Small Transaction Costs, Absence of Arbitrage and Consistent Price Systems

Yuri Kabanov

Laboratoire de Mathématiques, Université de Franche-Comté

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Introduction

For the discrete-time setting there is a plethora of criteria for various types of arbitrage, see Ch. 3 [1]. For continuous-time models only a few results on the no-arbitrage criteria are available. In a recent paper [2] it was established an interesting result in this direction. A question on sufficient and necessary conditions for the absence of arbitrage was formulated not for a single model but for a whole family of them. In GRS it was considered a family of **2-asset** models with a fixed **continuous** price process and **constant** transaction costs tending to zero. The no-arbitrage criterion is very simple : the NA^w-property holds for each model if and only if each model admits a consistent price system.

1. Kabanov Yu., Safarian M. *Markets with Transaction Costs. Mathematical Theory.* Springer, 2009.

2. Guasoni P., Rásonyi M., Schachermayer W. On fundamental theorem of asset pricing for continuous processes under small transaction costs. *Ann. Finance*, **6** (2010).

Let $K^{\varepsilon *} := \mathbf{R}_+(\mathbf{1} + U_\varepsilon) = \mathrm{cone}\,(\mathbf{1} + U_\varepsilon)$, where $U_\varepsilon := \{x \in \mathbf{R}^d : \ \max_i |x^i| \le \varepsilon\}$, $\varepsilon \in]0,1]$.
That is, $K^{\varepsilon *}$ is the closed convex cone in $\mathbf{R}^d$ generated by the max-norm ball of radius ε with center at $\mathbf{1} := (1,\dots,1)$. We denote by K^ε the (positive) dual cone of $K^{\varepsilon *}$.
Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis and let $S = (S_t)_{t \le T}$ be a **continuous** semimartingale with strictly positive components. We consider the linear controlled stochastic equation

$$dV_t^i = V_{t-}^i dY_t^i + dB_t^i, \quad V_0^i = 0, \qquad i \le d,$$

where Y^i is the stochastic logarithm of S^i, i.e. $dY_t^i = dS_t^i / S_t^i$, $Y_0^i = 1$, and the strategy B is a predictable càdlàg process of bounded variation with $\dot{B} \in -K^\varepsilon$. The notation $\dot{B}$ stands for (a measurable version of) the Radon–Nikodym derivative of B with respect to $||B||$, the total variation process of B.

Generalization, 2

A strategy B is ε-*admissible* if for the process $V = V^B$ there is a constant κ such that $V_t + \kappa S_t \in K^\varepsilon$ for all $t \leq T$. The set of processes V corresponding to ε-admissible strategies is denoted by $A_0^{T\varepsilon}$ while the notation $A_0^{T\varepsilon}(T)$ is reserved for the set of random variables V_T where $V \in A_0^{T\varepsilon}$.
Using the random operator

$$\phi_t : (x^1, ..., x^d) \mapsto (x^1/S_t^1, ..., x^d/S_t^d)$$

define the random cone $\widehat{K}_t^\varepsilon = \phi_t K^\varepsilon$ with the dual $\widehat{K}_t^{\varepsilon *} = \phi_t^{-1} K^{\varepsilon *}$. Put $\widehat{V}_t = \phi_t V_t$. We denote by $\mathcal{M}_0^T(\widehat{K}^{\varepsilon *} \setminus \{0\})$ the set of martingales Z such that $Z_t \in \widehat{K}_t^{\varepsilon *} \setminus \{0\}$.

Generalization : Main Theorem

Theorem

We have :

$$A_0^{T\varepsilon}(T) \cap L^0(\mathbf{R}_+^d, \mathcal{F}_T) = \{0\} \ \forall \varepsilon \quad \Leftrightarrow \quad \mathcal{M}_0^T(\widehat{K}^{\varepsilon *} \setminus \{0\}) \neq \emptyset \ \forall \varepsilon.$$

Comments on financial interpretation.
For $d = 2$ our model is the same as of GRS. The only difference is that we use the "old-fashion" definition of the value processes but it is not essential. In the financial interpretation the cones K^ε and $\widehat{K}^\varepsilon$ are the solvency regions in the terms of the numéraire and physical units, respectively, V and $\widehat{V}$ are value processes, elements of $\mathcal{M}_0^T(\widehat{K}^{\varepsilon *} \setminus \{0\})$ are ε-consistent price systems, etc. The condition "$A_0^{T\varepsilon}(T) \cap L^0(\mathbf{R}_+^d, \mathcal{F}_T) = \{0\}$ for all ε" can be referred to as the *universal NA^w-property*.

Applications to Financial Context

In the case $d > 2$ we have no financial interpretation for the considered objects. Nevertheless, our result can be applied to a wide class of financially meaningful models, even with varying transaction costs. To see this, let us consider the family of models of currency markets with the solvency cones

$$K(\Lambda^\varepsilon) = \mathrm{cone}\,\{(1+\lambda^\varepsilon_{ij})e_i - e_j,\ e_i,\ 1 \le i,j \le d\}.$$

Suppose that for every $\varepsilon \in]0,1]$ there is $\varepsilon' \in]0,1]$ such that $K(\Lambda^\varepsilon) \subseteq K^{\varepsilon'}$ and, vice versa, for any $\delta \in]0,1]$ there is $\delta' \in]0,1]$ such that $K^\delta \subseteq K(\Lambda^{\delta'})$. It is obvious that under this hypothesis Theorem ensures that for the currency markets the universal NA^w-property holds iif an ε-consistent price system does exist for every $\varepsilon > 0$. The hypothesis is fulfilled if $\Lambda^\varepsilon \to 0$ and the duals $K^*(\Lambda^\varepsilon)$ have interiors containing $\mathbf{1}$, e.g., if all $\lambda^\varepsilon_{ij} = \varepsilon$.

Application to a Model with Efficient Friction

Proposition

Suppose that $\Lambda^\varepsilon \to 0$ *and* $\operatorname{int} K^*(\Lambda^\varepsilon) \neq \emptyset$ *for all* $\varepsilon \in]0,1]$. *Then*

$$NA^w(\Lambda^\varepsilon) \quad \forall\, \varepsilon \in]0,1] \quad \Leftrightarrow \quad \mathcal{M}_0^T(\widehat{K}^*(\Lambda^\varepsilon) \setminus \{0\}) \neq \emptyset \quad \forall\, \varepsilon \in]0,1].$$

Proof. ($\Rightarrow$) Let $\delta \in]0,1]$ and $w \in K^*(\Lambda^\delta)$. Then $w^i/w^j \leq 1 + \lambda^\delta_{ij} \to 1$ as $\delta \to 0$. It follows that $K^*(\Lambda^{\delta'}) \subseteq K^{\delta *}$ for some $\delta' \in]0,1]$. For the primary cones the inclusion is opposite. Thus, the assumed no-arbitrage property implies the no-arbitrage property as in Theorem. Take $\varepsilon \in]0,1]$ and a vector $v \in \operatorname{int} K^*(\Lambda^\varepsilon) \cap U_1$. Put

$$\psi_v : (x^1, ..., x^d) \mapsto (v^1 x^1, ..., v^d x^d).$$

Choose $\delta \in]0,1]$ such that $\psi_v(\mathbf{1} + U_\delta) \subset K^*(\Lambda^\varepsilon)$. By Theorem there is $Z \in \mathcal{M}_0^T(\widehat{K}^{\delta *} \setminus \{0\})$. The process $\psi_v Z$ is a martingale. Since $\psi_v Z = \phi \psi_v \phi^{-1} Z$, it is an element of $\mathcal{M}_0^T(\widehat{K}^*(\Lambda^\varepsilon) \setminus \{0\})$.

Strategy of the Proof

To prove the nontrivial implication $(\Rightarrow)$ we exploit the fact that the universal NA^w-property holds for any imbedded discrete-time model. Using the criterion for NA^r-property we deduce from here the existence of a "universal chain", that is a sequence of stopping times τ_n increasing stationary to T and such that $\mathcal{M}_0^{\tau_n}(\widehat{K}^{\varepsilon *} \setminus \{0\}) \neq \emptyset$ for all $\varepsilon \in]0,1]$ and $n \geq 1$. In an analogy with GRS, we relate with this "universal chain" functions $F^i(\varepsilon)$, $i \leq d$, and check that there is, for each i, an alternative : either $F^i = 0$, or $F^i(0+) = 1$. This is the most involved part of the proof. If all $F^i = 0$, the sets $\mathcal{M}_0^{\tau_n}(\widehat{K}^{\varepsilon *} \setminus \{0\})$ are non-empty and we conclude. If there is a coordinate for which $F^i(0+) = 1$, there exists a strict arbitrage opportunity.

Universal Discrete-Time NA^w-property

A continuous-time model has *universal discrete-time NA^w-property* if for any $\varepsilon > 0$, $N \geq 2$, and stopping times $\sigma_1, \dots, \sigma_N \in \mathcal{T}_T$, such that $\sigma_n < \sigma_{n+1}$ on the set $\{\sigma_n < T\}$ we have that

$$L^0(\mathbf{R}^d_+, \mathcal{F}_T) \cap \sum_{n \leq N} L^0(-\phi_{\sigma_n} K^\varepsilon, \mathcal{F}_{\sigma_n}) = \{0\}.$$

Proposition

If the universal discrete-time NA^w-property holds, then there are strictly increasing stopping times τ_n with $P(\tau_n < T) \to 0$ as $n \to \infty$ such that $\mathcal{M}_0^{\tau_N}(\widehat{K}^{\varepsilon *} \setminus \{0\}) \neq \emptyset$ for every N and ε.

Proof. Define recursively the stopping times : $\sigma_0 = 0$,

$$\sigma_n = \sigma_n^\varepsilon := \inf\{t \geq \sigma_{n-1} : \max_{i \leq d} |\ln S^i_t - \ln S^i_{\sigma_{n-1}}| \geq \ln(1+\varepsilon/8)\}, \quad n \geq 1.$$

Lemma

For any integer $N \geq 1$ there is $Z \in \mathcal{M}_0^{\sigma_N}(\widehat{K}^{\varepsilon*} \setminus \{0\})$.

Proof. To avoid new notation we assume wlg that $S = S^{\sigma_N}$. Let $X_n := S_{\sigma_n}$. By our assumption and in virtue of the criterion for the NA^r-property there is a d.-t. martingale $(M_n)_{n \leq N}$ with $M_n \in L^\infty(\phi_{\sigma_n}^{-1} K^{\varepsilon/4*} \setminus \{0\})$. Put $Z_t := E(M_N|\mathcal{F}_t)$. Let us check that $Z \in \mathcal{M}_0^{\sigma_N}(\widehat{K}^{\varepsilon*} \setminus \{0\})$. On the set $\{t \in [\sigma_{n-1}, \sigma_n]\}$

$$\tilde{Z}_t := \phi_t Z_t = E(\phi_t \phi_{\sigma_n}^{-1} \tilde{Z}_{\sigma_n}|\mathcal{F}_t)$$

where $\tilde{Z}_{\sigma_n} := \phi_{\sigma_n} Z_{\sigma_n}$. Note that

$$(1+\varepsilon/8)^{-2} \leq S_t^i/S_{\sigma_n}^i = (S_t^i/S_{\sigma_{n-1}}^i)(S_{\sigma_{n-1}}^i/S_{\sigma_n}^i) \leq (1+\varepsilon/8)^2.$$

Therefore,

$$(1+\varepsilon/8)^{-2} E(\tilde{Z}_{\sigma_n}^i|\mathcal{F}_t) \leq \tilde{Z}_t^i \leq (1+\varepsilon/8)^2 E(\tilde{Z}_{\sigma_n}^i|\mathcal{F}_t).$$

But $E(\tilde{Z}_{\sigma_n}|\mathcal{F}_t) \in \mathrm{cone}\,(\mathbf{1} + U_{\varepsilon/4}) \setminus \{0\})$, i.e. the components of $E(\tilde{Z}_{\sigma_n}|\mathcal{F}_t)$ take values in the interval with the extremities $\lambda(1 \pm \varepsilon/4)$ where $\lambda > 0$ depends on n and ω. Thus,

$$1-\varepsilon \leq (1+\varepsilon/8)^{-2}(1-\varepsilon/4) \leq \tilde{Z}^i_t/\lambda \leq (1+\varepsilon/8)^2(1+\varepsilon/4) \leq 1+\varepsilon.$$

This implies the assertion of the lemma.
The end of proof is as in GRS. Take a sequence of $\varepsilon_k \downarrow 0$. For each $n \geq 1$ we find an integer $N_{n,k}$ such that

$$P(\sigma^{\varepsilon_k}_{N_{n,k}} < T) < 2^{-(n+k)}.$$

Wlg we assume that for each k the sequence $(N_{n,k})_{n\geq 1}$ is increasing. The increasing sequence of stopping times $\tau_n := \min_{k\geq 1} \sigma^{\varepsilon_k}_{N_{n,k}}$ converges to T stationary : $P(\tau_n < T) \leq 2^{-n}$. Applying the lemma with ε_k we obtain that for the process S stopped at $\sigma^{\varepsilon_k}_{N_{n,k}}$ there is an ε_k-consistent price system. The latter, being stopped at τ_n, is an ε_k-consistent price system for S^{τ_n}.

Properties of Universal Chains

We assume that $P(\tau_n < T) > 0$ for all n.
Let $\mathcal{T}_T \neq \emptyset$ be the set of stopping times σ such that $P(\sigma < T) > 0$ and, for some n, the inequality $\sigma \le \tau_n$ holds on $\{\sigma < T\}$.
Let $\sigma \in \mathcal{T}_T$ and let n be such that $\sigma \le \tau_n$ holds on $\{\sigma < T\}$.
We denote by $\mathcal{M}^i(\sigma, \varepsilon, n)$ the set of processes Z such that :
1) $Z = 0$ on $\{\sigma = T\}$;
2) Z is a martingale on $[\sigma, \tau_n]$, i.e. $E(Z_{\tau_n}|\mathcal{F}_\vartheta) = Z_\vartheta$ for any stopping time ϑ such that $\sigma \le \vartheta \le \tau_n$ on $\{\sigma < T\}$;
3) $Z_t(\omega) \in \operatorname{int} \widehat{K}_t^{\varepsilon *}(\omega)$ when $\sigma(\omega) < T$ and $t \in [\sigma(\omega), \tau_n(\omega)]$;
4) $EZ_\sigma^i I_{\{\sigma < T\}} = 1$.

The process $Z = \tilde{Z} I_{\{\sigma < T\}} / E\tilde{Z}_\sigma^i I_{\{\sigma < T\}}$ belongs to $\mathcal{M}^i(\sigma, \varepsilon, n)$ provided that $\tilde{Z} \in \mathcal{M}_0^{\tau_n}(\operatorname{int} \widehat{K}^{\varepsilon *})$.

Let $F^i(\varepsilon) := \sup_{\sigma\in\mathcal{T}_T} F^i(\sigma,\varepsilon)$ where

$$F^i(\sigma,\varepsilon) := \overline{\lim_n} \inf_{Z\in\mathcal{M}^i(\sigma,\varepsilon,n)} EZ^i_{\tau_n} I_{\{\tau_n<T\}}.$$

We also put

$$f^i(\sigma,\varepsilon,n) := \operatorname{ess}\inf_{Z\in\mathcal{M}^i(\sigma,\varepsilon,n)} E((Z^i_{\tau_n}/Z^i_\sigma) I_{\{\tau_n<T\}}|\mathcal{F}_\sigma).$$

Lemma

For any $Z\in\mathcal{M}^i(\sigma,\varepsilon,n)$ there is a process $\bar Z\in\mathcal{M}^i(\sigma,\varepsilon,n+1)$ such that $\bar Z^{\tau_n} = Z^{\tau_n}$.

Proof. Suppose first that $Z\in\mathcal{M}^i(\sigma,\varepsilon',n)$ for some $\varepsilon'<\varepsilon$. Take $\delta>0$ and $\tilde Z\in\mathcal{M}^i(\sigma,\delta,n+1)$. Define the process $\bar Z$ with components

$$\bar Z^j := Z^j I_{[0,\tau_n[} + \frac{Z^j_{\tau_n}}{\tilde Z^j_{\tau_n}} \tilde Z^j I_{[\tau_n,T]}.$$

Note that

$$\phi_t Z_t = \lambda_t(1+u_t^1,\dots,1+u_t^d), \quad t\in[\sigma,\tau_n],$$

$$\phi_t \tilde Z_t = \tilde\lambda_t(1+\tilde u_t^1,\dots,1+\tilde u_t^d), \quad t\in[\tau_n,\tau_{n+1}],$$

where $\max_j |u^j|\le \varepsilon'$, $\max_j |\tilde u^j|\le \delta$ and $\lambda_t,\tilde\lambda_t>0$. It follows that $\bar Z$ belongs to $\mathcal{M}^i(\sigma,\bar\varepsilon,n+1)$ with

$$\bar\varepsilon = \frac{(1+\varepsilon')(1+\delta)}{1-\delta}-1.$$

Since $\bar\varepsilon<\varepsilon$ for sufficiently small $\delta=\delta(\varepsilon')$, the result follows.
In the general case we consider the partition of the set $\{\sigma<T\}$ on $\mathcal{F}_{\tau_n}$-measurable subsets A_k, on which the process Z evolves, on the interval $[\sigma,\tau_n]$, in the cones $\widehat{K}^{\varepsilon_k *}$, where $\varepsilon_k:=(\varepsilon-1/k)\vee 0$. As above, take processes $\tilde Z^k\in\mathcal{M}^i(\sigma,\delta_k,n+1)$ with $\delta_k=\delta(\varepsilon_k)$.
Then we can take as $\bar Z$ the process with components

$$\bar Z^j := Z^j I_{[0,\tau_n[} + \sum_k \frac{Z^{kj}_{\tau_n}}{\tilde Z^j_{\tau_n}}\tilde Z^{kj} I_{A_k} I_{[\tau_n,T]}.$$

Lemma

The sequence $(f^i(\sigma,\varepsilon,n))_{n\geq 0}$ *decreases to* $f^i(\sigma,\varepsilon) \leq F^i(\varepsilon)$.

Proof. By Lemma 2 for any $Z \in \mathcal{M}^i(\sigma,\varepsilon,n)$ there is a process $\bar{Z} \in \mathcal{M}^i(\sigma,\varepsilon,n+1)$ such that $\bar{Z}^{\tau_n} = Z^{\tau_n}$. Using the martingale property of $\bar{Z}$ we get that

$$E((Z^i_{\tau_n}/Z^i_\sigma)I_{\{\tau_n<T\}}|\mathcal{F}_\sigma) = E((\bar{Z}^i_{\tau_n}/\bar{Z}^i_\sigma)I_{\{\tau_n<T\}}|\mathcal{F}_\sigma) \geq E((\bar{Z}^i_{\tau_{n+1}}/\bar{Z}^i_\sigma)I_{\{\tau_{n+1}<T\}}|\mathcal{F}_\sigma)$$

It follows that $f^i(\sigma,\varepsilon,n) \geq f^i(\sigma,\varepsilon,n+1)$.
Suppose that the inequality $f^i(\sigma,\varepsilon) \leq F^i(\varepsilon)$ fails. Then there is a non-null $\mathcal{F}_\sigma$-measurable set $A \subseteq \{\sigma < T\}$ and a constant $a > 0$ such that $f^i(\sigma,\varepsilon,n)I_A \geq (F^i(\varepsilon)+a)I_A$ for all sufficiently large n.
Put $\sigma_A := \sigma I_A + TI_{A^c}$. Then for any $Z \in \mathcal{M}^i(\sigma,\varepsilon,n)$ the process ZI_A/EZI_A is in $\mathcal{M}^i(\sigma_A,\varepsilon,n)$. Since $E(\xi|\mathcal{F}_\sigma)I_A = E(\xi|\mathcal{F}_{\sigma_A})I_A$,

$$f^i(\sigma_A,\varepsilon,n)I_A \geq f^i(\sigma,\varepsilon,n)I_A.$$

Thus, for any $Z \in \mathcal{M}^i(\sigma_A,\varepsilon,n)$ and large n

$$EZ^i_{\tau_n}I_{\{\tau_n<T\}} = EZ^i_{\sigma_A}E((Z^i_{\tau_n}/Z^i_{\sigma_A})I_{\{\tau_n<T\}}|\mathcal{F}_{\sigma_A}) \geq F^i(\varepsilon)+a$$

in contradiction with the definition of $F^i(\varepsilon)$.

Lemma

Let $\sigma \in \mathcal{T}_T$ be such that $\sigma \leq \tau_{n_0}$ on the set $\{\sigma < T\}$ and let $\varepsilon, \delta > 0$. Then there are $n \geq n_0$, $\Gamma \in \mathcal{F}_\sigma$ with $P(\Gamma) \leq \delta$, and $Z \in \mathcal{M}^i(\sigma, \varepsilon, n)$ such that $Z^i_\sigma = \eta := I_{\{\sigma<T\}}/EI_{\{\sigma<T\}}$ and

$$E(Z^i_{\tau_n} I_{\{\tau_n<T\}}|\mathcal{F}_\sigma) \leq \frac{I_{\{\sigma<T\}}}{EI_{\{\sigma<T\}}}[(F^i(\varepsilon)+\delta)I_{\Gamma^c} + I_\Gamma].$$

Proof. Recall that the essential infimum ξ of a family of random variables $\{\xi^\alpha\}$ is the limit of a decreasing sequence of random variables of the form $\xi^{\alpha_1} \wedge \xi^{\alpha_2} \wedge ... \wedge \xi^{\alpha_m}$, $m \to \infty$. Thus, for any $a > 0$ the sets $\{\xi^{\alpha_k} \leq \xi + a\}$ form a covering of Ω. Using the standard procedure, one can construct from this covering a measurable partition of Ω by sets A^k such that $\xi^{\alpha_k} \leq \xi + \gamma$ on A^k.

Thus, for any fixed $n \geq n_0$ there are a countable partition of the set $\{\sigma < T\}$ into $\mathcal{F}_\sigma$-measurable sets $A^{n,k}$ and a sequence of $Z^{n,k} \in \mathcal{M}^i(\sigma, \varepsilon, n)$ such that

$$E((Z^{n,k,i}_{\tau_n}/Z^{n,k,i}_\sigma) I_{\{\tau_n < T\}} | \mathcal{F}_\sigma) \leq f^i(\sigma, \varepsilon, n) + 1/n \quad \text{on } A^{n,k}.$$

Put, for $t \in [\sigma, \tau_n]$,

$$\tilde{Z}^n_t := \eta \sum_{k=1}^{\infty} \frac{1}{Z^{n,k,i}_\sigma} Z^{n,k}_t I_{A^{n,k}}.$$

Then $\tilde{Z}^n \in \mathcal{M}^i(\sigma, \varepsilon, n)$, $\tilde{Z}^{n,i}_\sigma = \eta$, and

$$E(\tilde{Z}^{n,i}_{\tau_n} I_{\{\tau_n < T\}} | \mathcal{F}_\sigma) = \eta E((\tilde{Z}^{n,i}_{\tau_n}/\eta) I_{\{\tau_n < T\}} | \mathcal{F}_\sigma) \leq \frac{I_{\{\sigma < T\}}}{E I_{\{\sigma < T\}}} [f^i(\sigma, \varepsilon, n) + 1/n$$

Note that $f^i(\sigma, \varepsilon, n) + 1/n$ decreases to $f^i(\sigma, \varepsilon) \leq F^i(\varepsilon)$. By the Egorov theorem the convergence is uniform outside of a set Γ of arbitrary small probability. The assertion of the lemma follows from here.

Proposition

For any $\varepsilon_1, \varepsilon_2$ we have the inequality

$$F^i(\varepsilon_1)F^i(\varepsilon_2) \geq F^i((1+\varepsilon_1)(1+\varepsilon_2)/(1-\varepsilon_2)-1). \qquad (1)$$

Either $F^i = 0$, or there is $c^i \geq 0$ such that $F^i(\varepsilon) \geq e^{-c^i\varepsilon^{1/3}}$.

Proof. Fix $\delta > 0$ and a stopping time $\sigma \leq \tau_{n_0}$ on the set $\{\sigma < T\}$. By the lemma there are $n \geq n_0$ and $Z^1 \in \mathcal{M}^i(\sigma, \varepsilon_1, n)$ such that

$$EZ^{1i}_{\tau_n} I_{\{\tau_n < T\}} \leq F^i(\varepsilon_1) + \delta.$$

Using the lemma again (now with τ_n playing the role of σ), we find $m > n$ and $Z^2 \in \mathcal{M}^i(\tau_n, \varepsilon_2, m)$ with $Z^{2i}_{\tau_n} = I_{\{\tau_n<T\}}/EI_{\{\tau_n<T\}}$ such that outside of a set $\Gamma \in \mathcal{F}_{\tau_n}$ with $P(\Gamma) \leq \delta$

$$E(Z^{2i}_{\tau_m} I_{\{\tau_m<T\}} | \mathcal{F}_{\tau_n}) \leq \frac{I_{\{\tau_n<T\}}}{EI_{\{\tau_n<T\}}}[(F^i(\varepsilon_2)+\delta)I_{\Gamma^c} + I_\Gamma].$$

Define on $[\sigma, \tau_m]$ the martingale Z with $Z^j_t := Z^{1j}_t$ on $[\sigma, \tau_n]$ and $Z^j_t := Z^{2j}_t Z^{1j}_{\tau_n}/Z^{2j}_{\tau_n}$ on $[\tau_n, \tau_m]$, $j = 1, \ldots, d$.

Then

$$\phi_t Z_t^1 = \lambda_t^1(1+u_t^{11},\dots,1+u_t^{1d}), \quad t\in[\sigma,\tau_n],$$
$$\phi_t Z_t^2 = \lambda_t^2(1+u_t^{21},\dots,1+u_t^{2d}), \quad t\in[\tau_n,\tau_m],$$

where $\max_j|u^{1j}|\le\varepsilon_1$, $\max_j|u^{2j}|\le\varepsilon_2$ and $\lambda_t^1,\lambda_t^2>0$. It follows that

$$Z\in\mathcal{M}^i(\sigma,(1+\varepsilon_1)(1+\varepsilon_2)/(1-\varepsilon_2)-1,m).$$

Note also that

$$\begin{aligned} EZ^i_{\tau_m}I_{\{\tau_m<T\}} &= P(\tau_n<T)EZ^{2i}_{\tau_m}Z^{1i}_{\tau_n}I_{\{\tau_m<T\}} \\ &\le P(\tau_n<T)EZ^{1i}_{\tau_n}I_{\{\tau_n<T\}}E(Z^{2i}_{\tau_m}I_{\{\tau_m<T\}}|\mathcal{F}_{\tau_n}). \end{aligned}$$

Hence,

$$EZ^i_{\tau_m}I_{\{\tau_m<T\}}\le(F^i(\varepsilon_1)+\delta)(F^i(\varepsilon_2)+\delta)+EZ^{1i}_{\tau_n}I_{\{\tau_n<T\}}I_\Gamma.$$

The inequality (1) follows from here. Note that for $\varepsilon_1,\varepsilon_2\in]0,1/4]$

$$\frac{(1+\varepsilon_1)(1+\varepsilon_2)}{1-\varepsilon_2}-1=\frac{\varepsilon_1+2\varepsilon_2+\varepsilon_1\varepsilon_2}{1-\varepsilon_2}\le 4(\varepsilon_1+\varepsilon_2).$$

Since F is decreasing, we obtain from (1) that $F^i(\varepsilon_1)F^i(\varepsilon_2)\ge F^i(4(\varepsilon_1+\varepsilon_2))$ for all $\varepsilon_1,\varepsilon_2\in]0,1/8]$. Using Lemma 5 below with $f=\ln F^i$, we get the needed bound.

Lemma

Let $f :]0, x_0] \to \mathbf{R}$ *be a decreasing function such that*

$$f(x_1) + f(x_2) \geq f(4(x_1 + x_2)), \qquad \forall\, x_1, x_2 \leq x_0. \tag{2}$$

Then there is $c > 0$ *such that* $f(x) \geq -cx^{1/3}$ *for* $x \in]0, x_0]$*.*

Proof. In the non-trivial case where $f(x_0) < 0$, the constant $\kappa = -\inf_{x \in]x_0/8, x_0]} f(x)/x$ is strictly greater than zero. Iterating the inequality $2f(x) \geq f(8x)$ we obtain that $2^n f(x) \geq f(2^{3n}x)$ for all $x \in]0, x_0 2^{-3n}]$ and all integers $n \geq 0$. Therefore,

$$\frac{f(x)}{x} \geq 2^{2n} \frac{f(2^{3n}x)}{2^{3n}x} = \frac{1}{4} x_0^{2/3} \left(\frac{2^{3(n+1)}}{x_0} \right)^{2/3} \frac{f(2^{3n}x)}{2^{3n}x}.$$

For $x \in]2^{-3(n+1)}x_0, 2^{-3n}x_0]$, the right-hand side dominates $-cx^{-2/3}$ with the constant $c := \kappa x_0^{2/3}/4$. Thus, the inequality $f(x)/x \geq -cx^{-2/3}$ holds on $]0, x_0]$.

($\Leftarrow$) The arguments are standard. For any $\xi \in \phi_t A_0^{T\varepsilon}(T)$ and $Z \in \mathcal{M}_0^T(\widehat{K}_t^{\varepsilon *} \setminus \{0\})$ we have $EZ_T\xi \leq 0$ and this inequality is impossible for $\xi \in L^0(\mathbf{R}_+^d, \mathcal{F}_T)$.

($\Rightarrow$) We need to consider only the case where the universal chain is such at that $P(\tau_n < T) > 0$ for every n and we can apply the results on functions F^i. The claim follows from the assertions below.

Proposition

If $\sum_i F^i(\varepsilon) = 0$ *for all* $\varepsilon > 0$, *then* $\mathcal{M}_0^T(\widehat{K}^{\varepsilon *} \setminus \{0\}) \neq \emptyset$.

Proof. Fix $\varepsilon \in]0,1]$ and define a sequence of $\varepsilon_k \downarrow 0$ such that $\bar{\varepsilon}_N \uparrow \varepsilon$ where

$$\bar{\varepsilon}_N := (1+\varepsilon_0) \prod_{k=1}^{N} \frac{1+\varepsilon_k}{1-\varepsilon_k} - 1.$$

We construct inductively an increasing sequence of integers $(n_N)_{N\geq 0}$ with $n_0 = 0$ and a sequence of $Z^{(N)} \in \mathcal{M}_0^{\tau_{n_N}}(\widehat{K}^{\bar{\varepsilon}_N *} \setminus \{0\})$ such that for $N = kd + r$ where $0 \leq r \leq d-1$

$$EZ^{(N)(r+1)}_{\tau_{n_N}} I_{\{\tau_{n_N} < T\}} \leq 2^{-N}. \qquad (3)$$

Since $F^1(\varepsilon) = 0$, Lemma 4 ensures the existence of $Z^1 \in \mathcal{M}^1(0, \varepsilon_1, n_1)$ with

$$EZ^{11}_{\tau_{n_1}} I_{\{\tau_{n_1} < T\}} \leq 2^{-1}.$$

Put $Z^{(1)} := Z^1$. Take now $\delta_1 > 0$ such that

$$EZ^{(1)2}_{\tau_{n_1}} I_{\{\tau_{n_1} < T\}} I_A \leq 2^{-3}$$

for every $A \in \mathcal{F}_{\tau_{n_1}}$ with $P(A) \leq \delta_1$. Using again Lemma 4 (now for the second coordinate), we find $n_2 > n_1$, the set $\Gamma_1 \in \mathcal{F}_{\tau_{n_1}}$ with $P(\Gamma_1) \leq \delta_1 \wedge 2^{-3}$, and $Z^2 \in \mathcal{M}^2(\tau_{n_1}, \varepsilon_2, n_2)$ such that $Z^{22}_{\tau_{n_1}} = 1$ and

$$E(Z^{22}_{\tau_{n_2}} I_{\{\tau_{n_2} < T\}} | \mathcal{F}_{\tau_{n_1}}) \leq [2^{-3} + I_{\Gamma_1}] I_{\{\tau_{n_1} < T\}} / P(\tau_{n_1} < T).$$

Put $Z_t^{(2)j} = Z_t^{(1)j}$ on $[0, \tau_{n_1}]$ and $Z_t^{(2)j} = Z_t^{2j} Z_{\tau_{n_1}}^{(1)j} / Z_{\tau_{n_1}}^{2j}$ on $]\tau_{n_1}, \tau_{n_2}]$, $j = 1, \dots, d$. Note that $Z^{(2)} \in \mathcal{M}_0^{\tau_{n_2}}(\phi^{-1}\mathrm{cone}\,\{\mathbf{1} + U_{\bar{\varepsilon}_2}\} \setminus \{0\})$ and

$$\begin{aligned} EZ_{\tau_{n_2}}^{(2)2} I_{\{\tau_{n_2} < T\}} &= P(\tau_{n_1} < T) EZ_{\tau_{n_2}}^{22} Z_{\tau_{n_1}}^{(1)2} I_{\{\tau_{n_2} < T\}} \\ &\le P(\tau_{n_1} < T) EZ_{\tau_{n_1}}^{(1)2} I_{\{\tau_{n_1} < T\}} E(Z_{\tau_{n_2}}^{22} I_{\{\tau_{n_2} < T\}} | \mathcal{F}_{\tau_{n_1}}) \le 2^{-2}. \end{aligned}$$

We continue this procedure passing at each step from the coordinate j to the coordinate $j+1$ for $j \le d-1$ and from the coordinate d to the first one.
Since $P(\tau_n = T) \uparrow 1$, there is a process Z such that $Z^{\tau_{n_N}} = Z^{(N)}$ for every N. The components of Z are strictly positive processes on $[0, T]$. The condition (3) ensures that they are martingales. Therefore, $Z \in \mathcal{M}_0^T(\widehat{K}^\varepsilon \setminus \{0\})$.

Proposition

Suppose that $\sum F^i \neq 0$. *Then there is* $\varepsilon \in]0, 1]$ *for which the property* NA_ε^w *does not hold.*

Proof. At least one of functions, say, F^1, is not equal identically to zero. So, we have the bound $F^1(\varepsilon) > e^{-c\varepsilon^{1/3}}$ for all sufficiently small ε. Hence, there is a stopping time σ dominated by certain τ_{n_0} such that

$$\inf_{Z\in\mathcal{M}^1(\sigma,\varepsilon,n)} EZ^1_{\tau_n} I_{\{\tau_n<T\}} > e^{-c\varepsilon^{1/3}}$$

for all sufficiently large n. Then for every $Z \in \mathcal{M}^1(\sigma,\varepsilon,n)$ we have that

$$E(Z^1 I_{\{\tau_n=T\}}|\mathcal{F}_\sigma) \le 1 - e^{-c\varepsilon^{1/3}}.$$

Let us consider the sequence of random variables $\xi^n \in L^0(\mathbf{R}^d, \mathcal{F}_{\tau_n})$ such that the components $\xi^{n2} = \cdots = \xi^{nd} = 0$ and

$$\xi^{n1} = -I_{\{\sigma<T\}} + (1 - e^{-c\varepsilon^{1/3}})^{-1} I_{\{\sigma<T,\tau_n=T\}}.$$

Clearly,

$$E(Z_{\tau_n}\xi^n|\mathcal{F}_\sigma) \le -I_{\{\sigma<T\}} + (1-e^{-c\varepsilon^{1/3}})^{-1} E(Z_{\tau_n} I_{\{\tau_n=T\}}|\mathcal{F}_\sigma) I_{\{\sigma<T\}} \le 0.$$

We have the inequality $EZ_{\tau_n}\xi^n \leq 0$, and, therefore, by the superhedging theorem (see Th. 3.6.3), ξ^n is the terminal value of an admissible process $\widehat{V} = \widehat{V}^B$ in the model having σ and τ_n as the initial and terminal dates, respectively. Note that on the non-null set $\{\sigma < T\}$ the limit of ξ^{n1} is strictly positive. To conclude we use the lemma below which one can get by applying, on each interval $[0, \tau_n]$, the Komlós-type result (Lemma 3.6.5) followed by the diagonal procedure.

Lemma

Suppose that $\xi^n = \widehat{V}^n_{\tau_n}$ where $\widehat{V}^n + \mathbf{1} \in \widehat{K}^\varepsilon$ and $\xi^n \to \xi$ a.s. as $n \to \infty$. Then there is a portfolio process $\widehat{V}$ such that $\widehat{V} + \mathbf{1} \in \widehat{K}^\varepsilon$ and $\xi = \widehat{V}_T$.