Smooth density field of catalytic super-Brownian motion

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Abstract

Given an (ordinary) super-Brownian motion (SBM) ϱ on R^d of dimension $d = 2, 3,$ we consider a (catalytic) SBM X^ϱ on R^d with "local branching rates" $\varrho_s(\mathrm{d}x)$.

We show that X_t^ϱ is absolutely continuous with a density function ξ_t^ϱ, say. Moreover, there exists a version of the map $(t, z) \mapsto \xi_t^\varrho(z)$ which is $\mathcal{C}^\infty$ and solves the heat equation off the catalyst ϱ, more precisely, off the (zero set of) closed support of the time-space measure $\mathrm{d}s\, \varrho_s(\mathrm{d}x)$.

Using self-similarity, we apply this result to answer the question of the long-term behavior of X^ϱ in dimension $d = 2$: If ϱ and X^ϱ start with a Lebesgue measure, then X_T^ϱ converges (persistently) as $T \to \infty$ towards a random multiple of Lebesgue measure.

1 Introduction

1.1 Motivation and sketch of results

Consider a continuous super-Brownian motion (SBM) $\varrho = (\varrho_t)_{t \geq 0}$ in R^d with a constant branching rate. Roughly speaking, the *catalytic SBM* $X^\varrho = (X_t^\varrho)_{t \geq 0}$ is a continuous SBM in R^d with local branching rate "proportional to" ϱ. A rigorous construction can be found in [DF97a].

In [DF97a] also the study of longtime behavior of X^ϱ was initiated, and then continued in [DF97b] and [EF98]. From these papers it is known that if both initial states ϱ_0 and X_0^ϱ are Lebesgue measures ℓ_{c} and ℓ_{r}, respectively, then X^ϱ is *persistent* in all three dimensions $d \leq 3$ of its non-trivial existence. (In $d = 3$ the catalyst process ϱ was actually started from its steady state rather than from ℓ_{c} at time zero; this simplification is of course not possible in lower dimensions where ϱ clusters in the longtime limit, hence dies out locally.) Here persistence means that all weak limit points of X_T^ϱ as $T \to \infty$ have the full intensity measure ℓ_{r} again. In dimensions one and three the stronger result of persistent *convergence* has been shown in ([DF97a, DF97b]). For dimension $d = 2$ persistence of X^ϱ was proved in ([EF98]). The approach of [EF98] was to show the relative compactness of the set of laws of random second moments by p.d.e. methods. However, uniqueness of the limit point, and hence convergence, remained open.

In dimension $d = 2$, the process X^ϱ has a self-similarity property that connects the long-term behavior of X^ϱ with local properties at a fixed time. Thus, as noted in [DF97b, Remark 14], persistent convergence of X_T^ϱ as $T \to \infty$ is equivalent to the existence of the limit $\xi_1^\varrho(0)$ of $(2\varepsilon)^{-2} X_1^\varrho\left((-\varepsilon, \varepsilon)^2\right)$ as $\varepsilon \downarrow 0$, with full expectation, and hence to the absolute continuity of X_1^ϱ. Our main objective in this paper is to show that X_1^ϱ is absolutely continuous.

It is well-known that the (continuous) SBM with constant branching rate has *absolutely continuous states* only in dimension one. In $d = 1$ actually "every" catalytic SBM has densities, at least at given times ([DFR91]). [DF95] construct *higher-dimensional* catalytic SBM (with finite variance branching) with absolutely continuous states where the branching rate is given by a certain class of additive functionals of Brownian motion. This class includes catalysts concentrated on hyperplanes. They show absolute continuity via constructing fundamental solutions of the related cumulant equation.

Recently Delmas [Del96] considered a class of time-independent catalysts in R^d with carrying Hausdorff dimension greater than $d-2$. He shows that the reactant has a smooth density off the catalyst. His technique is a refinement of the *method of Brownian excursions,* introduced by [FL95] for a single point-catalytic model in $d = 1$. The procedure in those two papers is first to determine the (singular) occupation density measures λ, say, on the (time-independent) catalyst, and then to represent the SBM by means of Brownian excursion densities off the catalyst (supported by a Lebesgue zero set) starting with random masses according to λ, and by densities of Brownian particles killed at the catalyst. Clearly, these densities are smooth and satisfy the heat equation. At least at a heuristic level, this makes clear that in these cases a smooth density field exists.

Our strategy is to first show in $d = 2, 3$ that X^ϱ has densities in an L^2–sense on the complement of the support of ϱ. Next we use a modification of Delmas' representation of catalytic SBM "off the catalyst" on a local level to derive our *main result.* Namely, we show that off the catalyst, X^ϱ has a *smooth density field* ξ^ϱ that solves the heat equation (Theorem 1 at page 5).

Finally we use this result to answer the open question mentioned above: In two dimensions, if we start ϱ and X^ϱ from Lebesgue measures, then X^ϱ_T *converges* in law to the random multiple $\xi^\varrho_1(0)\,\ell$ of the (normed) Lebesgue measure ℓ (Corollary 2 (b)).

1.2 Informal description of the model

We consider a stochastic model for a *chemical (or biological) diffusion-reaction system* of two substances (or species) C and R, say. While C evolves independently of R, the *reaction* of R is *catalyzed* by C, that is takes place locally only in the presence of C but without affecting C.

The mathematical model that we choose for the catalyst is the so-called *super-Brownian motion* (SBM) ϱ. It arises as the high density short lifetime limit of *branching Brownian motion.* The latter is an (infinite) particle system, where the particles move around in R^d according to independent Brownian motions. Moreover, the catalyst particles die with a constant rate γ, say, and are replaced at the location of their death by zero or two offspring, each possibility occurring with probability $\frac{1}{2}$ (critical binary branching). The offspring continue to evolve in the same manner as their parent. Now assign the mass $\varepsilon > 0$ to each particle and replace the branching rate γ by γ/ε. Then (see, e.g., [Daw93, Section 4.4]) SBM ϱ is the limiting process if we let $\varepsilon \downarrow 0$ (provided that the initial states converge). Summarizing, the *catalyst* ϱ arises as a diffusion approximation to a critical binary branching Brownian motion with constant branching rate. For background on SBM we recommend [Daw93].

The mathematical model X^ϱ for the *reactant* is also SBM, however the branching rate of an "infinitesimal reactant particle" is the local concentration of catalytic matter. Consequently, the heuristic picture is the same except that the reactant particles die only when they are in contact with the catalyst. The catalyst itself varies in time and space, and it concentrates in some localized regions if $d \le 2$.

The model is interesting only in dimensions $d \le 3$. Roughly speaking, the catalyst is a $(d \wedge 2)$–dimensional object in R^d, thus a reactant particle (which performs Brownian motion) cannot meet the catalyst if $d \ge 4$. Hence, in $d \ge 4$, the "reactant" X^ϱ is only the deterministic heat flow.

A mathematical approach to this *"one-way interaction"* model is possible by means of Dynkin's additive functional approach to superprocesses ([Dyn91]). In fact, given the medium ϱ, an intrinsic X^ϱ–particle (reactant) following a Brownian path W branches according to the clock given by the *collision local time,* $L_{[W,\varrho]}(\mathrm{d}s)$, of W with ϱ ([BEP91]). Somewhat more formally:

$$L_{[W,\varrho]}(\mathrm{d}s) \;=\; \mathrm{d}s \int \varrho_s(\mathrm{d}y)\, \delta_y(W_s). \tag{1}$$

For sufficiently nice initial states of ϱ, these collision local times $L_{[W,\varrho]}$ make sense non-trivially in dimensions $d \le 3$ ([EP94]), although the measures $\varrho_s(\mathrm{d}y)$ are singular for $d \ge 2$ ([DH79]). For this reason, in dimensions $d \le 3$ the *catalytic SBM* X^ϱ could be constructed in [DF97a] as a continuous measure-valued (time-inhomogeneous) Markov process $(X^\varrho, P^\varrho_{r,m})$, given the catalyst process ϱ (*quenched approach*). By standard notation, $P^\varrho_{r,m}$ denotes the law of the process X^ϱ (for ϱ fixed) if at time r we start X^ϱ in the measure m. The laws of the catalyst process ϱ will be denoted by $\mathbb{P}_\mu$ if $\varrho_0 = \mu$.

Averaging the random laws $P^\varrho_{0,m}$ by means of $\mathbb{P}_\mu$ gives the *annealed* distribution $\mathcal{P}_{\mu,m} := \mathbb{P}_\mu P^\varrho_{0,m}$ of X^ϱ.

Of particular interest is the case $\mu = \ell_\mathrm{c} := i_\mathrm{c}\ell$ and $m = \ell_\mathrm{r} := i_\mathrm{r}\ell$ for some positive constants i_c and i_r.

Consider for the moment the *critical dimension* $d = 2$ and initial states $(\varrho_0, X^\varrho_0) = (\ell_\mathrm{c}, \ell_\mathrm{r})$. Here the catalyst ϱ_T dies out locally in probability as $T \to \infty$. In the large regions without catalyst only the smoothing heat flow acts on the reactant X^ϱ. On the other hand, a finite window of observation will be visited by increasingly large catalytic clumps at arbitrarily large times (recall that the time averaged two-dimensional catalyst ϱ has a proper random limit despite local extinction, see, e.g., [FG86]). These clumps lead locally to a great variability of the concentration of reactant: In contact with the catalyst, huge amount of reactant mass piles up in relatively small areas, whereas large areas become vacant. But according to [EF98, Theorem 1], the smoothing effect in the large catalyst free regions wins this competition with the *"turbulence"* at the catalyst, leading to persistence: The intensity measure ℓ_{r} of X_T^{ϱ} is preserved also for all accumulation points (in law) as $T \to \infty$.

A formal description of the pair (ϱ, X^{ϱ}) will be given in Subsection 2.1.

1.3 Notation and regularity assumption

Let $\mathrm{p}_s(x) := (2\pi s)^{-d/2} \exp\Big[-\frac{|x|^2}{2s}\Big]$, $s > 0$, $x \in \mathsf{R}^d$, denote the standard *heat kernel,* and write q for the *potential kernel:*

$$\mathrm{q}_t(x) := \int_0^t \mathrm{d}s \; \mathrm{p}_s(x), \qquad t \geq 0, \quad x \in \mathsf{R}^d. \tag{2}$$

If μ is a measure on R^d and the function φ is integrable with respect to μ, put $\mu * \varphi(y) := \int \mu(\mathrm{d}x)\, \varphi(y-x)$. Introduce the spatial *shift* operators $\theta_z \varphi(y) := \varphi(y-z)$, $y, z \in \mathsf{R}^d$, defined on functions φ, and write $\langle \nu, \varphi \rangle$ for integral expressions as $\int \nu(\mathrm{d}y)\, \varphi(y)$.

The construction of our processes actually needs an integrability condition for the initial states μ and m. Namely, we will assume that $\mu, m \in \mathcal{M}_p$ for some $p > d$. Here $\mathcal{M}_p$ is the set of measures μ on R^d such that $\langle \mu, \phi_p \rangle < \infty$, where

$$\phi_p(x) := \frac{1}{(1+|x|^2)^{p/2}}\,, \qquad x \in \mathsf{R}^d. \tag{3}$$

$\mathcal{M}_p$ is endowed with the coarsest topology such that the map $\mu \mapsto \langle \mu, \varphi \rangle$ is continuous for $\varphi = \phi_p$ and for each φ in the cone $\mathcal{C}_+^{\mathrm{comp}}$ of all non-negative continuous functions on R^d with compact support.

In dimension $d = 1$, for each measure $\mu \in \mathcal{M}_p$, with $\mathbb{P}_\mu$–probability one, the collision local time $L_{[W,\varrho]}$ (recall (1)) makes sense non-trivially and is a "nice branching functional" (see [DF97a, Corollary 3, p.261], or Proposition 14 in the appendix). In higher dimensions however, we need an additional smoothness property for initial measures of the catalyst process ϱ. This will be introduced next. Write Z^d for the set of all points x in R^d having only integer-valued coordinates.

Definition (η–diffusive measures) Let $\eta \in (0, \frac{1}{4})$. A measure $\mu \in \mathcal{M}_p$ is called *η–diffusive* if there is a $p' \in (d, p)$ such that even $\mu \in \mathcal{M}_{p'}$ and that the map

$$(t, z) \mapsto \phi_{p'}(z)\, \mu * \mathrm{q}_t(z)$$

is *locally Hölder continuous* of order η with the following *uniformity* in the

Hölder constants: For each $N \ge 1$,

$$\sup_{n\in\mathsf{Z}^d}\ \sup_{\substack{0\le t_1,t_2\le N\\ z_1,z_2\in n+[0,1]^d\\ [t_1,z_1]\ne[t_2,z_2]}} \frac{\left|\phi_{p'}(z_1)\,\mu*\mathrm{q}_{t_1}(z_1)\ -\ \phi_{p'}(z_2)\,\mu*\mathrm{q}_{t_2}(z_2)\right|}{\left|[t_1,z_1]-[t_2,z_2]\right|^{\eta}} \tag{4}$$

is finite. ◇

Example (η–diffusive measures) $\mu\in\mathcal{M}_p$ is η–diffusive in each of the following cases:

- μ *absolutely continuous* with a density function f such that there is a $p'\in(d,p)$ satisfying $\sup_{x\in\mathsf{R}^d}\phi_{p'}(x)\,f(x)<\infty$. Here even $\eta=1$ can be taken (Lipschitz). For instance, μ with a bounded density function f belongs to this class.
- In particular, $\mu=i_{\mathrm{c}}\,\ell$, for some constant $i_{\mathrm{c}}>0$ (*Lebesgue* measures).
- $\mu=\varrho_\delta$ (a.s.) for $\delta>0$ if only ϱ_0 is η–diffusive (see Proposition 16 in the appendix).

Note that in $d\ge 2$ a measure μ with an atom is *not* η–diffusive. ◇

1.4 Results

The *key* to our main result (Theorem 1 below) is the following fact on ordinary SBM ϱ:

Let S^ϱ denote the *closed support* of the locally finite measure $\mathrm{d}s\,\varrho_s(\mathrm{d}x)$ on $(0,\infty)\times\mathsf{R}^d$. In dimensions $d\ge 2$, this S^ϱ is an $\ell^+\times\ell$–zero set (Proposition 6 (b) at page 10). Here ℓ^+ denotes the (normed) Lebesgue measure on $(0,\infty)$. Write $\mathbf{Z}^\varrho\subset(0,\infty)\times\mathsf{R}^d$ for the *complement* of S^ϱ in $(0,\infty)\times\mathsf{R}^d$. In $\mathbf{Z}^\varrho$ only the heat flow acts on X^ϱ. This suggests that here the catalytic SBM X^ϱ has densities satisfying the heat equation (recall that X^ϱ degenerates for $d\ge 4$) :

Theorem 1 (main result) *Let $d\le 3$. Take $r\ge 0$, $\mu,m\in\mathcal{M}_p$, $\eta\in(0,\frac14)$, and assume that μ is η–diffusive. Then for P_μ–almost all ϱ the following statements hold.*

(a) (absolute continuity) *$P^\varrho_{r,m}$–almost surely, for all $t>r$, the measure $X^\varrho_t(\mathrm{d}z)\,\mathbf{1}_{\mathbf{Z}^\varrho}\big((t,z)\big)$ is absolutely continuous with respect to Lebesgue measure. In particular, in $d=2$ or 3, the reactant X^ϱ lives on the set of all absolutely continuous measures in $\mathcal{M}_p$.*

(b) (smooth density field ξ^ϱ) *Denoting by $\xi^\varrho=\big\{\xi^\varrho_t(z):\,t>r,\ z\in\mathsf{R}^d\big\}$ the density field of $X^\varrho_t(\mathrm{d}z)\,\mathbf{1}\{(t,z)\in\mathbf{Z}^\varrho\}$, there is a version of ξ^ϱ such that $P^\varrho_{r,m}$–a.s. the mapping $(t,z)\mapsto\xi^\varrho_t(z)$, $(t,z)\in\mathbf{Z}^\varrho$, $t>r$, is of class $\mathcal{C}^\infty$ and solves the heat equation:*

$$\frac{\partial}{\partial t}\,\xi^\varrho_t(z)\ =\ \frac12\,\Delta\,\xi^\varrho_t(z),\qquad (t,z)\in\mathbf{Z}^\varrho,\quad t>r. \tag{5}$$

(c) (moments) *The* $\xi_t^\varrho(z)$ *belong to* $L^2 = L^2(P^\varrho_{r,m})$, *have expectation*

$$P^\varrho_{r,m}\xi_t^\varrho(z) \;=\; m * \mathrm{p}_{t-r}(z), \qquad (t,z) \in \mathbf{Z}^\varrho, \quad t > r, \tag{6}$$

and covariances

$$\begin{aligned} &\mathcal{C}\mathrm{ov}^\varrho_{r,m}\left[\xi^\varrho_{t_1}(z_1), \xi^\varrho_{t_2}(z_2)\right] \\ &\quad = \; 2\int_r^{t_1\wedge t_2} \mathrm{d}s \; \Big\langle \varrho_s\,,\; (m * \mathrm{p}_{s-r})\,(\theta_{z_1}\mathrm{p}_{t_1-s})\,(\theta_{z_2}\mathrm{p}_{t_2-s})\Big\rangle \;\ge\; 0, \end{aligned} \tag{7}$$

$(t_i\,, z_i) \in \mathbf{Z}^\varrho,\;\; t_i > r,\;\; i = 1,2.$

(d) (local L^2–Lipschitz continuity) *The field* $\{\xi_t^\varrho(z) : \,(t,z) \in \mathbf{Z}^\varrho,\;\; t > r\}$ *is locally* $L^2(P^\varrho_{r,m})$*–Lipschitz continuous: For every compact subset* C *of* $\mathbf{Z}^\varrho \cap \big((r,\infty) \times \mathsf{R}^d\big)$ *there is a constant* $c = c(\varrho, C)$ *such that*

$$\left\|\xi^\varrho_{t_1}(z_1) - \xi^\varrho_{t_2}(z_2)\right\|_2 \;\le\; c\,\left|(t_1, z_1) - (t_2\,, z_2)\right|, \tag{8}$$

$(t_1\,, z_1), (t_2\,, z_2) \in C.$

Remark (generalization) We have formulated our theorem for the case where the catalyst ϱ is SBM. However, all we really need is that ϱ is a (deterministic) $\mathcal{M}_p$–valued path such that the collision local time $L_{[W,\varrho]}$ exists and is a nice branching functional (see Proposition 14 (d) in the Appendix). In this case the theorem is still true (except, of course, the second sentence in (a)). In this sense our result can be viewed as a partial generalization of Delmas [Del96] for time dependent catalysts. ◇

Remark (genealogical interpretation) Formula (7) has the following genealogical interpretation. The covariance there measures the probability of two infinitesimal reactant particles at $(t_1\,, z_1)$ and $(t_2\,, z_2)$ to have a common ancestor. On the other hand, the integrand at the r.h.s. is the "distribution" of the time-space location (s,x) of a possible latest common ancestor of these infinitesimal particles. ◇

Remark (behavior at a given point) Since in $d \ge 2$ a given point (t,z) in $(r,\infty) \times \mathsf{R}^d$ belongs to $\mathbf{Z}^\varrho$ with $\mathbb{P}_\mu$–probability 1 (see Proposition 6 (b) below), $\xi_t^\varrho(z)$ is a well-defined $P^\varrho_{r,m}$–random variable, $\mathbb{P}_\mu$–a.s. (in $d = 2$ or 3). ◇

Remark (annealed model) Statement (a) of Theorem 1 implies that also with respect to the *annealed* law $\mathcal{P}_{\mu,m}$ the catalytic SBM X^ϱ lives on the set of *absolutely continuous* measures. Clearly, (7) and (6) yield that the $\mathcal{P}_{\mu,m}$–*covariances* of ξ^ϱ are given by

$$\begin{aligned} &\mathcal{C}\mathrm{ov}_{\mu,m}\left[\xi^\varrho_{t_1}(z_1), \xi^\varrho_{t_2}(z_2)\right] \\ &\quad = \; 2\int_0^{t_1\wedge t_2} \mathrm{d}s \,\Big\langle \ell\,,\; (\mu * \mathrm{p}_s)\,(m * \mathrm{p}_s)\,(\theta_{z_1}\mathrm{p}_{t_1-s})\,(\theta_{z_2}\mathrm{p}_{t_2-s})\Big\rangle \;<\; \infty \end{aligned} \tag{9}$$

$(t_i\,,z_i) \in (0,\infty)\times \mathsf{R}^d,\ \ i=1,2,\ \ (t_1\,,z_1)\neq(t_2\,,z_2)$. Hence (if $\mu,m\neq 0$), the covariance tends to infinity if $(t_2\,,z_2)\to(t_1\,,z_1)$. In particular,

$$\mathcal{V}\mathrm{ar}_{\mu,m}\,\xi_t^{\varrho}(z)\;\equiv\;\infty,\qquad (t,z)\in(0,\infty)\times\mathsf{R}^d. \qquad\diamond$$

Now we come back to the limiting behavior of X_T^{ϱ} as $T\uparrow\infty$ in $d=2$ with $(\varrho_0\,,X_0^{\varrho})=(\ell_{\mathrm{c}}\,,\ell_{\mathrm{r}})$. In this dimension, the long-term behavior of X^{ϱ} is connected to local properties (such as absolute continuity of states) by a *self-similarity* property. Proposition 13 in [DF97b] states that

$$X_T^{\varrho}\;\overset{\mathcal{L}}{=}\;K^{-1}X_{KT}^{\varrho}\Big(K^{1/2}\,\cdot\Big),\qquad T,K>0, \tag{10}$$

with respect to the random laws $P_{0,\ell_{\mathrm{r}}}^{\varrho}$. Here coincidence w.r.t. the random laws $P_{0,\ell_{\mathrm{r}}}^{\varrho}$ formally means that

$$\mathbb{P}_{\ell_{\mathrm{c}}}\Big[P_{0,\ell_{\mathrm{r}}}^{\varrho}[X_T^{\varrho}\in(\cdot)]\in(\cdot)\Big]\;=\;\mathbb{P}_{\ell_{\mathrm{c}}}\Big[P_{0,\ell_{\mathrm{r}}}^{\varrho}\Big[K^{-1}X_{KT}^{\varrho}(K^{1/2}\,\cdot\,)\in(\cdot)\Big]\in(\cdot)\Big]. \tag{11}$$

From this discussion the following corollary of Theorem 1 is immediate.

Corollary 2 **($d=2$)** *In dimension two, with respect to the random laws* $P_{0,\ell_{\mathrm{r}}}^{\varrho}$ *(with ϱ distributed according to $\mathbb{P}_{\ell_{\mathrm{c}}}$) the following two statements hold:*

(a) (self-similarity)

$$\xi_T^{\varrho}\;\overset{\mathcal{L}}{=}\;\xi_{KT}^{\varrho}(K^{1/2}\,\cdot\,),\qquad T,K>0. \tag{12}$$

(b) (persistent convergence) X_T^{ϱ} *converges in distribution to a random multiple of Lebesgue measure:*

$$X_T^{\varrho}\;\underset{T\uparrow\infty}{\longrightarrow}\;\xi_1^{\varrho}(0)\,\ell. \tag{13}$$

Note that the limit in (13) not only has full expectation, but has even locally finite conditional second moments (*persistence of second order*).

Coincidence in law in statement (a) is understood in the same way as in (11). Similarly, the assertion in (b) has the following formal meaning. Given ϱ (distributed according to $\mathbb{P}_{\ell_{\mathrm{c}}}$), let Q_T^{ϱ} and Q_∞^{ϱ} denote the laws of the random measures X_T^{ϱ} and $\xi_1^{\varrho}(0)\,\ell$, respectively. Set

$$\mathbf{Q}_T\;:=\;\mathbb{P}_{\ell_{\mathrm{c}}}\left[Q_T^{\varrho}\in(\cdot)\right],\qquad \mathbf{Q}_\infty\;:=\;\mathbb{P}_{\ell_{\mathrm{c}}}\left[Q_\infty^{\varrho}\in(\cdot)\right]. \tag{14}$$

Then the formal expression for the claim in (b) is

$$\mathbf{Q}_T\ \text{converges}\ \textit{weakly}\ \text{to}\ \mathbf{Q}_\infty\ \text{as}\ T\to\infty. \tag{15}$$

Note that for fixed medium ϱ one cannot expect convergence since ϱ_T itself does *not converge a.s.* as $T \to \infty$.

It is known from [DF97a, Theorem 51] that in dimension *one*

$$X_T^\varrho \underset{T\uparrow\infty}{\longrightarrow} \ell_{\mathrm{r}}\,, \quad \text{in } P^\varrho_{0,\ell_{\mathrm{r}}}\text{–probability}, \quad \text{for } \mathbb{P}_{\ell_{\mathrm{c}}}\text{–almost all } \varrho. \tag{16}$$

(It is still open whether this statement is true $P^\varrho_{0,\ell_{\mathrm{r}}}$–a.s.) The reason for this behavior is that in $d = 1$ the catalyst dies out locally *almost surely.* In contrast, in $d = 2$ the catalyst goes to local extinction only in $\mathbb{P}_{\ell_{\mathrm{c}}}$–probability. Hence, the reactant meets the catalyst at arbitrarily large times. The randomness in the limit in (13) reflects the random medium as experienced by the reactant at large times. In particular, X^ϱ does not converge for almost all ϱ. The almost sure properties of Theorem 1 get lost on the way to Corollary 2 (b) by using the self-similarity (a) that holds only in distribution.

Note that the two-dimensional reactant X^ϱ exhibits the following interesting phenomenon: Though started in a (spatially) ergodic state, the limit is *not* ergodic.

Remark 3 (annealed model) The self-similarity (10) holds also with respect to the annealed law $\mathcal{P}_{\ell_{\mathrm{c}},\ell_{\mathrm{r}}}$ ([DF97b, Proposition 13]). Hence, (12) and (13) are true also w.r.t. the annealed law. In other words, we have the following persistent weak convergence of averaged distributions:

$$\mathbb{P}_{\ell_{\mathrm{c}}} Q_T^\varrho \underset{T\uparrow\infty}{\longrightarrow} \mathbb{P}_{\ell_{\mathrm{c}}} Q_\infty^\varrho\,.$$

But recall that persistence of second order gets lost. ◇

Remark 4 (lattice model) In the model of two-dimensional simple branching random walk in the simple branching random medium, one can show a statement analogous to Corollary 2 (b): Here the reactant converges to a mixed Poisson system (homogeneous Poisson point process) with random intensity $\xi_1^\varrho(0)$ ([GKW97, Theorem 3]). The proof of this statement is based on our Theorem 1. However, since there is no scaling property in the lattice model, things become rather complicated. ◇

The rest of the *paper is laid out* as follows. In Section 2 we first recall the formal characterization of the catalytic SBM X^ϱ. We establish the fact that around $\ell^+ \times \ell$–almost all time-space points (t, z) there is no catalytic mass. The key step in Section in 2 is to show that at those (t, z) an asymptotic spatial $L^2(P^\varrho_{r,m})$–density $\xi_t^\varrho(z)$ of X_t^ϱ exists. Our theorem is proved in Section 3. The appendix is devoted to the notion of η–diffusive measures serving as initial states for the catalyst.

2 Preparations

2.1 Formal description of catalytic SBM

First we want to recall the formal characterization of the catalytic SBM X^{ϱ} in terms of its Laplace transition functional.

Write $\mathcal{B}^p$ for the set of all functions φ on R^d such that $|\varphi| \le c_\varphi\, \phi_p$ for some (finite) constant c_φ, and $\mathcal{B}^p_+$ for the subset of its non-negative members.

Fix a constant $\gamma > 0$. By definition, the *catalyst process* $\varrho = (\varrho_t)_{t\ge 0}$ is a continuous (critical) SBM with branching rate γ. This is the continuous $\mathcal{M}_p$–valued time-homogeneous Markov process $(\varrho, \mathbb{P}_\mu)$ with Laplace transition functional

$$\mathbb{P}_\mu \exp\langle \varrho_t\,, -\varphi\rangle \;=\; \exp\big\langle \mu, -u(t)\big\rangle\,, \qquad t\ge 0, \quad \mu\in\mathcal{M}_p\,, \quad \varphi\in\mathcal{B}^p_+\,. \tag{17}$$

Here $u = \{u(t):\, t\ge 0\} = \big\{u(t,x):\; t\ge 0,\; x\in\mathsf{R}^d\big\}$ is the unique non-negative (mild) solution to the *basic cumulant equation*

$$\frac{\partial}{\partial t}\, u \;=\; \frac{1}{2}\,\Delta u - \gamma\, u^2 \quad \text{on } (0,\infty)\times\mathsf{R}^d \tag{18}$$

with *initial* condition $u(0,x) = \varphi(x),\; x\in\mathsf{R}^d$.

The process ϱ serves as a random medium for a catalytic SBM X^{ϱ}. In order to characterize X^{ϱ}, roughly speaking, we have to replace the constant rate γ in (18) by the (randomly) varying rate $\varrho_t(x)$, where $\varrho_t(x)$ is the *generalized* derivative $\frac{\varrho_t(\mathrm{d}x)}{\mathrm{d}x}(x)$ of the measure $\varrho_t(\mathrm{d}x)$. Our aim is to define X^{ϱ} via its log-Laplace transition functionals v^{ϱ}_t that solve a certain cumulant equation. We do so by first making precise sense of this equation.

Because of time-inhomogeneity, it is convenient to write the formal cumulant equation in a *backward setting:*

$$-\frac{\partial}{\partial r}\, v^{\varrho}_t(r,x) \;=\; \frac{1}{2}\,\Delta v^{\varrho}_t(r,x) - \varrho_r(x)\, v^{\varrho}_t(r,x)^2, \tag{19}$$

$0\le r\le t,\;\; x\in\mathsf{R}^d$. Note that the initial condition has become a *terminal* condition: $v^{\varrho}_t(t) = \varphi$. After a formal integration, we can rewrite (19) rigorously and probabilistically as

$$v^{\varrho}_t(r,x) \;=\; \Pi_{r,x}\left[\varphi(W_t) - \gamma\int_r^t L_{[W,\varrho]}(\mathrm{d}s)\, v^{\varrho}_t(s,W_s)^2\right], \tag{20}$$

$0\le r\le t,\;\; x\in\mathsf{R}^d$, where $\Pi_{r,x}$ is the law of (standard) Brownian motion W starting at time r from x. For $d\le 3$ and finite μ with some regularity, [EP94, Theorem 4.1 and Proposition 4.7] show that for $\mathbb{P}_\mu$–a.a. ϱ the collision local time $L_{[W,\varrho]}$ makes sense as a non-trivial continuous additive functional of W. (In $d\ge 4$, actually $L_{[W,\varrho]} = 0$.) According to [DF97a, Theorem 4, p.259], this is still true if μ is random, the law of μ is shift-invariant and has finite moments of all orders, and μ is sufficiently smooth (a.s.).

In the present paper, we work with initial states μ of the catalyst process ϱ which are η–diffusive (recall the Definition in Subsection 1.3). Then, P_μ–a.s., the collision local time $L_{[W,\varrho]}$ exists and is a nice branching functional in the sense of Proposition 14 in the appendix. Therefore, according to ([DF97a, Proposition 1, p.225]), for P_μ–almost all ϱ and for t,φ fixed, there is a unique non-negative solution v_t^ϱ to (20). Finally, by ([DF97a, Proposition 12 (a), p.230, and Theorem 1 (b), p.235]), we get the following result:

Proposition 5 (existence of X^ϱ) *Let $d \le 3$, $\mu \in \mathcal{M}_p$. If $d = 2,3$ then let μ additionally be η–diffusive. Then, for P_μ–a.a. ϱ, there exists a continuous $\mathcal{M}_p$–valued time-inhomogeneous Markov process $(X^\varrho, P^\varrho_{r,m})$ with Laplace transition functional*

$$P^\varrho_{r,m} \exp \langle X^\varrho_t, -\varphi\rangle \;=\; \exp\left\langle m, -v^\varrho_t(r)\right\rangle, \tag{21}$$

$0 \le r \le t$, $m \in \mathcal{M}_p$, $\varphi \in \mathcal{B}^p_+$, *and* v^ϱ_t *the solution to* (20).

This is the *catalytic SBM* X^ϱ with catalyst ϱ, which was intuitively introduced in Subsection 1.2.

Since the branching mechanism is critical, X^ϱ has *expectation measure*

$$P^\varrho_{r,m} X^\varrho_t \;\equiv\; S_{t-r} m, \tag{22}$$

independent of the catalytic medium ϱ. Here $S = \{S_t : t \ge 0\}$ is the semigroup of Brownian motion. The *covariances* (given ϱ) related to (22) can be written as

$$\begin{aligned} &\mathrm{Cov}^\varrho_{r,m}\Big[\left\langle X^\varrho_{t_1}, \psi_1\right\rangle, \left\langle X^\varrho_{t_2}, \psi_2\right\rangle\Big] \\ &\quad=\; 2\int_r^{t_1\wedge t_2} \mathrm{d}s\,\Big\langle \varrho_s\,,\,(m * \mathrm{p}_{s-r})\,(S_{t_1-s}\psi_1)\,(S_{t_2-s}\psi_2)\Big\rangle, \end{aligned} \tag{23}$$

$0 \le r \le t_1, t_2$, $m \in \mathcal{M}_p$, $\psi_1,\psi_2 \in \mathcal{B}^p$; see [DF97a, formula (95)].

2.2 Catalyst free regions

Denote by $B_\delta(z)$ the open ball in R^d of radius δ centered at z. The starting point for our development is the following observation.

Proposition 6 (catalyst free regions close to time-space points) *Assume that $\mu \in \mathcal{M}_p$. For $t > 0$, denote by $\mathbf{Z}^\varrho_t$ the open set of all those $z \in \mathsf{R}^d$ such that there exists a $\delta = \delta(\varrho,t,z) \in (0,t)$ with*

$$\sup_{s\in[t-\delta,t+\delta]} \varrho_s\left(B_\delta(z)\right) \;=\; 0, \tag{24}$$

i.e. $\mathbf{Z}^\varrho = \Big\{(t,z) : t > 0,\ z \in \mathbf{Z}^\varrho_t\Big\}$. *Suppose* $d \ge 2$.

(a) (full measure) *Then,*

$$\sup_{t>0} \ell\left(\mathsf{R}^d \setminus \mathbf{Z}_t^{\varrho}\right) = 0, \quad \mathsf{P}_\mu\text{–}a.s., \tag{25}$$

hence, $\ell^+ \times \ell\Big(\big((0,\infty)\times\mathsf{R}^d\big)\setminus \mathbf{Z}^{\varrho}\Big) = 0,\ \mathsf{P}_\mu$*–a.s.*

(b) (absence at a given point) *In particular, for fixed* $t>0$ *and* $z\in\mathsf{R}^d$, *there is a* $\delta=\delta(\varrho,t,z)$ *such that* (24) *holds for* P_μ*–almost all* ϱ.

Remark 7 (polarity of points) Statement (b) has been known in the case of finite initial measures μ ([Dyn93, Theorem 11.2]). In potential-theoretical language, Dynkin shows that (in $d\geq 2$) a given time-space *point* (t,z) *is polar for the graph* of SBM ϱ. Note that this could also be concluded from [EP91, Theorems 3.1 and 3.3] and the fact that $(\mathrm{supp}\ \varrho_t)_{t\geq 0}$ is right-continuous in the Hausdorff metric (see [Per89, Theorem 1.4]). See also [LG94, Corollary 3]. ◇

Proof of Proposition 6 We only prove **(a)** since **(b)** is an immediate consequence of **(a)**. First note that by a countability argument it suffices to consider the supremum taken over $t\in(0,1]$ and to replace R^d by $B_1(0)$. Next we use the bounded support property of SBM and a Borel Cantelli argument to replace P_μ in (25) by P_{μ_N}, where we define $\mu_N(\mathrm{d}x) := \mathbf{1}_{B_N(0)}(x)\,\mu(\mathrm{d}x),\ N\geq 1$. In fact, there exists a constant $c>0$ such that for $x\in\mathsf{R}^d$,

$$\mathsf{P}_{\delta_x}\Big(B_1(0)\not\subset \mathbf{Z}_t^{\varrho} \text{ for some } t\leq 1\Big) \leq c\,\mathrm{e}^{-(\|x\|-1)^2/2} \tag{26}$$

(see [DIP89, Theorem 3.3 (a)]. Integrating this over $B_N(0)^{\mathrm{c}}$ with respect to $(\mu-\mu_N)(\mathrm{d}x)$ gives $\mathsf{P}_{\mu-\mu_N}\Big(B_1(0)\not\subset \mathbf{Z}_t^{\varrho} \text{ for some } t\leq 1\Big) \underset{N\uparrow\infty}{\longrightarrow} 0$. Hence, we only have to show

$$\mathsf{P}_{\mu_N}\Big(\ell\left(B_1(0)\setminus \mathbf{Z}_t^{\varrho}\right) = 0 \text{ for all } t\in(0,1]\Big) = 1. \tag{27}$$

This is known to be true if we replace $\mathbf{Z}_t^{\varrho}$ by $\widetilde{\mathbf{Z}}_t^{\varrho} := (\mathrm{supp}\ \varrho_t)^{\mathrm{c}} \subset \mathbf{Z}_t^{\varrho}$ (see [Per89, Corollary 1.3]). But from [Per90, Theorem 1.4] we know that P_{μ_N}–a.s. for almost all $t>0$, the set $\mathbf{Z}_t^{\varrho}\setminus\widetilde{\mathbf{Z}}_t^{\varrho}$ is at most a singleton. Hence (27) holds and we are done. ■

Remark 8 (dimension one) Properties as in Proposition 6 are *not* valid in $d=1$ since there ϱ has a jointly continuous density field on $(0,\infty)\times\mathsf{R}$ (see, e.g., [KS88]). ◇

2.3 Asymptotic L^2–densities of the reactant

Recall the definition of the reference function ϕ_p from (3). We will need the following trivial heat kernel estimates that we state here without proof.

Lemma 9 (estimates for the heat kernel) *For $d \ge 1$, let C be a compact subset of $(0,\infty)\times\mathsf{R}^d$, and let $k,n \ge 1$. Choose $\delta > 0$ such that*

$$C^\delta \;:=\; \bigcup_{(t,z)\in C} [t-\delta, t+\delta]\times B_\delta(z) \;\subset\; (0,\infty)\times\mathsf{R}^d. \tag{28}$$

Then there are constants $c_i = c_i(d,C,n,\delta)$, $i=1,2,3$, such that for $(t,z)\in C$ and $(s,x)\in\big((0,\infty)\times\mathsf{R}^d\big)\setminus C^\delta$ with $s\le t$, the following three statements hold:

$$\Big|\frac{\partial}{\partial r}\,\theta_z\,\mathrm{p}_r(x)\Big|_{r=n(t-s)}\Big| \;\le\; c_1\,\theta_z\,\mathrm{p}_{2n(t-s)}(x), \tag{29}$$

$$\Big|\frac{\partial}{\partial z}\,\theta_z\,\mathrm{p}_{n(t-s)}(x)\Big| \;\le\; c_2\,\theta_z\,\mathrm{p}_{2n(t-s)}(x), \tag{30}$$

$$\theta_z\,\mathrm{p}_{n(t-s)}(x) \;\le\; c_3\,\phi_p^k(x). \tag{31}$$

Since for η–diffusive μ the collision local time $L_{[W,\varrho]}$ is a nice branching functional in $d\le 3$ (see Proposition 14 (d) in the appendix), the following estimate is true:

Lemma 10 (uniform estimate) *Let $d\le 3$ and assume that $\mu\in\mathcal{M}_p$ is η–diffusive. Then, for $\mathbb{P}_\mu$–almost all ϱ, for every $T>0$ there exists a constant $c_{T,\varrho}$ such that*

$$\int_0^T \mathrm{d}s \int \varrho_s(\mathrm{d}y)\,\mathrm{p}_s(y-x)\,\phi_p^2(y) \;\le\; c_{T,\varrho}\,\phi_p(x), \qquad x\in\mathsf{R}^d. \tag{32}$$

The following L^2–result is the key of our development.

Proposition 11 (asymptotic L^2–densities at points in $\mathbf{Z}^\varrho$) *Let $d\le 3$. Take $r\ge 0$, $\mu, m\in\mathcal{M}_p$, and assume that μ is η–diffusive. Then for $\mathbb{P}_\mu$–almost all ϱ the following assertions hold.*

(a) (existence on $\mathbf{Z}^\varrho$) *For each $(t,z)\in\mathbf{Z}^\varrho$, $t>r$, there is an element $\xi_t^\varrho(z)\ge 0$ in the Lebesgue space $L^2 = L^2(P^\varrho_{r,m})$ such that the L^2–convergence*

$$X_t^\varrho * \mathrm{p}_\varepsilon(z) \;\underset{\varepsilon\downarrow 0}{\longrightarrow}\; \xi_t^\varrho(z) \tag{33}$$

takes place.

(b) (locally uniform convergence) *This convergence is uniform if (t,z) runs in a compact set $C = C(\varrho)\subset\mathbf{Z}^\varrho\cap\big((r,\infty)\times\mathsf{R}^d\big)$.*

(c) (moments) *$\xi_t^\varrho(z)$ has expectation $m*\mathrm{p}_{t-r}(z)$, and the covariances are given by (7).*

(d) (existence at a given point) *In particular, if* $d = 2$ *or* 3, *for* $t > r$ *and* $z \in \mathsf{R}^d$ *fixed,* $\xi_t^\varrho(z)$ *exists with those properties, for* $\mathbb{P}_\mu$*–a.a.* ϱ.

Remark 12 (basic solutions) The existence (in law) of the asymptotic density $\xi_t^\varrho(z)$ for $(t,z) \in \mathbf{Z}^\varrho$ is equivalent to the existence of the basic solution v_t^ϱ of (19) with terminal condition $v_t^\varrho(t,x) = \delta_z(x)$. This fact has been used in [DF95] for related models in higher dimensions to construct asymptotic densities. Here we make a direct approach to obtain the densities (in L^2). ◇

Proof of Proposition 11 Since ϱ_r is admissible $\mathbb{P}_\mu$–a.s. (recall the last example, listed in Subsection 1.3), using the Markov property, without loss of generality we may assume that $r = 0$. By the covariance formula (23),

$$\begin{aligned} &\Big\| X_t^\varrho * \mathrm{p}_{\varepsilon_1}(z) - X_t^\varrho * \mathrm{p}_{\varepsilon_2}(z) \Big\|_2^2 \\ &= \Big[m * \mathrm{p}_{t+\varepsilon_1}(z) - m * \mathrm{p}_{t+\varepsilon_2}(z) \Big]^2 \qquad (34) \\ &\qquad + 2 \int_0^t \mathrm{d}s \, \Big\langle \varrho_s \,,\, (m * \mathrm{p}_s) \big[\theta_z \mathrm{p}_{\varepsilon_1+t-s} - \theta_z \mathrm{p}_{\varepsilon_2+t-s} \big]^2 \Big\rangle, \end{aligned}$$

$t \geq 0$, $z \in \mathsf{R}^d$, $\varepsilon_1, \varepsilon_2 > 0$.

Fix a compact set $C \subset \mathbf{Z}^\varrho$ and $\delta > 0$ such that $C^\delta \subset \mathbf{Z}^\varrho$ (recall notation (28)). Further let $\tau := \sup\{(t: (t,z) \in C\}:$ Clearly, the first summand on the r.h.s. of (34) goes to 0 as $\varepsilon_1, \varepsilon_2 \to 0$, uniformly in $(t,z) \in C$.

We use the convention $\mathrm{p}_t := 0$ if $t < 0$. By (31) in Lemma 9, there exists a constant $c_3 < \infty$ such that

$$\begin{aligned} &\int_0^\infty \mathrm{d}s \, \Big\langle \varrho_s \,,\, (m * \mathrm{p}_s) \sup_{\substack{(t_1,z_1),(t_2,z_2) \in C \\ 0 < \varepsilon_1, \varepsilon_2 < \delta/2}} (\theta_{z_1} \mathrm{p}_{\varepsilon_1+t_1-s}) (\theta_{z_2} \mathrm{p}_{\varepsilon_2+t_2-s}) \Big\rangle \\ &\leq c_3^2 \int_0^{\tau+\delta} \mathrm{d}s \, \Big\langle \varrho_s \,,\, (m * \mathrm{p}_s)\, \phi_p^2 \Big\rangle. \end{aligned} \qquad (35)$$

By Lemma 10 the latter quantity is bounded by $c_4 \langle m, \phi_p \rangle < \infty$. Note that for all $(s,x) \in (C^\delta)^{\mathrm{c}}$,

$$\sup_{(t,z) \in C} \Big| \theta_z (\mathrm{p}_{\varepsilon_1+t-s} - \mathrm{p}_{\varepsilon_2+t-s})(x) \Big| \to 0 \quad \text{as} \quad \varepsilon_1, \varepsilon_2 \downarrow 0. \qquad (36)$$

If we combine (34), (35), and (36), the dominated convergence theorem yields that $(X_t^\varrho * \mathrm{p}_\varepsilon(z))_{\varepsilon > 0}$ is Cauchy in $L^2(P_{0,m}^\varrho)$ as $\varepsilon \downarrow 0$, uniformly in $(t,z) \in C$. Hence, the L^2-limit $\xi_t^\varrho(z)$, say, exists, and

$$\Big\| \sup_{(t,z) \in C,} X_t^\varrho * \mathrm{p}_\varepsilon(z) - \xi_t^\varrho(z) \Big\|_2 \to 0 \quad \text{as} \quad \varepsilon \downarrow 0. \qquad (37)$$

This proves **(a)** and **(b)**.

Since L^2–convergence implies L^1–convergence, $P^{\varrho}_{0,m}\xi^{\varrho}_t(z) = m*\mathrm{p}_t(z)$ follows from the expectation formula (22). Then also the covariance formula (7) can be derived from (23) and domination according to (35). This gives **(c)**.

Statement **(d)** is immediate (recall Proposition 6 (b)). ■

3 Proof of the theorem

In this section we prove Theorem 1. First we use Proposition 11 to show **(a)**, **(c)**, and **(d)**. Next we proceed similarly as in [Del96] to get the smoothness **(b)** of the density field. Delmas uses a representation of his catalytic SBM in terms of excursions started from his (time-independent) catalytic set. Our catalyst is *not* time-independent. However, it is not crucial in Delmas' argument to start the excursions from the catalyst. Our idea is to use a Delmas type representation of $X^{\varrho}_t(\mathrm{d}z)\,\mathbf{1}_{\mathbf{Z}^{\varrho}}\big((t,z)\big)$ on a local level with an occupation density measure Γ^{ϱ} concentrated on a nice set outside the catalyst.

Again we may assume that $r=0$. Recall that $d \le 3$.

(a) *Absolute continuity* of X^{ϱ} on $\mathbf{Z}^{\varrho}$ is immediate by the uniform convergence statement in Proposition 11. (An alternative argument for this fact will be given in the proof of part **(b)**.) Since $\xi^{\varrho}_t(z)$ has expectation $m*\mathrm{p}_t(z)$ on $\mathbf{Z}^{\varrho}_t$, and $\ell\big((\mathbf{Z}^{\varrho}_t)^{\mathrm{c}}\big) = 0$ in $d \ge 2$, we get the absolute continuity of X^{ϱ}_t on the whole space R^d by an exhaustion argument.

(c) (*moments*) This is Proposition 11 (c).

(d) (*local* $L^2(P^{\varrho}_{0,m})$*–Lipschitz continuity on* $\mathbf{Z}^{\varrho}$) We may assume that the compact set $C \subset \mathbf{Z}^{\varrho}$ is a closed box. Let $\delta > 0$ such that $C^{\delta} \subset \mathbf{Z}^{\varrho}$ (recall (28)). Set $\tau := \sup\{t:\ (t,z)\in C\}$, and let $(t_1,z_1),(t_2,z_2)\in C$ with $t_1 \le t_2$. From the moment formulas (6) and (7) we get

$$\big\|\xi^{\varrho}_{t_1}(z_1) - \xi^{\varrho}_{t_2}(z_2)\big\|_2^2 \;=\; I_1 + I_2 + I_3, \tag{38}$$

where

$$\begin{aligned} I_1 &:= \Big[m*\mathrm{p}_{t_1}(z_1) - m*\mathrm{p}_{t_2}(z_2)\Big]^2 \\ I_2 &:= 2\int_0^{t_1}\mathrm{d}s\,\Big\langle \varrho_s\,,\,(m*\mathrm{p}_s)\,\big[\theta_{z_1}\mathrm{p}_{t_1-s} - \theta_{z_2}\mathrm{p}_{t_2-s}\big]^2\Big\rangle \\ I_3 &:= 2\int_{t_1}^{t_2}\mathrm{d}s\,\Big\langle \varrho_s\,,\,(m*\mathrm{p}_s)\,\big[\theta_{z_2}\mathrm{p}_{t_2-s}\big]^2\Big\rangle. \end{aligned} \tag{39}$$

We use the bound of the partial derivatives of $\theta_z\,\mathrm{p}_{t-s}$ in Lemma 9 to derive the existence of a constant c^1 (depending only on C and ϱ) such that

$$I_2 \;\le\; \Big||t_1-t_2| + |z_1-z_2|\Big|^2\, c^1\int_0^{\tau}\mathrm{d}s\,\Big\langle \varrho_s\,,\,(m*\mathrm{p}_s)\Big[\sup_{(t,z)\in C}\theta_z\mathrm{p}^2_{2(t-s)}\Big]\Big\rangle.$$

By (31) and Lemma 10, this inequality can be continued with constants c^2, c^3 :

$$\leq \quad \left|(t_1, z_1) - (t_2\,, z_2)\right|^2 \, c^2 \int_0^\tau \mathrm{d}s \left\langle \varrho_s\,,\, (m * \mathrm{p}_s)\, \phi_p^2 \right\rangle$$

$$\leq \quad c^3 \left|(t_1, z_1) - (t_2\,, z_2)\right|^2.$$

Analogously we get the existence of c^4 such that

$$I_1 \;\leq\; c^4 \left|(t_1, z_1) - (t_2\,, z_2)\right|^2.$$

Similarly, we get $I_3 \leq c^5 \left|t_1 - t_2\right|^2$ for some constant c^5.

(b) (*smooth density field*) For the final part of proof we have the following strategy. We fix a cylinder $\mathcal{Z}$ contained in $\mathbf{Z}^\varrho$ and use the L^2–Proposition 11 to construct the occupation density measure Γ^ϱ of X^ϱ on the lateral area $\mathcal{A}$ of the cylinder. Next we use Delmas' representation of catalytic SBM in terms of Brownian excursions starting from $\mathcal{A}$ to derive the smoothness of ξ^ϱ in $\mathcal{Z}$.

Let ϱ be such that the assertions in Proposition 11 (a) – (c) and in Proposition 6 hold. Recall the characterization of $\mathbf{Z}^\varrho$ from Proposition 6 (a), and that $\mathbf{Z}^\varrho$ is open in $(0,\infty) \times \mathsf{R}^d$. Write $\overline{\ell}$ for the $(d-1)$–dimensional Lebesgue measure on the boundary $\partial B_\delta(z)$ of the open ball $B_\delta(z)$ around z of radius δ.

Fix $(t,z) \in \mathbf{Z}^\varrho$, and a $\delta > 0$ such that

$$[t-\delta, t+\delta] \times B_{2\delta}(z) \subset \mathbf{Z}^\varrho. \tag{40}$$

Define a measure Γ^ϱ on the *lateral area* $\mathcal{A} := [t-\delta, t+\delta] \times \partial B_\delta(z)$ of the *cylinder* $\mathcal{Z} := (t-\delta, t+\delta) \times B_\delta(z)$ by

$$\Gamma^\varrho(\mathrm{d}u, \mathrm{d}y) \;:=\; \mathrm{d}u \;\, \xi_u^\varrho(y)\, \overline{\ell}(\mathrm{d}y), \qquad (u,y) \in \mathcal{A}, \tag{41}$$

with ξ_u^ϱ from Proposition 11 (a).

First we show that Γ^ϱ is the *occupation density measure (super-local time)* of X^ϱ on $\mathcal{A}$. For this purpose, we define random measures $\Gamma_\varepsilon^\varrho$, $\varepsilon > 0$, on the lateral area $\mathcal{A}$, via their density functions

$$(u,y) \mapsto X_u^\varrho * \mathrm{p}_\varepsilon(y), \qquad (u,y) \in \mathcal{A},$$

with respect to the measure $\ell^+ \times \overline{\ell}$. The formal meaning of the statement that Γ^ϱ is the occupation density measure is that

$$\Gamma_\varepsilon^\varrho \quad \text{converges weakly to} \quad \Gamma^\varrho \quad \text{as } \varepsilon \downarrow 0, \quad P_{0,m}^\varrho\text{–a.s.} \tag{42}$$

To prove (42) it suffices to show that

$$\left\langle \Gamma_\varepsilon^\varrho, f \right\rangle \underset{\varepsilon\downarrow 0}{\longrightarrow} \left\langle \Gamma^\varrho, f \right\rangle, \quad P_{0,m}^\varrho\text{–a.s.}, \tag{43}$$

if f is a continuous function on $\mathcal{A}$. From the uniform convergence in Proposition 11 (b) we know that in $L^2\big(P^{\varrho}_{0,m}\big)$,

$$\begin{aligned}&\big\|\langle\Gamma^{\varrho}_{\varepsilon}, f\rangle - \langle\Gamma^{\varrho}, f\rangle\big\|_2\\ &\quad\le \int_{t-\delta}^{t+\delta} \mathrm{d}u \int_{\partial B_\delta(z)} \mathrm{d}y\ |f(u,y)|\ \Big\|X^{\varrho}_u * \mathrm{p}_\varepsilon(y) - \xi^{\varrho}_u(y)\Big\|_2 \underset{\varepsilon\downarrow 0}{\longrightarrow} 0,\end{aligned}$$

Hence, for every sequence $\varepsilon_n \downarrow 0$ as $n\uparrow\infty$, there exists a subsequence $\varepsilon_{n(k)}$ such that

$$\Big\langle \Gamma^{\varrho}_{\varepsilon_{n(k)}}, f\Big\rangle \underset{k\uparrow\infty}{\longrightarrow} \langle\Gamma^{\varrho}, f\rangle, \quad P^{\varrho}_{0,m}\text{–a.s.}$$

Since the mapping $\varepsilon \mapsto \langle\Gamma^{\varrho}_{\varepsilon}, f\rangle$, $\varepsilon>0$, is continuous, we have shown (43), and hence (42).

The aim is now to use Γ^{ϱ} to get a *representation* of ξ^{ϱ} as in Proposition 7.1 of [Del96] (see also Theorem 2 of [FL95]). This is Proposition 13 below. From this representation it is easily shown that ξ^{ϱ} is $\mathcal{C}^\infty$ and solves the heat equation (cf. Theorem 8.1 of [Del96]).

We start by introducing the ingredients of the representation formula. Recall that $(W,\Pi_{r,x})$ denotes the Brownian motion on R^d. Define the *exit time* $\tau^B := \inf\big\{s>0:\ W_s\notin B\big\}$ of the open ball $B := B_\delta(z)$, and the *exit density*

$$\mathrm{q}^B = \Big\{\mathrm{q}^B_t(x,y):\quad t>0,\ x\in B,\ y\in\partial B\Big\} \tag{44}$$

by

$$\Pi_{0,x}\, f\left(\tau^B, W_{\tau^B}\right) = \int_0^\infty \mathrm{d}t \int_{\partial B} \overline{\ell}(\mathrm{d}y)\, \mathrm{q}^B_t(x,y)\, f(t,y), \tag{45}$$

$f\in\mathcal{C}_b\left((0,\infty)\times\partial B\right)$, (that is f bounded and continuous). Clearly, for $y\in\partial B$ fixed, $(t,x)\mapsto \mathrm{q}^B_t(x,y)$ is of class $\mathcal{C}^\infty$ and solves the heat equation.

Fix $T>0$ and a compact set $D\subset B$. By a simple induction argument we derive from Lemma 9 that the partial derivatives of all orders are bounded, uniformly in $x\in D$, $y\in\partial B$, $t\in(0,T]$. Hence, for every finite measure ν on $(0,\infty)\times\partial B$, also the mixture

$$\nu\star\mathrm{q}^B(t,x) := \int_0^t\int_{\partial B} \nu(\mathrm{d}u,\mathrm{d}y)\ \mathrm{q}^B_{t-u}(x,y), \qquad t>0, \quad x\in B, \tag{46}$$

is of class $\mathcal{C}^\infty$ and solves the heat equation in $(0,\infty)\times B$.

Define the transition density $\mathrm{p}^B = \big\{\mathrm{p}^B_t(x,x'):\ t>0,\ \ x\in B,\ x'\in\mathsf{R}^d\big\}$ of *Brownian motion killed on* B^{c} :

$$\Pi_{0,x}\,\mathbf{1}_{\tau>t}\, f\left(W_t\right) = \int_B \mathrm{d}x'\ \mathrm{p}^B_t(x,x')\, f(x'), \qquad f\in\mathcal{C}_b\left(\mathsf{R}^d\right). \tag{47}$$

As above, $(t,x) \mapsto \mathrm{p}_t^B(x,x')$ is $\mathcal{C}^\infty$ and solves the heat equation. Further, for $n \in \mathcal{M}_p$, also the mixture

$$n * \mathrm{p}_t^B(x) := \int n(\mathrm{d}x')\, \mathrm{p}_t^B(x,x'), \qquad t>0, \quad x \in B, \tag{48}$$

is $\mathcal{C}^\infty$ and solves the heat equation.

Since $\nu \star \mathrm{q}^B$ and $n * \mathrm{p}^B$ are $\mathcal{C}^\infty$ and solve the heat equation, the same is true for ξ^ϱ by the following proposition. Hence, after verifying Proposition 13 below, the proof of Theorem 1 will be complete. ∎

The density of mass in the point (s,x) of the catalyst-free cylinder $\mathcal{Z} = (t-\delta, t+\delta)\times B$ decomposes in the reactant's mass $X_{t-\delta}^\varrho$ at time $t-\delta$ transported by the heat flow density p^B with absorption off $B = B_\delta(z)$, and the reactant's occupation density Γ^ϱ (recall (41)) at the lateral area $\mathcal{A}$ of $\mathcal{Z}$, transported by the excursion density q^B :

Proposition 13 (representation using smooth densities) *Let* $\mu, m \in \mathcal{M}_p$ *where* μ *is* η*–diffusive, and* $r \geq 0$. *Then for* $\mathbb{P}_\mu$*–almost all* ϱ *the following holds. Take* $(t,z) \in \mathbf{Z}^\varrho$, $t > r$. *Choose* $\delta \in (0, t-r)$ *such that the inclusion* (40) *holds. Then, with* $P_{r,m}^\varrho$*–probability one,*

$$\xi_s^\varrho(x) \;=\; X_{t-\delta}^\varrho * \mathrm{p}_{s-(t-\delta)}^B\,(x) \,+\, \Gamma^\varrho \star \mathrm{q}^B\left(s-(t-\delta), x\right), \qquad (s,x) \in \mathcal{Z}. \tag{49}$$

Proof Set $r=0$. We want to show the difference of both sides of (49) vanishes in $L^2(P_{0,m}^\varrho)$. Clearly,

$$P_{0,m}^\varrho \xi_s^\varrho(x) \;=\; m * \mathrm{p}_t(x) \;=\; P_{0,m}^\varrho \left[X_{t-\delta}^\varrho * \mathrm{p}_{s-(t-\delta)}^B\,(x) \,+\, \Gamma^\varrho \star \mathrm{q}^B\left(s-(t-\delta),x\right)\right],$$

thus, the expectation of that difference equals 0. Hence, it suffices to prove that the variance disappears. We use the covariance formulas (7) and (23) to deduce that

$$\begin{aligned}
&\mathrm{Var}_{0,m}^\varrho\left[\xi_s^\varrho(x) \,-\, X_{t-\delta}^\varrho * \mathrm{p}_{s-(t-\delta)}^B\,(x) \,-\, \Gamma^\varrho \star \mathrm{q}^B\left(s-(t-\delta),x\right)\right] \\
&\quad = 2\int_0^s \mathrm{d}s' \int \varrho_{s'}(\mathrm{d}x')\, m * \mathrm{p}_{s'}(x') \Big[\theta_x \mathrm{p}_{s-s'} \,-\, \mathbf{1}_{s'<t-\delta}\left(S_{t-\delta-s'}\, \mathrm{p}_{s-(t-\delta)}^B\right)(x') \\
&\qquad\qquad - \int_{s'\vee(t-\delta)}^s \mathrm{d}u \int_{\partial B} \overline{\ell}(\mathrm{d}y)\; \mathrm{p}_{u-s'}(y-x')\; \mathrm{q}_{s-u}^B\,(x,y)\Big]^2.
\end{aligned}$$

However, the integrand vanishes if $(s,x) \in \mathcal{Z}$ and $(s',x') \in \mathcal{Z}^{\mathrm{c}}$. In fact, we distinguish between the two cases whether the backward Brownian motion path leaves the cylinder $\mathcal{Z}$ at the base $\{t-\delta\}\times B$, or at the lateral area $\mathcal{A}$. This shows (49), hence the proof of the proposition is complete. ∎

Appendix: η–diffusive μ and collision local time

The purpose of this appendix is to show how to modify the argument in [DF97a] in order to get in the case of η–diffusive μ that for $\mathbb{P}_\mu$–a.a. ϱ the collision local time $L_{[W,\varrho]}$ exists and is a nice branching functional.

Consider (ordinary) SBM $(\varrho, \mathbb{P}_\mu)$. For $\varepsilon \in (0,1]$, introduce the continuous additive functional $L^{\varepsilon}_{[W,\varrho]}$ of W defined by $L^{\varepsilon}_{[W,\varrho]}(\mathrm{d}s) := \varrho_s * \mathrm{p}_\varepsilon(W_s)\,\mathrm{d}s$.

Proposition 14 (collision local time $L_{[W,\varrho]}$) *Let $d \le 3$ and $\mu \in \mathcal{M}_p$. If $d = 2,3$, assume additionally that μ is η–diffusive. Then, for $\mathbb{P}_\mu$–almost all ϱ, the collision local time $L_{[W,\varrho]}$ exists as a continuous additive functional of Brownian motion $(W, \Pi_{r,x})$ satisfying the following assertions:*

(a) (existence of $L_{[W,\varrho]}$) *Let $T > 0$, $\psi : [0,T] \times \mathsf{R}^d \to (0,+\infty)$ be continuous and uniformly dominated by ϕ_p. Then*

$$\Pi_{r,x} \sup_{r \le t \le T} \left| \int_r^t L^{\varepsilon}_{[W,\varrho]}(\mathrm{d}s)\,\psi(s,W_s) - \int_r^t L_{[W,\varrho]}(\mathrm{d}s)\,\psi(s,W_s) \right|^2$$

converges to 0 as $\varepsilon \downarrow 0$, uniformly on $(r,x) \in [0,T] \times \mathsf{R}^d$.

(b) (first two moments) *For $0 \le r < T$, $x \in \mathsf{R}^d$,*

$$\Pi_{r,x} \int_r^T L_{[W,\varrho]}(\mathrm{d}s)\,\psi(s,W_s) \;=\; \int_r^T \mathrm{d}s \int \varrho_s(\mathrm{d}y)\,\mathrm{p}_{s-r}(y-x)\,\psi(s,y),$$

$$\Pi_{r,x} \left[\int_r^T L_{[W,\varrho]}(\mathrm{d}s)\,\psi(s,W_s) \right]^2 \;=\; 2 \int_r^T \mathrm{d}s \int_s^T \mathrm{d}s' \int \varrho_s(\mathrm{d}y) \int \varrho_{s'}(\mathrm{d}y')$$
$$\mathrm{p}_{s-r}(y-x)\,\mathrm{p}_{s'-s}(y'-y)\,\psi(s,y)\,\psi(s',y'),$$

(c) (local admissibility) *For $r_0 \ge 0$,*

$$\sup_{x \in \mathsf{R}^d} \Pi_{r,x} \int_r^t L_{[W,\varrho]}(\mathrm{d}s)\,\phi_p(W_s) \;\underset{r,t \to r_0}{\longrightarrow}\; 0.$$

(d) (nice branching functional) *There exists a constant $c = c_{\varrho,T,\eta}$ such that*

$$\Pi_{r,x} \int_r^t L_{[W,\varrho]}(\mathrm{d}s)\,\phi_p^2(W_s) \;\le\; c\,|t-r|^{\eta}\,\phi_p(x),$$

$0 \le r \le t \le T, \quad x \in \mathsf{R}^d$.

Proof For $d = 1$ the statement is well known (see [DF97a, Corollary 3, p.261]). We only have to prove Proposition 14 for $d = 2,3$.

Theorem 4 of [DF97a] makes the same statement as this proposition but under different assumptions. Our notion of an η–diffusive initial state is new here. In fact, the proof of their Theorem 4 depends on their specific different assumption only via a statement concerning the Hölder continuity of the occupation time density, formulated in [DF97a, Theorem 3]. Below in Lemma 15 we give a new proof of the statement of that Theorem 3 under the assumption of η–diffusiveness. (Note that there is a misprint in [DF97a, Theorem 3, formula (4.29)]; the correct formulation is (A1) below.) This will conclude the proof. ■

Lemma 15 (Hölder continuity of occupation density) *Let* $d \le 3$, *and assume that* $\mu \in \mathcal{M}_p$ *is* η*–diffusive. Then, with respect to* $\mathbb{P}_\mu$, *there exists the occupation density field* $y = \{y_t(z) : t \ge 0, \ z \in \mathsf{R}^d\}$ *of* ϱ, *that is*

$$\mathbb{P}_\mu\biggl(\int_0^t \mathrm{d}s\, \varrho_s(\mathrm{d}z) \ = \ y_t(z)\,\mathrm{d}z \quad \textit{for all } t \ge 0\biggr) \ = \ 1,$$

which is Hölder continuous in the following sense. For $N \ge 1$,

$$\sup_{\substack{0 \le t_1, t_2 \le N \\ z_1, z_2 \in \mathsf{R}^d,\ |z_1 - z_2| \le N \\ [t_1, z_1] \ne [t_2, z_2]}} \frac{\bigl|y_{t_1}(z_1)\,\phi_p(z_1) \ - \ y_{t_2}(z_2)\,\phi_p(z_2)\bigr|}{\bigl|[t_1, z_1] - [t_2, z_2]\bigr|^\eta} \ < \ \infty, \ \ \mathbb{P}_\mu\text{–a.s.} \tag{A1}$$

Proof We follow the lines of the proof of [DF97a, Theorem 3, p.254]. We split up R^d into overlapping cells $C_n := [0, 2N]^d + nN$, $n \in \mathsf{Z}^d$. Now we take the supremum in (A1) separately for each n only over z_1 and z_2 in C_n :

$$C_{\eta,N,n} := \sup{}^n \ \frac{\bigl|y_{t_1}(z_1)\,\phi_p(z_1) \ - \ y_{t_2}(z_2)\,\phi_p(z_2)\bigr|}{\bigl|[t_1, z_1] - [t_2, z_2]\bigr|^\eta}\,, \tag{A2}$$

where $\sup^n$ denotes the supremum taken over

$$\Bigl\{0 \le t_1, t_2 \le N, \ \ z_1, z_2 \in C_n\,, \ \ [t_1, z_1] \ne [t_2, z_2]\Bigr\}. \tag{A3}$$

Using Chebyshev's inequality it is clear that it is enough to show that for some $k \ge 1$,

$$\sum_{n \in \mathsf{Z}^d} \bigl(\phi_p(n)\bigr)^{2k}\ \mathbb{P}_\mu(C_{\eta,N,n})^{2k} < \infty. \tag{A4}$$

Note that we have replaced here $\phi_p(z_1)$ and $\phi_p(z_2)$ by $\phi_p(n)$. This has no effect on the Hölder continuity that we want to prove since ϕ_p is (globally) Lipschitz and decays only algebraically fast.

By spatial homogeneity of SBM we have $\mathbb{P}_\mu(C_{\eta,N,n})^{2k} \ = \ \mathbb{P}_{\theta_{nN}\mu}(C_{\eta,N,0})^{2k}$. Note that (recall q from (2)) $\mathbb{P}_\mu\, y_t(z) \ = \ \int_0^t (\mathrm{p}_s * \mu)(z) \ = \ (\mathrm{q}_t * \mu)(z)$. Hence, from the assumption that μ is η–diffusive (see (4)), we get a bound for the $\mathbb{P}_\mu\, y_t(z)$:

$$\sup{}^n \ \frac{\bigl|\mathbb{P}_\mu\, y_{t_1}(z_1) - \mathbb{P}_\mu\, y_{t_2}(z_2)\bigr|}{\bigl|[t_1, z_1] - [t_2, z_2]\bigr|^\eta}\, \phi_{p'}(n) \ < \ \infty, \tag{A5}$$

for some $p' \in (d,p)$.

Consider the centered occupation times $Z_t(z) := y_t(z) - \mathbb{P}_\mu\, y_t(z)$. From [DF97a, last displayed equation in the proof of Theorem 2] we know that for $k > 2(d+1)/(1-4\eta)$ there exists a universal (i.e., independent of μ) constant $c = c(k)$ such that

$$\mathbb{P}_\mu \sup{}^n \left(\frac{\bigl|Z_{t_1}(z_1) - Z_{t_2}(z_2)\bigr|}{\bigl|[t_1,z_1] - [t_2,z_2]\bigr|^\eta} \right)^{2k} \;\le\; c\,(1 \vee \|\theta_{nN}\mu\|_p^{2k}). \tag{A6}$$

Combining (A5) and (A6) we get

$$\mathbb{P}_\mu (C_{\eta,N,n})^{2k} \;\le\; c'(k)\, \Bigl(\phi_{p'}(n)^{-2k} + \bigl(1 \vee \|\theta_{nN}\mu\|_p^{2k}\bigr)\Bigr). \tag{A7}$$

For $k > d/\bigl(2(p-p')\bigr)$ we have $\sum_{n\in\mathsf{Z}^d} \Bigl(\frac{\phi_p(n)}{\phi_{p'}(n)}\Bigr)^{2k} < \infty$. On the other hand

$$\sum_{n\in\mathsf{Z}^d} \phi_p(n)\,\|\theta_n\mu\|_p \;=\; \int \mu(\mathrm{d}a) \sum_n \phi_p(n)\,\phi_p(a-n).$$

But the series on n can be estimated from above by

$$2 \sum_{|n| \,\ge\, \frac{|a|}{2}} \phi_p(n)\,\phi_p(a-n) \;\le\; 2\,\phi_p(a/2) \sum_n \phi_p(n)$$

which is integrable with respect to $\mu(\mathrm{d}a)$. Hence we have shown (A4) and the proof is complete. ■

An important point is the fact that the property of η–diffusiveness for a certain $p > d$ is preserved under the dynamics of SBM. This is stated in the next proposition.

Proposition 16 (preservation of η–diffuseness) *Let $p > d$, $\eta \in (0, \frac14)$ and $\delta > 0$. If $\mu \in \mathcal{M}_p(\mathsf{R}^d)$ is η–diffusive, then $\mathbb{P}_\mu$–a.a. realizations ϱ_δ of SBM at time δ are also η–diffusive (with the same η and p).*

Proof By assumption there exists a $p' \in (d,p)$ such that μ is η–diffusive even w.r.t. p'. In particular $\mu \in \mathcal{M}_{p'}$, hence $\varrho_\delta \in \mathcal{M}_{p'}$ for $\mathbb{P}_\mu$–a.a. ϱ_δ. Thus we only have to check (4).

We adopt the notation from [DF97a] and write for $s \le t$ and $z \in \mathsf{R}^d$

$$y_{[s,t]}(z) \;:=\; y_t(z) - y_s(z).$$

From (A4) we know that for $k > (2d+1)/(1-4\eta)$ and $N > 0$,

$$\sum_{n\in\mathsf{Z}^d} \bigl(\phi_{p'}(n)\bigr)^{2k}\; \mathbb{P}_\mu \sup{}^n \left(\frac{\bigl|y_{[\delta,\delta+t_1]}(z_1) - y_{[\delta,\delta+t_2]}(z_2)\bigr|}{\bigl|[t_1,z_1] - [t_2,z_2]\bigr|^\eta} \right)^{2k} \;<\; \infty, \tag{A8}$$

where now $\sup^n$ is the abbreviation for the supremum taken over

$$\Big\{t_1, t_2 \in [0,N],\;\; z_1, z_2 \in n + [0,1]^d,\;\; [t_1, z_1] \neq [t_2, z_2]\Big\}.$$

However the expectations on the l.h.s. of (A8) equal (recall q from (2))

$$\begin{aligned}
&\mathbb{P}_\mu\, \mathbb{P}_\mu \left\{ \Big(\sup{}^n\, \frac{\big|y_{[\delta,\delta+t_1]}(z_1) - y_{[\delta,\delta+t_2]}(z_2)\big|}{\big|[t_1,z_1] - [t_2,z_2]\big|^\eta} \Big)^{2k} \,\Big|\, \varrho_\delta \right\} \\
&\geq\; \mathbb{P}_\mu \sup{}^n\, \mathbb{P}_\mu \left\{ \Big(\frac{\big|y_{[\delta,\delta+t_1]}(z_1) - y_{[\delta,\delta+t_2]}(z_2)\big|}{\big|[t_1,z_1] - [t_2,z_2]\big|^\eta} \Big)^{2k} \,\Big|\, \varrho_\delta \right\} \\
&\geq\; \mathbb{P}_\mu \sup{}^n\, \mathbb{P}_\mu \left\{ \Big(\frac{\big|y_{[\delta,\delta+t_1]}(z_1) - y_{[\delta,\delta+t_2]}(z_2)\big|}{\big|[t_1,z_1] - [t_2,z_2]\big|^\eta} \Big) \,\Big|\, \varrho_\delta \right\}^{2k} \\
&=\; \mathbb{P}_\mu \left(\sup{}^n\, \frac{\big|\mathrm{q}_{t_1} * \varrho_\delta(z_1) - \mathrm{q}_{t_2} * \varrho_\delta(z_2)\big|}{\big|[t_1,z_1] - [t_2,z_2]\big|^\eta} \right)^{2k}.
\end{aligned} \tag{A9}$$

Combining (A8) and (A9) we get that $\mathbb{P}_\mu$–a.s.

$$\sup_{n\in\mathbb{Z}^d} \sup{}^n\, \phi_{p'}(n)\, \frac{\big|\mathrm{q}_{t_1} * \varrho_\delta(z_1) - \mathrm{q}_{t_2} * \varrho_\delta(z_2)\big|}{\big|[t_1,z_1] - [t_2,z_2]\big|^\eta} \;<\; \infty. \tag{A10}$$

However this yields the claim. ■

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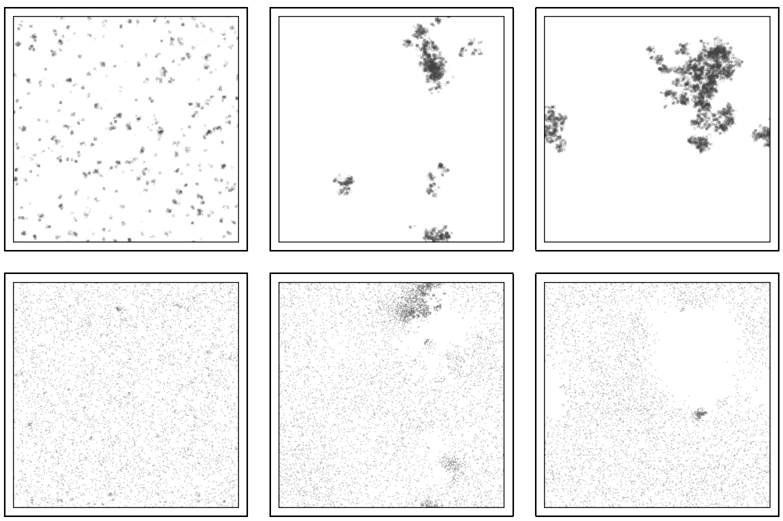
Simulation of the catalyst ϱ and the reactant X^ϱ



$t = 5$ $t = 900$ $t = 3500$

The first row shows a discrete version of ϱ (critical binary branching simple random walk) on a 250×250 grid with periodic boundaries, originally started from a "uniformly" distributed field. The number of particles per site is indicated by different grey scales. The "movie" clearly exhibits the well-known tendency of clustering in $d = 2$.

The second sequence of pictures shows a simulation of the analogous discrete version of X^ϱ, for the same realization of the branching rate ϱ. The figure illustrates that the reactant X^ϱ is uniformly spread out outside the catalytic clusters, except a few "hot spots" related to the catalyst, and that within the catalytic clumps mainly killing of the reactant happens.