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Smarandache $\mathcal{N}$–subalgebras(resp. filters) of CI–algebras

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Abstract. In this paper, we introduce the notions of $\mathcal{N}$-subalgebras and $\mathcal{N}$-filters based on Smarandache CI-algebra and give a number of their properties. The relationship between $\mathcal{N}(Q, f)$-subalgebras(filters) and $\mathcal{N}$-subalgebras(filters) are also investigated.

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1 Introduction

Some recent researchers led to generalizations of the notion of fuzzy set that introduced by Zadeh in 1965 [15]. The generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the point $\{1\}$ into the interval $[0, 1]$. In order to provide a mathematical tool to deal with negative information, Jun et. al. introduced $\mathcal{N}$-structures, based on negative-valued functions [6]. In 1966, Y. Imai and K. Iseki [3] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. H. S. Kim and Y. H. Kim defined a BE-algebra [5]. Biao Long Meng, defined notion of CI-algebra as a generalization of a BE-algebra [9]. It is Known that any BE-algebra is a CI-algebra. Hence, every BE-algebra is a weaker structure than CI-algebra, thus we can consider in any CI-algebra a weaker structure as BE-algebra. Jun et. al. discussed the notion of $\mathcal{N}$ -structures in $BCH/BCK/BCI$-algebras and investigated their properties in [6, 7]. They introduced the notions of $\mathcal{N}$-ideals of subtraction algebras and $\mathcal{N}$-closed ideals in BCK/BCI-algebras. We introduce the notions of $\mathcal{N}$-subalgebras and $\mathcal{N}$-filters in CI-algebras and give a number of their properties and The relationship between $\mathcal{N}$-subalgebras and $\mathcal{N}$-filters was discussed in [14]. Also, we discuss on Smarandache CI-algebra and investigated some of their useful properties in [2]. Beside, we introduced the notion of anti fuzzy set and stated the relationship with the $\mathcal{N}$-function of CI-algebra X. We showed that every anti fuzzy filter is an anti fuzzy subalgebra in [1]. K. J. Lee and Y. B. Jun introduced the notion of $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals based on a sub-BCK-algebra of a BCI-algebras and their relations/properties are investigated in [8].

In the present paper, we continue study of CI-algebras and apply the $\mathcal{N}$-structures to the filter theory in CI-algebras and Smarandache CI-algebras, also investigate the relationship between $\mathcal{N}$-subalgebra and $\mathcal{N}$-filters based on Smarandache CI-algebras. We show that any $\mathcal{N}(Q, f)$-closed filter is an $\mathcal{N}(Q, f)$-subalgebra. We give some conditions for $\mathcal{N}$-subalgebras(filters) to be $\mathcal{N}(Q, \varrho)$-subalgebras(resp. filters).

2 Preliminaries

In this section we review the basic definitions and some elementary aspects that are necessary for this paper.

Definition 2.1. [9] *An algebra* $(X; *, 1)$ *of type* $(2, 0)$ *is called a* CI*-algebra if it satisfying the following axioms:*

$(CI1)$ $x * x = 1$,

$(CI2)$ $1 * x = x$,

$(CI3)$ $x * (y * z) = y * (x * z)$, *for all* $x, y, z \in X$.

A CI–algebra X satisfying the condition $x * 1 = 1$ is called a BE-algebra. In any CI-algebra X one can define a binary relation "$\leq$" by $x \leq y$ if and only if $x * y = 1$.

A CI-algebra X has the following properties:

(i) $y * ((y * x) * x) = 1$,

(ii) $(x * 1) * (y * 1) = (x * y) * 1$,

(iii) if $1 \leq x$, then $x = 1$, for all $x, y \in X$.

A non-empty subset S of a CI-algebra X is called a subalgebra of X if $x * y \in S$ whenever $x, y \in S$. A mapping $f : X \to Y$ of CI -algebra is called a homomorphism if $f(x * y) = f(x) * f(y)$, for all $x, y \in X$. A non-empty subset F of CI-algebra X is called a filter of X if (1) $1 \in F$, (2) $x \in F$ and $x * y \in F$ implies $y \in F$. A filter F of CI-algebra X is said to closed if $x \in F$ implies $x * 1 \in F$. A nonempty subset S of a CI-algebra X is called a subalgebra of X if $x * y \in S$, for all $x, y \in S$. For our convenience, the empty set $\emptyset$ is regarded as a subalgebra of X. Denote by $Q(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $Q(X, [-1, 0])$ is a negative-valued function from X to $[-1, 0]$ (briefly, $\mathcal{N}$-function on X). By an $\mathcal{N}$-structure we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X.

In what follows, let X denote a CI-algebra and f an $\mathcal{N}$-function on X unless otherwise specified.

Definition 2.2. [14] *By a subalgebra of X based on $\mathcal{N}$-function f (briefly, $\mathcal{N}$-subalgebra of X), we mean an $\mathcal{N}$-structure (X, f) in which f satisfies the following assertion:*

$$f(x * y) \leq \max\{f(x), f(y)\}, \ for \ all \ x, y \in X.$$

Definition 2.3. [14] *By a filter of X based on $\mathcal{N}$-function f (briefly, $\mathcal{N}$-filter of X), we mean an $\mathcal{N}$-structure (X, f) in which f satisfies the following conditions:*

(i) $f(1) \leq f(y)$,

(ii) $f(y) \leq \max\{f(x * y), f(x)\}, \ for \ all \ x, y \in X.$

Definition 2.4. [2] *A Smarandache CI-algebra X is defined to be a CI–algebra X in which there exists a proper subset Q of X such that satisfies the following conditions:*

$(S1)$ $1 \in Q$ *and* $|Q| \geqslant 2$,

$(S2)$ *Q is a BE-algebra under the operation of X.*

Example 2.1. [2] Let $X := \{1, a, b, c, d\}$ be a set with the following table.

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	a	d
b	1	1	1	a	d
c	1	1	1	1	d
d	d	d	d	d	1

Then X is a CI-algebra and $Q = \{1, a, b, c\}$ is a BE-algebra.

Definition 2.5. [2] *A nonempty subset F of CI-algebra X is called a Smarandache filter of X related to Q (or briefly, Q-Smarandache filter of X) if it satisfies:*

- *(SF1)* $1 \in F$,
- *(SF2)* $(\forall y \in Q)(\forall x \in F)(x * y \in F \Rightarrow y \in F)$.

Definition 2.6. [11] *A fuzzy set $\mu : X \to [0, 1]$ is called an anti fuzzy subalgebra of X if it satisfy:*

$$\mu(x * y) \leq \max\{\mu(x), \mu(y)\}, \textit{ for all } x, y \in X.$$

Definition 2.7. [1] *A fuzzy set $\mu : X \to [0, 1]$ is called an anti fuzzy filter of X if it satisfies:*

- *(AFF1)* $\mu(1) \leq \mu(x)$,
- *(AFF2)* $\mu(y) \leq \max\{\mu(x * y), \mu(x)\}$, *for all* $x, y \in X$.

3 Smarandache $\mathcal{N}$-subalgebras

Definition 3.1. *Let X be a Q-Smarandache CI-algebra and $\varrho \in [-1, 0]$. An $\mathcal{N}$-structure (X, f) is called an $\mathcal{N}$-subalgebra of X based on Q and ϱ (briefly, $\mathcal{N}(Q, \varrho)$-subalgebra of X) if it is an $\mathcal{N}$-subalgebra of X such that satisfies the following condition:*

- *(type 1)* $(\forall x \in Q)$ $(\forall y \in X \setminus Q)$ $(f(x) \leq \varrho \leq f(y))$,
- *(type 2)* $(\forall x \in Q)$ $(\exists y \in X \setminus Q)$ $(f(x) \leq \varrho \leq f(y))$,
- *(type 3)* $(\exists x \in Q)$ $(\forall y \in X \setminus Q)$ $(f(x) \leq \varrho \leq f(y))$,

- *(type 4)* $(\exists x \in Q)$ $(\exists y \in X \setminus Q)$ $(f(x) \leq \varrho \leq f(y))$.

Note. If $\varrho := 0$, then $f(y) = 0$, for all $y \in X \setminus Q$. So, (Q, f) is an $\mathcal{N}$-subalgebra. If $\varrho := -1$, then $f(x) = -1$, for all $x \in Q$. And so $(X, f) = \mathcal{N}(Q, \varrho)$.

Example 3.1. a) In Example 2.1, an $\mathcal{N}$-structure (X, f) in which f is defined by $f(1) = f(a) = -0.7$, $f(b) = -0.4$, $f(c) = -0.6$ and $f(d) = -0.3$ is an $\mathcal{N}(Q, \varrho)$-subalgebra of all types on X, for $\varrho \in [-0.4, -0.3]$ and $Q = \{1, a, b, c\}$.

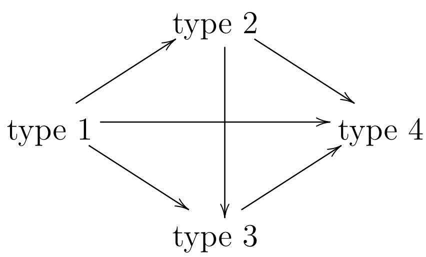
b) In Example 2.1, an $\mathcal{N}$-structure (X, g) in which g is defined by $g(1) = g(a) = -0.7$, $g(b) = -0.2$, $g(c) = -0.6$ and $g(d) = -0.3$ is not an $\mathcal{N}(Q, \varrho)$-subalgebra of X because $g(d) = -0.3 \ngeq g(b) = -0.2$.

c) In Example 2.1, an $\mathcal{N}$-structure (X, f) in which f is defined by $f(1) = f(a) = -0.7$, $f(b) = -0.4$, $f(c) = -0.5$ and $f(d) = -0.3$ is an $\mathcal{N}(Q, \varrho)$-subalgebra of type 2, type 3 and type 4 on X, for $\varrho \in [-0.4, -0.3]$ and $Q = \{1, a, b\}$, but it is not of type 1, because $f(c) \ngeq \varrho$.

d) In Example 2.1, an $\mathcal{N}$-structure (X, f) in which f is defined by $f(1) = f(a) = -0.7$, $f(b) = -0.2$, $f(c) = -0.3$ and $f(d) = -0.1$ is an $\mathcal{N}(Q, \varrho)$-subalgebra of type 3 and type 4 on X, for $\varrho \in [-0.7, -0.3]$ and $Q = \{1, a, b\}$, but it is not of type 1 and type 2 on X, because $f(b) \nleq \varrho$.

e) In Example 2.1, an $\mathcal{N}$-structure (X, f) in which f is defined by $f(1) = f(a) = -0.7$, $f(b) = -0.2$, $f(c) = -0.5$ and $f(d) = -0.3$ is an $\mathcal{N}(Q, \varrho)$-subalgebra of type 4 on X, for $\varrho \in [-0.7, -0.3]$ and $Q = \{1, a, b\}$, but it is not of type 1, type 2, type 3 on X.

Now, in the following diagram we summarize the results of this definition. The mark $A \to B$, means that A implies B.



In this paper, we focus on $\mathcal{N}(Q, \varrho)$-subalgebra of type 1 and from now on X is a Q-Smarandache CI-algebra.

The following example shows that there exists an $\mathcal{N}$-structure (X, f) in X such that it satisfies the condition (type 1), but it is not an $\mathcal{N}$-subalgebra of X.

Example 3.2. In Example 2.1, an $\mathcal{N}$–structure (X, f) in which f is defined by $f(1) = -0.7$, $f(a) = -0.2$, $f(b) = -0.4$, $f(c) = -0.6$ and $f(d) = -0.3$.

Then (X, f) satisfies the condition (2.1) for $\varrho \in [-0.2, -0.1]$, but it is not an $\mathcal{N}$-subalgebra. Because

$$f(b * c) = f(a) = -0.2 \not< -0.4 = \max\{f(b), f(c)\}.$$

Proposition 3.1. *If an* $\mathcal{N}$*–structure* (X, f) *satisfies the following condition:*

$$(\forall x \in Q)(\forall y \in X \setminus Q)(f(x) \leq f(y)),$$

then (X, f) *is an* (Q, ϱ)*–subalgebra of* X*, for every* $\varrho \in [\bigvee_{x \in Q} f(x), \bigwedge_{y \in X \setminus Q} f(y)]$.

Theorem 3.2. *Let* $\varrho \in [-1, 0]$*. If* (X, f) *is an* $\mathcal{N}(Q, \varrho)$*-subalgebra of* X*, then*

(*i*) $Q \subseteq C(f; \varrho)$,

(*ii*) $(\forall \beta \in [-1, 0])$ $(\beta < \varrho \Rightarrow C(f; \beta)$ *is a subalgebra of* $Q)$.

Proof. Let (X, f) be a $\mathcal{N}(Q, \varrho)$-subalgebra of X. Obviously, $Q \subseteq C(f; \varrho)$. If $\beta \in [-1, 0]$ be such that $\beta < \varrho$, then $C(f; \beta) \subseteq Q$. Let $x, y \in C(f; \beta)$. Then $f(x) \leq \beta$ and $f(x) \leq \beta$. Thus $f(x * y) \leq \max\{f(x), f(y)\} \leq \beta$, and so $x * y \in C(f; \beta)$. Thus $C(f; \beta)$ is a subalgebra of Q. □

In the following theorem we give some conditions for an $\mathcal{N}$–subalgebra to be an $\mathcal{N}(Q, \varrho)$-subalgebra.

Theorem 3.3. *Let* $\varrho \in [-1, 0]$*. If* (X, f) *is an* $\mathcal{N}$*-subalgebra of* X *satisfies the conditions* (*i*) *and* (*ii*) *in Theorem 3.2, then* (X, f) *is an* $\mathcal{N}(Q, \varrho)$*-subalgebra of* X*.*

Proof. Let $x \in Q$ and $y \in X \setminus Q$. Then by Theorem 3.2(i), $x \in C(f; \varrho)$, and so $f(x) \leq \varrho$. Let $f(y) = \beta$. If $\beta < \varrho$, then by Theorem 3.2(ii), $y \in C(f; \beta) \subseteq Q$, which is a contradiction. Hence $f(x) \leq \varrho \leq \beta = f(y)$. Thus (X, f) is an $\mathcal{N}(Q, \varrho)$-subalgebra of X. □

4 Smarandache $\mathcal{N}$-filters

Definition 4.1. *Let* X *be a* Q*-Smarandache* CI*-algebra and* $\varrho \in [-1, 0]$*. An* $\mathcal{N}$*-structure* (X, f) *is called an* $\mathcal{N}$*-filter of* X *based on* Q *and* ϱ *(briefly,* $\mathcal{N}(Q, \varrho)$*-filter of* X*) if it satisfies the following conditions:*

(*i*) $(\forall x \in Q)$ $(\forall y \in X \setminus Q)$ $(f(1) \leq f(x) \leq \varrho \leq f(y))$.

(ii) $(\forall x, y \in Q)$ $(f(y) \le \max\{f(x * y), f(x)\})$.

Example 4.1. In Example 2.1, an $\mathcal{N}$–structure (X, f) in which f is defined by $f(1) = -0.6$, $f(a) = -0.4$, $f(b) = -0.5$, $f(c) = -0.4$ and $f(d) = -0.3$ is an $\mathcal{N}(Q, \varrho)$–filter of X for $\varrho \in [-0.4, -0.3]$.

Theorem 4.1. *Let $\{\mathcal{N}(Q_i, \varrho_i) : i \in \Delta\}$ be a family of $\mathcal{N}(Q_i, \varrho_i)$-subalgebras (filters) of X where $\Delta \neq \emptyset$ and $\varrho_i \in [-1, 0]$, for all $i \in \Delta$.*
Then $\mathcal{N}(\cap Q_i, \min\{\varrho_i\})_{i\in\Delta}$, is a subalgebra (filter) of X, too.

Theorem 4.2. *Let $\varrho \in [-1, 0]$. If (X, f) is an $\mathcal{N}(Q, \varrho)$-filter of X, then*

(i) $Q \subseteq C(f; \varrho)$,

(ii) $(\forall \beta \in [-1, 0])$ $(\beta < \varrho \Rightarrow C(f; \beta)$ *is a filter of* $Q)$.

Proof. Let (X, f) be an $\mathcal{N}(Q, \varrho)$-filter of X. Obviously, $Q \subseteq C(f; \varrho)$. Let $\beta \in [-1, 0]$ be such that $\beta < \varrho$. If $x \in C(f; \beta)$, then $f(x) \le \beta < \varrho$, and so $x \in Q$. Hence $C(f; \beta) \subseteq Q$. by Definition 4.1(i), $f(1) \le f(x)$, for all $x \in X$. Hence $f(1) \le f(x) \le \beta$ for all $x \in C(f; \beta)$, and so $1 \in C(f; \beta)$. Let $x, y \in Q$ be such that $x * y \in C(f; \beta)$ and $x \in C(f, \beta)$. Then $f(x * y) \le \beta$ and $f(x) \le \beta$. If $x, y \in C(f; \beta)$, then $f(x) \le \beta$. Now by Definition 4.1(ii), $f(y) \le \max\{f(x * y), f(x)\} \le \beta$. Thus $y \in C(f; \beta)$. Therefore, $C(f; \beta)$ is a filter of Q. □

For a Q-Smarandache CI–algebra X and $\varrho \in [-1, 0]$, the following example shows that an $\mathcal{N}$-filter (X, f) of X may not be an $\mathcal{N}(Q, \varrho)$-filter of X.

Example 4.2. Let $X := \{1, a, b, c\}$ be a set with the following table.

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	c	c	c	1

Then X is a CI-algebra and $Q := \{1, a, b\}$ is a BE-algebra [13]. Define an $\mathcal{N}$-structure (X, f) in which f is defined by $f(1) = -0.7$, $f(a) = -0.2$, $f(b) = -0.4$, $f(c) = -0.2$. Then (X, f) is an $\mathcal{N}$-filter of X. But it is not an $\mathcal{N}(Q, \varrho)$ of X for $\varrho \in [-0.7, -0.3]$. Because $f(a) = -0.2 > \varrho$.

In the following theorem we give conditions for an $\mathcal{N}$-filter to be an $\mathcal{N}(Q, \varrho)$-filter.

Theorem 4.3. *Let $\varrho \in [-1, 0]$ and (X, f) be an $\mathcal{N}$-filter of X satisfies the conditions (i) and (ii) of Theorem 4.2. Then (X, f) is an $\mathcal{N}(Q, \varrho)$–filter of X.*

Proof. Let $x \in Q$ and $y \in X \backslash Q$. Then by Theorem 4.2(i), $x \in C(f; \varrho)$, and so $f(x) \leq \varrho$. Let $f(y) = \beta$. If $\beta < \varrho$, then by Theorem 4.2(ii), $y \in C(f; \beta) \subseteq Q$, which is a contradiction. Hence $\varrho \leq \beta = f(y)$. Since $f(1) \leq f(x)$ for all $x \in X$, it follows that $f(1) \leq f(x) \leq \varrho \leq \beta = f(y)$ so that condition (i) of Definition 4.1 is valid. Since f is an $\mathcal{N}$-filter of X, the condition (ii) of Definition 4.1 is obvious. Therefore, (X, f) is an $\mathcal{N}(Q, \varrho)$-filter of X. □

The following example shows that an $\mathcal{N}(Q, \varrho)$-subalgebra may not be an $\mathcal{N}(Q, \varrho)$-filter.

Example 4.3. Let $X := \{1, a, b, c, d\}$ be a set with the following table.

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	a	d
b	1	1	1	a	d
c	1	1	1	1	d
d	d	d	d	d	1

Then X is a CI-algebra and $Q = \{1, a, b, c\}$ is a BE-algebra. Define an $\mathcal{N}$-structure (X, f) in which f is defined by $f(1) = -0.4$, $f(a) = -0.4$, $f(b) = -0.3$, $f(c) = -0.2$ and $f(d) = -0.1$. Then (X, f) is an $\mathcal{N}$-subalgebra, for $\varrho \in [-0.2, 0]$, but it is not an $\mathcal{N}$-filter because

$$f(c) = -0.2 \not< -0.3 = \max\{f(b * c), f(b)\}.$$

Definition 4.2. *An $\mathcal{N}$-function on X is called closed $\mathcal{N}$-filter if f satisfies:*

$$f(x * 1) \leq f(x) \leq \max\{f(y * x, f(y)\}, \ for\ all\ x, y \in X.$$

Example 4.4. Let $X := \{1, a, b\}$ be a set with the following table:

$*$	1	a	b
1	1	a	b
a	a	1	1
b	a	1	1

Then X is a CI-algebra [10]. Define an $\mathcal{N}$-function $f : X \to [0, 1]$ by $f(1) = -0.7$, $f(a) = -0.3$ and $f(b) = -0.4$. Then (X, f) is an $\mathcal{N}$-filter of X. But it is not an $\mathcal{N}$-closed filter because

$$f(b * 1) = f(a) = -0.3 \not\leq f(b) = -0.4.$$

Example 4.5. In Example 4.4, if define $\mathcal{N}$-function $f : X \to [0, 1]$ by $f(1) = -0.7$, $f(a) = -0.4$ and $f(b) = -0.4$. Then (X, f) is an $\mathcal{N}$-closed filter of X.

Proposition 4.4. *Let (X, f) be an $\mathcal{N}$-closed filter. Then $f(1) \leq f(x)$, for all $x \in X$.*

Proof. Let $x \in X$. Now, by Definition 4.2, we have

$$f(1) \leq \max\{f(x * 1), f(x)\} \leq \max\{f(x), f(x)\} = f(x).$$

□

Theorem 4.5. *Let (X, f) be an closed $\mathcal{N}$-filter and $\varrho \in [-1, 0]$. Then every $\mathcal{N}(Q, \varrho)$-filter is $\mathcal{N}(Q, \varrho)$-subalgebra of X.*

Proof. Let (X, f) be $\mathcal{N}(Q, \varrho)$-filter and $x, y \in X$. Then by $(CI3)$ and Definition 4.2, we have

$$\begin{aligned} f(x * y) &\leq \max\{f(y * (x * y)), f(y)\} \\ &= \max\{f(x * (y * y)), f(y)\} \\ &= \max\{f(x * 1), f(y)\} \\ &\leq \max\{f(x), f(y)\}. \end{aligned}$$

Therefore, (X, f) is an $\mathcal{N}$-subalgebra of X. □

Theorem 4.6. *Let (X, f) and (X, g) be $\mathcal{N}(Q_1, \varrho_1)$ and $\mathcal{N}(Q_2, \varrho_2)$-subalgebra (filter) of X respectively. Then $(X \times X, f \times g)$ is an $\mathcal{N}(Q_1 \times Q_2, \max\{\varrho_1, \varrho_2\}$-subalgebra(filter) of $X \times X$.*

Proof. Let $(x, y) \in (Q_1 \times Q_2)$ and $(z, t) \in (X \times X) \setminus (Q_1 \times Q_2)$. Then we have

$$\begin{aligned} (f \times g)(1, 1) = \max\{f(1), g(1)\} &\leq \max\{f(x), g(y)\} \\ &\leq \max\{\varrho_1, \varrho_2\} \\ &\leq \max\{f(z), f(t)\} = (f \times g)(z, t). \end{aligned}$$

Now, let $(x_1, x_2), (y_1, y_2) \in (Q_1 \times Q_2)$. Then

$$\begin{aligned} (f \times g)((x_1, x_2) * (y_1, y_2)) &= (f \times g)((x_1 * y_1), (x_2 * y_2)) \\ &= \max\{f(x_1 * y_1)), g(x_2 * y_2)\} \\ &\leq \max\{\max\{f(x_1), f(y_1)\}, \max\{g(x_2), g(y_2)\}\} \\ &= \max\{\max\{f(x_1), g(x_2)\}, \max\{f(y_1), g(y_2)\}\} \\ &= \max\{(f \times g)(x_1, x_2), (f \times g)(y_1, y_2)\}. \end{aligned}$$

Hence $(X \times X, f \times g)$ is an $\mathcal{N}(Q_1 \times Q_2, \max\{\varrho_1, \varrho_2\})$-subalgebra(resp. filter) of $X \times X$. □

Proposition 4.7. *Let Q_1 and Q_2 be two BE-algebras which are properly contained in X, $Q_1 \subseteq Q_2$ and $\varrho \in [-1, 0]$. Then every $\mathcal{N}(Q_2, \varrho)$-subalgebra(filter) of X is an $\mathcal{N}(Q_1, \varrho)$-subalgebra(filter) of X.*

Note. By the following example we show that the converse of above theorem is not correct in general.

Example 4.6. Let $X := \{1, a, b, c\}$ be a set with the following table.

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	c	c	c	1

Then $Q_1 = \{1, a\}$, $Q_2 = \{1, a.b\}$ are BE-algebras which are properly contained in X and $f(1) = -0.7$, $f(a) = -0.4$, $f(b) = -0.2$ and $f(c) = -0.1$. Then (X, f) is an $\mathcal{N}(Q_1, \varrho)$-subalgebra, for all $\varrho \in [-0.4, 0]$, but it is not an $\mathcal{N}(Q_2, \varrho)$-subalgebra, because, if $\varrho := -0.3$, then $f(b) = -0.2 \not< -0.3$.

5 Conclusion

A Smarandache structure on a set A means a week structure W on A such that there exist a proper subset B of A which is embedded with a strong structure S. It is that any BE-algebra is a CI-algebra. Hence, every BE-algebra is a weaker structure than CI-algebra, thus we can consider in any CI-algebra a weaker structure as BE-algebra.

In this paper, we have introduced the concept of $\mathcal{N}$-subalgebra (filter) based on Smarandache CI-algebras and some related properties are investigated. We show that any $\mathcal{N}(Q, f)$-closed filter is an $\mathcal{N}(Q, f)$-subalgebra. We give some conditions for an $\mathcal{N}$ -subalgebras (filters) to be $\mathcal{N}(Q, \varrho)$-subalgebras (filters).

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