon(v) s_\varepsilon(w) dv dw\right) \\
&= \int_{y-\delta}^{y+\delta} e^{-\lambda_\varepsilon t} 2^{-1} \|\varphi\|_\infty t^2 \int_{-\infty}^{\infty} s_\varepsilon(v) \mathbb{E}\left(\mathbf{1}_{\{U_t^\varepsilon \ge c\}} s_\varepsilon(u - X_t^\varepsilon - v)\right) dv du =: \int_{y-\delta}^{y+\delta} \bar{P}_t^{2,1}(u) du,
\end{aligned} \tag{A.10}
$$

and, hence,

$$\sup_u \bar{P}_t^{2,1}(u) \leq e^{-\lambda_\varepsilon t}2^{-1}\|\varphi\|_\infty t^2\lambda_\varepsilon a_\varepsilon \mathbb{P}\left(U_t^\varepsilon \geq c\right) \leq e^{-\lambda_\varepsilon t}2^{-1}\|\varphi\|_\infty \lambda_\varepsilon a_\varepsilon C_1(c,\varepsilon)t^3.$$

Similarly,

$$\begin{aligned}
A_{2,3}(t) &= e^{-\lambda_\varepsilon t}2^{-1}\|\varphi\|_\infty t^2\mathbb{E}\left(\iint \mathbf{1}_{\{\bar{X}_t^\varepsilon+v+w\geq b \text{ or } \underline{X}_t^\varepsilon+v+w\leq a\}}\mathbf{1}_{\{y-\delta<X_t^\varepsilon+w+v<y+\delta\}}s_\varepsilon(v)s_\varepsilon(w)dvdw\right)\\
&= \int_{y-\delta}^{y+\delta} e^{-\lambda_\varepsilon t}2^{-1}\|\varphi\|_\infty t^2\int \mathbb{E}\left(\mathbf{1}_{\{\bar{X}_t^\varepsilon-X_t^\varepsilon+u\geq b \text{ or } \underline{X}_t^\varepsilon-X_t^\varepsilon+u\leq a\}}s_\varepsilon(u-X_t^\varepsilon-v)\right)s_\varepsilon(v)dvdu\\
&=: \int_{y-\delta}^{y+\delta} \bar{P}_t^{2,3}(u)du, \qquad\qquad (A.11)
\end{aligned}$$

and, thus, as in (A.8),

$$\sup_{u\in[a+\delta_0,b-\delta_0]} \bar{P}_t^{2,3}(u) \leq e^{-\lambda_\varepsilon t}2^{-1}\|\varphi\|_\infty t^2\lambda_\varepsilon a_\varepsilon\mathbb{P}\left(X_t^\varepsilon-\underline{X}_t^\varepsilon\geq\delta_0 \text{ or } \bar{X}_t^\varepsilon-X_t^\varepsilon\geq\delta_0\right) \leq e^{-\lambda_\varepsilon t}2^{-1}\|\varphi\|_\infty\lambda_\varepsilon a_\varepsilon C_1(\delta_0/2,\varepsilon)t^3.$$

Finally, we provide an upper bound for $A_{2,2}(t)$. First, we use the bound $\varphi(X_\tau^\varepsilon+Y_1)\leq\varphi(Y_1)+\|\varphi\|_{\mathrm{Lip}}U_t^\varepsilon$ and again the independence of N^ε, the Y_i's, and X^ε to get

$$A_{2,2}(t)\leq e^{-\lambda_\varepsilon t}(t^2/2)\lambda_\varepsilon^2\mathbb{E}\left(\left\{\varphi(Y_1)+\|\varphi\|_{\mathrm{Lip}}U_t^\varepsilon\right\}\mathbf{1}_{\{\bar{X}_t^\varepsilon+Y_1\geq b \text{ or } \underline{X}_t^\varepsilon+Y_1\leq a;X_t^\varepsilon+Y_1+Y_2\in I_\delta(y)\}}\right).$$

Next, by conditioning on $\sigma(X_s^\varepsilon:s\geq 0)\vee\sigma(Y_1)$, we may write[2]

$$\begin{aligned}
A_{2,2}(t) &\leq e^{-\lambda_\varepsilon t}(t^2/2)\lambda_\varepsilon\mathbb{E}\left(\left\{\varphi(Y_1)+\|\varphi\|_{\mathrm{Lip}}U_t^\varepsilon\right\}\mathbf{1}_{\{\bar{X}_t^\varepsilon+Y_1\geq b \text{ or } \underline{X}_t^\varepsilon+Y_1<a\}}\int_{y-\delta-X_t^\varepsilon-Y_1}^{y+\delta-X_t^\varepsilon-Y_1}s_\varepsilon(w)dw\right)\\
&= e^{-\lambda_\varepsilon t}(t^2/2)\mathbb{E}\left(\int_{(a-\underline{X}_t^\varepsilon,b-\bar{X}_t^\varepsilon)^c}\left\{\varphi(v)+\|\varphi\|_{\mathrm{Lip}}U_t^\varepsilon\right\}s_\varepsilon(v)\int_{y-\delta-X_t^\varepsilon-v}^{y+\delta-X_t^\varepsilon-v}s_\varepsilon(w)dwdv\right) \qquad (A.12)\\
&= \int_{y-\delta}^{y+\delta}\underbrace{e^{-\lambda_\varepsilon t}2^{-1}t^2\mathbb{E}\left(\int_{(a-\underline{X}_t^\varepsilon,b-\bar{X}_t^\varepsilon)^c}\left\{\varphi(v)+\|\varphi\|_{\mathrm{Lip}}U_t^\varepsilon\right\}s_\varepsilon(v)s_\varepsilon(u-X_t^\varepsilon-v)dv\right)}_{\bar{P}_t^{2,2}(u)}du.
\end{aligned}$$

In order to find a lower bound for $A_2(t)$, note that

$$\begin{aligned}
\varphi(X_\tau)\mathbf{1}_{\{\tau\leq t,X_t\in I_\delta(y),N_t^\varepsilon=2\}} &\geq \varphi(X_\tau^\varepsilon+Y_1)\mathbf{1}_{\{\tau_1\leq\tau<\tau_2,X_t^\varepsilon+Y_1+Y_2\in I_\delta(y),N_t^\varepsilon=2\}}\\
&\geq \varphi(X_\tau^\varepsilon+Y_1)\mathbf{1}_{\{Y_1+\underline{X}_t^\varepsilon\geq b \text{ or } \bar{X}_t^\varepsilon+Y_1\leq a\}}\mathbf{1}_{\{\bar{X}_t^\varepsilon<b,\underline{X}_t^\varepsilon>a\}}\mathbf{1}_{\{X_t^\varepsilon+Y_1+Y_2\in I_\delta(y),N_t^\varepsilon=2\}}.
\end{aligned}$$

Using the previous inequality and the lower bound $\varphi(X_\tau^\varepsilon+Y_1)\geq\varphi(Y_1)-\|\varphi\|_{\mathrm{Lip}}U_t^\varepsilon$ together with the independence of N^ε, the Y_i's, and X^ε, it follows that

$$\begin{aligned}
A_2(t) &\geq e^{-\lambda_\varepsilon t}\frac{(\lambda_\varepsilon t)^2}{2}\mathbb{E}\left(\left\{\varphi(Y_1)-\|\varphi\|_{\mathrm{Lip}}U_t^\varepsilon\right\}\mathbf{1}_{\{Y_1\in(a-\bar{X}_t^\varepsilon,b-\underline{X}_t^\varepsilon)^c,\bar{X}_t^\varepsilon<b,\underline{X}_t^\varepsilon>a,X_t^\varepsilon+Y_1+Y_2\in I_\delta(y)\}}\right)\\
&= \int_{y-\delta}^{y+\delta}\underbrace{e^{-\lambda_\varepsilon t}2^{-1}t^2\mathbb{E}\left(\mathbf{1}_{\{\bar{X}_t^\varepsilon<b,\underline{X}_t^\varepsilon>a\}}\int_{(a-\bar{X}_t^\varepsilon,b-\underline{X}_t^\varepsilon)^c}\left\{\varphi(v)-\|\varphi\|_{\mathrm{Lip}}U_t^\varepsilon\right\}s_\varepsilon(v)s_\varepsilon(u-X_t^\varepsilon-v)dv\right)}_{\underline{P}_t^2(u)}du.
\end{aligned}$$

As it will be proved in Lemma A.1 below, $\bar{P}_t^{2,2}(u)$ and $\underline{P}_t^2(u)$ are such that

$$\lim_{t\to 0}\sup_{u\in(a+\delta_0,b-\delta_0)}\left|\frac{1}{t^2}\bar{P}_t^{2,2}(u)-\frac{1}{2}\int_{(a,b)^c}\varphi(v)s_\varepsilon(v)s_\varepsilon(u-v)dv\right|=0, \qquad (A.13)$$

$$\lim_{t\to 0}\sup_{u\in(a+\delta_0,b-\delta_0)}\left|\frac{1}{t^2}\underline{P}_t^2(u)-\frac{1}{2}\int_{(a,b)^c}\varphi(v)s_\varepsilon(v)s_\varepsilon(u-v)dv\right|=0. \qquad (A.14)$$

[2] Here and below we use the convention $(x,y)=\emptyset$ and $(x,y)^c=(-\infty,\infty)$ for $x>y$.

Using (A.9), (A.14) and the corresponding bounds for $\bar{P}_t^{2,1}(u)$ and $\bar{P}_t^{2,3}(u)$, it follows that the nonnegative function $P_t^2(u)$ defined in (A.3) is such that

$$\lim_{t\to 0}\sup_{u\in(a+\delta_0,b-\delta_0)}\left|\frac{1}{t^2}P_t^2(u)-\frac{1}{2}\int_{(a,b)^c}\varphi(v)s_\varepsilon(v)s_\varepsilon(u-v)dv\right|=0. \tag{A.15}$$

(4) Three or more big jumps. As before, we have the following bound

$$\begin{aligned}0\le\mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\{\tau\le t,X_t\in I_\delta(y),N_t^\varepsilon=n\}}\right)&\le\|\varphi\|_\infty\mathbb{E}\left(\mathbf{1}_{\{X_t^\varepsilon+\sum_{i=1}^n Y_i\in I_\delta(y),N_t^\varepsilon=n\}}\right)\\&=\|\varphi\|_\infty\mathbb{P}(N_t^\varepsilon=n)\int_{y-\delta}^{y+\delta}\mathbb{E}\left(s_\varepsilon^{*n}(u-X_t^\varepsilon)\right)du,\end{aligned}$$

Using the previous inequality and (A.3), we have

$$A_3(t)=\int_{y-\delta}^{y+\delta}P_t^3(u)du\le\int_{y-\delta}^{y+\delta}\left[\sum_{n=3}^\infty e^{-\lambda_\varepsilon t}\frac{t^n}{n!}\|\varphi\|_\infty\mathbb{E}\left(s_\varepsilon^{*n}(u-X_t^\varepsilon)\right)\right]du=:\int_{y-\delta}^{y+\delta}\bar{P}_t^3(u)du.$$

Since $\|s_\varepsilon^{*n}\|_\infty\le\lambda_\varepsilon^{n-1}a_\varepsilon$,

$$\sup_u\bar{P}_t^3(u)\le e^{-\lambda_\varepsilon t}a_\varepsilon\|\varphi\|_\infty\sum_{n=3}^\infty\frac{t^n}{n!}\lambda_\varepsilon^{n-1}\le C(\varepsilon)t^3, \tag{A.16}$$

for some constant $C(\varepsilon)<\infty$, and we conclude that $0\le P_t^3(u)\le C(\varepsilon)t^3$ for $\mathcal{L}$-a.e. u.

Putting the four previous steps together, we conclude that $\mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\{\tau\le t,X_t\in(y-\delta,y+\delta)\}}\right)=\int_{y-\delta}^{y+\delta}P_t(u)du$, for a function $P_t(u)$ such that

$$\lim_{t\to 0}\sup_{u\in(a+\delta_0,b-\delta_0)}\left|\frac{1}{t^2}P_t(u)-\frac{1}{2}\int_{(a,b)^c}\varphi(v)s_\varepsilon(v)s_\varepsilon(u-v)dv\right|=0.$$

Finally, it is easy to see that for any $u\in(a+\delta_0,b-\delta_0)$ and $a<0<b$, there exists an $\varepsilon_0>0$ small enough such that $\int_{(a,b)^c}\varphi(v)s_\varepsilon(v)s_\varepsilon(u-v)dv=\int_{(a,b)^c}\varphi(v)s(v)s(u-v)dv$, for all $0<\varepsilon<\varepsilon_0$. This concludes the proof. $\square$

Lemma A.1. *Verification of (A.13) and (A.14).*

Proof. Let $0<\varepsilon<1$ and $M_t^\varepsilon:=X_t^\varepsilon-\mu_\varepsilon t$ be the martingale component of X^ε. We shall analyze the expressions appearing inside the absolute values on the right hand side of equations (A.13) and (A.14). Define the random intervals $\bar{I}:=(a-\underline{X}_t^\varepsilon,b-\bar{X}_t^\varepsilon)$, $\underline{I}:=(a-\bar{X}_t^\varepsilon,b-\underline{X}_t^\varepsilon)$, and the corresponding limiting interval $J=(a,b)$, under the convention $(x,y)=\emptyset$ if $y<x$. Denote

$$\begin{aligned}D_t^1(u)&=\mathbb{E}\bigg(\int_{\bar{I}^c}\{\varphi(v)+\|\varphi\|_{\mathrm{Lip}}U_t^\varepsilon\}\,s_\varepsilon(v)s_\varepsilon(u-X_t^\varepsilon-v)dv\bigg)-\int_{J^c}\varphi(v)s_\varepsilon(v)s_\varepsilon(u-v)dv,\\D_t^2(u)&=\mathbb{E}\bigg(\mathbf{1}_{\{\bar{X}_t^\varepsilon<b,\underline{X}_t^\varepsilon>a\}}\int_{\underline{I}^c}\{\varphi(v)-\|\varphi\|_{\mathrm{Lip}}U_t^\varepsilon\}\,s_\varepsilon(v)s_\varepsilon(u-X_t^\varepsilon-v)dv\bigg)-\int_{J^c}\varphi(v)s_\varepsilon(v)s_\varepsilon(u-v)dv.\end{aligned}$$

Let us first analyze D_t^1. Clearly,

$$\begin{aligned}D_t^1(u)=&\;\|\varphi\|_{\mathrm{Lip}}\,\mathbb{E}\bigg(U_t^\varepsilon\int_{\bar{I}^c}s_\varepsilon(v)s_\varepsilon(u-X_t^\varepsilon-v)dv\bigg)+\mathbb{E}\bigg(\int_{\bar{I}^c\setminus J^c}\varphi(v)s_\varepsilon(v)s_\varepsilon(u-X_t^\varepsilon-v)dv\bigg)\\&+\mathbb{E}\bigg(\int_{J^c}\varphi(v)s_\varepsilon(v)[s_\varepsilon(u-X_t^\varepsilon-v)-s_\varepsilon(u-v)]dv\bigg),\end{aligned}$$

and, therefore, using that $\bar{I}^c\setminus J^c\subset(a,a-\underline{X}_t^\varepsilon)\cup(b-\bar{X}_t^\varepsilon,b)$, under the convention $(-\infty,-\infty)=(\infty,\infty)=\emptyset$,

$$\begin{aligned}|D_t^1(u)|&\le a_\varepsilon\lambda_\varepsilon\|\varphi\|_{\mathrm{Lip}}\mathbb{E}U_t^\varepsilon+a_\varepsilon^2\|\varphi\|_\infty\mathbb{E}(\bar{X}_t^\varepsilon-\underline{X}_t^\varepsilon)+\lambda_\varepsilon\|\varphi\|_\infty\|s_\varepsilon'\|_\infty\mathbb{E}|X_t^\varepsilon|\\&\le(a_\varepsilon\lambda_\varepsilon\|\varphi\|_{\mathrm{Lip}}+2\|\varphi\|_\infty a_\varepsilon^2)\big(\mathbb{E}\sup_{s\le t}|M_s^\varepsilon|+|\mu_\varepsilon|t\big)+\|\varphi\|_\infty\lambda_\varepsilon\|s_\varepsilon'\|_\infty\left(\mathbb{E}|M_t^\varepsilon|+|\mu_\varepsilon|t\right)\end{aligned}$$

Using the trivial inequalities $(\mathbb{E}\sup_{s\le t}|M_s^\varepsilon|)^2 \le \mathbb{E}\sup_{s\le t}(M_s^\varepsilon)^2$ and $(\mathbb{E}|M_s^\varepsilon|)^2 \le \mathbb{E}(M_s^\varepsilon)^2$, together with Doob's inequality, we then get the bound

$$\begin{aligned}\left|D_t^1(u)\right| \le& \left[2a_\varepsilon\lambda_\varepsilon\|\varphi\|_{\mathrm{Lip}} + 4\|\varphi\|_\infty a_\varepsilon^2 + \|\varphi\|_\infty\lambda_\varepsilon\|s'_\varepsilon\|_\infty\right]\sigma_\varepsilon t^{1/2}\\ &+ \left[a_\varepsilon\lambda_\varepsilon\|\varphi\|_{\mathrm{Lip}} + 2\|\varphi\|_\infty a_\varepsilon^2 + \|\varphi\|_\infty\lambda_\varepsilon\|s'_\varepsilon\|_\infty\right]|\mu_\varepsilon|t,\end{aligned} \tag{A.17}$$

where $\sigma_\varepsilon^2 := \sigma^2 + \int \bar{c}_\varepsilon(x)x^2\nu(dx)$. For $D_t^2(u)$, note that

$$\begin{aligned}D_t^2(u) =& -\mathbb{E}\left(\mathbf{1}_{\{\bar{X}_t^\varepsilon\ge b \text{ or } \underline{X}_t^\varepsilon\le a\}}\int_{\underline{I}^c}\varphi(v)s_\varepsilon(v)s_\varepsilon(u-X_t^\varepsilon-v)dv\right) + \|\varphi\|_{\mathrm{Lip}}\mathbb{E}\left(\mathbf{1}_{\{\bar{X}_t^\varepsilon<b,\underline{X}_t^\varepsilon>a\}}U_t^\varepsilon\int_{\underline{I}^c}s_\varepsilon(v)s_\varepsilon(u-X_t^\varepsilon-v)dv\right)\\ &+\mathbb{E}\left(\int_{\underline{I}^c}\varphi(v)s_\varepsilon(v)s_\varepsilon(u-X_t^\varepsilon-v)dv\right) - \int_{J^c}\varphi(v)s_\varepsilon(v)s_\varepsilon(u-v)dv.\end{aligned}$$

Defining $c = |a|\wedge b$ and following the same steps as above, it is easy to verify that $|D_t^2(u)|$ admits the following upper bound:

$$\begin{aligned}\left|D_t^2(u)\right| \le& \|\varphi\|_\infty\lambda_\varepsilon a_\varepsilon\mathbb{P}(U_t^\varepsilon\ge c) + a_\varepsilon\lambda_\varepsilon\|\varphi\|_{\mathrm{Lip}}\mathbb{E}U_t^\varepsilon + 2a_\varepsilon^2\|\varphi\|_\infty\mathbb{E}U_t^\varepsilon + \lambda_\varepsilon\|\varphi\|_\infty\|s'_\varepsilon\|_\infty\mathbb{E}|X_t^\varepsilon|\\ \le& \|\varphi\|_\infty\lambda_\varepsilon a_\varepsilon C_1(c,\varepsilon)t + \left[2a_\varepsilon\lambda_\varepsilon\|\varphi\|_{\mathrm{Lip}} + 4\|\varphi\|_\infty a_\varepsilon^2 + \|\varphi\|_\infty\lambda_\varepsilon\|s'_\varepsilon\|_\infty\right]\sigma_\varepsilon t^{1/2}\\ &+ \left[a_\varepsilon\lambda_\varepsilon\|\varphi\|_{\mathrm{Lip}} + 2\|\varphi\|_\infty a_\varepsilon^2 + \|\varphi\|_\infty\lambda_\varepsilon\|s'_\varepsilon\|_\infty\right]|\mu_\varepsilon|t,\end{aligned} \tag{A.18}$$

where we had used the tail probability bound in (2.4).

$\square$

A.2 Proof of Proposition 2.4

We use the notation introduced at the beginning of Section A.1 above and, as before, we assume without loss of generality that φ is nonnegative. As it was done in (A.2), by partitioning the space into the different values that N_t^ε can take on, we can decompose $\mathbb{E}(\varphi(X_\tau)\mathbf{1}_{\{X_t\in I_\delta(y)\}})$ into three terms: no big jumps, one big jump, and two or more big jumps. These terms can in turn be expressed as integrals of the form (A.3) using a procedure similar to (A.4). The term with no big jumps is such that

$$\int_{y-\delta}^{y+\delta}P_t^0(u)du := \mathbb{E}(\varphi(X_\tau)\mathbf{1}_{\{X_t\in I_\delta(y),N_t^\varepsilon=0\}}) \le \|\varphi\|_\infty\mathbb{P}\left(X_t^\varepsilon\in I_\delta(y), N_t^\varepsilon=0\right) = \int_{y-\delta}^{y+\delta}e^{-\lambda_\varepsilon t}\|\varphi\|_\infty f_t^\varepsilon(u)du,$$

which yields an upper bound for $P_t^0(u)$ of the form $\bar{P}_t^0(u) := e^{-\lambda_\varepsilon t}\|\varphi\|_\infty f_t^\varepsilon(u)$. Using (2.4-ii), we can further upper bound $\bar{P}_t^0(u)$ by $\|\varphi\|_\infty c_2(c,\varepsilon)t^2$ uniformly in $(a-\delta_0, b+\delta_0)^c$. The term with two or more big jumps can be bounded similarly to the term with three or more big jumps in the previous section. Concretely, this term satisfies

$$\int_{y-\delta}^{y+\delta}P_t^2(u)du := \mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\{X_t\in I_\delta(y),N_t^\varepsilon\ge2\}}\right) \le \int_{y-\delta}^{y+\delta}\left[\sum_{n=2}^\infty e^{-\lambda_\varepsilon t}\frac{t^n}{n!}\|\varphi\|_\infty\mathbb{E}\left(s_\varepsilon^{*n}(u-X_t^\varepsilon)\right)\right]du =: \int_{y-\delta}^{y+\delta}\bar{P}_t^2(u)du,$$

and, using that $\|s_\varepsilon^{*n}\|_\infty \le \lambda_\varepsilon^{n-1}a_\varepsilon$, we can further upper bound $\bar{P}_t^2(u)$ by $C(\varepsilon)t^2$ for a constant $C(\varepsilon)<\infty$. The term with exactly one jump is decomposed as follows:

$$\int_{y-\delta}^{y+\delta}P_t^1(u)du := \mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\{X_t\in I_\delta(y),N_t^\varepsilon=1\}}\right) = \mathbb{E}\left(\varphi(X_\tau^\varepsilon)\mathbf{1}_{\{X_t\in I_\delta(y);\tau<\tau_1;N_t^\varepsilon=1\}}\right) + \mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\{X_t\in I_\delta(y);\tau\ge\tau_1;N_t^\varepsilon=1\}}\right),$$

where τ_1 is the time of the first big jump. Out of these two terms, the first one satisfies

$$\begin{aligned}\mathbb{E}\left(\varphi(X_\tau^\varepsilon)\mathbf{1}_{\{X_t\in I_\delta(y);\tau<\tau_1;N_t^\varepsilon=1\}}\right) &\le \|\varphi\|_\infty\mathbb{P}[\exists s\in[0,t]: X_s^\varepsilon\notin D; X_t^\varepsilon+Y_1\in I_\delta; N_t^\varepsilon=1]\\ &= e^{-\lambda_\varepsilon t}\lambda_\varepsilon t\|\varphi\|_\infty\mathbb{P}[\exists s\in[0,t]: X_s^\varepsilon\notin D; X_t^\varepsilon+Y_1\in I_\delta]\\ &\le \int_{y-\delta}^{y+\delta}e^{-\lambda_\varepsilon t}t\|\varphi\|_\infty\mathbb{E}\left[\mathbf{1}_{U_t^\varepsilon\ge c}s_\varepsilon(u-X_t^\varepsilon)\right]du,\end{aligned}$$

where the integrand $\bar{P}_t^{1,1}(u) := e^{-\lambda_\varepsilon t} t \|\varphi\|_\infty \mathbb{E}\left[\mathbf{1}_{U_t^\varepsilon \geq c} s_\varepsilon(u - X_t^\varepsilon)\right]$ is uniformly bounded by $\|\varphi\|_\infty a_\varepsilon C_1(c,\varepsilon)t^2$. As for the second term $\mathbb{E}\left(\varphi(X_\tau)\mathbf{1}_{\{X_t \in I_\delta(y);\tau\geq\tau_1;N_t^\varepsilon=1\}}\right) = \mathbb{E}\left(\varphi(X_\tau^\varepsilon + Y_1)\mathbf{1}_{\{X_t^\varepsilon+Y_1\in I_\delta(y);\tau\geq\tau_1;N_t^\varepsilon=1\}}\right)$, it can be bounded from above by

$$
\begin{aligned}
&\mathbb{E}(\varphi(Y_1)\mathbf{1}_{\{X_t^\varepsilon+Y_1\in I_\delta(y);N_t^\varepsilon=1\}}) + \|\varphi\|_{\mathrm{Lip}}\mathbb{E}(U_t^\varepsilon \mathbf{1}_{\{X_t^\varepsilon+Y_1\in I_\delta(y);N_t^\varepsilon=1\}}) \\
&= \int_{y-\delta}^{y+\delta} \left\{e^{-\lambda_\varepsilon t} t\mathbb{E}\left[\varphi(u-X_t^\varepsilon)s_\varepsilon(u-X_t^\varepsilon)\right] + e^{-\lambda_\varepsilon t} t\|\varphi\|_{\mathrm{Lip}}\mathbb{E}\left[U_t^\varepsilon s_\varepsilon(u-X_t^\varepsilon)\right]\right\} du \\
&\leq \int_{y-\delta}^{y+\delta} \left\{t\varphi(u)s_\varepsilon(u) + t(\|\varphi\|_{\mathrm{Lip}} a_\varepsilon + \|\varphi\|_\infty \|s_\varepsilon'\|_\infty)\mathbb{E}[|X_t^\varepsilon|] + t\|\varphi\|_{\mathrm{Lip}} a_\varepsilon \mathbb{E}[U_t^\varepsilon]\right\} du.
\end{aligned}
$$

Similarly, this can be bounded from below by

$$
\begin{aligned}
&\mathbb{E}(\varphi(Y_1)\mathbf{1}_{\{X_t^\varepsilon+Y_1\in I_\delta(y);\bar{X}_t^\varepsilon<b;\underline{X}_t^\varepsilon>a;N_t^\varepsilon=1\}}) - \|\varphi\|_{\mathrm{Lip}}\mathbb{E}(U_t^\varepsilon \mathbf{1}_{\{X_t^\varepsilon+Y_1\in I_\delta(y);N_t^\varepsilon=1\}}) \\
&= \int_{y-\delta}^{y+\delta} \left\{e^{-\lambda_\varepsilon t} t\mathbb{E}(\varphi(u-X_t^\varepsilon)s_\varepsilon(u-X_t^\varepsilon)\mathbf{1}_{\{\bar{X}_t^\varepsilon<b,\underline{X}_t^\varepsilon>a\}}) - e^{-\lambda_\varepsilon t} t\|\varphi\|_{\mathrm{Lip}}\mathbb{E}(U_t^\varepsilon s_\varepsilon(u-X_t^\varepsilon))\right\} du \\
&\geq \int_{y-\delta}^{y+\delta} \left\{t\varphi(u)s_\varepsilon(u) - \|\varphi\|_\infty a_\varepsilon \lambda_\varepsilon t^2 - t(\|\varphi\|_{\mathrm{Lip}} a_\varepsilon + \|\varphi\|_\infty \|s_\varepsilon'\|_\infty)\mathbb{E}[|X_t^\varepsilon|] - t\|\varphi\|_{\mathrm{Lip}} a_\varepsilon \mathbb{E}[U_t^\varepsilon] - t\|\varphi\|_\infty a_\varepsilon C_1(c,\varepsilon)t\right\} du.
\end{aligned}
$$

To conclude, we estimate $\mathbb{E}[|X_t^\varepsilon|]$ and $\mathbb{E}[U_t^\varepsilon]$ as in the proof of Lemma A.1 above. □

B Proofs of Section 3

In this part, we provide the building blocks to develop an upper bound for the remainder $\mathcal{R}_t(u)$ appearing in (2.5).

B.1 Proof of Lemma 3.1

Let us first assume that $\mu \geq 0$ so that $X_t := M_t + \mu t$ is a submartingale. By Doob's inequality, for all $c > 0$,

$$
\mathbb{P}\left(\sup_{s\leq t} X_s \geq \eta\right) = \mathbb{P}\left(\sup_{s\leq t} e^{cX_s} \geq e^{c\eta}\right) \leq \frac{\mathbb{E}[e^{cX_t}]}{e^{c\eta}} = e^{t\psi(c)-c\eta} \tag{B.1}
$$

with $\psi(c) = \mu c + \frac{\sigma^2 c^2}{2} + \int_{|z|\leq\varepsilon}(e^{cz} - 1 - cz)\nu(dz)$. Minimizing the right-hand side over all $c > 0$, we get, as in [38] (see p. 87 therein),

$$
\inf_{c>0} e^{t\psi(c)-c\eta} = \exp\left(-t\int_{\psi'(0)}^{\eta/t} \tau(z)dz\right) = \exp\left(-t\int_{\mu}^{\eta/t} \tau(z)dz\right), \tag{B.2}
$$

where we are taking $t < \eta/\mu$ and $\tau : [0,\infty) \to \mathbb{R}$ is the inverse function of

$$
\psi'(x) = \mu + \sigma^2 x + \int_{|z|\leq\varepsilon} z(e^{zx} - 1)\nu(dz). \tag{B.3}
$$

As in [25], note that, for $x \geq 0$,

$$
\begin{aligned}
0 \leq \int_{|z|\leq\varepsilon} z(e^{zx} - 1)\nu(dz) &\leq \int_{|z|\leq\varepsilon} |z|(e^{|z|x} - 1)\nu(dz) \leq \int_{|z|\leq\varepsilon} |z|\sum_{k=1}^{\infty}\frac{(|z|x)^k}{k!}\nu(dz) \\
&\leq \int_{|z|\leq\varepsilon} |z|^2\nu(dz)\sum_{k=1}^{\infty}\frac{\varepsilon^{k-1}x^k}{k!} = \int_{|z|\leq\varepsilon} |z|^2\nu(dz)\frac{1}{\varepsilon}\left(e^{\varepsilon x} - 1\right).
\end{aligned}
$$

From the previous inequality, for $x \geq 0$,

$$0 \leq \psi'(x) \leq \mu + \sigma^2 x + \int_{|z|\leq\varepsilon} |z|^2 \nu(dz) \frac{1}{\varepsilon} \left(e^{\varepsilon x} - 1\right) \leq \mu + \frac{e^{\varepsilon x} - 1}{\varepsilon} \sigma_\varepsilon^2,$$

where we used the fact that $\sigma_\varepsilon^2 = \sigma^2 + \int_{|z|\leq\varepsilon} |z|^2 \nu(dz)$. This implies that

$$\tau(z) \geq \frac{1}{\varepsilon} \log \left\{ 1 + \frac{z-\mu}{\sigma_\varepsilon^2} \varepsilon \right\},$$

and therefore, substituting this into (B.1) and (B.2) and using that $v\ln(v) \leq (1+v)\ln(1+v)$ and $e^{-v\log v} \leq e^{e^{-1}}$ for all $v > 0$, we have

$$\begin{aligned}
\mathbb{P}\left[\sup_{s\leq t} X_s \geq \eta\right] &\leq \exp\left\{ -\frac{t\sigma_\varepsilon^2}{\varepsilon^2} \int_0^{\frac{\varepsilon(\eta-\mu t)}{t\sigma_\varepsilon^2}} \log(1+s)ds \right\} \\
&= \exp\left\{ -\frac{t\sigma_\varepsilon^2}{\varepsilon^2} \left(\left(1 + \frac{\varepsilon(\eta-\mu t)}{t\sigma_\varepsilon^2}\right) \log\left(1 + \frac{\varepsilon(\eta-\mu t)}{t\sigma_\varepsilon^2}\right) - \frac{\varepsilon(\eta-\mu t)}{t\sigma_\varepsilon^2} \right) \right\} \\
&\leq \exp\left\{ -\frac{\eta-\mu t}{\varepsilon} \log\left(\frac{\varepsilon(\eta-\mu t)}{e\sigma_\varepsilon^2 t}\right) \right\} \leq t^{\frac{\eta}{\varepsilon}} e^{\frac{\mu}{\varepsilon} e^{-1}} \exp\left\{ -\frac{\eta-\mu t}{\varepsilon} \log\left(\frac{\varepsilon(\eta-\mu t)}{e\sigma_\varepsilon^2}\right) \right\},
\end{aligned}$$

The above inequality proves the statement (2-i) for the case $\mu = 0$. Next, it is easy to check that the function $u \to (u/\varepsilon)\log(\varepsilon u/e\sigma_\varepsilon^2)$ is strictly convex in $(0,\infty)$ and reaches its global minimum value of $-\sigma_\varepsilon^2/\varepsilon^2$ at $u = \sigma_\varepsilon^2/\varepsilon$. Hence, whenever $\eta - \mu t \geq 0$,

$$\mathbb{P}\left[\sup_{s\leq t} X_s \geq \eta\right] \leq t^{\frac{\eta}{\varepsilon}} e^{\frac{\mu}{\varepsilon} e^{-1}} e^{\frac{\sigma_\varepsilon^2}{\varepsilon^2}},$$

which proves the statement (1) for $\mu \geq 0$. Also, if $\mu > 0$, $t < \eta/\mu$, and $\eta < \sigma_\varepsilon^2/\varepsilon$, we have that

$$\exp\left\{ -\frac{\eta-\mu t}{\varepsilon} \log\left(\frac{\varepsilon(\eta-\mu t)}{e\sigma_\varepsilon^2}\right) \right\} \leq \sup_{0\leq u\leq\eta} \exp\left\{ -\frac{u}{\varepsilon} \log\left(\frac{\varepsilon u}{e\sigma_\varepsilon^2}\right) \right\} = e^{-\frac{\eta}{\varepsilon}\log\left(\frac{\varepsilon\eta}{e\sigma_\varepsilon^2}\right)} = \left(\frac{e\sigma_\varepsilon^2}{\varepsilon\eta}\right)^{\frac{\eta}{\varepsilon}},$$

which proves the statement (2-ii). Finally, we consider the case $\mu < 0$. In that case, obviously, $M_t + \mu t \leq M_t$ and

$$\mathbb{P}\left(\sup_{s\leq t}(M_s + \mu s) \geq \eta\right) \leq \mathbb{P}\left(\sup_{s\leq t} M_s \geq \eta\right) \leq t^{\frac{\eta}{\varepsilon}} \left(\frac{e\sigma_\varepsilon^2}{\varepsilon\eta}\right)^{\frac{\eta}{\varepsilon}} \leq t^{\frac{\eta}{\varepsilon}} e^{\frac{\sigma_\varepsilon^2}{\varepsilon^2}},$$

where in the second inequality we used the case (2-i) with $\mu = 0$ that was proved above. The previous inequality proves the bounds (2-i) and (1) for $\mu < 0$. □.

B.2 Proof of Theorem 3.2.

To prove the estimate (3.11) for the remainder $\mathcal{R}_t(y)$, we analyze each of the four terms in (A.2) contributing to it.

(No big jump) The first component of the error is due to P_t^0 which, as seen in (A.5), can be bounded by

$$e^{(0)}(0,y,t) := e^{-\lambda_\varepsilon t} \|\varphi\|_\infty \sup_{0<u\leq t} \sup_{x\in(a,b)^c} f_u^\varepsilon(y-x) = e^{-\lambda_\varepsilon t} \|\varphi\|_\infty \sup_{0<u\leq t} \sup_{z\in(y-b,y-a)^c} f_u^\varepsilon(z).$$

Next, recalling the notation $\Delta_y = (b-y)\wedge(y-a) > 0$ and employing our hypothesis that X_t^ε has unimodal distribution, we can further apply the bound (3.10) to get

$$e^{(0)}(0,y,t) \leq e^{-\lambda_\varepsilon t} \frac{4\|\varphi\|_\infty}{\Delta_y} \sup_{0<u\leq t} \mathbb{P}\left[|X_u^\varepsilon| \geq \frac{\Delta_y}{2}\right] \leq \frac{8e^{-\lambda_\varepsilon t}\|\varphi\|_\infty}{\Delta_y} C(\Delta_y/4,\varepsilon) t^{\frac{\Delta_y}{4\varepsilon}},$$

for $t < t_0(\varepsilon, \Delta_y/2) \wedge t_1(\varepsilon, \Delta_y/2)$.

(One big jump) There are two sub-components to the error in this case. The first is due to $\bar{P}^{1,1}$ in (A.6). This term can be bounded by

$$e^{(1,1)}(0,y,t) := \|\varphi\|_\infty e^{-\lambda_\varepsilon t} t \mathbb{E}\left(\mathbf{1}_{\{U_t^\varepsilon \geq c\}} s_\varepsilon(y - X_t^\varepsilon)\right) \leq \|\varphi\|_\infty e^{-\lambda_\varepsilon t} t a_\varepsilon \mathbb{P}(U_t^\varepsilon \geq c) \leq 2\|\varphi\|_\infty e^{-\lambda_\varepsilon t} a_\varepsilon C(c/2,\varepsilon) t^{1+\frac{c}{2\varepsilon}},$$

for $t < t_0(\varepsilon, c/2)$. The other sub-component is due to $\bar{P}^{1,2}$ in (A.7), which can be bounded, for $t < t_0(\varepsilon, \Delta_y/2)$, as follows:

$$\begin{aligned} e^{(1,2)}(0,y,t) &:= \|\varphi\|_\infty e^{-\lambda_\varepsilon t} t \mathbb{E}\left(\mathbf{1}_{\{\bar{X}_t^\varepsilon - X_t^\varepsilon + y \geq b \text{ or } \underline{X}_t^\varepsilon - X_t^\varepsilon + y \leq a\}} s_\varepsilon(y - X_t^\varepsilon)\right) \\ &\leq \|\varphi\|_\infty e^{-\lambda_\varepsilon t} t a_\varepsilon \mathbb{P}\left(\sup_{u \leq t} |X_u^\varepsilon| \geq \frac{\Delta_y}{2}\right) \leq 2\|\varphi\|_\infty e^{-\lambda_\varepsilon t} a_\varepsilon C(\Delta_y/4, \varepsilon) t^{1+\frac{\Delta_y}{4\varepsilon}}. \end{aligned}$$

(Three or more big jumps) This component can be bounded as in (A.16):

$$e^{(3)}(0,y,t) := \|\varphi\|_\infty e^{-\lambda_\varepsilon t} \sum_{n=3}^{\infty} \frac{t^n}{n!} \mathbb{E}\left(s_\varepsilon^{*n}(u - X_t^\varepsilon)\right) \leq \|\varphi\|_\infty a_\varepsilon \lambda_\varepsilon^{-1} \left(1 - e^{-\lambda_\varepsilon t}[1 + \lambda_\varepsilon t + (\lambda_\varepsilon t)^2/2]\right). \tag{B.4}$$

(Two big jumps) There are three sub-components to the error in this case. From (A.10),

$$e^{(2,1)}(0,y,t) := \|\varphi\|_\infty e^{-\lambda_\varepsilon t} \frac{t^2}{2} \int_{-\infty}^{\infty} s_\varepsilon(v) \mathbb{E}\left\{\mathbf{1}_{\{U_t^\varepsilon \geq c\}} s_\varepsilon(y - X_t^\varepsilon - v)\right\} dv \leq \|\varphi\|_\infty e^{-\lambda_\varepsilon t} a_\varepsilon \lambda_\varepsilon C(c/2,\varepsilon) t^{2+\frac{c}{2\varepsilon}}, \tag{B.5}$$

for $t < t_0(\varepsilon, c)$. Similarly, from (A.11),

$$\begin{aligned} e^{(2,3)}(0,y,t) &:= \|\varphi\|_\infty e^{-\lambda_\varepsilon t} \frac{t^2}{2} \int_{-\infty}^{\infty} s_\varepsilon(v) \mathbb{E}\left\{\mathbf{1}_{\{\bar{X}_t^\varepsilon - X_t^\varepsilon + y \geq b \text{ or } \underline{X}_t^\varepsilon - X_t^\varepsilon + y \leq a\}} s_\varepsilon(y - X_t^\varepsilon - v)\right\} dv \\ &\leq \|\varphi\|_\infty e^{-\lambda_\varepsilon t} \frac{t^2}{2} a_\varepsilon \lambda_\varepsilon \mathbb{P}\left(\sup_{u \leq t} |X_u^\varepsilon| \geq \frac{\Delta_y}{2}\right) \leq \|\varphi\|_\infty e^{-\lambda_\varepsilon t} a_\varepsilon \lambda_\varepsilon C(\Delta_y/4, \varepsilon) t^{2+\frac{\Delta_y}{4\varepsilon}}, \end{aligned} \tag{B.6}$$

for $t < t_0(\varepsilon, \Delta_y/2)$. Next, we consider the error due to the limits (A.13-A.14). These were bounded in Lemma A.1. Hence, by taking the maximum of (A.17) and (A.18), after some simplification, we get the following expression for the error term $e^{(2,2)}(0,y,t)$:

$$e^{-\lambda_\varepsilon t} t^2 \left(\|\varphi\|_\infty \lambda_\varepsilon a_\varepsilon C(c/2,\varepsilon) t^{\frac{c}{2\varepsilon}} + \left[a_\varepsilon \lambda_\varepsilon \|\varphi\|_{\mathrm{Lip}} + 2\|\varphi\|_\infty a_\varepsilon^2 + \|\varphi\|_\infty \lambda_\varepsilon \|s_\varepsilon'\|_\infty\right] (\sigma_\varepsilon t^{1/2} + \frac{|\mu_\varepsilon|}{2} t) \right).$$

Finally, we also need to take into account the error due to approximating $e^{-\lambda_\varepsilon t} \frac{t^2}{2} \int_{(a,b)^c} \varphi(v) s_\varepsilon(v) s_\varepsilon(y - v) dv$ by $\frac{t^2}{2} \int_{(a,b)^c} \varphi(v) s_\varepsilon(v) s_\varepsilon(y - v) dv$, which is of order $\|\varphi\|_\infty \lambda_\varepsilon^2 a_\varepsilon t^3/2$. Putting all the previous bounds together, we obtain the overall bound (3.11). $\square$

C Finding the estimate $e_f(0,y,t)$ for the Cauchy process

In this paragraph our aim is to find an explicit bound for the Cauchy process with Lévy density $\nu(x) = \frac{c}{|x|^2}$ (and no drift), which is used in the numerical illustrations. For simplicity, we shall only consider the one-sided case ($a = -\infty$). Setting $c_\varepsilon(x) = \mathbf{1}_{|x| > \varepsilon}$, we get $\mu_\varepsilon = 0$ for all ε, and the law of the process is symmetric, which means that $t_0(\varepsilon, \eta) = t_1(\varepsilon, \eta) = +\infty$ for all $\varepsilon > 0$ and $\eta > 0$. Moreover, $\sigma_\varepsilon^2 = 2c\varepsilon$ and Lemma 3.1 implies that $\mathbb{P}[\sup_{s \leq t} X_t \geq \eta] \leq t^{\frac{\eta}{\varepsilon}} C(\eta, \varepsilon)$ and $\mathbb{P}[\sup_{s \leq t} |X_t| \geq \eta] \leq 2t^{\frac{\eta}{\varepsilon}} C(\eta, \varepsilon)$ with $C(\eta, \varepsilon) = \left(\frac{2ce}{\eta}\right)^{\frac{\eta}{\varepsilon}}$. The results of the above section can now be improved to

$$\begin{aligned} &e^{(0)}(0,y,t) \leq \|\varphi\|_\infty \frac{4e^{-\lambda_\varepsilon t}}{b - y} C(\varepsilon, (b-y)/2) t^{\frac{b-y}{2\varepsilon}} \qquad e^{(1,1)}(0,y,t) \leq \|\varphi\|_\infty e^{-\lambda_\varepsilon t} a_\varepsilon C(\varepsilon, b) t^{1+\frac{b}{\varepsilon}} \\ &e^{(1,2)}(0,y,t) \leq 2\|\varphi\|_\infty e^{-\lambda_\varepsilon t} a_\varepsilon C(\varepsilon, (b-y)/2) t^{1+\frac{b-y}{2\varepsilon}}, \qquad e^{(2,1)}(0,y,t) \leq \frac{\|\varphi\|_\infty}{2} e^{-\lambda_\varepsilon t} a_\varepsilon \lambda_\varepsilon C(b, \varepsilon) t^{2+\frac{b}{\varepsilon}} \\ &e^{(2,3)}(0,y,t) \leq \|\varphi\|_\infty e^{-\lambda_\varepsilon t} a_\varepsilon \lambda_\varepsilon C((b-y)/2, \varepsilon) t^{2+\frac{b-y}{2\varepsilon}} \end{aligned}$$

To estimate $e^{(2,2)}$ more precisely, let $\varepsilon_0 < \frac{b-y-\varepsilon}{2} \wedge (b-\varepsilon)$. Then,

$$
\begin{aligned}
|D^1_t(y)| &\leq 2\|\varphi\|_\infty a_\varepsilon \lambda_\varepsilon \mathbb{P}\left(U^\varepsilon_t \geq \varepsilon_0\right) + 2\|\varphi\|_{\mathrm{Lip}} \mathbb{E}\left[U^\varepsilon_t \int_b^\infty s_\varepsilon(v - \bar{X}^\varepsilon_t) s_\varepsilon(y - v + \bar{X}^\varepsilon_t - X^\varepsilon_t) dv\right] \\
&\quad + \|\varphi\|_\infty \mathbb{E}\left[\mathbf{1}_{U^\varepsilon_t < \varepsilon_0} \int_b^\infty (s_\varepsilon(v - \bar{X}^\varepsilon_t) s_\varepsilon(y - v + \bar{X}^\varepsilon_t - X^\varepsilon_t) - s_\varepsilon(v) s_\varepsilon(y - v)) dv\right] \\
&\leq 2a_\varepsilon \lambda_\varepsilon (\|\varphi\|_\infty \mathbb{P}\left(U^\varepsilon_t \geq \varepsilon_0\right) + \|\varphi\|_{\mathrm{Lip}} \mathbb{E}[U^\varepsilon_t]) - \|\varphi\|_\infty \mathbb{E}\left[U^\varepsilon_t\right] \int_b^\infty s'_\varepsilon(v - \varepsilon_0) s_\varepsilon(y - v + 2\varepsilon_0) dv \\
&\quad + 2\|\varphi\|_\infty \mathbb{E}\left[U^\varepsilon_t\right] \int_b^\infty s_\varepsilon(v) s'_\varepsilon(y - v + \frac{b-y}{2}) dv \\
&\leq 2a_\varepsilon \lambda_\varepsilon (\|\varphi\|_\infty \mathbb{P}\left(U^\varepsilon_t \geq \varepsilon_0\right) + \|\varphi\|_{\mathrm{Lip}} \mathbb{E}[U^\varepsilon_t]) + 2\|\varphi\|_\infty \mathbb{E}\left[U^\varepsilon_t\right] s_\varepsilon\left(b - \varepsilon_0\right) s_\varepsilon\left(b - y - 2\varepsilon_0\right).
\end{aligned}
$$

A similar argument shows that

$$
|D^2_t(y)| \leq s_\varepsilon(b) \lambda_\varepsilon (2\|\varphi\|_{\mathrm{Lip}} \mathbb{E}[U^\varepsilon_t] + \|\varphi\|_\infty \mathbb{P}[\bar{X}^\varepsilon_t \geq b]) + \|\varphi\|_\infty \mathbb{E}\left[U^\varepsilon_t\right] s_\varepsilon\left(b\right) s_\varepsilon\left(b - y\right),
$$

which means that the bound for $|D^1_t(y)|$ always dominates. Using the former bound, we finally find the following upper bound for $e^{2,2}(0, y, t)$:

$$
2\|\varphi\|_\infty e^{-\lambda_\varepsilon t} a_\varepsilon \lambda_\varepsilon C(\varepsilon_0, \varepsilon) t^{2 + \frac{\varepsilon_0}{\varepsilon}} + 2e^{-\lambda_\varepsilon t} t^{\frac{5}{2}} \sigma_\varepsilon \{s_\varepsilon\left(b - \varepsilon_0\right) s_\varepsilon\left(b - y - 2\varepsilon_0\right) \|\varphi\|_\infty + a_\varepsilon \lambda_\varepsilon \|\varphi\|_{\mathrm{Lip}}\}.
$$

To specialize the estimate $e^{(3)}$, we upper bound $\lambda^n_\varepsilon \mathbb{P}(\bar{X}_t \geq b, X_t \in I_\delta(y) | N^\varepsilon_t = n)$ by

$$
\lambda^n_\varepsilon \mathbb{P}\Bigg(\bar{X}^\varepsilon_t + \max_{0 \leq k \leq n} \sum_{i=1}^k Y_i \geq b, X^\varepsilon_t + \sum_{i=1}^n Y_i \in I_\delta(t)\Bigg) \leq \lambda^n_\varepsilon \sum_{k=0}^n \mathbb{P}\Bigg(\bar{X}^\varepsilon_t + \sum_{i=1}^k Y_i \geq b, X^\varepsilon_t + \sum_{i=1}^n Y_i \in I_\delta(t)\Bigg).
$$

The cases $k = 0$ and $k = n$ are treated separately:

$$
\lambda^n_\varepsilon \mathbb{P}\left(\bar{X}^\varepsilon_t \geq b, X^\varepsilon_t + \sum_{i=1}^n Y_i \in I_\delta(t)\right) \leq \delta \mathbb{P}\left(\sup_{u \leq t} X^\varepsilon_u \geq b\right) \sup_x s^{*n}_\varepsilon(x) \leq \delta C(b, \varepsilon) t^{\frac{b}{\varepsilon}} \sup_x s^{*n}_\varepsilon(x).
$$

$$
\lambda^n_\varepsilon \mathbb{P}\left(\bar{X}^\varepsilon_t + \sum_{i=1}^n Y_i \geq b, X^\varepsilon_t + \sum_{i=1}^n Y_i \in I_\delta(t)\right) \leq \delta \mathbb{P}\left(\bar{X}^\varepsilon_t - X^\varepsilon_t + y + \delta \geq b\right) \sup_x s^{*n}_\varepsilon(x) \leq 2\delta C((b-y)/2, \varepsilon) t^{\frac{b-y}{2\varepsilon}} \sup_x s^{*n}_\varepsilon(x).
$$

For $0 < k < n$,

$$
\begin{aligned}
\mathbb{P}\left(\bar{X}^\varepsilon_t + \sum_{i=1}^k Y_i \geq b, X^\varepsilon_t + \sum_{i=1}^n Y_i \in I_\delta(t)\right) &= \mathbb{E}\left[\int_y^{y+\delta} du \int_{b - \bar{X}^\varepsilon_t}^\infty dv\, s^{*k}_\varepsilon(v) s^{*(n-k)}_\varepsilon(u - v - X^\varepsilon_t)\right] \\
&\leq \delta \sup_x s^{*n}_\varepsilon(x) \mathbb{P}\left(U^\varepsilon_t \geq \varepsilon_0\right) + \delta \bar{s}^{*k}_\varepsilon(b - \varepsilon_0) \int_b^\infty dv\, \bar{s}^{*(n-k)}_\varepsilon(y - v + 2\varepsilon_0 + \delta)
\end{aligned}
$$

where $\bar{s}_\varepsilon$ is any function which is increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$ and satisfies $\bar{s}_\varepsilon(x) \geq s_\varepsilon(x)$ for all x. For the Cauchy process one can take

$$
\bar{s}_\varepsilon(x) = \frac{2c}{x^2 + \varepsilon^2} \quad \text{so that} \quad \bar{s}^{*k}_\varepsilon(x) = \frac{1}{\pi}\left(\frac{2\pi c}{\varepsilon}\right)^k \frac{\varepsilon k}{(\varepsilon k)^2 + x^2}, \quad \int_b^\infty \bar{s}^{*k}_\varepsilon(v) dv = \frac{1}{\pi}\left(\frac{2\pi c}{\varepsilon}\right)^k \arctan \frac{\varepsilon k}{b}.
$$

Assembling all the estimates together, we finally get

$$
e^{(3)}(0, y, t) \leq \frac{\|\varphi\|_\infty}{6} a_\varepsilon \lambda^2_\varepsilon t^3 (C(b, \varepsilon) t^{\frac{b}{\varepsilon}} + 2C((b-y)/2, \varepsilon) t^{\frac{b-y}{2\varepsilon}} + 2C(\varepsilon_0, \varepsilon) t^{\frac{\varepsilon_0}{\varepsilon}}) + \frac{16\pi c^3 t^3 \|\varphi\|_\infty}{3\varepsilon(b - \varepsilon_0)^2 (b - y - 2\varepsilon_0)} e^{\frac{2\pi c t}{\varepsilon} - \lambda_\varepsilon t}.
$$

The above estimates satisfy condition (2.6) for $\varepsilon < \frac{b-y}{4} \wedge b$. In the numerical examples discussed in the paper we have taken $\varepsilon = \frac{b-y}{8} \wedge \frac{b}{2}$ and $\varepsilon_0 = \frac{b-y}{4} \wedge \frac{b}{2}$.

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