Smolyak's Algorithm, Sampling on Sparse Grids and Function Spaces of Dominating Mixed Smoothness

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Abstract

We investigate the rate of convergence in $\| \cdot |L_p\|$, $1 \le p \le \infty$, of the d-dimensional Smolyak algorithm, associated to a sequence of sampling operators in the framework of periodic Sobolev and Besov spaces with dominating mixed smoothness.

Key Words. Trigonometric interpolation, sampling operators, blending operators, Smolyak algorithm, rate of convergence, function spaces of dominating mixed smoothness, approximate recovery.

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1 Introduction

Let $(a_j^1)_{j=0}^\infty, \ldots , (a_j^d)_{j=0}^\infty$ be convergent sequences of complex numbers. The respective limits are denoted by $a^1, \ldots , a^d$. In addition we put $a_{-1}^\ell = 0$, $\ell = 1, \ldots , d$. Then $a^\ell = \sum_{j=0}^\infty (a_j^\ell - a_{j-1}^\ell)$ and hence

$$a^1 \cdot \ldots \cdot a^d = \sum_{j_1,\ldots,j_d=0}^{\infty} \prod_{\ell=1}^{d} (a_{j_\ell}^\ell - a_{j_\ell-1}^\ell)\,.$$

It has been the idea of Smolyak [32] to use the sequence

$$\sum_{j_1+\ldots+j_d\le m} \prod_{\ell=1}^{d} (a_{j_\ell}^\ell - a_{j_\ell-1}^\ell)\,, \qquad m = 0, 1, \ldots ,$$

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to approximate the product $a^1 \cdot \ldots \cdot a^d$. Now, if

$$a_j^1 = a_j^2 = \ldots = a_j^d = I_j f(x)\,, \qquad x \in \mathbb{T}\,,$$

where I_j denotes a sampling operator with respect to a certain set $\mathcal{T}_j$ of sample points, then the suggested approximation procedure results in an operator which uses samples from a sparse grid in $\mathbb{T}^d$ only, cf. Section 3 for details. Furthermore, the sequence of sampling operators constructed in such a way, should yield good approximations of tensor products $f_1 \otimes \ldots \otimes f_d$ of functions $f_\ell : \mathbb{T} \to \mathbb{C}$. In this paper we investigate the approximation power of these sampling operators for functions belonging to periodic Sobolev and Besov spaces of dominating mixed smoothness, see the Appendix for a definition. Let $A(\mathbb{T}^d)$ be either a Sobolev or a Besov space of dominating mixed smoothness on $\mathbb{T}^d$. Then the norm in these classes is a cross-norm, i.e.

$$\| f_1 \otimes \ldots \otimes f_d \,|A(\mathbb{T}^d)\| = \prod_{\ell=1}^{d} \| f_\ell \,|A(\mathbb{T})\|\,.$$

Hence, the function spaces under consideration here, are sufficiently close to the tensor product of function spaces defined on $\mathbb{T}$. Based on the approximation power of I_j on the torus $\mathbb{T}$ we shall derive sharp estimates for the order of convergence of Smolyak's algorithm on $\mathbb{T}^d$.

The present article continues investigations of the approximation properties of trigonometric interpolation with respect to uniform grids, see [14, 15, 34, 36, 27], where we now study the d-variate situation with respect to a sparse grid. More precisely, we investigate the rate of convergence of the Smolyak algorithm (applied to a sampling operator) for functions belonging to a Besov space of dominating mixed smoothness. This continues earlier work of Smolyak [32], Temlyakov [34], Wasilkowski, Woźniakowski [42] and one of the authors [28, 29]. It turns out that the Smolyak algorithm applied to a sampling operator yields a worst case within a wider class of Smolyak algorithms. In particular, the Smolyak algorithm applied to the partial sum of the Fourier series behaves better in approximation order than the Smolyak algorithm with respect to a sampling operator, see Subsection 3.2 for details. Let us mention that the algorithm applied to the partial sum of the Fourier series results in approximation from dyadic hyperbolic crosses, a subject, widely treated in the literature, see e.g. [2, 3], [1], [5], [7], [9], [12], [17], [18], [20], [24], [25], [30], [32], [37], [36] and [42].

The paper is organised as follows. In Section 2 we deal with interpolation on the torus including the discussion of some examples (de la Vallée-Poussin kernels, periodic spline interpolation, and Dirichlet kernels). Then we switch to the d-dimensional case in Section 3. To begin with we recall the construction of the Smolyak algorithm (Subsection

3.1), discuss a few more or less elementary properties of it and then we formulate our main results on the approximation power of this algorithm in a rather general frame (Subsection 3.2). Also consequences of our estimates for the problem of optimal recovery (sampling numbers) are discussed. Afterwards a few examples are presented based on what has been done in Section 2. Section 4 contains the proofs. The definitions and a few properties of Sobolev and Besov spaces of dominating mixed smoothness will be recalled in the Appendix.

The symbol I is reserved for identity operators (we do not indicate the space where I is considered, hoping this will be clear from the context). I_n and $I(\Lambda_n^\pi, \cdot)$ denote special sampling operators defined in Section 2. We also use the notation $a \asymp b$ if there exists a constant $c > 0$ (independent of the context dependent relevant parameters) such that

$$c^{-1}\, a \le b \le c\, a\, .$$

Constants will change their value from line to line, indicated by adding subscripts. Sometimes a constant will represent a fixed value for the paper. This is indicated by capital letters like $C_1, C_2, \ldots$. Finally, if $x \in \mathbb{R}^d$ then $|x|$ is used for the Euclidean distance (norm in ℓ_2^d) and $|x|_1$ denotes the norm in ℓ_1^d, respectively.

2 Interpolation on the Torus

In this first section we give a short survey about certain aspects of trigonometric interpolation.

2.1 Periodic Fundamental Interpolants

As usual, $\mathbb{N}$ is reserved for the natural numbers, by $\mathbb{N}_0$ we denote the natural numbers including 0 and by $\mathbb{Z}$ the set of all integers. Let $\mathbb{T}$ denote the torus, represented in $\mathbb{R}$ by the interval $\mathbb{T} = [0, 2\pi]$, where opposite points are identified. The functions f considered in this section will always be defined on the torus, i.e. they will be complex-valued and 2π-periodic. As usual, let

$$c_k(f) = (2\pi)^{-1} \int_{\mathbb{T}} f(t)\, e^{-ikt}\, dt\, , \qquad k \in \mathbb{Z}\, ,$$

denote the Fourier coefficient of $f \in L_1(\mathbb{T})$. Further, let

$$\mathcal{D}_m(t) := \sum_{|k| \le m} e^{ikt}\, , \qquad t \in \mathbb{T}, \quad m \in \mathbb{N}_0\, ,$$

be the Dirichlet kernel and let

$$I_m f(t) := \frac{1}{2m+1} \sum_{\ell=0}^{2m} f(t_\ell)\, \mathcal{D}_m(t - t_\ell)\,, \qquad t_\ell = \frac{2\pi\ell}{2m+1}\,. \tag{1}$$

Then I_m is the unique trigonometric polynomial of degree less than or equal to m which interpolates f at the nodes t_ℓ. This is the prototype for the class of sampling operators on $\mathbb{T}$ we have in mind. To generalise this concept we proceed as follows.
Let $n \in \mathbb{N}$. We put

$$K_n := \left\{ \ell \in \mathbb{Z} :\ -\frac{n}{2} \le \ell < \frac{n}{2} \right\} \qquad \text{and} \qquad J_n := \left\{ t_\ell = \frac{2\pi\ell}{n} \,:\, \ell \in K_n \right\}. \tag{2}$$

Obviously, the cardinality $|J_n|$ of J_n is equal to n. Here we are interested in periodic fundamental interpolants with respect to this grid J_n, i.e. we consider continuous 2π-periodic functions Λ_n such that

$$\Lambda_n(t_\ell) = \delta_{0,\ell}\,, \qquad \ell \in K_n\,.$$

Here $\delta_{0,\ell}$ is the Kronecker symbol. As in case of the trigonometric interpolation we associate to such a fundamental interpolant a linear operator given by

$$I(\Lambda_n, f)(t) := \sum_{\ell \in K_n} f(t_\ell)\, \Lambda_n(t - t_\ell)\,.$$

In this section our aim consists in deriving some sufficient conditions on Λ_n such that we can estimate the error $f - I(\Lambda_n, f)$ in the L_p-norm for functions from Nikol'skij-Besov spaces. For us it will be convenient to construct a sequence $(\Lambda_n)_n$ from one given function $\Lambda : \mathbb{R} \to \mathbb{R}$. In this context the Fourier transform represents a tool with some advantages. For $\Lambda \in L_1(\mathbb{R})$ we put

$$\mathcal{F}\Lambda(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\xi}\, \Lambda(t)\, dt, \qquad \xi \in \mathbb{R}\,.$$

Then the following lemma is known, cf. e.g. [30].

Lemma 1 *Let $\Lambda : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$(E1) \qquad \Lambda(2\pi\ell) = \delta_{0\,\ell}\,, \qquad \ell \in \mathbb{Z}\,,$$

$$(E2) \qquad \sum_{k \in \mathbb{Z}} |\Lambda(x + 2\pi k)| \quad \textit{is uniformly convergent on} \quad [0, 2\pi]\,.$$

(i) *Then, for $n \in \mathbb{N}$,*

$$\Lambda_n^\pi(t) := \sum_{\ell \in \mathbb{Z}} \Lambda(nt + 2\pi\ell n)\,, \qquad t \in \mathbb{R}\,, \tag{3}$$

is a continuous 2π-periodic fundamental interpolant with respect to the grid J_n.
(ii) *The Fourier coefficients of these functions are given by*

$$c_\ell(\Lambda_n^\pi) = \frac{1}{n\sqrt{2\pi}}\, \mathcal{F}\Lambda(\ell/n)\,, \qquad \ell \in \mathbb{Z}\,, \quad n \in \mathbb{N}\,.$$

2.2 The Rate of Convergence

Given an appropriate function Λ we shall investigate the error $f - I(\Lambda_n, f)$ in the L_p-norm. For us it will be sufficient to manage this for functions f belonging to some Nikol'skij-Besov space $B^r_{p,\infty}(\mathbb{T})$, see Subsection 5.1.
By $\psi : \mathbb{R} \to \mathbb{R}$ we denote a smooth cut-off function, i.e. $\psi \in C^\infty(\mathbb{R})$, $\psi(t) = 1$ if $|t| \le 1$, and $\psi(t) = 0$ if $|t| \ge 2$.

Proposition 1 *Let Λ be a continuous function satisfying the hypothesis (E1) and (E2). Let Λ^π_n be defined as in (3). Further we assume that for some numbers $0 \le \beta < \alpha$ the function $\mathcal{F}\Lambda$ satisfies:*

(E3) $\mathcal{F}\Lambda(\ell) = \sqrt{2\pi}\,\delta_{0,\ell} \quad , \quad \ell \in \mathbb{Z}$;

(E4) the functions

$$\begin{aligned} A(\xi) &:= \psi\left(\frac{\xi}{2}\right)|\xi|^{-\alpha}\left(1 - \frac{\mathcal{F}\Lambda(\xi)}{\sqrt{2\pi}}\right), \\ B_\ell(\xi) &:= \psi\left(\frac{\xi}{2}\right)|\xi|^{-\alpha}\mathcal{F}\Lambda(\xi + l), \qquad \ell \in \mathbb{Z}\setminus\{0\} \\ C_\ell(\xi) &:= \Big(1 - \psi(2\xi)\Big)|\xi|^{-\beta}\mathcal{F}\Lambda(\xi - \ell), \qquad \ell \in \mathbb{Z}, \end{aligned}$$

belong to $L_1(\mathbb{R})$,

(E5) the integrals

$$\begin{aligned} \int_{-\infty}^{\infty} |\mathcal{F}^{-1}A(w)|\,dw &< \infty, \\ \sum_{\ell\neq 0}\int_{-\infty}^{\infty} |\mathcal{F}^{-1}B_\ell(w)|\,dw &< \infty, \\ \text{and} \qquad \sum_{\ell\in\mathbb{Z}}\int_{-\infty}^{\infty} |\mathcal{F}^{-1}C_\ell(w)|\,dw &< \infty \end{aligned}$$

are finite.

If $\beta < r < \alpha$ and $1 \le p \le \infty$, then we have

$$\| I - I(\Lambda^\pi_n, \cdot)\,|B^r_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T})\| \asymp n^{-r}. \tag{4}$$

Remark 1 *A proof of the estimate from above in (4) can be found in [30], at least if $1 < p < \infty$. The necessary modifications, to include the limiting cases, are straightforward. For full details we refer to [39]. For $p = 2$ the conditions can be simplified, see [31]. The estimate from below can be deduced from the behaviour of the linear widths (approximation numbers) of the embeddings $B^r_{p,\infty}(\mathbb{T}) \hookrightarrow L_p(\mathbb{T})$, see Remark 2 below.*

Remark 2 *Linear widths. For two Banach spaces X, Y such that $X \hookrightarrow Y$ we define*

$$\lambda_n(I, X, Y) := \inf \Big\{ \| \, I - L \, | \mathcal{L}(X,Y) \| \, : \quad L \in \mathcal{L}(X,Y), \ \operatorname{rank} L \le n \Big\} .$$

Since our operator $I(\Lambda_n^\pi, \cdot)$ has rank $\le n$ we obtain

$$\lambda_n(I, B^r_{p,\infty}(\mathbb{T}), L_p(\mathbb{T})) \le \| \, I - I(\Lambda_n^\pi, \cdot) \, | \mathcal{L}(B^r_{p,\infty}(\mathbb{T}), L_p(\mathbb{T})) \, \| \, .$$

Since $\lambda_n(I, B^r_{p,\infty}(\mathbb{T}), L_p(\mathbb{T})) \asymp n^{-r}$, cf. e.g. [36, 1.4], it is clear that our interpolation operators yield optimal in the order of approximation.

2.3 Interpolation with de la Vallée-Poussin Means

For $0 < \mu < 1/2$ we consider the functions

$$\Lambda_\mu(t) := 2 \, \frac{\sin(t/2) \, \sin(\mu t)}{\mu \, t^2} \, , \qquad t \in \mathbb{R} \, . \tag{5}$$

Then the Fourier transform is given by

$$\mathcal{F}\Lambda_\mu(\xi) = \sqrt{2\pi} \begin{cases} 1 & \text{if } \ |\xi| \le \frac{1}{2} - \mu \, , \\ \frac{1}{2\,\mu} \left(\frac{1}{2} + \mu - |\xi| \right) & \text{if } \ \frac{1}{2} - \mu < |\xi| < \frac{1}{2} + \mu \, , \\ 0 & \text{if } \ \frac{1}{2} + \mu \le |\xi| \, , \end{cases}$$

i.e. a piecewise linear function.

Lemma 2 *Let $0 < \mu < 1/2$. Then the function Λ_μ satisfies the restrictions in Proposition 1 with $\beta = 1$ and $\alpha > 0$ arbitrary.*

Proof A proof has been given in [30]. ■

Corollary 1 *Let $0 < \mu < 1/2$ and Λ_μ be defined as in (5). Let further $1 \le p \le \infty$ and $r > 1/p$. Then we have*

$$\| \, I - I(\Lambda^\pi_{\mu,n}, \cdot) \, | B^r_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T}) \| \asymp n^{-r} \, .$$

Proof The corollary becomes a consequence of Proposition 1, Lemma 2 and complex interpolation for the estimate from above, cf. [30] for details, and Remark 2 for the estimate from below. ■

Remark 3 *Let*

$$v_{2n-1}(t) := \frac{1}{n} \sum_{j=n}^{2n-1} D_j(t)\,, \qquad t \in \mathbb{R}\,, \quad n \in \mathbb{N}\,,$$

denote the de la Vallée-Poussin kernels of odd order. Then

$$c_k(v_{2n-1}) = \begin{cases} 1 & \text{if} \quad |k| \le n\,, \\ 2\,(1-|k|/(2n)) & \text{if} \quad n < |k| < 2n\,, \\ 0 & \text{if} \quad |k| \ge 2n\,. \end{cases}$$

From Lemma 1(ii) we conclude the identity

$$\Lambda^{\pi}_{\mu,3n} = \frac{v_{2n-1}}{3n}\,, \qquad \mu = \frac{1}{6}\,, \quad n \in \mathbb{N}\,.$$

Hence

$$I(\Lambda^{\pi}_{\mu,3n}, f)(t) = \frac{1}{3n} \sum_{\ell \in K_{3n}} f(t_\ell)\, v_{2n-1}(t - t_\ell)\,, \qquad \mu = \frac{1}{6}\,.$$

In contrast to our treatment Temlyakov [36, 1.6] considered the sequence of sampling operators

$$\mathcal{R}_n f(t) := \frac{1}{4n} \sum_{\ell \in K_{4n}} f(t_\ell)\, v_{2n-1}(t - t_\ell)\,, \qquad t_\ell \in J_{4n}\,,$$

and proved that these operators also satisfy

$$\|\, I - \mathcal{R}_n \,|B^r_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T})\| \asymp n^{-r}\,,$$

if $1 \le p \le \infty$ *and let* $r > 1/p$*.*

2.4 Periodic Spline Interpolation

There is no need to concentrate on operators $I(\Lambda_n, \cdot)$ such that the range space is a subset of the set of trigonometric polynomials. The cardinal centralized B–spline $\mathcal{M}_\lambda$ of order $\lambda \in \mathbb{N}$ is defined as

$$\mathcal{M}_\lambda(x) \;:=\; (\underbrace{\mathcal{M}_1 * \ldots * \mathcal{M}_1}_{\lambda\text{-fold}})(x)\,, \qquad x \in \mathbb{R}\,, \quad \lambda \in \mathbb{N}\,.$$

Here $\mathcal{M}_1$ denotes the characteristic function of the interval $[-1/2, 1/2]$. The Fourier transform is given by

$$\mathcal{F}\mathcal{M}_\lambda(\xi) = \frac{1}{\sqrt{2\pi}} \left(\frac{\sin \frac{\xi}{2}}{\frac{\xi}{2}} \right)^{\lambda}\,, \qquad \xi \in \mathbb{R}\,.$$

To construct a fundamental interpolant on $\mathbb{R}$, we follow a standard procedure, cf. e.g. Jetter [16]. The symbol corresponding to $\mathcal{M}_\lambda$ is given by

$$\widetilde{\mathcal{M}}_\lambda(\xi) := \sum_{m\in\mathbb{Z}} \mathcal{F}\mathcal{M}_\lambda(2\pi\xi + 2\pi m)\,, \qquad \xi \in \mathbb{R}\,.$$

For λ being an even number these functions are known to be strictly positive. We define a family of fundamental interpolants on $\mathbb{R}$ as follows:

$$\mathcal{F}\Lambda_{2\lambda}(\xi) = \sqrt{2\pi}\,\frac{\mathcal{F}\mathcal{M}_{2\lambda}(2\pi\xi)}{\widetilde{\mathcal{M}}_{2\lambda}(\xi)}\,, \qquad \xi \in \mathbb{R}\,, \quad \lambda \in \mathbb{N}\,.$$

The function $\mathcal{F}\Lambda_{2\lambda}$ belongs to $L_1(\mathbb{R})$, hence $\Lambda_{2\lambda}$ is at least continuous. It is easy to check that these functions $\Lambda_{2\lambda}$ satisfy the conditions (E1), (E2) in Lemma 1.

Lemma 3 *Let $\lambda \in \mathbb{N}$. The function $\Lambda_{2\lambda}$ satisfies the restrictions in Proposition 1 with $\alpha < 2\lambda$ and $\beta > 1$.*

Proof A proof has been given in [30]. ■

Now, similarly as in the previous subsection, one derives from Lemma 3 and Remark 2 consequences for periodic spline interpolation.

Corollary 2 *Let $\lambda \in \mathbb{N}$, $1 \le p \le \infty$, and suppose*

$$\frac{1}{p} < r < 2\lambda\,.$$

Then periodic spline interpolation has the following property: it holds

$$\|\, I - I(\Lambda^{\pi}_{2\lambda,n},\cdot)\,|B^r_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T})\| \asymp n^{-r}\,.$$

Remark 4 *Results as stated in the Corollary 2 are not new, cf. e.g. Oswald [23] and [30].*

2.5 Interpolation with the Dirichlet Kernel

The classical case of trigonometric interpolation requires some modifications. It is not covered by Proposition 1, however well-known in the literature. Recall, I_n has been defined in (1). Then the following is known, see [14, 15, 34, 36, 27].

Proposition 2 *Let $1 < p < \infty$ and let $r > 1/p$. Then we have*

$$\|\, I - I_n\,|B^r_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T})\| \asymp n^{-r}\,.$$

2.6 Some other Means

For better comparison we recall some well-known classical approximation properties of the Fourier partial sums and de la Vallée-Poussin means, cf. e.g. [26, Chapt. 3] or [36, Chapt. 1]. Let

$$\sigma(t) := \begin{cases} 1 & \text{if} \quad |t| \le 1\,, \\ 2 - |t| & \text{if} \quad 1 \le |t| \le 2\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

For $f \in L_1(\mathbb{T})$ we put

$$\begin{aligned} S_n f(t) \quad &:= \quad \sum_{k=-n}^{n} c_k(f)\, e^{ikt}\,, \qquad (6) \\ V_{2n-1} f(t) \quad &:= \quad \sum_{k=-\infty}^{\infty} \sigma(k/n)\, c_k(f)\, e^{ikt}\,, \qquad n \in \mathbb{N}_0\,. \end{aligned}$$

Proposition 3 *Let $r > 0$.*
(i) *Let $1 < p < \infty$. Then we have*

$$\| \, I - S_n \, | B^r_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T}) \| \asymp n^{-r}\,.$$

(ii) *Let $1 \le p \le \infty$. Then we have*

$$\| \, I - V_{2n-1} \, | B^r_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T}) \| \asymp n^{-r}\,.$$

3 Approximation on Sparse Grids

$\mathbb{R}^d$ denotes the Euclidean d-space and $\mathbb{Z}^d$ means those elements in $\mathbb{R}^d$ having integer components. The symbol $\mathbb{T}^d$ is used for the d-dimensional torus represented in $\mathbb{R}^d$ by $[0, 2\pi]^d$. For $f \in L_1(\mathbb{T}^d)$ and $k \in \mathbb{Z}^d$ we put

$$c_k(f) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x)\, e^{-ikx}\, dx\,.$$

Then the Fourier series Sf of f is given by

$$Sf(x) := \sum_{k \in \mathbb{Z}^d} c_k(f)\, e^{ikx}\,.$$

Here $kx = \sum_{\ell=1}^{d} k_\ell\, x_\ell$, $k = (k_1, \ldots, k_d)$, $x = (x_1, \ldots, x_d)$.

3.1 The Smolyak Algorithm

3.1.1 The Definition and General Properties

Let $d \geq 2$. Let X and Y be Banach spaces such that $X, Y \hookrightarrow L_1(\mathbb{T})$. Further we assume that $P_1, \ldots, P_d : X \to Y$ are continuous linear operators. Then we define its tensor product $P_1 \otimes \ldots \otimes P_d$ to be the linear operator such that:

$$(P_1 \otimes \ldots \otimes P_d)(e^{ik_1 \cdot} \cdot \ldots \cdot e^{ik_d \cdot})(x_1, ..., x_d) := \prod_{\ell=1}^{d} P_\ell(e^{ik_\ell \cdot})(x_\ell)$$

$x_\ell \in \mathbb{T}, \quad k_\ell \in \mathbb{Z}, \quad \ell = 1, \ldots, d$. Formally this operator is defined on trigonometric polynomials only. If X is either $L_p(\mathbb{T})$, $1 \leq p < \infty$, or if $X = C(\mathbb{T})$, then, because of the density of trigonometric polynomials, there exists a unique continuous extension of $P_1 \otimes \ldots \otimes P_d$ to either $L_p(\mathbb{T}^d)$ or $C(\mathbb{T}^d)$, respectively. For this extension we shall use the same symbol.

Let either $L_j : L_p(\mathbb{T}) \to L_p(\mathbb{T})$, $1 \leq p < \infty$, or $L_j : C(\mathbb{T}) \to L_p(\mathbb{T})$, $1 \leq p \leq \infty$, $j \in \mathbb{N}_0$, be a sequence of continuous linear operators, denoted by L. Then we put

$$\Delta_j(L) := \begin{cases} L_j - L_{j-1} & \text{if} \quad j \in \mathbb{N}\,, \\ L_0 & \text{if} \quad j = 0\,. \end{cases}$$

Definition 1 *Let $m \in \mathbb{N}_0$. The Smolyak-Algorithm $A(m, d, \vec{L})$ relative to the d sequences $L^1 := (L^1_j)_{j=0}^\infty, \ldots, L^d := (L^d_j)_{j=0}^\infty$, is the linear operator*

$$A(m, d, \vec{L}) := \sum_{j_1+\ldots+j_d \leq m} \Delta_{j_1}(L^1) \otimes \ldots \otimes \Delta_{j_d}(L^d)\,.$$

Remark 5 *Originally introduced in [32] there are now hundreds of references dealing with this construction. A few basics and some references can be found in [22] and [42]. In particular the following formula is proved in [42]:*

$$A(m, d, \vec{L}) = \sum_{m-d+1 \leq |j|_1 \leq m} (-1)^{m-|j|_1} \binom{d-1}{m-|j|_1} L^1_{j_1} \otimes \ldots \otimes L^d_{j_d}\,. \tag{7}$$

This will be used later on.

3.1.2 Some Properties of Sampling Operators

In this article we shall restrict ourselves to sequences $(L_j)_j$ of linear operators having some additional properties. Recall, the symbol I is reserved for identity operators.

Let $1 \leq p \leq \infty$ and $r > 0$ be fixed. We suppose

(H1) For any $j \in \mathbb{N}_0$ we have $L_j \in \mathcal{L}(L_p(\mathbb{T}), L_p(\mathbb{T}))$.

(H1') For any $j \in \mathbb{N}_0$ we have $L_j \in \mathcal{L}(C(\mathbb{T}), L_p(\mathbb{T}))$.

(H2) There exists a positive number λ such that

$$L_j(e^{ik\cdot})(t) = e^{ikt}\,, \qquad t \in \mathbb{R}\,, \quad |k| \le \lambda\, 2^j\,, \quad k \in \mathbb{Z}\,, \quad j \in \mathbb{N}_0\,.$$

(H3) For all $0 < s \le r$ there exists a positive constant $C_0(s)$ such that

$$\sup_{j=0,1,\ldots} 2^{js}\, \|\, I - L_j\, |B^s_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T})\| = C_0(s) < \infty\,. \tag{8}$$

(H3') For all $1/p < s \le r$ there exists a positive constant $C_0(s)$ such that

$$\sup_{j=0,1,\ldots} 2^{js}\, \|\, I - L_j\, |B^s_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T})\| = C_0(s) < \infty\,. \tag{9}$$

We shall say that $\vec{L}$ satisfies the hypothesis (Hn) if each sequence L^i, $i = 1, \ldots, d$, satisfies (Hn).

We collect some properties of related sequences $A(m, d, \vec{L})$.

Lemma 4 *Let*

$$H(m,d,\lambda) := \Big\{ \ell \in \mathbb{Z}^d : \exists u_1, ..., u_d \in \mathbb{N}_0 \quad s.t. \quad |\ell_k| \le 2^{u_k}\lambda \text{ and } \sum_{k=1}^d u_k = m \Big\} \tag{10}$$

be a dyadic hyperbolic cross. Suppose that $\vec{L}$ satisfies (H2) for some $\lambda > 0$. Then

$$A(m,d,\vec{L})\, e^{ik\cdot} = e^{ik\cdot}\,, \quad k \in H(m,d,\lambda)\,.$$

Remark 6 *A special case of Lemma 4 can be found in [34].*

Of particular interest for us are sampling operators. Here we shall work with the following hypothesis:

(H4) For any $j \in \mathbb{N}_0$ there exist a natural number N_j, fixed functions $\psi^j_1, \ldots, \psi^j_{N_j} : C(\mathbb{T}) \to \mathbb{C}$ and sampling points $t^j_1, \ldots t^j_{N_j}$ such that

$$L_j f(x) = \sum_{\ell=1}^{N_j} f(t^j_\ell)\, \psi^j_\ell(x)\,, \quad f \in C(\mathbb{T})\,, \quad x \in \mathbb{T}.$$

Furthermore, we assume the existence of two positive constants C_1 and C_2 such that

$$C_1\, 2^j \le N_j \le C_2\, 2^j\,, \qquad j \in \mathbb{N}\,.$$

Sometimes we also need the assumption **(H5)**: let $\mathcal{T}_j := \{t_1^j, \dots t_{N_j}^j\}$. Then we suppose

$$\Big|\mathcal{T}_{n+1} \setminus \bigcup_{j=0}^{n} \mathcal{T}_j\Big| \geq C_3\, 2^{n+1}\,, \qquad n \in \mathbb{N}\,,$$

with some constant $0 < C_3 \leq C_2$.

Let $\vec{L}$ consist of sequences of sampling operators. The set of sampling points used by L_j^i will be denoted by $\mathcal{T}_j^i$. Then we put

$$\mathcal{G}(m,d,\vec{L}) := \bigcup_{m-d+1 \leq |j|_1 \leq m} \mathcal{T}_{j_1}^1 \times \dots \times \mathcal{T}_{j_d}^d \tag{11}$$

By (7) the operator $A(m,d,\vec{L})$ uses only samples from the grid $\mathcal{G}(m,d,\vec{L})$. To begin with we state a simple property of the standard grid induced by the choice $\mathcal{T}_j^i = J_{2^j}$.

Lemma 5 *Let $\vec{L}$ be a sequence of operators such that L_j^i uses samples from the grid $\mathcal{T}_j^i = J_{2^j}$, $i = 1, \dots, d$, $j \in \mathbb{N}_0$ (see (2)). Then the cardinality $S(m,d)$ of the grid $\mathcal{G}(m,d,\vec{L})$ is given by*

$$S(m,d) = \sum_{j=0}^{d-1} \binom{d-1}{j} 2^{m-j} \binom{m}{j}\,, \tag{12}$$

where we put $\binom{m}{j} := 0$ in case $m < j$.

This Lemma can be generalised to the following assertion.

Lemma 6 (i) *The hypothesis (H4) should be fulfilled. Then the cardinality $|\mathcal{G}(m,d,\vec{L})|$ of the grid $\mathcal{G}(m,d,\vec{L})$ satisfies*

$$|\mathcal{G}(m,d,\vec{L})| \leq (2C_2)^d\, S(m,d)\ .$$

(ii) *If the hypothesis (H4) and (H5) are fulfilled then*

$$\min(C_1,C_3)^d S(m,d) \leq |\mathcal{G}(m,d,\vec{L})| \leq (2C_2)^d\, S(m,d)\ , \quad m \in \mathbb{N}_0\ . \tag{13}$$

holds.

In addition we need the cardinality of certain subsets of $\mathbb{Z}^d$, especially the hyperbolic cross $H(m,d,1)$, defined in (10). For $m \in \mathbb{N}$ we consider also the sets

$$P_0(m,d) = \Big\{(n_1,...,n_d) \in \mathbb{N}_0^d : \sum_{i=1}^{d} n_i = m\Big\}$$

$$\text{and} \qquad P_1(m,d) = \Big\{(n_1,...,n_d) \in \mathbb{N}^d : \sum_{i=1}^{d} n_i = m\Big\}.$$

Lemma 7 *For $m \in \mathbb{N}$ it holds*
(i)

$$|P_0(m,d)| = \binom{m+d-1}{d-1}, \qquad |P_1(m,d)| = \binom{m-1}{d-1},$$

(ii) *and*

$$2^d\, S(m,d) \le |H(m,d,1)| \le 3^d\, S(m,d).$$

Remark 7 (i) *Obviously, for fixed d we have*

$$|P_0(m,d)| \asymp m^{d-1} \quad , \quad |P_1(m,d)| \asymp m^{d-1} \quad \text{and} \quad S(m,d) \asymp 2^m\, m^{d-1}\,, \qquad m \in \mathbb{N}\,.$$

We call the grids $\mathcal{G}(m,d,\vec{L})$ sparse because their cardinality is growing only with a logarithmic order with respect to d.
(ii) *Estimates of the cardinality of grids related to Smolyak algorithms are given at various places. What concerns the dependence on d we refer e.g. to [19].*

Lemma 8 *The hypothesis (H4) should be fulfilled. In addition to (6) we assume that*

$$L^i_j f(t^i_j) = f(t^i_j)\,, \qquad t^i_j \in \mathcal{T}^i\,, \qquad i = 1, \ldots, d\,,$$

for all $j \le m$ and all $f \in C(\mathbb{T})$. If the grids $\mathcal{T}^i_j$ are nested, i.e.

$$\mathcal{T}^i_j \subset \mathcal{T}^i_{j+1}\,, \qquad i = 1, 2, \ldots, d\,, \quad j \in \mathbb{N}_0\,,$$

then $A(m,d,\vec{L})$ interpolates on $\mathcal{G}(m,d,\vec{L})$, more precisely

$$A(m,d,\vec{L})f(x) = f(x)\,, \qquad x \in \mathcal{G}(m,d,\vec{L})\,, \quad f \in C(\mathbb{T}^d)\,.$$

3.1.3 Smolyak's Algorithm and Convolution Operators

Two other examples of Smolyak algorithms are interesting for us. In the first case we consider

$$L^i_j := S_{2^j}\,, \qquad j \in \mathbb{N}_0\,, \quad i = 1, \ldots d,$$

cf. (6). To indicate this special choice we write $A(m,d,S)$ instead of $A(m,d,\vec{L})$. In the second case we choose ψ as in Subsection 5.4.3 and define for $f \in L_1(\mathbb{T})$

$$L^i_j f(t) := \sum_{k \in \mathbb{Z}} \psi(2^{-j}k)\, c_k(f)\, e^{ikt}, \qquad t \in \mathbb{T}\,, \quad j \in \mathbb{N}_0\,, \quad i = 1, \ldots d.$$

We shall call them smooth de la Vallée-Poussin means. This time we write $A(m,d,\psi)$ instead of $A(m,d,\vec{L})$.

Lemma 9 *For $f \in L_1(\mathbb{T}^d)$ we have*

$$A(m,d,S)f(x) = \sum_{k \in H(m,d,1)} c_k(f)\, e^{ikx} \quad \textit{and} \quad A(m,d,\psi)f(x) = \sum_{|j|_1 \le m} f_j^\psi(x)\,, \tag{14}$$

cf. (68). Furthermore

$$\operatorname{rank} A(m,d,S)\,,\ \operatorname{rank} A(m,d,\psi) \asymp m^{d-1}\, 2^m\,, \qquad m \in \mathbb{N}\,. \tag{15}$$

By investigating basic properties of $A(m,d,\psi)$ we meet an interesting difference between the univariate case and the d-dimensional situation, already observed in [36, Lem. 3.1.2]. Recall, on the torus we have uniform boundedness of the operators S_n and V_{2n-1}, i.e.

$$\sup_{n=1,2,\ldots} \|S_n : L_p(\mathbb{T}) \to L_p(\mathbb{T})\| < \infty\,, \qquad 1 < p < \infty\,,$$
$$\sup_{n=1,2,\ldots} \|V_{2n-1} : L_p(\mathbb{T}) \to L_p(\mathbb{T})\| \le 3\,, \qquad 1 \le p \le \infty\,.$$

The picture is different for the multivariate situation.

Lemma 10 (i) *Let $1 < p < \infty$. Then*

$$\sup_{m=1,2,\ldots} \|A(m,d,S) : L_p(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| < \infty\,,$$

and

$$\sup_{m=1,2,\ldots} \|A(m,d,\psi) : L_p(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| < \infty\,.$$

(ii) *For the limiting situations it holds*

$$\|A(m,d,\psi) : L_1(\mathbb{T}^d) \to L_1(\mathbb{T}^d)\| \asymp \|A(m,d,\psi) : L_\infty(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)\| \asymp m^{d-1}\,.$$

Remark 8 *The first part in (i) may be found in [21, 1.5.2]. The second one follows from a Littlewood-Paley argument, see Lemma 22 below. However, a complete proof has been given in [36, Lem. 3.1.2].*

3.2 Approximation on Sparse Grids and Besov Spaces of Dominating Mixed Smoothness

This section contains the main results of this paper. As mentioned in the Introduction we will study the approximation power of the Smolyak algorithm for functions taken from Besov spaces of dominating mixed smoothness $S^r_{p,q}B(\mathbb{T}^d)$, see Subsection 5.4 for a definition. Our first result is the following general estimate for sampling operators. Here we shall use that functions from $S^r_{p,q}B(\mathbb{T}^d)$ with $r > 1/p$ have a continuous representative, see Lemma 20.

Theorem 1 *Let $1 \le p, q \le \infty$ and $r > 1/p$. Let further $\vec{L}$ satisfy the hypotheses (H1'), (H2), and (H3'). Then there exists a constant $c > 0$ such that*

$$\| I - A(m, d, \vec{L}) \, | S^r_{p,q} B(\mathbb{T}^d) \to L_p(\mathbb{T}^d) \| \le c \, m^{(d-1)(1-1/q)} \, 2^{-mr} \tag{16}$$

holds for all $m \in \mathbb{N}_0$.

However, in the case of convolution operators the additional assumption $r > 1/p$ is not necessary.

Theorem 2 *Let $1 \le p, q \le \infty$ and $r > 0$. Let further $\vec{L}$ satisfy the hypotheses (H1), (H2), and (H3). Then there exists a constant $c > 0$ such that*

$$\| I - A(m, d, \vec{L}) \, | S^r_{p,q} B(\mathbb{T}^d) \to L_p(\mathbb{T}^d) \| \le c \, m^{(d-1)(1-1/q)} \, 2^{-mr}$$

holds for all $m \in \mathbb{N}_0$.

Remark 9 *Theorems 1 and 2 generalise the results obtained in [28] in various directions. In [28] we have investigated only the bivariate case and restricted us to $1 < p < \infty$. In addition the admissible operators $A(m, d, \vec{L})$ are more general now.*

In case of specific sampling operators with respect to uniform grids the obtained estimates are unimprovable.

Theorem 3 *Let $1 \le p, q \le \infty$ and $r > 1/p$. Let $\vec{L}$ satisfy the hypothesis (H1'), (H2), (H3') and (H4). Furthermore we assume that L^i_j has the following structure*

$$L^i_j f(t) = \sum_{\ell \in K_{N_j}} f(t^j_\ell) \, \Lambda_j(t - t^j_\ell), \qquad i = 1, \ldots, d, \quad f \in C(\mathbb{T}) \, ,$$

with some sequence of functions $\Lambda_j \in C(\mathbb{T})$, and $t^j_\ell \in J_{N_j}$, $\ell \in K_{N_j}$, $j \in \mathbb{N}_0$, see (2). Finally, we assume that $1 < \frac{N_{j+1}}{N_j} \in \mathbb{N}$ for all $j \in \mathbb{N}_0$. Then

$$\| I - A(m, d, \vec{L}) \, | S^r_{p,q} B(\mathbb{T}^d) \to L_p(\mathbb{T}^d) \| \asymp m^{(d-1)(1-1/q)} \, 2^{-mr}$$

holds.

Remark 10 *Let L^i_j be given by*

$$L^i_j := \mathcal{R}_{2^j} \, , \qquad j \in \mathbb{N}_0 \, , \quad i = 1, \ldots, d \, ,$$

cf. Remark 3. For the corresponding algorithm $A(m, d, \vec{L})$ and $q = \infty$ Theorem 3 has been proved by Temlyakov in [36, Chapt. 4, Thm. 5.1].

Using a simple bump function argument we can show that starting with sampling on non equidistant points does not help to improve the approximation properties of the related Smolyak algorithm, at least if $q = 1$.

Theorem 4 *Let $1 \le p \le \infty$ and $r > 1/p$. Let $\vec{L}$ satisfy the hypothesis (H1'), (H2), (H3'), (H4) and (H5). Then*

$$\|I - A(m, d, \vec{L})|S^r_{p,1}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \asymp 2^{-rm}\,.$$

However, for the operators of convolution type $A(m, d, S)$ as well as $A(m, d, \psi)$ we have better results.

Theorem 5 *Let $1 < p < \infty$, $1 \le q \le \infty$ and $r > 0$. Then*

$$\begin{aligned} &\|\, I - A(m, d, S)\,|S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \\ &\qquad \asymp \begin{cases} m^{(d-1)(\frac{1}{p}-\frac{1}{q})}\, 2^{-mr} & \text{if } \ 1 < p \le 2 \text{ and } \ p \le q \le \infty\,, \\ m^{(d-1)(\frac{1}{2}-\frac{1}{q})}\, 2^{-mr} & \text{if } \ 2 < p < \infty \text{ and } \ 2 < q \le \infty\,, \\ 2^{-mr} & \text{otherwise}\,, \end{cases} \end{aligned}$$

$m \in \mathbb{N}_0$.

Remark 11 *In the periodic setting Theorem 5 with $q = \infty$ was known for a long time, cf. Bugrov [5] ($p = 2$), Nikol'skaya [20] and Temlyakov [36, Thm. 3.3.3]. The case $q \neq \infty$ has been treated in Dinh Dung [9], Galeev [12] and Romanyuk [24]. For the nonperiodic case we refer to Lizorkin, Nikol'skij [18], Bazarkhanov [2, 3], and [25]. The analogous problem for spaces defined on the unit cube and spline approximation has been treated by Kamont [17]. Many times the problem has been investigated in connection with best approximation from hyperbolic crosses. So this remark applies also with respect to Corollary 3 below.*

Theorem 5 can be extended to the limiting cases $p = 1$ and $p = \infty$ by switching from $A(m, d, S)$ to $A(m, d, \psi)$.

Theorem 6 *Let $1 \le p \le \infty$, $1 \le q \le \infty$ and $r > 0$. Then*

$$\begin{aligned} &\|\, I - A(m, d, \psi)\,|S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \\ &\qquad \asymp \begin{cases} m^{(d-1)(\frac{1}{p}-\frac{1}{q})}\, 2^{-mr} & \text{if } \ 1 \le p \le 2 \text{ and } \ p \le q \le \infty\,, \\ m^{(d-1)(\frac{1}{2}-\frac{1}{q})}\, 2^{-mr} & \text{if } \ 2 < p < \infty \text{ and } \ 2 < q \le \infty\,, \\ m^{(d-1)(1-\frac{1}{q})}\, 2^{-mr} & \text{if } \ p = \infty \text{ and } \ 1 \le q \le \infty\,, \\ 2^{-mr} & \text{otherwise}\,, \end{cases} \end{aligned}$$

$m \in \mathbb{N}_0$.

Remark 12 *We compare the multi-dimensional situation with the univariate one by considering the quantities*

$$\begin{aligned} E_m^S(r,p,q) &:= \| I - A(m,d,S) : S^r_{p,q}B(\mathbb{T}^d) \mapsto L_p(\mathbb{T}^d)\| \\ E_m^\psi(r,p,q) &:= \| I - A(m,d,\psi) : S^r_{p,q}B(\mathbb{T}^d) \mapsto L_p(\mathbb{T}^d)\|. \\ E_m^{\vec{L}}(r,p,q) &:= \| I - A(m,d,\vec{L}) : S^r_{p,q}B(\mathbb{T}^d) \mapsto L_p(\mathbb{T}^d)\| \end{aligned}$$

There are at least three new phenomena in the multi-dimensional case.

- *The sampling operators $A(m,d,\vec{L})$, where $\vec{L}$ is as in Theorem 3, and the Fourier partial sum operator $A(m,d,S)$ do not have the same approximation power. This has to be compared with Remark 2 and Propositions 1, 2, 3.*

- *Observe that the case $p = \infty$ plays a particular role. Obviously, for fixed $p \geq 2$ and m sufficiently large we have*

$$\frac{E_m^\psi(r,\infty,q)}{E_m^\psi(r,p,q)} \asymp \begin{cases} m^{\frac{d-1}{2}} & \text{if} \quad 2 < q \leq \infty\,, \\ m^{(d-1)(1-\frac{1}{q})} & \text{if} \quad 1 \leq q \leq 2\,. \end{cases}$$

  So if p is approaching infinity we have a jump in the order of convergence except $q = 1$.

- *Let $1 < p < \infty$. The microscopic index q of the Besov space enters the order of approximation for all operators we have considered here. Let $\vec{L}$ be as in Theorem 3. Then also the ratio*

$$\frac{E_m^{\vec{L}}(r,p,q)}{E_m^S(r,p,q)} \asymp \begin{cases} m^{(d-1)(1-\frac{1}{p})} & \text{if} \quad 1 < p \leq 2 \text{ and } p \leq q \leq \infty\,, \\ m^{\frac{d-1}{2}} & \text{if} \quad 2 < p < \infty \text{ and } 2 < q \leq \infty\,, \\ m^{(d-1)(1-\frac{1}{q})} & \text{if} \quad 1 < p < \infty \text{ and } 1 \leq q \leq \min(2,p)\,, \end{cases}$$

  depends on q. For the limiting situations we replace $A(m,d,S)$ by $A(m,d,\psi)$ and obtain

$$\frac{E_m^{\vec{L}}(r,1,q)}{E_m^S(r,1,q)} \asymp 1 \qquad \text{as well as} \qquad \frac{E_m^{\vec{L}}(r,\infty,q)}{E_m^S(r,\infty,q)} \asymp 1\,.$$

Best Approximation with Respect to Hyperbolic Crosses

We define for $1 \leq p,q \leq \infty$ and $r > 0$

$$\mathcal{E}_m(S^r_{p,q}B(\mathbb{T}^d))_p := \sup_{\|f|S^r_{p,q}B\|=1} \inf \Big\{ \| f - g\,|L_p(\mathbb{T}^d)\| \;:\; g \quad \text{is a trigonometric polynomial s.t.} \quad c_k(g) = 0 \quad \text{for all} \quad k \notin H(m,d,1) \Big\}$$

As an immediate consequence of Theorem 6 we obtain the following.

Corollary 3 *Let $1 \le p \le \infty$, $1 \le q \le \infty$ and $r > 0$. Then*

$$\mathcal{E}_m(S^r_{p,q}B(\mathbb{T}^d))_p \asymp \begin{cases} m^{(d-1)(\frac{1}{p}-\frac{1}{q})}\, 2^{-mr} & \text{if} \quad 1 \le p \le 2 \text{ and } p \le q \le \infty\,, \\ m^{(d-1)(\frac{1}{2}-\frac{1}{q})}\, 2^{-mr} & \text{if} \quad 2 < p < \infty \text{ and } 2 < q \le \infty\,, \\ m^{(d-1)(1-\frac{1}{q})}\, 2^{-mr} & \text{if} \quad p = \infty \text{ and } 1 \le q \le \infty\,, \\ 2^{-mr} & \text{otherwise}\,, \end{cases} \tag{17}$$

holds for $m \in \mathbb{N}_0$.

Remark 13 *The given estimates for $\mathcal{E}_m(f, L_p(\mathbb{T}^d))$ do not characterise the classes $S^r_{p,q}B(\mathbb{T}^d)$ in $L_p(\mathbb{T}^d)$, see [25] for more details in this direction. Function spaces defined by best approximation from hyperbolic crosses have been investigated in [8], [18] and [25].*

3.3 Examples

First we consider interpolation with de la Vallée-Poussin means, cf. Subsection 2.3. We put

$$A(m,d,\mu) := A(m,d,\vec{L})\,, \qquad L^i_j := I(\Lambda^\pi_{\mu,2^j},\,\cdot\,)\,, i = 1,\dots d\,, j \in \mathbb{N}_0\,. \tag{18}$$

As an immediate consequence of Corollary 1 we obtain that $\vec{L}$ satisfies (H3'). A simple calculation shows that (H2) is satisfied with $\lambda = 1/2-\mu$. Since $I(\Lambda^\pi_{\mu,2^j},\,\cdot\,)$ uses function values from the standard grid J_{2^j} also (H4) and (H5) are fulfilled. Altogether Theorem 3 and Lemma 8 yield the following.

Corollary 4 *Let $1 \le p, q \le \infty$, $r > 1/p$, and $0 < \mu < 1/2$. Then*

$$\|\, I - A(m,d,\mu)\,|S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \asymp m^{(d-1)(1-1/q)}\, 2^{-mr}\,, \qquad m \in \mathbb{N}_0\,.$$

Furthermore, with

$$\mathcal{G}(m,d) := \Big\{ \Big(\frac{2\pi\ell_1}{2^{j_1}}, \dots, \frac{2\pi\ell_d}{2^{j_d}}\Big) : \\ -2^{j_i-1} \le \ell_i < 2^{j_i-1}, i = 1,\dots,d, \quad m-d+1 \le |j|_1 \le m \Big\}$$

the operator $A(m,d,\mu)$ interpolates on $\mathcal{G}(m,d)$, i.e.

$$A(m,d,\mu)f(x) = f(x)\,, \qquad x \in \mathcal{G}(m,d)\,,$$

for all $f \in C(\mathbb{T}^d)$.

As a second example we consider interpolation by means of the Dirichlet kernel, i.e. we put

$$A(m,d,D) := A(m,d,\vec{L})\,, \qquad L_j^i := I_{2^j}\,, \qquad i=1,\dots d\,, \quad j \in \mathbb{N}_0\,,$$

cf. (1). Here Theorem 3 is not applicable in the stated form. With specific modifications of the family of testfunctions used in the proof, see (49), it is also possible to obtain a sharp estimate from below, see [29] for the bivariate case. Nevertheless Theorem 1 gives the following.

Corollary 5 *Let $1 < p < \infty$, $1 \le q \le \infty$, and $r > 1/p$. Then there is a constant $c > 0$ such that*

$$\| \, I - A(m,d,D)\,|S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \le c\, m^{(d-1)(1-1/q)}\, 2^{-mr}\,, \qquad m \in \mathbb{N}_0\,.$$

Furthermore, with

$$\mathcal{G}^*(m,d) := \Big\{ \Big(\frac{2\pi\ell_1}{2^{j_1+1}+1}, \dots, \frac{2\pi\ell_d}{2^{j_d+1}+1} \Big) : \\ 0 \le \ell_i \le 2^{j_i+1}, i=1,\dots,d, \quad m-d+1 \le |j|_1 \le m \Big\}$$

the operator $A(m,d,D)$ uses function values from $\mathcal{G}^*(m,d)$, but in general $A(m,d,D)$ is not interpolating.

Remark 14 (i) *From Lemma 6 and Lemma 9 it follows that $A(m,d,\mu), A(m,d,D)$ as well as $A(m,d,S)$ and $A(m,d,\psi)$ are all of the same complexity. However, from Theorems 3, 5, 6 it follows that they are not of the same efficiency.*
(ii) *To see that $A(m,d,D)$ is not interpolating it is sufficient to consider the operator $A(1,2,D)$ applied to the function $f(x_1,x_2) = e^{i3x_1}$ at the point $\big(2\pi/3, 0\big) \in \mathcal{G}^*(1,2)$. Of course, the reason for this consists in*

$$\Big\{ \frac{2\pi\ell}{2^{j+1}+1} : \quad 0 \le \ell \le 2^{j+1} \Big\} \not\subset \Big\{ \frac{2\pi\ell}{2^j+1} : \quad 0 \le \ell \le 2^j \Big\}\,.$$

3.4 Optimal Recovery of Functions from Besov Spaces of Dominating Mixed Smoothness

Let

$$\Psi_M(f,\xi)(x) := \sum_{j=1}^{M} f(\xi^j)\, \psi_j(x)$$

denote a general sampling operator for a class F of continuous, periodic functions defined on $\mathbb{T}^d$, where

$$\xi := \Big\{ \xi^1, ..., \xi^M \Big\}\,, \qquad \xi^i \in \mathbb{T}^d\,, \quad i=1,2,\dots,M\,,$$

is a fixed set of sampling points and $\psi_j : \mathbb{T}^d \to \mathbb{C}$, $j = 1, \ldots, M$, are fixed, continuous, periodic functions. Then the quantity

$$\rho_M(F, L_p(\mathbb{T}^d)) := \inf_{\xi} \inf_{\psi_1,\ldots,\psi_M} \sup_{\|f|F\|\le 1} \| f - \Psi_M(f,\xi) \,|L_p(\mathbb{T}^d)\|$$

measures the optimal rate of approximate recovery of the functions taken from F. We are interested in the case, when $F = S^r_{p,q}B(\mathbb{T}^d)$, $1 \le p, q \le \infty$, $r > 1/p$. Observe that the operator $A(m, d, \mu)$, see (18), uses $M = M(m, d) \asymp 2^m\, m^{d-1}$ function values from its argument, see Remark 7. Therefore $m \le c \log M$ with some c independent of m and hence

$$2^{-rm} m^{(d-1)(1-1/q)} \le M^{-r} (c \log M)^{(d-1)(r+1-1/q)} .$$

In view of Theorem 3 this implies the upper bound given below.

Corollary 6 *Let $1 < p < \infty$, $1 \le q \le \infty$ and $r > 1/p$. Then there exist positive constants c_1 and c_2 such that for all $M \in \mathbb{N}$*

$$\begin{aligned} c_1\, M^{-r} (\log M)^{(d-1)r} \eta(M, d, p, q) &\le \rho_M(S^r_{p,q}B(\mathbb{T}^d), L_p(\mathbb{T}^d)) \\ &\le c_2\, M^{-r} (\log M)^{(d-1)(r+1-1/q)} , \end{aligned}$$

where

$$\eta(M, d, p, q) := \begin{cases} (\log M)^{(d-1)(\frac{1}{2}-\frac{1}{q})} & \text{if} \quad 2 \le p, q\,, \\ (\log M)^{(d-1)(\frac{1}{p}-\frac{1}{q})} & \text{if} \quad 1 < p < 2 \quad \text{and} \quad p \le q\,, \\ 1 & \text{otherwise}\,. \end{cases} \qquad (19)$$

Remark 15 (i) *The Smolyak algorithm uses samples of a very specific structure. Corollary 6 tells us that allowing arbitrary sets of sampling points of the same cardinality we can not do much better. The difference is at most $(\log M)^{(d-1)/2}$ if $1 < p < \infty$.*
(ii) *In case $q = \infty$ Temlyakov proved the estimate from above in Corollary 6, cf. [34] and [36, 4.5]. The given estimates from below are a consequence of the known behaviour of related Kolmogorov numbers, cf. e.g. Galeev [13].*

3.5 Sobolev Spaces of Dominating Mixed Smoothness and the Problem of Best Recovery

By using (60) one can derive immediately some consequences for the quality of the approximation by $A(m, d, \vec{L})$. The conclusions, obtained in that way, turn out to be unimprovable if $1 < p \le 2$.

Theorem 7 *Let $1 < p \le 2$ and $r > 1/p$. Let further $\vec{L}$ satisfy the hypothesis (H1'), (H2), (H3') and (H4). Furthermore we assume that L_j^i has the following structure*

$$L_j^i f(t) = \sum_{\ell \in K_{N_j}} f(t_\ell^j)\, \Lambda_j(t - t_\ell^j), \qquad i = 1, \ldots, d, \quad f \in C(\mathbb{T}),$$

with some sequence of functions $\Lambda_j \in C(\mathbb{T})$, and $t_\ell^j \in J_{N_j}$, $\ell \in K_{N_j}$, $j \in \mathbb{N}_0$, see (2). Finally, we assume that $1 < \frac{N_{j+1}}{N_j} \in \mathbb{N}$ for all $j \in \mathbb{N}_0$. Then

$$\| I - A(m, d, \vec{L}) \,|S_p^r W(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \asymp m^{(d-1)/2}\, 2^{-mr}, \qquad m \in \mathbb{N},$$

holds.

Theorem 8 *Let $1 < p < \infty$ and $r > 0$. Then*

$$\| I - A(m, d, S) \,|S_p^r W(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \asymp 2^{-mr}, \qquad m \in \mathbb{N}_0,$$

holds.

Remark 16 *In a forthcoming paper one of the authors will prove optimal estimates in case $2 < p < \infty$. This approach will be based on a different method, see [29] for the bivariate situation.*

A consequence of Theorem 1, applied to the algorithm $A(m, d, D)$, is

Corollary 7 *Let $1 < p \le 2$ and $r > 1/p$. Then there exist positive constants c_1, c_2 such that*

$$c_1\, M^{-r} (\log M)^{(d-1)r} \le \rho_M(S_p^r W(\mathbb{T}^d), L_p(\mathbb{T}^d)) \le c_2\, M^{-r} (\log M)^{(d-1)(r+1/2)}, \qquad M \in \mathbb{N}.$$

holds.

4 Proofs

To begin with we discuss several types of test functions which will be used later on. Then we continue with proofs of the statements in Subsections 3.1.2 and 3.1.3. After these preparations we shall proof our main results stated in Subsection 3.2.

4.1 Test Functions

We shall investigate different types of test functions. On the one hand we study Dirichlet kernels with respect to the hyperbolic annuli

$$\mathcal{H}(m) := H(m+1,d,1) \setminus H(m,d,1)$$

and some modifications by using de la Vallée-Poussin kernels instead of the Dirichlet kernels, on the other hand lacunary series and bump functions. In addition to that we introduce special kinds of test functions for treating the limiting cases in Corollary 3.

Lemma 11 *Suppose* $1 < p < \infty$*. Then*

(i)

$$\Big\| \sum_{k \in P_j} e^{ikx} \Big| L_p(\mathbb{T}^d) \Big\| \asymp 2^{|j|_1(1-1/p)}\, , \qquad j \in \mathbb{N}_0^d\, , \tag{20}$$

(ii)

$$\Big\| \sum_{k \in H(m,d,1)} e^{ikx} \Big| L_p(\mathbb{T}^d) \Big\| \asymp m^{(d-1)/p}\, 2^{m(1-1/p)}\, , \qquad m \in \mathbb{N}\, , \tag{21}$$

(iii) *and*

$$\Big\| \sum_{k \in \mathcal{H}(m)} e^{ikx} \Big| L_p(\mathbb{T}^d) \Big\| \asymp m^{(d-1)/p}\, 2^{m(1-1/p)}\, , \qquad m \in \mathbb{N}\, ,$$

hold.

Proof *Step 1.* Proof of (i). The function in part (i) is a tensor product. So the needed estimates can be traced from the well-known behaviour of the Dirichlet kernel in the one-dimensional setting.

Step 2. Proof of (ii) and (iii). In (21) the estimate from above follows from Lemma 1.1 in Chapter 3 in [36]. For the estimate from below we derive with the help of Lemma 7 and Remark 7

$$\Big\| \sum_{k \in H(m,d,1)} e^{ikx} \Big| L_2(\mathbb{T}^d) \Big\|^2 \asymp m^{(d-1)}\, 2^m$$

and use

$$\| f \,| L_2\|^2 \le \|f|L_p\| \cdot \|f|L_{p'}\|\, , \qquad \text{where} \quad \frac{1}{p} + \frac{1}{p'} = 1\, , \tag{22}$$

in connection with the estimates from above. By means of the same arguments one can prove also (iii). ■

Remark 17 *The main ideas of the proof are taken from [36, Lem. 3.1.1].*

Based on this lemma it is relatively easy to obtain some more sophisticated estimates. Recall, the rectangles $\mathcal{P}_j$ have been defined in (62). Then

$$\mathcal{H}(m) = \bigcup_{|j|_1=m+1} \mathcal{P}_j \, .$$

Lemma 12 *Suppose* $1 < p < \infty$ *and* $1 \le q \le \infty$.
(i) *There exist positive constants* c_1 *and* c_2 *such that*

$$\begin{aligned} c_1\, m^{\frac{d-1}{p}}\, 2^{m(1-1/p)} \quad &\le \quad \Big\| \sup_{|j|_1=m} \Big| \sum_{k\in\mathcal{P}_j} e^{ikx} \Big| \, \Big| L_p(\mathbb{T}^d) \Big\| \\ &\le \quad \Big\| \sum_{|j|_1=m} \Big| \sum_{k\in\mathcal{P}_j} e^{ikx} \Big| \, \Big| L_p(\mathbb{T}^d) \Big\| \le c_2\, m^{\frac{d-1}{p}}\, 2^{m(1-1/p)} \end{aligned}$$

holds for all $m \in \mathbb{N}$.
(ii) *Define*

$$g_m(x) := \sum_{|j|_1=m} \sum_{k\in\mathcal{P}_j} e^{ikx} \, , \qquad m \in \mathbb{N} \, . \tag{23}$$

Then, if $r > 0$,

$$\| \, g_m \, | S^r_{p,q} B(\mathbb{T}^d) \| \asymp m^{\frac{d-1}{q}}\, 2^{m(1-1/p)}\, 2^{rm} \, , \qquad m \in \mathbb{N} \, , \tag{24}$$

and for all $r \ge 0$

$$\| \, g_m \, | S^r_p W(\mathbb{T}^d) \| \asymp m^{\frac{d-1}{p}}\, 2^{m(1-1/p)}\, 2^{rm} \, , \qquad m \in \mathbb{N} \, , \tag{25}$$

hold.

Proof *Step 1.* Let us first prove the third inequality in part (i). Let $1 < q < p$. By means of [36, Lem. 2.2.1] and (20) we obtain

$$\begin{aligned} \Big\| \sum_{|j|_1=m} \Big| \sum_{k\in\mathcal{P}_j} e^{ikx} \Big| \, \Big| L_p(\mathbb{T}^d) \Big\| &\le c_1 \Big(\sum_{|j|_1=m} 2^{m(1/q-1/p)p} \Big\| \sum_{k\in\mathcal{P}_j} e^{ikx} \Big| L_q(\mathbb{T}^d) \Big\|^p \Big)^{1/p} \\ &\le c_2\, m^{\frac{d-1}{p}}\, 2^{m(1-1/p)} \, . \end{aligned} \tag{26}$$

Here we have to mention that Temlyakov stated a weaker result than he had proved. Although we start with

$$\sum_{|j|_1=m} \Big| \sum_{k\in\mathcal{P}_j} e^{ikx} \Big| \qquad \text{instead of} \qquad \sum_{|j|_1=m} \sum_{k\in\mathcal{P}_j} e^{ikx}$$

on the left-hand side, the relation remains valid.
The first inequality in (i) becomes a consequence of (26) and

$$
\begin{aligned}
2^m\, m^{d-1} &\asymp \Big\| \sum_{|j|_1=m} \sum_{k\in\mathcal{P}_j} e^{ikx} \Big| L_2(\mathbb{T}^d)\Big\|^2 \\
&= \sum_{|j|_1=m} \Big\| \sum_{k\in\mathcal{P}_j} e^{ikx} | L_2(\mathbb{T}^d)\Big\|^2 \\
&= \int_{\mathbb{T}^d} \sum_{|j|_1=m} \Big| \sum_{k\in\mathcal{P}_j} e^{ikx}\Big| \cdot \Big| \sum_{k\in\mathcal{P}_j} e^{ikx}\Big|\, dx \\
&\le \int_{\mathbb{T}^d} \sup_{|j|_1=m} \Big| \sum_{k\in\mathcal{P}_j} e^{ikx}\Big| \cdot \sum_{|j|_1=m} \Big| \sum_{k\in\mathcal{P}_j} e^{ikx}\Big|\, dx \\
&\le \Big\| \sup_{|j|_1=m} \Big| \sum_{k\in\mathcal{P}_j} e^{ikx}\Big| \Big| L_p(\mathbb{T}^d)\Big\| \cdot \Big\| \sum_{|j|_1=m} \Big| \sum_{k\in\mathcal{P}_j} e^{ikx}\Big| \Big| L_{p'}(\mathbb{T}^d)\Big\| \\
&\le c\, m^{\frac{d-1}{p'}}\, 2^{m(1-1/p')} \Big\| \sup_{|j|_1=m} \Big| \sum_{k\in\mathcal{P}_j} e^{ikx}\Big| \Big| L_p(\mathbb{T}^d)\Big\| .
\end{aligned}
$$

Step 2. Let us turn to part (ii). Together with Lemma 23 and Remark 7 we derive

$$
\begin{aligned}
\| \, g_m \, | S^r_{p,q} B(\mathbb{T}^d)\| &\asymp \Big(\sum_{|j|_1=m} 2^{rmq} \Big\| \sum_{k\in P_j} e^{ikx} \Big| L_p(\mathbb{T}^d)\Big\|^q\Big)^{1/q} \\
&\asymp m^{(d-1)/q}\, 2^{rm}\, 2^{m(1-1/p)}
\end{aligned}
$$

(modification if $q=\infty$) and

$$
\begin{aligned}
\| \, g_m \, | S^r_{p} W(\mathbb{T}^d)\| &\asymp \Big\| \Big(\sum_{|j|_1=m} 2^{rm2} \Big| \sum_{k\in P_j} e^{ikx} \Big|^2\Big)^{1/2} \Big| L_p(\mathbb{T}^d)\Big\| \\
&\asymp m^{(d-1)/p}\, 2^{rm}\, 2^{m(1-1/p)} ,
\end{aligned}
$$

which completes our proof. ■

Remark 18 *In the bivariate nonperiodic situation a proof of Lemma 12(i) has been given in [25]. However, the main ideas are taken from [36, Lem. 3.1.1].*

Recall, Θ_j has been defined in (66).

Lemma 13 *Suppose* $1 \le p \le \infty$.
(i) *We have*

$$
\Big\| \sum_{k\in\mathbb{Z}^d} \Theta_j(k)\, e^{ikx} \Big| L_p(\mathbb{T}^d)\Big\| \asymp 2^{|j|_1(1-1/p)} , \qquad j \in \mathbb{N}_0^d , \tag{27}
$$

(ii) *Suppose* $1 \le p < \infty$. *Then*

$$\Big\| \sum_{|j|_1=m} \sum_{k\in\mathbb{Z}^d} \Theta_j(k)\, e^{ikx} \Big| L_p(\mathbb{T}^d)\Big\| \asymp m^{(d-1)/p}\, 2^{m(1-1/p)} \tag{28}$$

and

$$\Big\| \sum_{|j|_1\le m} \sum_{k\in\mathbb{Z}^d} \Theta_j(k)\, e^{ikx} \Big| L_p(\mathbb{T}^d)\Big\| \asymp m^{(d-1)/p}\, 2^{m(1-1/p)} \tag{29}$$

holds for $m \in \mathbb{N}$.

(iii) *We have for* $m \in \mathbb{N}$

$$\Big\| \sum_{|j|_1=m} \sum_{k\in\mathbb{Z}^d} \Theta_j(k)\, e^{ikx} \Big| L_\infty(\mathbb{T}^d)\Big\| \asymp m^{d-1}\, 2^{m} \tag{30}$$

and

$$\Big\| \sum_{|j|_1\le m} \sum_{k\in\mathbb{Z}^d} \Theta_j(k)\, e^{ikx} \Big| L_\infty(\mathbb{T}^d)\Big\| \asymp m^{d-1}\, 2^{m} \tag{31}$$

Proof *Step 1.* We show (27) and deal with $p = \infty$ and $p = 1$ first. For $p = \infty$ we have

$$\Big\| \sum_{k\in\mathbb{Z}^d} \Theta_j(k)\, e^{ikx} \Big| L_\infty(\mathbb{T}^d)\Big\| = \sum_{k\in\mathbb{Z}^d} \Theta_j(k) \asymp |\mathcal{P}_j| \asymp 2^{|j|_1}\, .$$

Let us turn to $p = 1$. Poisson's summation formula, cf. [33, Cor. 7.2.6], yields

$$\sum_{k\in\mathbb{Z}^d} \Theta_j(k)\, e^{ikx} = (2\pi)^{d/2} \sum_{\ell\in\mathbb{Z}^d} \mathcal{F}^{-1}\Theta_j(x + 2\pi\ell)\, .$$

From this identity we conclude

$$\begin{aligned} \Big\| \sum_{k\in\mathbb{Z}^d} \Theta_j(k)\, e^{ikx} \Big| L_1(\mathbb{T}^d)\Big\| \;&\le\; (2\pi)^{d/2} \| \mathcal{F}^{-1}\Theta_j \,|L_1(\mathbb{R}^d)\| \\ &=\; (2\pi)^{d/2} \| \mathcal{F}^{-1}\Theta_{(1,1,\ldots,1)} \,|L_1(\mathbb{R}^d)\| \end{aligned} \tag{32}$$

with an obvious modification if $\min_i j_i = 0$. In the case $1 < p < \infty$ we shall use the representation (cf. [26, 3.3.4])

$$\sum_{k\in Z} \Theta_j(k) e^{ikx} = (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} \mathcal{F}^{-1}\Theta_j(y) \tilde{\mathcal{D}}_j(x-y)\, dy\, , \qquad \tilde{\mathcal{D}}_j(x) := \sum_{k\in P_j} e^{ikx}\, .$$

Clearly,

$$\Big\| \sum_{k\in\mathbb{Z}^d} \Theta_j(k)\, e^{ikx} \Big| L_p(\mathbb{T}^d)\Big\| \le (2\pi)^{-d/2}\, \| \mathcal{F}^{-1}\Theta_j \,|L_1(\mathbb{R}^d)\|\; \| \tilde{\mathcal{D}}_j \,|L_p(\mathbb{T}^d)\|\, .$$

Applying (20) and a homogeneity argument we obtain the desired estimate from above. The estimates from below follow as in proof of Lemma 12 from the L_2 result, (22), and the estimate from above.

Step 2. We show (28) and (30). Obviously

$$\Big\| \sum_{|j|_1=m} \sum_{k\in\mathbb{Z}^d} \Theta_j(k)\, e^{ikx} \Big| L_\infty(\mathbb{T}^d)\Big\| = \sum_{|j|_1=m} \sum_{k\in\mathbb{Z}^d} \Theta_j(k) \asymp |\mathcal{H}(m)| \asymp m^{d-1}\, 2^m\, .$$

The argument used in case $p=1$ is a little bit different from this one. The estimate from above follows by triangle inequality and (27)

$$\Big\| \sum_{|j|_1=m} \sum_{k\in\mathbb{Z}^d} \Theta_j(k)\, e^{ikx} \Big| L_1(\mathbb{T}^d)\Big\| \le \sum_{|j|_1=m} \Big\| \sum_{k\in\mathbb{Z}^d} \Theta_j(k)\, e^{ikx} \Big| L_1(\mathbb{T}^d)\Big\| \le c\, m^{d-1}\, .$$

The estimate from below uses a Bernstein-Nikol'skij inequality with respect to the hyperbolic cross. We are going to apply Theorem 3.2.2 in [36]. But this needs a further comment, since we are going to use this result with $r=0$ which is excluded in the conditions of this theorem. However the estimate of the L_∞-norm by the L_1-norm requires an application of Lemma 2.4 in [36] which is true for $r=0$. Taking this into account we find

$$2^m\, m^{d-1} \asymp \Big\| \sum_{|j|_1=m} \sum_{k\in\mathbb{Z}^d} \Theta_j(k)\, e^{ikx} \Big| L_\infty(\mathbb{T}^d)\Big\| \le c\, 2^m \Big\| \sum_{|j|_1=m} \sum_{k\in\mathbb{Z}^d} \Theta_j(k)\, e^{ikx} \Big| L_1(\mathbb{T}^d)\Big\|\, .$$

For the case $1<p<\infty$ we can argue as in proof of (26).

Step 3. It remains to verify (29) and (31). The estimates from above (except the case $p=1$) are a simple consequence of the triangle inequality and the results of Step 2. In case $p=1$ we use (14), (66), as well as (7) and obtain

$$\begin{aligned} &\Big\| \sum_{|j|_1\le m} \sum_{k\in\mathbb{Z}^d} \Theta_j(k)\, e^{ikx} \Big| L_1(\mathbb{T}^d)\Big\| \\ &= \Big\| \sum_{m-d+1\le |j|_1\le m} (-1)^{m-|j|_1} \binom{d-1}{m-|j|_1} \sum_{k\in\mathbb{Z}^d} \Theta_{(0,\ldots,0)}(2^{-j_1}k_1, \ldots, 2^{-j_d}k_d) e^{ikx} \Big| L_1(\mathbb{T}^d)\Big\|\, . \end{aligned}$$

This identity, the triangle inequality and a homogeneity argument imply the upper bound in (29). To prove the lower bound we proceed as in Step 2. ■

Remark 19 *Replacing our differences of smooth de la Vallée-Poussin kernels, see (66), by differences of the classical de la Vallée-Poussin kernels, see Remark 3, analogous estimates have been proved in [36, Lem. 3.1.2].*

Lemma 14 *Suppose* $1 \le p,q \le \infty$. *Define*

$$h_m(x) := \sum_{|j|_1=m} \sum_{k\in\mathbb{Z}^d} \Theta_j(k)\, e^{ikx}\, , \qquad m\in\mathbb{N}\, . \tag{33}$$

Then, if $r > 0$,

$$\| \, h_m \, | S^r_{p,q} B(\mathbb{T}^d) \| \asymp m^{(d-1)/q} \, 2^{rm} \, 2^{m(1-1/p)} \, , \qquad m \in \mathbb{N}, \tag{34}$$

and, if $1 < p < \infty$ *and* $r \ge 0$,

$$\| \, h_m \, | S^r_{p} W(\mathbb{T}^d) \| \asymp m^{(d-1)/p} \, 2^{rm} \, 2^{m(1-1/p)} \, , \qquad m \in \mathbb{N}, \tag{35}$$

hold.

Proof To prove (34) we apply the Fourier multiplier assertion

$$\Big\| \sum_{k \in \mathbb{Z}^d} \Theta_\ell(k) \, \Theta_j(k) \, e^{ikx} \, \Big| L_p(\mathbb{T}^d) \Big\| \le c \begin{cases} \Big\| \sum_{k \in \mathbb{Z}^d} \Theta_j(k) \, e^{ikx} \, \Big| L_p(\mathbb{T}^d) \Big\| & \text{if} \quad |\ell - j|_1 \le d \, , \\ & \\ 0 & \text{otherwise} \, . \end{cases}$$

For the proof of (35) we refer to the proof of (28) in case $1 < p < \infty$. ■

Even more simple are the following lacunary series. Let

$$p_j := (\frac{3}{2} \, 2^{j_1 - 1}, \ldots , \frac{3}{2} \, 2^{j_d - 1}) \qquad \text{if} \quad \min_{i=1,\ldots,d} j_i > 0 \, .$$

If $j_i = 0$ then $3 \, 2^{j_i - 1}$ has to be replaced by 0. We define

$$f_\alpha(x) = \sum_{j \in \mathbb{N}_0^d} \alpha_j \, e^{i p_j x} \tag{36}$$

for a given sequence $\alpha = \{\alpha_j\}_j$ of complex numbers.

Lemma 15 (i) *Let* $1 \le p, q \le \infty$ *and* $r > 0$. *We have*

$$\| \, f_\alpha \, | S^r_{p,q} B(\mathbb{T}^d) \| \asymp \Big(\sum_{j \in \mathbb{N}_0^d} 2^{r|j|_1 q} \, |\alpha_j|^q \Big)^{1/q} .$$

(ii) *Let* $1 < p < \infty$ *and* $r \ge 0$. *We have*

$$\| \, f_\alpha \, | S^r_{p} W(\mathbb{T}^d) \| \asymp \Big(\sum_{j \in \mathbb{N}_0^d} 2^{r|j|_1 2} \, |\alpha_j|^2 \Big)^{1/2} .$$

Proof Since p_j is the centre of $\mathcal{P}_j$ we may select a system $\varphi \in \Phi$ (see Subsections 5.2, 5.4 and (66), (67)) such that

$$\varphi_j(p_\ell) = \delta_{j,\ell} \, , \qquad j, \ell \in \mathbb{N}_0^d.$$

From this the claim follows. ■

Two further types of test functions are needed for treating the limiting cases in Theorem 6 and Corollary 3. Again we make use of ideas of Temlyakov, see [36, Sect. 3.3].

The construction of the first family (up to a small modification) can be found in the proof of Theorem 3.3.3 on pages 247/248. The Fejer - kernel of order n is given by

$$\mathcal{F}_n(t) := \frac{1}{n}\sum_{k=0}^{n-1} \mathcal{D}_k(t) = \frac{\sin(nt/2)}{n\sin^2(t/2)}\,, \qquad t \in \mathbb{T}\,.$$

For fixed m we define the set

$$M_m = \Big\{ \ell \in \mathbb{N}_0^d : \quad |\ell|_1 = m\,,\ \ell_i \ge \frac{m}{2d}\,,\ i = 1, \ldots, d \Big\}.$$

Clearly, $|M_m| \asymp m^{d-1}$. Next we define a natural number n by

$$n := \Big[\big|M_{m+1}\big|^{1/d} \Big] \asymp m^{1-1/d} \qquad ([\,\cdot\,] \text{ integer part})\,.$$

Now we divide the cube $[-\pi, \pi]^d$ into n^d subcubes with sidelength $2\pi/n$. Let $N_m \subset M_{m+1}$ be a subset which contains n^d elements. Obviously, there is a bijection between the set N_m and the set of subcubes. Let $\{x^\ell : \ell \in N_m\}$ denote the collection of the centres of them. Next we introduce a particular subset of N_m. Let $\mathcal{N}_m$ be a large subset of N_m such that

$$\ell\,, j \in \mathcal{N}_m \quad \text{implies} \quad \max_{i=1,\ldots,d} |\ell_i - j_i| \ge 3\,.$$

There exist subsets $\mathcal{N}_m$ satisfying $|\mathcal{N}_m| \asymp |N_m|$. Such a sequence of subset will be taken to define our family of test functions:

$$\psi_m(x) := \sum_{\ell \in \mathcal{N}_m} \prod_{j=1}^{d} e^{i2^{\ell_j} x_j}\, \mathcal{F}_n(x_j - x_j^\ell)\,, \qquad x \in \mathbb{T}^d\,. \tag{37}$$

Lemma 16 (i) *There exist positive constants* c_1, c_2 *such that*

$$\|\psi_m|L_\infty(\mathbb{T}^d)\| \le c_1\, m^{d-1} \qquad \text{and} \qquad \|\psi_m|L_2(\mathbb{T}^d)\| \ge c_2\, m^{d-1}$$

holds for all $m \in \mathbb{N}$.

(ii) *For* $r > 0$ *and* $1 \le q \le \infty$ *we have*

$$\|\,\psi_m\,|S^r_{1,q}B(\mathbb{T}^d)\| \asymp 2^{rm}\, m^{\frac{d-1}{q}}\,.$$

(iii) *For sufficiently large* $m \in \mathbb{N}$ *we have*

$$c_k(\psi_m) = 0 \qquad \text{for all} \quad k \in H(m, d, 1)\,.$$

Proof *Step 1.* Parts (i) and (iii) are proved in [36, pp. 247/248] (up to the small modification that Temlyakov works with N_m instead of $\mathcal{N}_m$; but this does not influence the argument).
Step 2. Proof of (ii). For m sufficiently large we have $2^{\ell_j-1} > n$, $j = 1, \ldots, d$. Concerning the spectra of the functions $e^{i2^{\ell_j}x_j}\, \mathcal{F}_n(x_j - x_j^{\ell_j})$ it follows

$$c_k\Big(e^{i2^{\ell_j}x_j}\, \mathcal{F}_n(x_j - x_j^{\ell_j})\Big) = 0 \qquad \text{if} \quad k \not\in \Big[2^{\ell_j-1}, 2^{\ell_j+1}\Big]\, , \qquad j = 1, \ldots, d\, .$$

Let $(\varphi_j)_j$ be a properly chosen smooth dyadic decomposition of unity, see (59). For given $j \in \mathcal{N}_m$ we have then

$$\begin{aligned}
&\Big\| \sum_{k\in\mathbb{Z}^d} \varphi_j(k)\, c_k(\psi_m)\, e^{ikx} \Big| L_1(\mathbb{T}^d)\Big\| \\
&\qquad = \Big\| \sum_{k\in\mathbb{Z}^d} \varphi_j(k)\, c_k\Big(\prod_{u=1}^d e^{i2^{j_u}y_u}\, \mathcal{F}_n(y_u - x_u^j)\Big)\, e^{ikx} \Big| L_1(\mathbb{T}^d)\Big\| \\
&\qquad = \Big\| \prod_{u=1}^d e^{i2^{j_u}y_u}\, \mathcal{F}_n(y_u - x_u^j) \Big| L_1(\mathbb{T}^d)\Big\| \\
&\qquad = \|\, \mathcal{F}_n\, |L_1(\mathbb{T})\|^d\, ,
\end{aligned}$$

whereas in case $j \not\in \mathcal{N}_m$

$$\Big\| \sum_{k\in\mathbb{Z}^d} \varphi_j(k)\, c_k(\psi_m)\, e^{ikx} \Big| L_1(\mathbb{T}^d)\Big\| = 0$$

follows. Hence

$$\begin{aligned}
\|\, \psi_m\, |S^r_{1,q}B(\mathbb{T}^d)\| &= \Big(\sum_{\ell\in\mathcal{N}_m} 2^{r|\ell|_1 q} \Big\| \prod_{u=1}^d e^{i2^{\ell_u}y_u}\, \mathcal{F}_n(y_u - x_u^\ell) \Big| L_1(\mathbb{T}^d)\Big\|^q \Big)^{1/q} \\
&\asymp 2^{rm}\, m^{\frac{d-1}{q}}\, ,
\end{aligned}$$

which completes the proof. ■

As mentioned above also the next example is taken from [36], see formula (3.47) on page 251 for the bivariate case. We consider two families of periodic functions:

$$f_m(x_1, ..., x_d) := \sum_{|\ell|_1=m} \cos(2^{d\ell_1}x_1) \cdot \ \ldots\ \cdot \cos(2^{d\ell_d}x_d)\, , \tag{38}$$

and

$$\Phi_m(x_1, ..., x_d) := -1 + \prod_{|\ell|_1=m} \Big(1 + \cos(2^{d\ell_1}x_1) \cdot \ \ldots\ \cdot \cos(2^{d\ell_d}x_d)\Big)\, . \tag{39}$$

The f_m are lacunary series but it is the combination with the functions Φ_m which makes them interesting.

Lemma 17 (i) *The Fourier coefficients of* f_m *satisfy*

$$c_k(f_m) = 0 \qquad \text{if} \quad k \notin H(dm, d, 1) \setminus H(dm-1, d, 1)\,.$$

(ii) *We have* $\| f_m \,|L_2(\mathbb{T}^d)\|^2 \asymp m^{d-1}$.

(iii) *For* $r > 0$, $1 \le p \le \infty$ *and* $1 \le q \le \infty$ *we have*

$$\| f_m \,|S^r_{p,q}B(\mathbb{T}^d)\| \asymp 2^{rdm}\, m^{(d-1)/q}\,.$$

(iv) *The functions* $t_m := \Phi_m - f_m$ *are trigonometric polynomials such that*

$$c_k(t_m) = 0 \qquad \text{if} \quad k \in H(dm, d, 1)\,.$$

(v) *The* L_1*-norms of the functions* Φ_m *is uniformly bounded. It holds*

$$\| \Phi_m \,|L_1(\mathbb{T}^d\| \le 2(2\pi)^d\,.$$

Proof The parts (i)-(iv) are more or less obvious. We make a short comment to (v). In view of

$$\begin{aligned}\|\Phi_m|L_1(\mathbb{T}^d)\| &\le \|1|L_1\| + \Big\| \prod_{|\ell|_1=m} \big(1+\cos(2^{d\ell_1}x_1)\cdot \ldots \cdot \cos(2^{d\ell_d}x_d)\big)\Big| L_1(\mathbb{T}^d)\Big\| \\ &= (2\pi)^d + \int_{\mathbb{T}^d} \prod_{|\ell|_1=m} \big(1+\cos(2^{d\ell_1}x_1)\cdot \ldots \cdot \cos(2^{d\ell_d}x_d)\big)\, dx_1 \,\cdots\, dx_d \\ &= 2(2\pi)^d\,,\end{aligned}$$

it is enough to notice that the $\int_0^{2\pi} \cos nt\, dt = 0$ for any natural number n. ■

The last concept of test functions we want to introduce, are so called bump functions. Let $\widetilde{B}$ be a $C_0^\infty(\mathbb{R}^d)$ function such that $\operatorname{supp} \widetilde{B} \subset \{x \in \mathbb{R}^d : |x| \le 1\}$. Its 2π-periodic extension is denoted by B. Obviously, $B \in S^r_{p,q}B(\mathbb{T}^d)$ for all $1 \le p \le \infty$, $1 \le q \le \infty$ and $r \ge 0$. Furthermore, if $\lambda = (\lambda_1, \ldots, \lambda_d)$, $\lambda_i > 0$, $i = 1, \ldots, d$, is given then $B(\lambda\,\cdot)$ denotes the 2π-periodic extension of $\widetilde{B}(\lambda\cdot)$.

Lemma 18 *Let* $1 \le p \le \infty$, $1 \le q \le \infty$ *and* $\lambda = (\lambda_1, ..., \lambda_d)$. *Let* $r > 1/p$. *Then there exists a positive constant* c *such that*

$$\| B(\lambda\cdot) \,|S^r_{p,q}B(\mathbb{T}^d)\| \le c\, \lambda_1^{r-1/p} \cdot \ \ldots\ \cdot \lambda_d^{r-1/p}\, \| B \,|S^r_{p,q}B(\mathbb{T}^d)\| \tag{40}$$

and

$$\| B(\lambda\cdot) \,|L_p(\mathbb{T}^d)\| = \lambda_1^{-1/p} \cdot \ldots \cdot \lambda_d^{-1/p}\, \| B(\cdot) \,|L_p(\mathbb{T}^d)\|$$

holds for all λ, $1 \le \lambda_i < \infty$, $i = 1, ..., d$. *If* $1 < p < \infty$, *then also*

$$\| B(\lambda\cdot) \,|S^r_pW(\mathbb{T}^d)\| \le c\, \lambda_1^{r-1/p} \cdot \ldots \cdot \lambda_d^{r-1/p}\, \| B(\cdot) \,|S^r_pW(\mathbb{T}^d)\|$$

holds for all λ, $1 \le \lambda_i < \infty$, $i = 1, ..., d$ *(with* c *independent of* λ*).*

Proof Both parts are simple consequences of characterisations of $S^r_{p,q}B(\mathbb{T}^d)$ and $S^r_pW(\mathbb{T}^d)$ by differences. For this subject we refer to [40, Thm. 4.6.1] as well as [26, Thm. 2.3.4/2] $(d=2)$. By $\Delta^m_{h,\ell}$ we denote the m-th order difference with respect to the ℓ-th variable and translation h, see also Subsection 5.1. For $m \in \mathbb{N}$, $m > r$, the following expression is an equivalent norm in $S^r_{p,q}B(\mathbb{T}^d)$:

$$\|\, f \,|S^r_{p,q}B(\mathbb{T}^d)\|^{\Delta} := \|\, f \,|L_p(\mathbb{T}^d)\| + \sum_{n=1}^{d} \sum_{\beta\in\{0,1\}^d, |\beta|_1=n} S^{\Delta}_{\beta}(f)\,,$$

where for $|\beta|_1 = n \geq 1$

$$S^{\Delta}_{\beta}(f) := \Big[\int_{\mathbb{R}^n} \big(\prod_{i=1}^{n} |h_i|^{-rq}\big) \big\| \big(\triangle^m_{h_1,\delta_1} \circ \cdots \circ \triangle^m_{h_n,\delta_n} f\big)(x)\big| L_p(\mathbb{T}^d)\big\|^q \, \frac{dh_1}{|h_1|} \cdots \frac{dh_n}{|h_n|} \Big]^{1/q} \quad ,$$

and $\delta = (\delta_1, \ldots, \delta_n)$ is defined by means of $\beta_{\delta_i} = 1,\ i = 1, ..., n$.
For abbreviation we put $B_\lambda(x) = B(\lambda x)$. It is not difficult to recognize

$$\big(\triangle^m_{h_1,\delta_1} \circ \cdots \circ \triangle^m_{h_d,\delta_d} B_\lambda\big)(x) = \big(\triangle^m_{\lambda_{\delta_1}\cdot h_1,\delta_1} \circ \cdots \circ \triangle^m_{\lambda_{\delta_d}\cdot h_d,\delta_d} B\big)(\lambda_1 x_1, ..., \lambda_d x_d)\ ,$$

which corresponds to the well-known formula $(\triangle^M_h f(\lambda\cdot)(t) = \triangle^M_{\lambda h} f(\lambda t)$, $t \in \mathbb{R}$. Finally, a change of variable yields

$$S^{\Delta}_{\beta}(B_\lambda) = \lambda_{\delta_1}^{r-1/p} \cdot \ldots \cdot \lambda_{\delta_d}^{r-1/p} S^{\Delta}_{\beta}(B)\ .$$

Now (40) is a consequence of $\lambda_i \geq 1$ and $r > 1/p$. ∎

4.2 Proof of the Lemmata in Subsection 3.1

4.2.1 Proof of Lemma 4

We consider the linear operators

$$T := \sum_{0\le j_1\le m} \cdots \sum_{0\le j_d\le m} \bigotimes_{k=1}^{d} \Delta^k_{j_k} \qquad \text{and} \qquad R := \sum_{|j|_1\ge m+1} \bigotimes_{k=1}^{d} \Delta^k_{j_k}\,,$$

where we put $\Delta^k_{j_k} := \Delta_{j_k}(L^k)$. Then $A(m,d,\vec{L}) = T - R$. Since $\sum_{j=0}^{m} \Delta^k_j = L^k_m$ we obtain

$$T = \bigotimes_{k=1}^{d} L^k_m\,, \qquad m \in \mathbb{N}_0\,.$$

Obviously, if $\ell \in H(m,d,\lambda)$, i.e. $|\ell_u| \leq \lambda\, 2^m$ for all $1 \leq u \leq d$, then

$$\big(Te^{i\ell\cdot}\big)(x) = \prod_{u=1}^{d} (L^u_m e^{i\ell_u\cdot})(x_u) = e^{i\ell x}\,, \qquad x \in \mathbb{T}^d\,,$$

because of (H2). It remains to prove $Re^{i\ell\cdot} \equiv 0$. Let $j = (j_1, ..., j_d)$ be such that $|j|_1 \geq m+1$. Because of $\ell \in H(m,d,\lambda)$ there exist nonnegative integers u_k, $k = 1, ..., d$ satisfying $\sum_{k=1}^d u_k = m$ and $|\ell_k| \leq 2^{u_k}\lambda$. Thanks to $|j|_1 \geq m+1 > m$ there is at least one component j_k of j with $j_k > u_k$. It follows

$$|\ell_k| \leq 2^{u_k}\,\lambda \leq 2^{j_k-1}\,\lambda < 2^{j_k}\,\lambda\,.$$

Hence, using again (H2), we find

$$\Delta_{j_k}^k e^{i\ell_k t} = L_{j_k}^k e^{i\ell_k t} - L_{j_k-1}^k e^{i\ell_k t} = e^{i\ell_k t} - e^{i\ell_k t} = 0\,, \qquad t \in \mathbb{T}\,.$$

By definition of R this proves the claim. ■

4.2.2 Proof of Lemma 5

Step 1. For abbreviation we write $\mathcal{G}(m,d)$ instead of $\mathcal{G}(m,d,\vec{L})$. By using the nestedness of the sequence J_{2^n} we obtain the following recursion formula (see also [19])

$$\begin{aligned}\mathcal{G}(m,d+1) &= \bigcup_{0\leq j_1+...+j_{d+1}\leq m} J_{2^{j_1}} \times ... \times J_{2^{j_{d+1}}} \\ &= \bigcup_{n=0}^{m} \mathcal{G}(m-n,d) \times J_{2^n} \\ &= (\mathcal{G}(m,d) \times J_1) \cup \Big(\bigcup_{n=1}^{m} \mathcal{G}(m-n,d) \times \big(J_{2^n} \setminus J_{2^{n-1}}\big)\Big)\,,\end{aligned}$$

where $\mathcal{G}(m,1) = J_{2^m}$. Therefore $\mathcal{G}(m,d+1)$ is decomposed into a disjoint union of subsets. This yields

$$S(m,d+1) = S(m,d) + \sum_{n=1}^{m} S(m-n,d)\,2^{n-1}\,,$$

with $S(m,1) = 2^m$.

Step 2. We proceed by induction with respect to d. From the induction hypothesis (12) and our recursion formula we derive

$$\begin{aligned}S(m,d+1) &= \sum_{j=0}^{d-1}\binom{d-1}{j}2^{m-j}\binom{m}{j} + \sum_{j=0}^{d-1}\binom{d-1}{j}\sum_{n=1}^{m}2^{m-n-j}\binom{m-n}{j}2^{n-1} \\ &= \sum_{j=0}^{d-1}\binom{d-1}{j}2^{m-j}\binom{m}{j} + \sum_{j=0}^{d-1}\binom{d-1}{j}2^{m-j-1}\sum_{n=1}^{m}\binom{m-n}{j}\,.\end{aligned}$$

Using the identity

$$\sum_{n=j}^{m-1}\binom{n}{j} = \binom{m}{j+1}\,, \qquad j \in \mathbb{N}_0\,, \tag{41}$$

we obtain

$$S(m,d+1) = \sum_{j=0}^{d-1} \binom{d-1}{j} 2^{m-j} \binom{m}{j} + \sum_{j=1}^{d} \binom{d-1}{j-1} 2^{m-j} \binom{m}{j}$$
$$= \sum_{j=0}^{d} \binom{d}{j} 2^{m-j} \binom{m}{j},$$

which proves our claim. ■

4.2.3 Proof of Lemma 6

The same arguments as used in Step 1 of the proof of Lemma 5 lead to a decomposition of $\mathcal{G}(m, d+1)$ into disjoint subsets

$$\mathcal{G}(m,d+1) = (\mathcal{G}(m,d) \times \mathcal{T}_0^{d+1}) \cup \left(\bigcup_{n=1}^{m} \mathcal{G}(m-n,d) \times \left(\mathcal{T}_n^{d+1} \setminus \bigcup_{\ell=0}^{n-1} \mathcal{T}_\ell^{d+1} \right) \right).$$

In view of (H4) and (H5) this yields

$$\min(C_1, C_3)\Big(|\mathcal{G}(m,d)| + \sum_{n=1}^{m} |\mathcal{G}(m-n,d)|\, 2^{n-1} \Big) \le |\mathcal{G}(m,d+1)|$$
$$\le 2C_2 \left(|\mathcal{G}(m,d)| + \sum_{n=1}^{m} |\mathcal{G}(m-n,d)|\, 2^{n-1} \right).$$

Induction with respect to d by taking (13) as induction hypothesis yields the desired result. ■

4.2.4 Proof of Lemma 7

Part (i) is an easy consequence of the recursion formulas

$$|P_0(m,d+1)| = \sum_{n=0}^{m} |P_0(m-n,d)| \quad \text{and} \quad |P_1(m,d+1)| = \sum_{n=1}^{m-d} |P_0(m-n,d)|$$

with $P_0(m,1) = P_1(m,1) = 1$, $m \in \mathbb{N}$ and induction with respect to d using (41). The same arguments as used in the proof of the Lemmata 5, 6 imply

$$2\Big(|H(m,d)| + \sum_{n=1}^{m} |H(m-n,d)|\, 2^{n-1} \Big) \le |H(m,d+1)|$$
$$\le 3 \left(|H(m,d)| + \sum_{n=1}^{m} |H(m-n,d)|\, 2^{n-1} \right)$$

with $2 \cdot 2^m \le H(m,1) \le 3 \cdot 2^m$. Induction with respect to d yields the result.

4.2.5 Proof of Lemma 8

The proof is similar to the proof of Lemma 4. Observe that the nestedness of the grids $\mathcal{T}_j^i$ implies

$$\mathcal{G}(m,d,\vec{L}) = \bigcup_{|j|_1 \le m} \mathcal{T}_{j_1}^1 \times \ldots \times \mathcal{T}_{j_d}^d = \bigcup_{|j|_1 = m} \mathcal{T}_{j_1}^1 \times \ldots \times \mathcal{T}_{j_d}^d .$$

We employ the same notation and decomposition of $A(m,d,\vec{L}) = T - R$ as in proof of Lemma 4. Since L_m^i interpolates on $\mathcal{T}_m^i$ the operator T interpolates on $\mathcal{T}_m^1 \times \ldots \times \mathcal{T}_m^d$. Hence, it is enough to prove

$$Rf(x) = 0 \qquad \text{for all} \quad x \in \mathcal{T}_{k_1}^1 \times \ldots \times \mathcal{T}_{k_d}^d , \qquad |k|_1 = m ,$$

and all $f \in C(\mathbb{T}^d)$. We shall prove even more, namely

$$\Big(\Delta_{j_1}^1 \otimes \ldots \otimes \Delta_{j_d}^d\Big) f(x) = 0 , \qquad x \in \mathcal{T}_{k_1}^1 \times \ldots \times \mathcal{T}_{k_d}^d , \quad |k|_1 = m ,$$

$f \in C(\mathbb{T}^d)$ and $|j|_1 > m$.
Let j, $|j|_1 > m$, k, $|k|_1 = m$ and $x \in \mathcal{G}(m,d,\vec{L})$ be given. For $f \in C(\mathbb{T}^d)$ and $1 \le u \le d$ we put $g_u(t) := f(x_1, \ldots, x_{u-1}, t, x_{u+1}, \ldots, x_d)$, $t \in \mathbb{T}$. Furthermore, there exists at least one component u such that $k_u < j_u$. But this implies $L_{j_u}^u g_u(x_u) = L_{j_u-1}^u g_u(x_u)$ which proves the claim. ■

4.2.6 Proof of Lemma 9

Step 1. We prove (14) for $A(m,d,S)$. Since for $j \in \mathbb{N}_0^d$

$$\big(\Delta_{j_1} \otimes ... \otimes \Delta_{j_d}\big) f(x) = \sum_{k \in \mathcal{P}_j} c_k(f) e^{ikx} \qquad \text{and} \qquad H(m,d,1) = \bigcup_{|j|_1 \le m} \mathcal{P}_j ,$$

Definition 1 leads to the desired representation. For the second identity observe

$$(\Delta_\ell) f(t) = \sum_{k \in \mathbb{Z}} \Big(\psi(2^{-\ell}k) - \psi(2^{-\ell+1}k)\Big)\, c_k(f)\, e^{ikt} , \qquad t \in \mathbb{T} , \quad \ell \in \mathbb{N} .$$

Step 2. Proof of (15). Step 1 yields that both operators map their arguments to trigonometric polynomials with respect to the hyperbolic cross $H(m,d,1)$ and $H(m+d,d,1)$, respectively. Hence (15) is a consequence of Lemma 7. ■

4.2.7 Proof of Lemma 10

Let

$$g_m(x) := \sum_{|j|_1 \le m} \sum_{k \in \mathbb{Z}^d} \Theta_j(k)\, e^{ikx}\,, \qquad m \in \mathbb{N}_0\,,$$

then $A(m,d,\psi)f = 1/(2\pi)^d\, g_m * f$. Consequently

$$\begin{aligned} \| \, A(m,d,\psi) \,|\, L_1(\mathbb{T}^d) \to L_1(\mathbb{T}^d)\| &= \| \, A(m,d,\psi) \,|\, L_\infty(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)\| \\ &= 1/(2\pi)^d \, \| \, g_m \,|L_1(\mathbb{T}^d)\| \,. \end{aligned}$$

Now Lemma 10(ii) follows from Lemma 13(ii). ■

4.3 Proof of Theorems 1 and 2

4.3.1 Tensor Products of Operators

One of our tools consists in the following estimate for tensor product operators.

Lemma 19 *Let* $1 \le p \le \infty$ *and* $r > 0$. *Suppose* $P_j \in \mathcal{L}(B^r_{p,p}(\mathbb{T}), L_p(\mathbb{T}))$, $j = 1, \ldots, d$. *Then*

$$\| \, P_1 \otimes \, \ldots \, \otimes P_d \, f \,|L_p(\mathbb{T}^d)\| \le \Big(\prod_{j=1}^d \| \, P_j \,|B^r_{p,p}(\mathbb{T}) \to L_p(\mathbb{T})\| \Big) \, \| \, f \,|S^r_{p,p}B(\mathbb{T}^d)\|$$

holds for all trigonometric polynomials f.

Proof Let $f = \sum_{k \in \mathbb{Z}^d} c_k(f)\, e^{ikx}$ be a trigonometric polynomial. We define $k = (k_1, k')$, $k_1 \in \mathbb{Z}$, $k' \in \mathbb{Z}^{d-1}$, $x = (x_1, x')$, $x_1 \in \mathbb{T}$, $x' \in \mathbb{T}^{d-1}$, and

$$g_{k_1}(x') := \sum_{k' \in \mathbb{Z}^{d-1}} c_k(f) \Big(\prod_{n=2}^d P_n(e^{ik_n \cdot})(x_n) \Big)\,, \qquad x' \in \mathbb{R}^{d-1}\,, \quad k_1 \in \mathbb{Z}\,.$$

Then

$$\begin{aligned} \| \, (P_1 \otimes \, \ldots \, \otimes P_d) f \,|L_p(\mathbb{T}^d)\|^p &= \int_{T^{d-1}} \Big\| \, P_1 \Big(\sum_{k_1 \in \mathbb{Z}} g_{k_1}(x')\, e^{ik_1 \cdot} \Big)(x_1) \, \Big| L_p(\mathbb{T}, x_1) \Big\|^p \, dx' \\ &\le \| \, P_1 \, \|^p \int_{T^{d-1}} \Big\| \Big(\sum_{k_1 \in \mathbb{Z}} g_{k_1}(x')\, e^{ik_1 \cdot} \Big)(x_1) \, \Big| B^r_{p,p}(\mathbb{T}, x_1) \Big\|^p \, dx'\,. \end{aligned} \tag{42}$$

Now, let $(\varphi_j)_j \in \Phi$ be an appropriate decomposition of unity, see Subsection 5.2. Then

$$\begin{aligned} \Big\| \Big(\sum_{k_1 \in \mathbb{Z}} g_{k_1}(x')\, e^{ik_1 x_1} \Big) \, \Big| B^r_{p,p}(\mathbb{T}) \Big\|^p &= \sum_{j_1=0}^\infty 2^{j_1 r p} \Big\| \sum_{k_1 \in \mathbb{Z}} \varphi_{j_1}(k_1)\, g_{k_1}(x')\, e^{ik_1 x_1} \, \Big| L_p(\mathbb{T}, x_1) \Big\|^p \\ = \sum_{j_1=0}^\infty 2^{j_1 r p} \Big\| \, P_2 \Big(\sum_{k \in \mathbb{Z}^d} c_k(f)\, \varphi_{j_1}(k_1)\, e^{ik_1 x_1} \Big(\prod_{n=3}^d & P_n(e^{ik_n \cdot})(x_n) \Big) \, e^{ik_2 \cdot} \Big)(x_2) \, \Big| L_p(\mathbb{T}, x_1) \Big\|^p \end{aligned}$$

This identity will be inserted into (42). Then we interchange the order of integration and proceed as above:

$$
\begin{aligned}
\| (P_1 \otimes \ldots \otimes P_d) f \,|L_p(\mathbb{T}^d)\|^p \le \| P_1 \|^p \sum_{j_1=0}^{\infty} 2^{j_1 r p} \int_{\mathbb{T}^{d-1}} \int_0^{2\pi} \\
\Big| P_2 \Big(\sum_{k \in \mathbb{Z}^d} c_k(f)\, \varphi_{j_1}(k_1)\, e^{ik_1 x_1} \Big(\prod_{n=3}^{d} P_n(e^{ik_n \cdot})(x_n) \Big)\, e^{ik_2 \cdot} \Big)(x_2) \Big|^p dx_2\, dx_1\, dx_3 \ldots dx_d \\
\le \;\; \| P_1 \|^p \ldots \| P_d \|^p \\
\times \sum_{j \in \mathbb{N}_0^d} 2^{r|j|p} \int_{\mathbb{T}^d} \Big| \sum_{k \in \mathbb{Z}^d} c_k(f)\, \varphi_{j_1}(k_1)\, e^{ik_1 x_1} \ldots \varphi_{j_d}(k_d)\, e^{ik_d x_d} \Big|^p dx_1 \ldots dx_d \\
= \;\; \| P_1 \|^p \ldots \| P_d \|^p \, \| f \,|S^r_{p,p}B(\mathbb{T}^d)\|^p \,.
\end{aligned}
$$

This proves the claim. ■

4.3.2 Proof of Theorems 1 and 2

The proofs of Theorem 1 and Theorem 2 do not differ very much. So we shall prove both theorems simultaneously. For later use we will take care of the constants and their dependence on the dimension d in all inequalities.

Step 1. The aim of this first step consists in a description of the decomposition we are going to use. It will a be bit more sophisticated than the one used in the proof of Theorem 5, cf. (53) and (54). The main difference will be the necessity of a further decomposition of the set Q_1^m. In addition one has to modify in order of taking λ into account, see (H2).

First of all recall the decomposition (58) of $f \in S^r_{p,q}B(\mathbb{T}^d)$ into the pieces f_ℓ. Because of $r > 0$ ($r > 1/p$) we have convergence in $L_p(\mathbb{T}^d)$ ($C(\mathbb{T}^d)$), see Lemma 20. Next we need to fix a natural number n_λ such that $2^{-n_\lambda} \le \lambda$. Now we suppose that m is larger than $d(n_\lambda + 1)$. Further we put $s_m := m - d(n_\lambda + 1) \ge 0$ (we drop the parameter λ in all other notations). Let $I_0^m := [0, s_m]$ and $I_1^m := (s_m, \infty)$, respectively. For $b = (b_1, \ldots b_d)$, $b_i \in \{0, 1\}$, $i = 1, \ldots, d$, we define

$$
Q_b^m := \{\ell \in \mathbb{N}_0^d :\, \ell_n \in I^m_{b_n},\, n = 1, \ldots, d,\, |\ell|_1 > s_m\} \,,
$$

This leads to the decomposition

$$
f(x) = h(x) + \sum_{b \in \{0,1\}^d} f^b(x) \;,
$$

where

$$f^b(x) := \sum_{\ell \in Q_b^m} f_\ell(x)\,.$$

The function $h(x)$ is a trigonometric polynomial given by

$$h(x) := \sum_{|\ell|_1 \le s_m} f_\ell(x)\,.$$

Observe $c_k(h) \neq 0$ implies $c_k(f_\ell) \neq 0$ for some $\ell \in \mathbb{Z}^d$ satisfying $|\ell|_1 \le s_m$. Hence $|k_n| \le 2^{\ell_n+1} \le 2^{\ell_n+1+n_\lambda}\lambda$ for $n = 1, ..., d$. Therefore $k \in H(m, d, \lambda)$ and consequently $A(m, d, \vec{L})\, h = h$ follows, see Lemma 4.

Step 2. Estimation (first part). By means of the invariance of h under the application of $A(m, d, \vec{L})$ we find

$$\| f - A(m,d,\vec{L}) f \,|L_p(\mathbb{T}^d)\| \le \sum_{b \in \{0,1\}^d} \| f^b - A(m,d,\vec{L}) f^b \,|L_p(\mathbb{T}^d)\|\,.$$

Obviously, there exists a number $M_\ell \in \mathbb{N}$, such that the trigonometric polynomial f_ℓ has all its harmonics in the hyperbolic cross $H(M_\ell, d, \lambda)$. For $M_\ell > |\ell|_1 + d(1 + n_\lambda)$ Lemma 4 implies

$$\begin{aligned}
\| f_\ell - A(m,d,\vec{L}) f_\ell \,|L_p(\mathbb{T}^d)\| &= \| A(M_\ell,d,\vec{L}) f_\ell - A(m,d,\vec{L}) f_\ell \,|L_p(\mathbb{T}^d)\| \\
&= \Big\| \sum_{m < |j|_1 \le M_\ell} \Big(\bigotimes_{n=1}^{d} \Delta^n_{j_n} \Big) f_\ell \,\Big|\, L_p(\mathbb{T}^d) \Big\| \\
&= \Big\| \sum_{j \in \Lambda_\ell^m} \Big(\bigotimes_{n=1}^{d} \Delta^n_{j_n} \Big) f_\ell \,\Big|\, L_p(\mathbb{T}^d) \Big\|\,,
\end{aligned}$$

where

$$\Lambda_\ell^m := \Big\{ j = (j_1, ..., j_d): \quad |j|_1 > m, \quad j_n \le \ell_n + 1 + n_\lambda,\ n = 1, ..., d \Big\}\,.$$

In order to keep the notation simple we used again $\Delta^n_{j_n}$ instead of $\Delta_{j_n}(L^n)$, $n = 1, ..., d$, $j_n = 0, 1, 2, ...$ The last step here is a consequence of (H2), the definition of the tensor product and the choice of M_ℓ. We continue by using Lemma 19. Let us choose r_0 such that $0 < r_0 < r$ (this condition has to be replaced by $1/p < r_0 < r$ in the case of Theorem 1). Then

$$\Big\| \Big(\bigotimes_{n=1}^{d} \Delta^n_{j_n} \Big) f_\ell \,\Big|\, L_p(\mathbb{T}^d) \Big\| \le \| f_\ell \,|S^{r_0}_{p,p}B(\mathbb{T}^d)\| \prod_{n=1}^{d} \| \Delta^n_{j_n} \,|B^{r_0}_{p,p}(\mathbb{T}) \to L_p(\mathbb{T})\|\,.$$

Using hypothesis (H3), the triangle inequality and (57) this gives

$$\Big\| \Big(\bigotimes_{n=1}^{d} \Delta^n_{j_n} \Big) f_\ell \,\Big|\, L_p(\mathbb{T}^d) \Big\| \le C_4^d\, 2^{-m r_0}\, \| f_\ell \,|S^{r_0}_{p,p}B(\mathbb{T}^d)\|\,,$$

where $C_4 := C_0(r_0)\cdot(1+2^{r_0})$, see (8) and (9). Furthermore

$$\begin{aligned}\| f_\ell \,|S^{r_0}_{p,p}B(\mathbb{T}^d)\| &\le \Big(\sum_{\substack{|j_k-\ell_k|\le 1\\ k=1,\ldots,d}} 2^{r_0|j|_1 p}\sup_{j\in\mathbb{N}_0^d}\|(2\pi)^{-d/2}\mathcal{F}^{-1}\varphi_j|L_1(\mathbb{R}^d)\|^p\cdot\|f_\ell|L_p(\mathbb{T}^d)\|^p\Big)^{1/p}\\ &\le C_5\, 2^{r_0|\ell|_1}\,\| f_\ell\,|L_p(\mathbb{T}^d)\|\,,\end{aligned}$$

where

$$C_5 := 2^{r_0 d}3^{d/p}(2\pi)^{-d/2}\max_{n=0,1,\ldots,d}\|\mathcal{F}^{-1}\varphi_0\,|L_1(\mathbb{R})\|^n\|\mathcal{F}^{-1}\varphi_1\,|L_1(\mathbb{R})\|^{d-n}\,.$$

Here we used the standard convolution inequality and an argument like in (32). Altogether we obtain

$$\begin{aligned}&\| f^b - A(m,d,\vec{L})f^b\,|L_p(\mathbb{T}^d)\| \le C_4^d\, C_5\sum_{\ell\in Q_b^m} 2^{r_0(|\ell|_1-m)}\,|\Lambda_\ell^m|\;\| f_\ell\,|L_p(\mathbb{T}^d)\|\\ &= C_4^d\,C_5\,2^{-r_0 m}\sum_{\ell\in Q_b^m}2^{(r_0-r)|\ell|_1}\,|\Lambda_\ell^m|\;2^{r|\ell|_1}\,\| f_\ell\,|L_p(\mathbb{T}^d)\|\,. \qquad (43)\end{aligned}$$

Of course, $|\Lambda_\ell^m|$ denotes the cardinality of the set Λ_ℓ^m. We need to estimate this quantity. Obviously

$$\Lambda_\ell^m\subset\Big[m-\sum_{\substack{n=1\\ n\neq 1}}^d(\ell_n+1+n_\lambda),\ell_1+1+n_\lambda\Big]\times\cdots\times\Big[m-\sum_{\substack{n=1\\ n\neq d}}^d(\ell_n+1+n_\lambda),\ell_d+1+n_\lambda\Big]\,.$$

This implies

$$|\Lambda_\ell^m|\le\min\Big((|\ell|_1+d(1+n_\lambda)+1-m)^d\;,\;\prod_{n=1}^d(\ell_n+2+n_\lambda)\Big)\,. \qquad (44)$$

Step 3. Estimation (second part). Depending on the size of $|b|_1$ we continue.
Step 3.1. Let $|b|_1\le 1$. Without loss of generality we may assume that $b_1=|b|_1$. For given q let q' be such that $(1/q)+(1/q')=1$. Then we find

$$\begin{aligned}&2^{-mr_0}\Big(\sum_{\ell\in Q_b^m}2^{q'(r_0-r)|\ell|_1}\,|\Lambda_\ell^m|^{q'}\Big)^{1/q'}\\ &\le 2^{-mr_0}\Big(\sum_{\ell\in Q_b^m}2^{q'(r_0-r)|\bar{\ell}|_1}\,(|\ell|_1+d(n_\lambda+1)+1-m)^{dq'}\Big)^{1/q'}\\ &\le 2^{-mr_0}\Big(\sum_{\ell_2,\ldots,\ell_d=0}^{s_m}\sum_{u=0}^\infty 2^{q'(r_0-r)(u+m-d(n_\lambda+1)-1)}u^{dq'}\Big)^{1/q'}\\ &\le 2^{-mr}\Big(m^{d-1}\sum_{u=0}^\infty 2^{q'(r_0-r)(u-d(n_\lambda+1)-1)}u^{dq'}\Big)^{1/q'}\\ &\le C_6\,2^{-mr}\,m^{(d-1)(1-1/q)}\,, \qquad (45)\end{aligned}$$

where

$$C_6 := 2^{(r-r_0)(d(n_\lambda+1)+1)} \Big(\sum_{u=0}^{\infty} 2^{q'(r_0-r)u}\, u^{dq'}\Big)^{1/q'} .$$

In this case Hölder's inequality and (45) lead to

$$\begin{aligned} &\| f^b - A(m,d,\vec{L}) f^b) \,|L_p(\mathbb{T}^d)\| \\ \le\ & C_4^d\, C_5\, 2^{-mr_0} \Big(\sum_{\ell \in Q_b^m} 2^{q'(r_0-r)|\ell|_1}\, |\Lambda_\ell^m|^{q'}\Big)^{1/q'} \Big(\sum_{\ell\in Q_b^m} 2^{r|\ell|_1 q}\, \| f_\ell \,|L_p(\mathbb{T}^d)\|^q\Big)^{1/q} \\ \le\ & C_4^d\, C_5\, C_6\, 2^{-mr}\, m^{(d-1)(1-1/q)}\, \| f \,|S^r_{p,q}B(\mathbb{T}^d)\| . \end{aligned} \tag{46}$$

Step 3.2. Let $|b|_1 \ge 2$. In this case the estimate becomes easier. We use

$$2^{-mr_0} \sum_{\ell\in Q_b^m} 2^{(r_0-r)|\ell|_1}\, |\Lambda_\ell^m| \le 2^{-mr_0} \prod_{n=1}^{d} \Big(\sum_{\ell_n \in I_{b_n}^m} 2^{(r_0-r)\ell_n} (\ell_n + 2 + n_\lambda)\Big) ,$$

see (44), as well as

$$\begin{aligned} &\sum_{\ell_n = s_m+1}^{\infty} 2^{(r_0-r)\ell_n} (\ell_n + 2 + n_\lambda) \\ &\quad = 2^{m(r_0-r)} 2^{(r-r_0)(d(n_\lambda+1)-1)} \sum_{u=0}^{\infty} 2^{(r_0-r)u}\, (u + m - d(n_\lambda+1) + 3) \\ &\quad \le 2^{m(r_0-r)}\, 2^{(r-r_0)(d(n_\lambda+1)-1)} \sum_{u=0}^{\infty} 2^{(r_0-r)u}\, (u+m+3) \\ &\quad \le C_7\, 2^{m(r_0-r)}\, m \end{aligned}$$

with

$$C_7 := 2\, 2^{(r-r_0)(d(n_\lambda+1)-1)} \Big(\sum_{u=0}^{\infty} 2^{(r_0-r)u}\, (u+3)\Big) ,$$

and

$$\sum_{\ell_n=0}^{s_m} 2^{(r_0-r)\ell_n} (\ell_n + 2 + n_\lambda) \le m \sum_{u=0}^{\infty} 2^{(r_0-r)u} \le C_7\, m .$$

Altogether this leads to

$$\begin{aligned} &\| f_b - A(m,d,\vec{L}) f_b \,|L_p(\mathbb{T}^d)\| \le C_4^d\, C_5\, \| f \,|S^r_{p,\infty}B(\mathbb{T}^d)\|\, 2^{-mr_0} \sum_{\ell\in Q_b^m} 2^{(r_0-r)|\ell|_1} |\Lambda_\ell^m| \\ \le\ & C_4^d\, C_5\, C_7^d\, \| f \,|S^r_{p,\infty}B(\mathbb{T}^d)\|\, 2^{-mr} m^d 2^{m(r_0-r)(|b|_1-1)} \\ \le\ & C_4^d\, C_5\, C_7^d\, C_8\, 2^{-mr}\, \| f \,|S^r_{p,\infty}B(\mathbb{T}^d)\| , \end{aligned} \tag{47}$$

where

$$C_8 := \max_{n=1,\ldots,d-1}\ \sup_{m\in\mathbb{N}}\ m^d\, 2^{m(r_0-r)n} ,$$

see (43). It remains to sum up over $|b|_1 \le 1$ in (46) and over $2 \le |b|_1 \le d$ in (47), respectively. This completes the proof of Theorem 1 and Theorem 2. ■

4.4 Proof of Theorem 3

Step 1. Preliminaries. Let

$$Lf(t) := \sum_{\ell \in K_N} f(t_\ell)\, \Lambda(t - t_\ell)\,, \qquad t_\ell \in J_N\,.$$

If f belongs to the Wiener algebra $\mathcal{A}(\mathbb{T})$, then the Fourier coefficients of Lf are given by the formula

$$c_k(Lf) = N\, c_k(\Lambda) \sum_{\ell \in \mathbb{Z}} c_{k+\ell N}(f)\,, \qquad k \in \mathbb{Z}\,.$$

In particular, if $f(t) = e^{imt}$ one obtains

$$c_0(L(e^{im\cdot})) = N\, c_0(\Lambda) \cdot \begin{cases} 1 & \text{if } \frac{m}{N} \in \mathbb{Z}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{48}$$

The admissible operators L reproduce the function $f(t) = 1$. Hence

$$1 = c_0(Lf) = N\, c_0(\Lambda)\,.$$

Step 2. Test functions. Only the estimate from below is of interest. For this reason we construct a sequence of test function which are similar to the lacunary series studied in (36). For $m \geq d^2$ we put

$$f_m(x_1, ..., x_d) := \sum_{\substack{u_k \geq d \\ |u|_1 = m}} e^{iN_{u_1}x_1 + \ldots + iN_{u_d}x_d}\,, \tag{49}$$

where N_j, $j \in \mathbb{N}_0$, is the given sequence of natural numbers appearing in hypothesis (H4). Minor modifications of the arguments used in Lemma 15 yield

$$\|\, f_m\, |S^r_{p,q}B(\mathbb{T}^d)\| \asymp 2^{rm}\, m^{(d-1)/q}\,. \tag{50}$$

Again we refer to Remark 7.

Step 3. Calculation of $c_{(0,\ldots,0)}(A(m,d,\vec{L})f_m)$. Let us first study the number $c_0(\Delta^k_{j_k}(e^{iN_{u_k}\cdot}))$. Putting

$$d_M(N) = \begin{cases} 1 & \text{if } \frac{N}{M} \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}\,,$$

we derive from (48)

$$c_0(\Delta^k_{j_k}(e^{iN_{u_k}\cdot})) = \begin{cases} d_{N_{j_k}}(N_{u_k}) - d_{N_{j_k-1}}(N_{u_k}) & \text{if } \; j_k \geq 1 \\ d_{N_0}(N_{u_k}) & \text{if } \; j_k = 0 \end{cases}\,.$$

Now we employ our assumption $N_{j+1}/N_j \in \mathbb{N}$, $j = 0, 1, \ldots$, and obtain

$$c_0(\Delta^k_{j_k}(e^{iN_{u_k}\cdot})) = \begin{cases} -1 & \text{if} \quad j_k = u_k + 1 \\ \;\;1 & \text{if} \quad j_k = 0 \\ \;\;0 & \text{otherwise} \end{cases} . \tag{51}$$

This yields

$$\begin{aligned} c_0\big(A(m,d,\vec{L})\, f_m\big) &= \sum_{\substack{u_k \geq d \\ |u|_1 = m}} c_0\big[A(m,d,\vec{L})\big(e^{iN_{u_1}\cdot + \ldots + iN_{u_d}\cdot}\big)\big] \\ &= \sum_{\substack{u_k \geq d \\ |u|_1 = m}} \sum_{j \in T_u} c_0\Big[\Big(\bigotimes_{k=1}^{d} \Delta^k_{j_k}\Big)\big(e^{iN_{u_k}\cdot + \ldots + iN_{u_d}\cdot}\big)\Big] \\ &= \sum_{\substack{u_k \geq d \\ |u|_1 = m}} \sum_{j \in T_u} c_0[\Delta^1_{j_1}(e^{iN_{u_1}\cdot})] \cdot \ldots \cdot c_0[\Delta^d_{j_d}(e^{iN_{u_d}\cdot})] \ , \end{aligned} \tag{52}$$

where

$$T_u \;=\; \Big\{(j_1, ..., j_d) \in \mathbb{N}_0^d : \quad |j|_1 \leq m \quad \text{and either} \quad j_k = u_k + 1 \quad \text{or} \quad j_k = 0, \quad k = 1, ..., d\Big\} .$$

Clearly, T_u does not contain $(u_1+1, \ldots, u_d+1)$ because of $|u|_1 = m$. Let us decompose the index set T_u into the disjoint subsets $T_u = \bigcup_{\ell=1}^{d} T_u^\ell$, where

$$T_u^\ell = \{(j_1, ..., j_d) \in T_u : \quad \text{exactly } \ell \text{ components of } j \text{ vanish}\} , \qquad \ell = 1, ..., d.$$

The set T_u^ℓ contains exactly $\binom{d}{\ell}$ elements for every u and because of (51) we have

$$c_0[\Delta^1_{j_1}(e^{iN_{u_1}\cdot})] \cdot \ldots \cdot c_0[\Delta^d_{j_d}(e^{iN_{u_d}\cdot})] = (-1)^{d-\ell} \ , \qquad j \in T_u^\ell .$$

This together with (52) yields

$$\begin{aligned} c_0\big(A(m,d,\vec{L})\, f_m\big) &= \sum_{\substack{u_k \geq d \\ |u|_1 = m}} \sum_{\ell=1}^{d} \sum_{j \in T_u^\ell} (-1)^{d-\ell} \\ &= \sum_{\substack{u_k \geq d \\ |u|_1 = m}} \sum_{\ell=1}^{d} (-1)^{d-\ell} \binom{d}{\ell} \\ &= \sum_{\substack{u_k \geq d \\ |u|_1 = m}} \sum_{\ell=0}^{d-1} (-1)^{\ell} \binom{d}{\ell} . \end{aligned}$$

Because of

$$\sum_{\ell=0}^{d-1}(-1)^\ell\binom{d}{\ell} = \binom{d-1}{0}+\sum_{\ell=1}^{d-1}(-1)^\ell\Big(\binom{d-1}{\ell-1}+\binom{d-1}{\ell}\Big) = (-1)^{d-1}\binom{d-1}{d-1} = (-1)^{d-1}$$

we conclude in view of Remark 7

$$\big|\, c_0\big(f_m - A(m,d,\vec{L})f_m\big)\big| = \big|\, c_0\big(A(m,d,\vec{L})\, f_m\big)\big| = \Big|\sum_{\substack{u_k\ge d\\ |u|_1=m}} (-1)^{d-1}\Big| \asymp m^{d-1}\,.$$

Since we know the behaviour of $\| \, f_m \,|S^r_{p,q}B(\mathbb{T}^d)\|$, see (50), we finally get

$$\begin{aligned}\| \, I - A(m,d,\vec{L}) \,|S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \ &\ge \ \frac{\| \, f_m - A(m,d)f_m \,|L_p(\mathbb{T}^d)\|}{\| \, f_m \,|S^r_{p,q}B(\mathbb{T}^d)\|}\\ &\ge \ c\, 2^{-rm}\, m^{(d-1)(1-1/q)}\end{aligned}$$

with some positive constant c independent of $m \in \mathbb{N}$. ■

4.5 Proof of Theorem 4

Only the estimate from below is of interest. Associated to $L = (L^1, ..., L^d)$ is the sequence of grids $\mathcal{G}(m,d,\vec{L})$, $m \in \mathbb{N}_0$, see (11). For simplicity we concentrate on the first component for a moment. Because of (H4) we find

$$\Big|\bigcup_{j=0}^{m} \mathcal{T}_j^1\Big| \le C_2\, 2^{m+1}\,.$$

Consequently, for every $m \in \mathbb{N}_0$ there exists an open interval $\mathcal{I}_m \subset [-\pi,\pi]$, $|\mathcal{I}_m| = \frac{1}{C_2}2^{-(m+1)}$, such that

$$\mathcal{I}_m \cap \Big(\bigcup_{j=0}^{m} \mathcal{T}_j^1\Big) = \emptyset\,.$$

Therefore we can find a rectangle $R_m := \mathcal{I}_m \times [-\pi,\pi] \times ... \times [-\pi,\pi]$ such that

$$R_m \cap \mathcal{G}(m,d,\vec{L}) = \emptyset\,.$$

Let B denote the function investigated in Lemma 18. We choose $\lambda_1 = C_2\, 2^{m+1}$ and $\lambda_2 = ... = \lambda_d = 1$. If x^m denotes the centre of R_m the function $B(\lambda(\cdot\, - x^m))$ vanishes in $\mathcal{G}(m,d,\vec{L})$. In view of Lemma 18 this implies

$$\frac{\| \, B(\lambda(\cdot\, - x^m)) - A(m,d,\vec{L})(B(\lambda(\cdot\, - x^m))) \,|L_p(\mathbb{T}^d)\|}{\| \, B(\lambda(\cdot\, - x^m)) \,|S^r_{p,1}B(\mathbb{T}^d)\|} \ge c\lambda_1^{-r}$$

where the corresponding constants do not depend on m. ■

4.6 Proof of Theorems 5, 6 and Corollary 3

4.6.1 The Estimates from Above

The proof will be subdivided into several steps. Step 1-4 is devoted to the corresponding part of the proof of Theorem 5. Further, in Step 5 we deal with the upper estimate in Theorem 6 and finally, Step 6 covers the upper estimate in Corollary 3.
For $m \in \mathbb{N}_0^d$ we put

$$Q_1^m := \Big\{\ell \in \mathbb{N}_0^d : \quad \exists k \quad \text{s.t.} \quad \ell_k > m\Big\}, \tag{53}$$

$$Q_0^m := \Big\{\ell \in \mathbb{N}_0^d : \quad \ell_k \le m\,,\; k = 1, \ldots, d, \quad \text{and} \quad |\ell|_1 > m\Big\}. \tag{54}$$

Let $\widetilde{f}_\ell$ be defined as in (63). Then the following identities will play the major role in our estimates

$$A(m,d,S)f = \sum_{|\ell|_1 \le m} \widetilde{f}_\ell\,,$$

cf. Lemma 9, and

$$f - A(m,d,S)f = \sum_{\ell \in Q_0^m} \widetilde{f}_\ell + \sum_{\ell \in Q_1^m} \widetilde{f}_\ell\,.$$

They hold for all periodic distributions f, at least in the weak sense.
Step 1. Estimate of $\| \sum_{\ell \in Q_1^m} \widetilde{f}_\ell \,|L_p(\mathbb{T}^d)\|$. The triangle inequality and (65) yield

$$\begin{aligned} \| \sum_{\ell \in Q_1^m} \widetilde{f}_\ell \,|L_p(\mathbb{T}^d)\| &\le \sum_{\ell \in Q_1^m} \| \widetilde{f}_\ell \,|L_p(\mathbb{T}^d)\| \\ &\le c \,\| f|S^r_{p,\infty}B(\mathbb{T}^d)\| \Big(\sum_{\ell \in Q_1^m} 2^{-r|\ell|_1}\Big)\,. \end{aligned}$$

Since

$$\sum_{\ell \in Q_1^m} 2^{-r|\ell|_1} \le c\, 2^{-rm}$$

with c independent of m and $S^r_{p,q}B(\mathbb{T}^d) \hookrightarrow S^r_{p,\infty}B(\mathbb{T}^d)$ the desired inequality follows in all three cases.
Step 2. Estimate in case $q \le \min(2,p)$. By Step 1 it is enough to deal with
$\| \sum_{\ell \in Q_0^m} \widetilde{f}_\ell \,|L_p(\mathbb{T}^d)\|$. Our assumptions imply $S^r_{p,q}B(\mathbb{T}^d) \hookrightarrow S^r_pW(\mathbb{T}^d)$, see (60). Hence

$$\begin{aligned} \| \sum_{\ell \in Q_0^m} \widetilde{f}_\ell \,|L_p(\mathbb{T}^d)\| &\asymp \Big\| \Big(\sum_{\ell \in Q_0^m} |\widetilde{f}_\ell(x)|^2\Big)^{1/2} \Big| L_p(\mathbb{T}^d)\Big\| \\ &\le c_1\, 2^{-rm} \Big\| \Big(\sum_{\ell \in Q_0^m} 2^{2r|\ell|_1}\, |\widetilde{f}_\ell(x)|^2\Big)^{1/2} \Big| L_p(\mathbb{T}^d)\Big\| \\ &\le c_2\, 2^{-rm}\, \| f\,|S^r_{p,q}B(\mathbb{T}^d)\|\,. \end{aligned}$$

Step 3. Estimate in case $1 < p \le 2$ and $p \le q \le \infty$. Let $(1/p) = (1/q) + (1/u)$. The monotonicity of the ℓ_q-norms, the Littlewood-Paley characterisation (64), and Hölder's inequality yield

$$\begin{aligned} \| \sum_{\ell \in Q_0^m} \widetilde{f}_\ell \,|L_p(\mathbb{T}^d)\| &\asymp \Big\| \Big(\sum_{\ell \in Q_0^m} |\widetilde{f}_\ell(x)|^2 \Big)^{1/2} \Big| L_p(\mathbb{T}^d) \Big\| \\ &\le c_1 \Big(\sum_{\ell \in Q_0^m} \|\widetilde{f}_\ell \,|L_p(\mathbb{T}^d)\|^p \Big)^{1/p} \\ &\le c_1 \Big(\sum_{\ell \in Q_0^m} 2^{r|\ell|_1 q} \,\|\widetilde{f}_\ell \,|L_p(\mathbb{T}^d)\|^q \Big)^{1/q} \Big(\sum_{\ell \in Q_0^m} 2^{-r|\ell|_1 u} \Big)^{1/u} . \end{aligned}$$

Next we observe that

$$\Big(\sum_{\ell \in Q_0^m} 2^{-r|\ell|_1 u} \Big)^{1/u} \le c\, m^{(d-1)/u}\, 2^{-rm} \tag{55}$$

holds with a constant c independent of m. Altogether we have found

$$\| \sum_{\ell \in Q_0^m} \widetilde{f}_\ell \,|L_p(\mathbb{T}^d)\| \le c\, m^{(d-1)(\frac{1}{p}-\frac{1}{q})}\, 2^{-rm}\, \|\, f\, |S^r_{p,q}B(\mathbb{T}^d)\| \, .$$

Step 4. Estimate in case $2 < p < \infty$ and $2 < q \le \infty$. Let $(1/2) = (1/q) + (1/u)$. Similarly to Step 3 we obtain

$$\begin{aligned} \| \sum_{\ell \in Q_0^m} \widetilde{f}_\ell \,|L_p(\mathbb{T}^d)\| &\asymp \Big\| \Big(\sum_{\ell \in Q_0^m} |\widetilde{f}_\ell(x)|^2 \Big)^{1/2} \Big| L_p(\mathbb{T}^d) \Big\| \\ &\le c_1 \Big(\sum_{\ell \in Q_0^m} \|\widetilde{f}_\ell \,|L_p(\mathbb{T}^d)\|^2 \Big)^{1/2} \\ &\le c_1 \Big(\sum_{\ell \in Q_0^m} 2^{r|\ell|_1 q} \,\|\widetilde{f}_\ell \,|L_p(\mathbb{T}^d)\|^q \Big)^{1/q} \Big(\sum_{\ell \in Q_0^m} 2^{-r|\ell|_1 u} \Big)^{1/u} \\ &\le c_2\, m^{(d-1)(\frac{1}{2}-\frac{1}{q})}\, 2^{-rm}\, \|\, f\, |S^r_{p,q}B(\mathbb{T}^d)\| \, , \end{aligned}$$

where c_2 does not depend on m.
Step 5. By replacing the pieces $\widetilde{f}_\ell$ by f_ℓ^ψ, see Lemma 22, the upper estimate in Theorem 6 is derived. So it remains to deal with $p = 1$ and $p = \infty$.
First of all we proceed as in Step 1 and obtain

$$\begin{aligned} \| \sum_{\ell \in Q_1^m} f_\ell^\psi \,|L_p(\mathbb{T}^d)\| &\le \sum_{\ell \in Q_1^m} \|\, f_\ell^\psi \,|L_p(\mathbb{T}^d)\| \\ &\le c\, 2^{-rm}\, \|\, f |S^r_{p,\infty}B(\mathbb{T}^d)\| \, , \end{aligned}$$

which remains true for the limiting cases. Next we deal with $\sum_{\ell\in Q_0^m} f_\ell^\psi$. For any $1 \le p \le \infty$ we have

$$\begin{aligned}\| \sum_{\ell\in Q_0^m} f_\ell^\psi \,|L_p(\mathbb{T}^d)\| &\le \sum_{\ell\in Q_0^m} \| f_\ell^\psi \,|L_p(\mathbb{T}^d)\| \\ &\le c\, m^{(d-1)(1-\frac{1}{q})}\, 2^{-rm} \, \| f \,|S^r_{p,q}B(\mathbb{T}^d)\| \,,\end{aligned}$$

where we used (55).

Step 6. Because of

$$c_k(A(m,d,\psi)f) = 0 \quad \text{if} \quad k \notin H(m,d,2)$$

the upper estimate in Corollary 3 is a direct consequence of Theorem 6. ■

4.6.2 The Estimates from Below

Step 1. Let g_{m+1} be the function defined in (23). Since $A(m,d,S)g_m = 0$ we obtain

$$\frac{\| (I - A(m,d,S))\, g_{m+1} \,|L_p(\mathbb{T}^d)\|}{\| g_{m+1} \,|S^r_{p,q}B(\mathbb{T}^d)\|} = \frac{\| g_{m+1} \,|L_p(\mathbb{T}^d)\|}{\| g_{m+1} \,|S^r_{p,q}B(\mathbb{T}^d)\|} \ge c\, \frac{m^{(d-1)/p}}{2^{rm}\, m^{(d-1)/q}} \,, \qquad m \in \mathbb{N}\,,$$

see Lemmata 11, 12(ii). Next we shall use the functions

$$f_{m+1}(x) = \sum_{|j|_1 = m+1} e^{ip_j x} \qquad \text{(see (36))}.$$

Again $A(m,d,S)f_{m+1} = 0$ and hence

$$\frac{\| (I - A(m,d,S))\, f_{m+1} \,|L_p(\mathbb{T}^d)\|}{\| f_{m+1} \,|S^r_{p,q}B(\mathbb{T}^d)\|} = \frac{\| f_{m+1} \,|L_p(\mathbb{T}^d)\|}{\| f_{m+1} \,|S^r_{p,q}B(\mathbb{T}^d)\|} \ge c\, \frac{m^{(d-1)/2}}{m^{(d-1)/q}\, 2^{rm}} \,, \qquad m \in \mathbb{N}\,,$$

see Lemma 15. Finally we test with the functions $e^{ip_{(m+1,0,\ldots,0)}x_1}$. Then

$$A(m,d,S)e^{ip_{(m+1,0,\ldots,0)}x_1} = 0 \,, \qquad \| e^{ip_{(m+1,0,\ldots,0)}x_1} \,|L_p(\mathbb{T}^d)\| = (2\pi)^{1/p} \,,$$

and $\| e^{ip_{(m+1,0,\ldots,0)}x_1} \,|S^r_{p,q}B(\mathbb{T}^d)\| \asymp 2^{rm}$, where we used Lemma 15. This proves the claim.

Step 2. Let us turn to $A(m,d,\psi)$. In case $1 < p < \infty$ we can argue more or less as in Step 1. Let h_{m+2d} be a function defined in (33). Since $A(m,d,\psi)h_{m+2d} = 0$ we obtain

$$\frac{\| (I - A(m,d,\psi))\, h_{m+2d} \,|L_p(\mathbb{T}^d)\|}{\| h_{m+2d} \,|S^r_{p,q}B(\mathbb{T}^d)\|} \ge c\, \frac{m^{(d-1)/p}}{2^{rm}\, m^{(d-1)/q}} \,, \qquad m \in \mathbb{N}\,, \tag{56}$$

see Lemma 14. Next we shall use the functions

$$f_{m+2d}(x) = \sum_{|j|_1 = m+2d} e^{ip_j x} \qquad \text{(see (36))}.$$

Again $A(m,d,\psi)f_{m+2d}=0$, see (67), and hence

$$\frac{\|\,(I-A(m,d,\psi))\,f_{m+2d}\,|L_p(\mathbb{T}^d)\|}{\|\,f_{m+2d}\,|S^r_{p,q}B(\mathbb{T}^d)\|}\geq c\,\frac{m^{(d-1)/2}}{m^{(d-1)/q}\,2^{rm}}\,,\qquad m\in\mathbb{N}\,,$$

see Lemma 15. Finally we test with the functions $e^{ip_{(m+2,0,\ldots,0)}x_1}$. Then

$$A(m,d,\psi)e^{ip_{(m+2,0,\ldots,0)}x_1}=0\,,\qquad \|\,e^{ip_{(m+2,0,\ldots,0)}x_1}\,|L_p(\mathbb{T}^d)\|=(2\pi)^{1/p}\,,$$

and $\|\,e^{ip_{(m+2,0,\ldots,0)}x_1}\,|S^r_{p,q}B(\mathbb{T}^d)\|\asymp 2^{rm}$, see Lemma 15.

Step 3. Let $p=1$. The inequality (56) is still applicable and that is enough here.

Step 4. Let $p=\infty$. Instead of (56) we have

$$\frac{\|\,(I-A(m,d,\psi))\,h_{m+2d}\,|L_\infty(\mathbb{T}^d)\|}{\|\,h_{m+2d}\,|S^r_{\infty,q}B(\mathbb{T}^d)\|}\geq c\,\frac{m^{(d-1)}}{2^{rm}\,m^{(d-1)/q}}\,,\qquad m\in\mathbb{N}\,,$$

see Lemmata 13(iii) and 14. This proves the claim.

Step 5. Let g be a trigonometric polynomial with frequencies in the hyperbolic cross $H(m,d,1)$. For $1<p<\infty$ we use $A(m,d,S)g=g$, the inequality

$$\|\,f-A(m,d,S)f\,|L_p(\mathbb{T}^d)\|\leq(1+\|\,A(m,d,S)\,:\mathcal{L}(L_p(\mathbb{T}^d))\|)\;\|\,f-g\,|L_p(\mathbb{T}^d)\|$$

and the uniform boundedness of $\|\,A(m,d,S)\,:\,\mathcal{L}(L_p(\mathbb{T}^d))\|$ (see e.g. [21, 1.5.2]) to prove the desired estimate from below in (17).

Step 6. Finally we turn to the estimates from below in case $p=1$ and $p=\infty$. As it is obvious from Lemma 10(ii) the previous argument is not applicable.

Substep 6.1. For $p=1$ we use the functions defined in (37). For g being as in Step 5 and m large enough we conclude by Lemma 16

$$\begin{aligned}c_2^2\,m^{2(d-1)}\leq\|\psi_m|L_2(\mathbb{T}^d)\|^2&=(\psi_m,\psi_m)\\&=(\psi_m-g,\psi_m)\\&\leq\|\psi_m-g|L_1(\mathbb{T}^d)\|\cdot\|\psi_m|L_\infty(\mathbb{T}^d)\|\\&\leq c_1\,m^{d-1}\,\|\psi_m-g|L_1(\mathbb{T}^d)\|\,.\end{aligned}$$

The claim follows from Lemma 16(iii).

Substep 6.2. In case $p=\infty$ we argue as follows. Let g be a trigonometric polynomial such that $c_k(g)=0$ if $k\notin H(dm-1,d,1)$. Let f_m and Φ_m be the functions defined in (38) and (39), respectively. Then Lemma 17 implies

$$\begin{aligned}c_1\,m^{d-1}\leq\|f_m|L_2(\mathbb{T}^d)\|^2=(f_m,f_m)&=(f_m,\Phi_m)\\&=(f_m-g,\Phi_m)\\&\leq\|f_m-g|L_\infty(\mathbb{T}^d)\|\cdot\|\Phi_m|L_1(\mathbb{T}^d)\|\\&\leq 2\,(2\pi)^d\,\|f_m-g|L_\infty(\mathbb{T}^d)\|\,.\end{aligned}$$

Taking into account Lemma 17(iii) we have finished. ∎

4.7 Proof of Corollary 6

The estimate from above follows from Corollary 4. For the estimate from below we shall use some well-known results about Kolmogorov numbers of those embedding operators. Recall, for a Banach space $F \hookrightarrow L_p(\mathbb{T}^d)$ we put

$$d_M(F, L_p(\mathbb{T}^d)) := \inf_{\{u_i\}_{i=1}^M \subset L_p(\mathbb{T}^d)} \sup_{\|f|F\| \le 1} \inf_{c_1,\cdots,c_M} \Big\| f - \sum_{i=1}^M c_i\, u_i \Big| L_p(\mathbb{T}^d) \Big\| .$$

Hence $d_M \le \rho_M$. In case of $F = S^r_{p,\infty}B(\mathbb{T}^d)$ one has the convenient references [38, 11.4.11] and [36, Thm. 3.4.5], but with some additional restrictions what concerns r and p. For the general case we refer to [13]. Galeev considered a bit different spaces. However, by some standard arguments his estimates carry over to our situation, see e.g. [36, Introduction to Chapt. 3]. For $1 < p < \infty$, $1 \le q \le \infty$ and $r > 0$ this leads to

$$d_M(I, S^r_{p,q}B(\mathbb{T}^d), L_p(\mathbb{T}^d)) \asymp M^{-r} (\log M)^{(d-1)r} \eta(M,d,p,q) ,$$

where $\eta(M,d,p,q)$ is defined in (19). ■

Remark 20 *For the estimate from below one can also use entropy and approximation numbers. For a definition of these quantities we refer, e.g., to [11], [38] or [41]. Let $e_n(I,X,Y)$ denote the n-th dyadic entropy number of the embedding operator I which maps the Banach space X into the Banach space Y and let $\lambda_n(I,X,Y)$ denote the n-th approximation number (linear width, see Remark 2) of this embedding. Then trivially $\lambda_M \le \rho_M$ and furthermore $e_n \le c\,\lambda_n$ under certain weak conditions on X and Y which are satisfied in our context, see Theorem 1.3.3 in [11]. So, entropy numbers can be used as well for deriving lower bounds of ρ_M. The estimates*

$$e_M(I, S^r_{p,q}B(\mathbb{T}^d), L_p(\mathbb{T}^d)) \ge c \begin{cases} M^{-r} (\log M)^{(d-1)(r+\frac12-\frac1q)_+} & \text{if} \quad 1 < p < \infty , \\ M^{-r} (\log M)^{(d-1)(r-1/q)_+} & \text{if} \quad p = 1, \infty , \end{cases}$$

with some positive constant c (independent of M) are known, at least in a situation very close to ours. For (non-periodic) function spaces on domains it has been proved in [41, Thm. 4.11]. This can be transferred to the periodic situation. In our case it is enough to construct a bounded linear extension operator from $S^r_{p,q}B((-1,1)^d)$ to $S^r_{p,q}B(\mathbb{T}^d)$ and to apply the multiplicativity of the entropy numbers, see [11, 1.3.1]. We omit details and refer to [6] where a similar situation is investigated. Under additional restrictions on p and q entropy numbers of the embeddings $I : S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d))$ are studied in [4], [10] and [35].

4.8 Proof of Theorems 7 and 8

The estimate from above in Theorem 7 follows from $S^r_pW(\mathbb{T}^d) \hookrightarrow S^r_{p,2}B(\mathbb{T}^d)$, see (60), in combination with Theorem 1. For Theorem 8 it is contained in the proof of Theorem 5, see Subsection, 4.6.1 Step 1/2.

Step 1. Estimate from below in Theorem 8. We proceed as in proof of Theorem 5, cf. Subsection 4.6.2, Step 1, and observe that

$$\| f_{m+1} \,|S^r_pW(\mathbb{T}^d)\| \asymp 2^{rm} m^{(d-1)/2}\,, \qquad m \in \mathbb{N}\,,$$

see (36) and Lemma 15(ii).

Step 2. Estimate from below in Theorem 7. We employ the same strategy as in proof of Theorem 3, see Subsection 4.4. The test functions, defined in (49), satisfy

$$\| f_m \,|S^r_pW(\mathbb{T}^d)\| \asymp 2^{rm}\, m^{(d-1)/2}\,,$$

see Lemma 15(ii). ∎

4.9 Proof of Corollary 7

The estimate from above follows from Corollary 6. For the estimate from below we argue as there. It is enough to use

$$e_M(I:\; S^r_pW(\mathbb{T}^d) \to L_p(\mathbb{T}^d)) \geq c\, M^{-r} (\log M)^{(d-1)r}$$

with some positive constant c (independent of M), see [4] and [41].

5 Appendix - Function Spaces

Let $D(\mathbb{T}^d)$ denote the collection of all infinitely differentiable, complex-valued and in each component 2π-periodic functions equipped with the topology generated by

$$\| f \|_\alpha := \sup_{x\in\mathbb{T}^d} |D^\alpha f(x)|\,, \qquad \alpha \in \mathbb{N}^d_0\,.$$

The elements of the topological dual $D'(\mathbb{T}^d)$ (equipped with the weak topology) are called periodic distributions. All function spaces considered here in this paper will be continuously embedded into $D'(\mathbb{T}^d)$.

5.1 Nikol'skij-Besov Spaces on the Torus

Let $1 \le p \le \infty$ and $r > 0$. Let M be a natural number such that $M-1 \le r < M$. A function $f \in L_p(\mathbb{T})$ belongs to the Nikol'skij-Besov space $B^r_{p,\infty}(\mathbb{T})$ if

$$\| f \,|B^r_{p,\infty}(\mathbb{T})\| := \| f \,|L_p(\mathbb{T})\| + \sup_{t>0} t^{-r}\, \omega^M(f,t)_p \le \infty \,.$$

Here

$$\omega^M(f,t)_p := \sup_{|h|<t} \Big\| \sum_{j=0}^{M} \binom{M}{j} (-1)^j f(x+(M-j)h) \Big| L_p(\mathbb{T}) \Big\| \,.$$

Our general references for these spaces are [21, 26]. Of certain interest is the following property: we have

$$B^r_{p,\infty}(\mathbb{T}) \hookrightarrow C(\mathbb{T}) \qquad \Longleftrightarrow \qquad r > 1/p\,,$$

cf. [21, 6.3, 6.10.1] or [26, Rem. 3.5.5/3].

5.2 Besov Spaces on the Torus

For us it is convenient to introduce Besov spaces by making use of a smooth dyadic decomposition of unity. Let $C_0^\infty(\mathbb{R})$ denote the set of all compactly supported, complex-valued and infinite differentiable functions on the real line and Φ the collection of all systems $\varphi = \{\varphi_j(x)\}_{j=0}^\infty \subset C_0^\infty(\mathbb{R})$ satisfying

(i) supp $\varphi_0 \subset \{x : |x| \le 2\}$,

(ii) supp $\varphi_j \subset \{x : 2^{j-1} \le |x| \le 2^{j+1}\}$, $j = 1, 2, ...,$

(iii) $\forall\, \ell \in \mathbb{N}_0$ we have $\sup_{x,j} 2^{j\ell}\, |\varphi_j^{(\ell)}(x)| \le c_\ell < \infty$,

(iv) $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ for all $x \in \mathbb{R}$.

Let $1 \le p \le \infty$ and $r > 0$. Then $f \in L_p(\mathbb{T})$ belongs to $B^r_{p,p}(\mathbb{T})$ if

$$\| f \,|B^r_{p,p}(\mathbb{T})\| := \Big(\sum_{j=0}^{\infty} 2^{jrp} \,\| \sum_{k\in\mathbb{Z}} \varphi_j(k)\, c_k(f)\, e^{ikt} \,|L_p(\mathbb{T})\|^p \Big)^{1/p} < \infty \,.$$

Different elements of Φ lead to equivalent norms. For this and other properties, for instance

$$B^r_{p,p}(\mathbb{T}) \hookrightarrow B^r_{p,\infty}(\mathbb{T})\,, \tag{57}$$

we refer e.g. to [21] and [26, Chapt. 3].

5.3 Sobolev Spaces of Dominating Mixed Smoothness

If r is a natural number and $1 \le p \le \infty$, then the Sobolev space $S^r_pW(\mathbb{T}^d)$ of dominating mixed smoothness of order r is defined as the collection of all $f \in L_p(\mathbb{T}^d)$ such that

$$D^\alpha f \in L_p(\mathbb{T}^2)\,, \qquad \alpha = (\alpha_1, \ldots, \alpha_d)\,, \quad 0 \le \alpha_\ell \le r\,, \quad \ell = 1, \ldots, d\,.$$

Derivatives have to be understood in the weak sense. For general $r > 0$ and $1 < p < \infty$ one may use

$$\sum_{k\in\mathbb{Z}^d} c_k(f)(1+|k_1|^2)^{r/2} \ldots (1+|k_d|^2)^{r/2}\, e^{ikx} \in L_p(\mathbb{T}^d)\,.$$

In case $r \in \mathbb{N}$ this leads to an equivalent characterisation. For $r \in \mathbb{N}$ we endow these classes with the norm

$$\| f\,|S^r_pW(\mathbb{T}^d)\| := \sum_{\alpha\le r} \| D^\alpha f\,|L_p(\mathbb{T}^d)\|\,.$$

For $r > 0$, $r \notin \mathbb{N}$, and $1 < p < \infty$ we shall use

$$\| f\,|S^r_pW(\mathbb{T}^d)\| := \Big\| \sum_{k\in\mathbb{Z}^d} c_k(f)(1+|k_1|^2)^{r/2} \ldots (1+|k_d|^2)^{r/2}\, e^{ikx} \Big| L_p(\mathbb{T}^d)\Big\|\,.$$

Sometimes we use the symbol $S^0_pW(\mathbb{T}^d)$ instead of $L_p(\mathbb{T}^d)$.

5.4 Besov Spaces of Dominating Mixed Smoothness

These smooth dyadic decompositions of unity on $\mathbb{R}$ can be used to construct decompositions on $\mathbb{R}^d$ by means of tensor products. Let $\ell = (\ell_1, \ldots \ell_d) \in \mathbb{N}_0^d$. Then we put

$$\varphi_\ell(x) := \varphi_{\ell_1}(x_1) \cdot \ldots \cdot \varphi_{\ell_d}(x_d)\,.$$

Hence

$$\sum_{\ell\in\mathbb{N}_0^d} \varphi_\ell(x) = 1 \qquad \text{for all} \quad x \in \mathbb{R}^d\,.$$

As an abbreviation we shall use

$$f_\ell(x) := \sum_{k\in\mathbb{Z}^d} c_k(f)\,\varphi_\ell(k)\,e^{ikx}\,, \qquad x \in \mathbb{T}^d, \quad \ell \in \mathbb{N}_0^d\,, \tag{58}$$

which results in

$$f = \sum_{\ell\in\mathbb{N}_0^d} f_\ell\,,$$

at least in the sense of periodic distributions.
Let $\varphi \in \Phi$, $1 \le p \le \infty$, $1 \le q \le \infty$, and $r > 0$. Then the Besov space $S^r_{p,q}B(\mathbb{T}^d)$ of dominating mixed smoothness is the collection of all functions $f \in L_p(\mathbb{T}^d)$ such that

$$\| f \,|S^r_{p,q}B(\mathbb{T}^d)\| := \Big(\sum_{\ell \in \mathbb{N}_0^d} 2^{r|\ell|_1 q} \, \| f_\ell \,|L_p(\mathbb{T}^d)\|^q \Big)^{1/q} < \infty \,. \tag{59}$$

These classes are Banach spaces independent of the chosen system Φ (in the sense of equivalent norms), cf. [26, Chapt. 2,3] for $d = 2$. Below we shall recall a few facts about these classes. Let $1 < p < \infty$. Then

$$S^r_{p,\min(p,2)}B(\mathbb{T}^d) \hookrightarrow S^r_p W(\mathbb{T}^d) \hookrightarrow S^r_{p,\max(p,2)}B(\mathbb{T}^d)\,, \tag{60}$$

see [36, pp. 20/21] or [26, 2.2.3].

Lemma 20 *Let $1 \le p \le \infty$ and $1 \le q \le \infty$. Let $r > 1/p$. Then*

$$S^r_{p,q}B(\mathbb{T}^d) \hookrightarrow S^{1/p}_{p,1}B(\mathbb{T}^d) \hookrightarrow C(\mathbb{T}^d)$$

holds. Furthermore, if $f \in S^{1/p}_{p,1}B(\mathbb{T}^d)$ then

$$\lim_{m\to\infty} \Big\| f - \sum_{|\ell|_1 \le m} f_\ell \Big| C(\mathbb{T}^d) \Big\| = 0\,.$$

Proof *Step 1.* The embedding $S^r_{p,q}B(\mathbb{T}^d) \hookrightarrow S^{1/p}_{p,1}B(\mathbb{T}^d)$ is obvious. Using the Nikol'skij inequality, cf. e.g. [26, 3.3.2], we find

$$\begin{aligned} \| f \,|L_\infty(\mathbb{T}^d)\| &= \Big\| \sum_{\ell \in \mathbb{N}_0^d} f_\ell \Big| L_\infty(\mathbb{T}^d) \Big\| \\ &\le \sum_{\ell \in \mathbb{N}_0^d} \| f_\ell \,|L_\infty(\mathbb{T}^d)\| \\ &\le c \sum_{\ell \in \mathbb{N}_0^d} 2^{|\ell|/p} \, \| f_\ell \,|L_p(\mathbb{T}^d)\| \\ &= c\, \| f \,|S^{1/p}_{p,1}B(\mathbb{T}^d\| \,. \end{aligned} \tag{61}$$

This proves the boundedness of the elements in $S^{1/p}_{p,1}B(\mathbb{T}^d)$. The continuity follows from the continuity of the pieces f_ℓ, the convergence of $\sum_{|\ell|_1 \le m} f_\ell$ in $C(\mathbb{T}^d)$, employing the same arguments as in (61), and

$$\lim_{m\to\infty} \sum_{|\ell|_1 \le m} f_\ell = f \qquad \text{(convergence in} \quad D'(\mathbb{T}^d))\,.$$

■

Remark 21 *A proof of the nonperiodic counterpart to $S^{1/p}_{p,1}B(\mathbb{T}^2) \hookrightarrow C(\mathbb{T}^2)$ can be found in [26, 2.4.1].*

5.4.1 Fourier Multipliers

First of all we need some spaces of functions on $\mathbb{R}^d$. By $\mathcal{F}$ and $\mathcal{F}^{-1}$ we denote the Fourier transform and its inverse on $L_2(\mathbb{R}^d)$, respectively. Let $\kappa \geq 0$. Then a function $f \in L_2(\mathbb{R}^d)$ belongs to $S_2^\kappa H(\mathbb{R}^d)$ if

$$\| f \,|S_2^\kappa H(\mathbb{R}^d)\| := \Big(\int_{\mathbb{R}^d} (1+|\xi_1|^2)^\kappa \, \ldots \, (1+|\xi_1|^2)^\kappa \, |\mathcal{F}(\xi)|^2 \, d\xi \Big)^{1/2} < \infty \,.$$

Let $(b^j)_j$ be a sequence in $(0,\infty)^d$ and let $\Lambda = (\Lambda_j)_j$ be a sequence of subsets of $\mathbb{Z}^d$ s.t.

$$\Lambda_j \subset \{\ell \in \mathbb{Z}^d \, : \quad |\ell_i| \leq b_i^j \,, \quad i = 1,\ldots,d\} \,, \qquad j \in \mathbb{N}_0^d \,.$$

We say that a sequence $(g_j)_j$ of trigonometric polynomials belongs to $L_p^\Lambda(\mathbb{T}^d, \ell_q)$ if

$$\Big\| (g_j)_j \,|L_p(\mathbb{T}^d, \ell_q)\Big\| := \Big(\int_{\mathbb{T}^d} \Big(\sum_{j \in \mathbb{N}_0^d} |g_j(x)|^q \Big)^{p/q} dx \Big)^{1/p} < \infty$$

and

$$c_k(g_j) = 0 \qquad \text{for all} \quad k \notin \Lambda_j \,, \quad j \in \mathbb{N}_0^d \,.$$

Lemma 21 *Let $1 \leq p < \infty$, $1 \leq q \leq \infty$ and let*

$$\kappa > \frac{1}{\min(p,q)} + \frac{1}{2} \,.$$

If $(M_j)_j$ is a sequence in $S_2^\kappa H(\mathbb{R}^d)$, then there exists a constant c such that

$$\Big\| \sum_{k \in \mathbb{Z}^d} M_j(k) \, c_k(g_j) \, e^{ikx} \Big| L_p(\mathbb{T}^d, \ell_q) \Big\| \leq c \sup_{j \in \mathbb{N}_0^d} \|M_j(b^j \,\cdot\,) \,|S_2^\kappa H(\mathbb{R}^d)\| \; \| \, g_j \,|L_p(\mathbb{T}^d, \ell_q)\|$$

holds for all $(g_j)_j \in L_p^\Lambda(\mathbb{T}^d, \ell_q)$. Here c neither depends on $(g_j)_j$ nor on $(M_j)_j$.

Remark 22 *A nonperiodic counterpart to Lemma 21 is proved in [26, 1.8.3]. The proof in the periodic situation is similar. Details will be published elsewhere.*

5.4.2 Further Littlewood-Paley Characterizations

Similar to isotropic Sobolev spaces also the classes $S_p^r W(\mathbb{T}^d)$ allow a Littlewood-Paley characterisation.

Lemma 22 *Let $1 < p < \infty$ and $r \geq 0$. Let f_ℓ be as in (58). Then*

$$\| \, f \,|S_p^r W(\mathbb{T}^d)\| \asymp \Big\| \Big(\sum_{\ell \in \mathbb{N}_0^d} 2^{2r|\ell|_1} |f_\ell(x)|^2 \Big)^{1/2} \Big| L_p(\mathbb{T}^d) \Big\|$$

holds for all $f \in L_p(\mathbb{T}^d)$.

Remark 23 *For $r = 0$ this can be found in Nikol'skij [21, 1.5.2/(13)]. For $r > 0$ one has to use a lifting property, we refer to [26, 2.2.6] for the nonperiodic counterpart.*

Now we turn to the Lizorkin representation of Besov as well as Sobolev spaces. We need a special covering of $\mathbb{R}^d$. Let

$$\begin{aligned} P_0 &:= [-1,1]\,, \qquad P_j := \{x \in \mathbb{R}^d :\, 2^{j-1} < |x| \le 2^j\}\,, \quad j \in \mathbb{N}\,, \\ \mathcal{P}_j &:= P_{j_1} \times \ldots \times P_{j_d}\,, \qquad j \in \mathbb{N}_0^d\,. \end{aligned} \tag{62}$$

Then

$$\mathbb{R}^d = \bigcup_{j \in \mathbb{N}_0^d} \mathcal{P}_j \qquad \text{and} \qquad \mathcal{P}_j \cap \mathcal{P}_\ell = \emptyset \quad \text{if} \quad j \neq \ell\,.$$

Hence, with

$$\widetilde{f}_\ell(x) := \sum_{k \in \mathcal{P}_\ell} c_k(f)\, e^{ikx}\,, \qquad x \in \mathbb{T}^d,\ \ell \in \mathbb{N}_0^d\,, \tag{63}$$

we find

$$f = \sum_{\ell \in \mathbb{N}_0^d} \widetilde{f}_\ell$$

(convergence in the sense of periodic distributions), compare with (58).

Lemma 23 *Let $1 < p < \infty$, $1 \le q \le \infty$. Then, if $r \ge 0$*

$$\|\, f\,|S^r_p W(\mathbb{T}^d)\| \asymp \Big\| \Big(\sum_{\ell \in \mathbb{N}_0^d} 2^{2r|\ell|_1}\, |\widetilde{f}_\ell(x)|^2 \Big)^{1/2} \Big| L_p(\mathbb{T}^d) \Big\|\,, \tag{64}$$

and if $r > 0$

$$\|\, f\,|S^r_{p,q} B(\mathbb{T}^d)\| \asymp \Big(\sum_{\ell \in \mathbb{N}_0^d} 2^{r|\ell|_1 q}\, \|\, \widetilde{f}_\ell\,|L_p(\mathbb{T}^d)\|^q \Big)^{1/q} \tag{65}$$

holds for all $f \in L_p(\mathbb{T}^d)$.

Remark 24 *The Lemma can be proved by making use of a periodic version of a vector-valued Fourier multiplier theorem of Lizorkin, see [26, Thm. 3.4.3/3], in combination with Lemma 21 and Lemma 22.*

5.4.3 Smooth de la Vallée-Poussin Means

Finally we consider a special decomposition of unity. Let $\psi \in C_0^\infty(\mathbb{R})$ be an even function such that $\psi(t) = 1$ if $|t| \le 1$ and $\operatorname{supp} \psi \subset [-3/2, 3/2]$. Furthermore we assume that ψ is nonincreasing on $[0,\infty)$. Then we put

$$\begin{aligned} \vartheta_0(t) &:= \psi(t)\,, \\ \vartheta_j(t) &:= \psi(2^{-j}t) - \psi(2^{-j+1}t)\,, \qquad j \in \mathbb{N}\,, \\ \Theta_j(x) &:= \vartheta_{j_1}(x_1) \cdot \ldots \cdot \vartheta_{j_d}(x_d)\,, \qquad x \in \mathbb{T}^d, \quad j \in \mathbb{N}_0^d\,. \end{aligned} \tag{66}$$

Clearly, $(\vartheta_j)_j \in \Phi$. It holds

$$\vartheta_j(t) = 1 \qquad \text{if} \qquad \frac{3}{2}\, 2^{j-1} \le |t| \le 2^j \tag{67}$$

as well as

$$\sum_{j\in\mathbb{N}_0^d} \Theta_j(x) = 1 \qquad \text{for all} \quad x \in \mathbb{R}^d\,.$$

In addition we have $\Theta_j \ge 0$ and

$$\sum_{k\in\mathbb{Z}^d} \Theta_j(k) \asymp |\mathcal{P}_j|\,, \qquad j \in \mathbb{N}_0^d\,.$$

For $f \in L_1(\mathbb{T}^d)$ we introduce the decomposition

$$f_\ell^\psi(x) := \sum_{k\in\mathbb{Z}^d} \Theta_\ell(k)\, c_k(f)\, e^{ikx}\,, \qquad x \in \mathbb{T}^d\,, \quad \ell \in \mathbb{N}_0^d\,. \tag{68}$$

As a particular case of the definition we obtain

$$\|\, f\, |S^r_{p,q}B(\mathbb{T}^d)\| \asymp \Big(\sum_{\ell\in\mathbb{N}_0^d} 2^{r|\ell|_1 q}\, \|\, f_\ell^\psi\, |L_p(\mathbb{T}^d)\|^q \Big)^{1/q}\,.$$

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