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SMOOTH GROUP ACTIONS ON DEFINITE 4-MANIFOLDS AND MODULI SPACES

IAN HAMBLETON AND RONNIE LEE

In this paper we give an application of equivariant moduli spaces to the study of smooth group actions on certain 4-manifolds. A rich source of examples for such actions is the collection of algebraic surfaces (compact and nonsingular) together with their groups of algebraic automorphisms. From this collection, further examples of smooth but generally nonalgebraic actions can be constructed by an equivariant connected sum along an orbit of isolated points. Given a smooth oriented 4-manifold X which is diffeomorphic to a connected sum of algebraic surfaces, we can ask: (i) which (finite) groups can act smoothly on X preserving the orientation, and (ii) how closely does a smooth action on X resemble some equivariant connected sum of algebraic actions on the algebraic surface factors of X?

For the purposes of this paper we will restrict our attention to the simplest case, namely $X = P^2(\mathbf{C}) \# \cdots \# P^2(\mathbf{C})$, a connected sum of n copies of the complex projective plane (arranged so that X is simply connected). Furthermore,

ASSUMPTION.   *All actions will be assumed to induce the identity on $H_*(X, \mathbf{Z})$.*

In previous works [17], [18], [19], we considered problem (i) and a variant of problem (ii) when $X = P^2(\mathbf{C})$. It turned out that the only finite groups which could act as above on $P^2(\mathbf{C})$ were the subgroups of $PGL_3(\mathbf{C})$ ([18] and [23] independently). For problem (ii) there are 2 interesting notions weaker than smooth equivalence. If (X, π) is a smooth action, then the isotropy group $\pi_x = \{g \in \pi | gx = x\}$, $x \in X$, acts linearly on the tangent space $T_x X$ and we can ask the following.

Question (iii) a.   Given an action (X, π), is there an equivariant connected sum of actions on $P^2(\mathbf{C})$ with the same fixed point data and tangential isotropy representations?

Question (iii) b.   Given an action (X, π), is there an equivariant connected sum of actions on $P^2(\mathbf{C})$ which is π-homotopy equivalent or π-equivariantly homeomorphic to (X, π)?

Partial results were obtained on these questions in [17] and [10]: if π acts smoothly on $P^2(\mathbf{C})$, inducing the identity on homology, and the action has an

isolated fixed point, then $(P^2(\mathbf{C}), \pi)$ is equivariantly homeomorphic to a linear action. We observed in [17] that this is false in general for actions of noncyclic groups π on $P^2(\mathbf{C})$, due to the existence of actions with knotted 2-spheres in the singular set. There are related results for topological actions of cyclic groups on more general 4-manifolds in [9], [11], which will be summarized in §1, §2.

For the remainder of this paper, we will assume that $X = \#_1^n P^2(\mathbf{C})$ and concentrate on the case $n > 1$. Our main results are as follows.

THEOREM A. *Let π be a finite group acting smoothly on $X = \#_1^n P^2(\mathbf{C})$. If π induces the identity on homology, then π is isomorphic to a subgroup of $PGL_3(\mathbf{C})$. If in addition $n > 1$, then π is abelian of rank $\leqslant 2$.*

In the statement of the next result, we will let $[F_1], [F_2], \ldots, [F_r]$ denote the homology classes in X of the oriented 2-dimensional components of the fixed-point set, denoted by $\mathrm{Fix}(X, \pi)$, and $e_1, e_2, \ldots, e_n$ the standard basis of $H_2(X, \mathbf{Z})$ given by the complex embeddings $P^1(\mathbf{C}) \subset P^2(\mathbf{C})$ in each factor of the connected sum.

THEOREM B. *Let $\pi = \mathbf{Z}/p$ be a cyclic group of prime order acting smoothly on $X = \#_1^n P^2(\mathbf{C})$, inducing the identity on homology. Then the given orientation on X induces an orientation on $\mathrm{Fix}(X, \pi)$. Furthermore, there exist disjoint, nonempty subsets $S_1, S_2, \ldots, S_r$ of the set $\{1, 2, \ldots, n\}$ such that $[F_i] = \sum \{e_k | k \in S_i\}$ for $1 \leqslant i \leqslant r$. If $p = 2$, the subsets $S_1, S_2, \ldots, S_r$ form a partition of the set $\{1, 2, \ldots, n\}$.*

Here is a partial answer to Question (iii) a (see also [15], [16]).

THEOREM C. *Let (X, π) be a smooth cyclic group action on $X = \#_1^n P^2(\mathbf{C})$, inducing the identity on homology. Then there is an equivariant connected sum of actions of $P^2(\mathbf{C})$ with the same fixed-point data and tangential isotropy representations.*

By combining Theorem C with the results of Edmonds and Ewing [11], we obtain the following.

THEOREM D. *There exist locally-linear topological group actions (X, π) of cyclic groups on connected sums of $P^2(\mathbf{C})$, inducing the identity on homology, which are not smoothable.*

Before giving the proofs, we will use standard methods to collect some preliminary results about group actions on 4-manifolds. These include (§1) P. A. Smith theory, and (§2) the Atiyah-Singer index formula. Our new ingredient is in (§3), namely equivariant gauge theory [19].

1. P. A. Smith theory.   In this section we recall some results of A. Edmonds [9] and G. Bredon [4]. For convenience we have included proofs which apply to our special case. The idea is to study the Borel cohomology $H_\pi^*(X) = H^*(E\pi \times_\pi X, \mathbf{Z})$ to obtain information about the singular set. It is easy to see that the spectral sequence for the fibration $X \to X \times_\pi E\pi \to B\pi$ collapses [19, 6.1], [9, 2.3].

COROLLARY 1. *Every class in $H^2(X, \mathbf{Z})$ is $c_1(L)$ for some π-equivariant $U(1)$-bundle $L \to X$.*

COROLLARY 2. *If $\pi = \mathbf{Z}/\mathbf{p}$ (p prime), then the fixed point set $F(X, \pi)$ is a disjoint union of isolated points and 2-spheres.*

Proof. For $\pi = \mathbf{Z}/\mathbf{p}$, X is totally nonhomologous to 0 in X_π, and it follows that

$$(*) \qquad \sum_i \dim H^i(X, \mathbf{Z}/\mathbf{p}) = \sum_i \dim H^i(F, \mathbf{Z}/\mathbf{p}).$$

Since the action of π is orientation-preserving, $F = F(X, \pi)$ is the disjoint union of n_P isolated points $\{p_i\}$ and n_S surfaces $\{F_k\}$ of genus $g(F_k)$. If p is odd, the normal bundles to the surfaces F_k have a natural orientation from the complex structure and so each F_k is orientable. If $p = 2$, we can use the fact that $H^{2l}(X_\pi, \mathbf{Z}/2) \cong H^{2l}(F_\pi, \mathbf{Z}/2)$ for large l to see that Sq^1 is trivial on $H^1(F, \mathbf{Z}/2)$, implying again that each surface F_k is orientable (compare [4, page 379]).

Now the Euler characteristic

$$\chi(F) = \sum_k (2 - 2g(F_k)) + n_P = n + 2,$$

and $(*)$ implies that $\sum (2 + 2g(F_k)) + n_P = n + 2$, hence $g(F_k) = 0$ for all k. ∎

LEMMA 3. *If $\pi = \mathbf{Z}/\mathbf{p}$, the map $H^2(\pi) \to H^2(\partial_i(X - N)/\pi)$ induced by the classifying map of the π-covering is injective for some component $\partial_i(X - N)/\pi$ of $\partial(X - N)/\pi$, where N is a tubular neighbourhood of F in X.*

Proof. We have just seen that $2n_S + n_P = n + 2$. For $\pi = \mathbf{Z}/\mathbf{p}$ and $\mathbf{Z}/\mathbf{p}$-coefficients, there is an exact sequence

$$0 \to H^i(X_\pi) \to H^i(F_\pi) \to H^{i+1}_{cp}((X - F)/\pi) \to 0.$$

Since $\mathrm{rank}_{\mathbf{Z}/\mathbf{p}} H^2(X_\pi) = n + 1$, we get $H^3_{cp}((X - F)/\pi) = \mathbf{Z}/\mathbf{p}$. By Poincaré duality, $H_1((X - F)/\pi) = \mathbf{Z}/\mathbf{p}$, and the result is clear if $n_P \neq 0$ since the link of an isolated singular point in the quotient X/π is a lens space.

If $n_P = 0$, then $n_S \geq 2$ since we can assume $n > 1$. Then we claim that each 2-sphere component F_i of F has a nontrivial normal bundle in X, implying that the link of F_i in the quotient is a homology lens space, and we finish as before. If not, note first that suitable multiples (prime to p) of the circle fibres for the normal bundle in the quotient are homologous in $(X - F)/\pi$. Completing a 2-chain connecting these fibres by adjoining normal 2-disks in N produces a 2-cycle in X having a nontrivial intersection number with F_i. Therefore $0 \neq [F_i] \in H_2(X, \mathbf{Z})$. Since X has definite intersection form, the Euler class of the normal bundle is $[F_i] \cdot [F_i] \neq 0$. ∎

COROLLARY 4.   *If $\pi = \mathbf{Z}/\mathbf{p}$, then the restriction map $H^2(X, \mathbf{Z}/\mathbf{p}) \to H^2(F, \mathbf{Z}/\mathbf{p})$ is surjective. Then 2-dimensional components of F represent linearly independent elements in $H_2(X, \mathbf{Z}/\mathbf{p})$.*
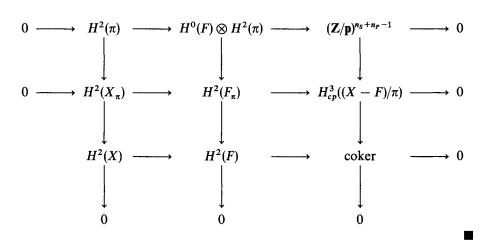
Proof.   Comparison of the spectral sequences for Borel cohomology of $F \subset X$ gives a commutative diagram

$$
\begin{array}{ccc}
H^0(F) \otimes H^2(\pi) & \longrightarrow & H^2(\partial(X - N)/\pi) \\
\downarrow & & \downarrow \\
H^2(F_\pi) & \longrightarrow & H^3_{cp}((X - F)/\pi) \; .
\end{array}
$$

From Lemma 3 it follows that the composition

$$
H^2(\pi) \to H^2(\partial(X - N)/\pi) \xrightarrow{\delta} H^3_{cp}((X - F)/\pi) = \mathbf{Z}/\mathbf{p}
$$

is surjective (for $\mathbf{Z}$ or $\mathbf{Z}/\mathbf{p}$ coefficients) and a diagram chase finishes the proof:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^2(\pi) & \longrightarrow & H^0(F) \otimes H^2(\pi) & \longrightarrow & (\mathbf{Z}/\mathbf{p})^{n_S + n_P - 1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^2(X_\pi) & \longrightarrow & H^2(F_\pi) & \longrightarrow & H^3_{cp}((X - F)/\pi) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H^2(X) & \longrightarrow & H^2(F) & \longrightarrow & \text{coker} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

∎

This result leads to the following question.

Question (iv).   *If F_i is an oriented 2-sphere component of $F = \mathrm{Fix}(X, \mathbf{Z}/\mathbf{p})$, what are the possibilities for $[F_i] \in H_2(X, \mathbf{Z})$ expressed in terms of the standard basis of $H_2(X, \mathbf{Z})$ given by the embeddings $P^1(\mathbf{C}) \subset P^2(\mathbf{C})$ in each factor?*

Partial answers were provided by [4, 7.4] when $p = 2$, and [9, 2.6] for any p, without assuming that the action induces the identity on homology or restricting the 4-manifold to be a connected sum of $P^2(\mathbf{C})$'s. Note that in these results, the action is not necessarily smooth. In the smooth case, by applying gauge theory we obtain a complete answer to Question (iv) in the setting of Theorem B.

2. Atiyah-Singer formulas. These formulas [1, §6] relate the tangential isotropy representations of $g \in \pi$ to the equivariant signature, when π acts locally-linearly on a 4-manifold X. If $p \in X$ is an isolated fixed point, and t denotes a fixed 1-dimensional faithful complex representation of $\langle g \rangle$, then $T_p(X) = t^a + t^b$ for some integers (a, b) relatively prime to $|g|$. If F_k is a 2-sphere in X fixed by g, then the normal bundle has the slice representation $\nu(F_k)|_x = t^{\theta_k}$, $x \in F_k$.

(i) For $|g|$ odd,

$$\text{Sign}\,(g, X) = n = -\sum_i \cot\left(\frac{a_i}{2}\right)\cot\left(\frac{b_i}{2}\right) + \sum_k \text{cosec}^2\left(\frac{\theta_k}{2}\right) \cdot [F_k]^2$$

where the sums correspond to the sets of isolated fixed points and fixed 2-spheres respectively.

(ii) For $|g| = 2$,

$$\text{Sign}\,(g, X) = F \circ F,$$

and in particular, isolated fixed points do not contribute.

A partial converse is provided by the following result of Edmonds and Ewing.

THEOREM 5 [11]. *If p is an (odd) regular prime, and the set of rotation numbers $\{(a_i, b_i)\}$ satisfies the Atiyah-Singer formula, then there exists a locally linear pseudofree topological $\mathbf{Z}/\mathbf{p}$-action on X with these representations at the isolated fixed points.*

In contrast, we will see in Theorem D that some of the actions constructed by Edmonds and Ewing are *not* smoothable, and so extra restrictions on the fixed-point data exist for smooth actions.

3. Equivariant moduli spaces. In [19, §2] we adapted the theory of Yang-Mills moduli spaces [7], [8] to the equivariant setting. Let $P \to X$ be a principal SU(2)-bundle and suppose that (X, π) is a smooth action with π finite. We fix a real analytic structure on X compatible with the π-action, and a real analytic equivariant metric. Then we constructed an equivariant perturbation of the Yang-Mills equations (using the method of Bierstone [2]) to construct a general position moduli space of self-dual connections $(\mathcal{M}, \pi)$ with an action of π. The main features are as follows.

(i) $(\mathcal{M}, \pi)$ is a Whitney stratified space [22] with an effective π-action and open manifold strata

$$\mathcal{M}^*_{(\pi')} = \left\{x \in \mathcal{M}^* | \pi_x = \pi' \subseteq \pi\right\},$$

where $\mathcal{M}^* \subseteq \mathcal{M}$ is the subset of irreducible connections (up to gauge equivalence).

(ii) For $\pi' \subseteq \pi$ each component of the fixed point set $\mathcal{M}^{\pi'}$ is the moduli space of $\tilde{\pi}'$-invariant connections on P with respect to a $\tilde{\pi}'$-SU(2) bundle structure on P (compare [14], [13]). Here $\tilde{\pi}'$ is the preimage of π' under the surjection $\mathcal{G}(\pi) \to \pi$, where $\mathcal{G}(\pi)$ is the group of SU(2)-bundle automorphisms of P which cover an isometry $g: X \to X$ given by the action of some element $g \in \pi$.

(iii) The strata have topologically locally trivial equivariant cone bundle neighbourhoods in $(\mathcal{M}, \pi)$.

(iv) $(\mathcal{M}, \pi)$ has an equivariant compactification $(\bar{\mathcal{M}}, \pi)$. When $c_2(P) = -1$, $(\bar{\mathcal{M}}, \pi)$ $= (\mathcal{M} \cup X \times [0, 1), \pi)$, where $\tau: X \times 0 \subset \bar{\mathcal{M}}$ is the Taubes embedding of X as the set of ideal "concentrated" connections and $X \times [0, 1)$ is embedded as an equivariant collar of the given π-action on X.

(v) The dimensions of the strata $\mathcal{M}_{(\pi')}$ can be computed from the π'-fixed set of the fundamental elliptic complex. A stratum $\mathcal{M}_{(\pi')}$ is nonempty only when its formal dimension is positive.

In [3], Bierstone gave a refinement of the definition of equivariant general position and showed that it is equivalent to a notion of equivariant stability or infinitesimal equivariant stability. We can modify our construction of equivariant moduli spaces to take this refinement into account. An advantage of this approach is that nearby perturbations result in smoothly equivalent stratified moduli spaces. Moreover, the internal structure of normal cone bundles along each stratum is now smoothly locally trivial.

We will consider the instanton number 1 moduli space (i.e., $c_2(P) = -1$) and discuss 4 additional features.

(a) Reducible connections. In general, reducible connections correspond to splittings $E = L \oplus L^{-1}$ of the associated $\mathbf{C}^2$-bundle to P, as a sum of line bundles. In our case, there are $n > 1$ reductions from line bundles $\{L_i\}$ with $c_1(L_i) = e_i$, $1 \leqslant i \leqslant n$, where $e_i \in H^2(X, \mathbf{Z})$ is the Poincaré dual of the ith standard basis element. By Corollary 1, we can assume that each L_i is a π-U(1) bundle over X, and so the associated reducible connection $[D_i]$ is fixed by the π-action on $\mathcal{M}$.

Next we consider neighbourhoods of a reducible connection $[D]$ in $\mathcal{M}$ and in $\mathcal{B}$. Recall that $[D]$ has a neighbourhood $\mathcal{U}^* \subset \mathcal{B}^*$ which is the weak homotopy type of $\mathbf{CP}^\infty$ [8, 5.1.18]. We may assume that $\mathcal{U}^*$ is π-invariant, and then for $\pi = \mathbf{Z}/\mathbf{p}$, p prime, it follows that the action $(\mathcal{U}^*, \mathbf{Z}/\mathbf{p})$ is weakly $\mathbf{Z}/\mathbf{p}$-equivariantly homotopy equivalent to a linear action on $T^*_{D,\varepsilon}/S^1$ where $T_{D,\varepsilon} \subset \mathcal{A}$ is a transverse slice to the $\mathcal{G}$-orbit.

LEMMA 6. *If $\pi = \mathbf{Z}/\mathbf{p}$, then $\mathrm{Fix}(\mathcal{U}^*, \mathbf{Z}/\mathbf{p})$ is a disjoint union of p subspaces $\mathcal{U}^*(\chi)$, each weakly homotopy equivalent to $\mathbf{CP}^\infty$, associated to the decomposition of $T_{D,\varepsilon}$ into complex linear subspaces $T_{D,\varepsilon}(\chi)$ on which $\mathbf{Z}/\mathbf{p}$ acts via a fixed character $\chi: \pi \to S^1$. In addition, if $\mathcal{F}_0$ is a component of $\mathrm{Fix}(\mathcal{U}^*, \mathbf{Z}/\mathbf{p})$, then the restriction map $H^*(\mathcal{U}^*, \mathbf{Z}/\mathbf{p}) \to H^*(\mathcal{F}_0, \mathbf{Z}/\mathbf{p})$ is surjective.*

Finally, we have the following result of Donaldson.

PROPOSITION 7 [8, 5.1.21].   *For any $a \in H_2(X . \mathbf{Z})$, the restriction of $\mu(a)$ to the copy of $\mathbf{CP}^\infty$ which links the reducible connection $[D_i]$ is given by*

$$\mu(a)|_{\mathbf{CP}^\infty} = - \langle c_1(L_i), a \rangle \cdot h$$

where $h \in H^2(\mathbf{CP}^\infty, \mathbf{Z})$ is the positive generator, and $\mu \colon H_2(X, \mathbf{Z}) \to H^2(\mathscr{B}^*, \mathbf{Z})$ is the map defined in [8, 5.1.11]. The composition

$$H_2(X, \mathbf{Z}) \overset{\mu}{\to} H^2(\mathscr{B}^*, \mathbf{Z}) \overset{\iota^*}{\to} H^2(X, \mathbf{Z})$$

is Poincaré duality [8, 5.3.3].

(b) *Orientation.*   Donaldson shows that the real determinant line bundle $\Lambda(P)$ associated to the elliptic complex

$$\Omega^*_\pm(\mathrm{Ad}\ P) \colon 0 \to \Omega^0(\mathrm{Ad}\ P) \to \Omega^1(\mathrm{Ad}\ P) \to \Omega^2_-(\mathrm{Ad}\ P) \to 0$$

has a canonical trivialization over $\mathscr{B}$ which induces the given orientation on X times the inward pointing normal, where X is embedded as the Taubes boundary in $\mathscr{M}$ [8, §5.4].

LEMMA 8.   *Let $\pi = \mathbf{Z}/\mathbf{p}$ for p an odd prime, and $C \subset \mathscr{M}^*_{(\pi)}$ a connected component. Then C is an orientable manifold.*

Proof.   If $[D] \in C$, we can split the elliptic complex

$$\Omega^*_\pm(\mathrm{Ad}\ P) = \Omega^*_\pm(\mathrm{Ad}\ P)^\pi \oplus [\Omega^*_\pm(\mathrm{Ad}\ P)^\pi]^\perp$$

into a fixed subcomplex and a perpendicular complex. It follows that the line bundle $\Lambda(P) = \Lambda_{t(C)} \otimes \Lambda_{n(C)}$, where $\Lambda_{t(C)}$ is the determinant line bundle of $\Omega^*_\pm(\mathrm{Ad}\ P)^\pi$ and $\Lambda_{n(C)}$ is for the complementary part of $\Omega^*_\pm(\mathrm{Ad}\ P)$. Since π is odd-order cyclic, the action of π on $[\Omega^*_\pm(\mathrm{Ad}\ P)^\pi]^\perp$ induces a complex structure and hence an orientation on $\Lambda_{n(C)}$. Then $\Lambda_{t(C)}$ is orientable also. However, the moduli space is locally modelled on the zero set of a smooth map $f \colon H^1_D \to H^2_D$ in Bierstone general position. Furthermore, the fixed set $(\mathrm{coker}\ df_0)^\pi = 0$ and $\ker df_0$ (which is fixed under π) is the tangent space to the manifold stratum C at $[D]$. Therefore C is orientable.   ∎

(c) *Connectedness.*   Since our equivariant moduli space $(\mathscr{M}, \pi)$ is constructed by general position, it is unique only up to a suitable notion of cobordism. In particular, we can try to arrange for it to be as connected as possible.

LEMMA 9.   *For each reducible connection $[D]$, there exists an equivariant neighbourhood $(\mathscr{U}^*, \pi) \subset (\mathscr{B}^*, \pi)$ such that the closure of the free stratum in $\mathscr{M}^* \cap \mathscr{U}^*$ contains all singular strata of dimension <5. Moreover, for each singular point, the intersection of its link with the free stratum is connected.*

Proof. Let $\Sigma \subseteq \partial \mathscr{U} \cap \mathscr{M}^*$ be the link of $[D]$ in the closure of the free stratum of $(\mathscr{M}, \pi)$. We will identify $\mathscr{U} \cap \mathscr{M} \approx \phi^{-1}(0)/S^1$, where $\phi: H_D^1 \to H_D^2$ is a local chart for the moduli space. Here the spaces H_D^i are complex π representations and ϕ is equivariant with respect to the $\pi \times S^1$ actions. Note that the S^1-action inducing the complex structure is free except at the origin. Then $\Sigma = \tilde{\Sigma}/S^1$ is the quotient space of the intersection $\tilde{\Sigma}$ of a sphere (of small radius) in H_D^1 with $\phi^{-1}(0)$.

By the index formula, the dimension of the free stratum equals 5. For a singular stratum in $\phi^{-1}(0)/S^1$, its normal cone structure can be determined by applying Bierstone general position to $\phi: H_D^1 \to H_D^2$ with respect to the isotropy subgroup $\pi' \subset \pi$ of the stratum. From this it follows that the closure of the free stratum is the union of the free stratum with singular strata of dimension < 5. It remains to show that the closure of the free stratum is connected in a neighbourhood of the reducible connection $[D]$.

Since the strata have even codimension, if p, q are in distinct connected components of Σ, we may assume that p lies in the free stratum and that there is an arc ℓ_{pq} joining p and q, such that $\ell_{pq}(t)$ lies *in the free stratum* of Σ for $0 \leqslant t < 1$. Now we can assume that the image of the path $\ell(t)$ under ϕ is a simple closed path such that the interior points, $\ell(t)$ for $0 < t < 1$, lie in the free stratum of H_D^2/S^1. Since the π-action H_D^1 is the cone over an action on the unit sphere, we can join p and q radially to the origin, and this closed path in H_D^1/S^1 bounds an embedded 2-disk B whose interior lies in the free stratum. Then we deform $\phi = \phi_0$ in H_D^2 through general position maps ϕ_s, $0 \leqslant s \leqslant 1$, so that $\phi_1(B) = 0$ and the new zero set $\phi_1^{-1}(0)/S^1 \approx \phi_0^{-1}(0)/S^1 \cup B$ in a neighbourhood of the origin. Note that the above construction can be made in the smooth category. Since a Bierstone general position map is determined by its jets up to some level n_0, i.e., by all its partial derivatives $(\partial/\partial x^{i_1} \cdots x^{i_k})\phi$ of degree $\leqslant n_0$, it follows that the closure of the free stratum is connected. ∎

(d) Domination and links of reducibles. After the last section we can assume that the link $\Sigma \subseteq \partial \mathscr{U} \cap \mathscr{M}^*$ of a reducible connection $[D]$ inside the closure of the free stratum is connected. In order to obtain information about its topology, we can compare our equivariant moduli space to the nearby generic moduli spaces without group action. By integrating a suitable vector field in a compact subset of a local chart, we can construct a continuous map $f: P^2(\mathbf{C}) \to \Sigma$ such that $f_*: H_4(P^2(\mathbf{C}), \mathbf{Z}) \to H_4(\Sigma, \mathbf{Z})$ is degree 1. Here $P^2(\mathbf{C})$ is the link of $[D]$ in a generic 5-dimensional smooth moduli space near $(\mathscr{M}, \pi)$, and Σ carries a 4-dimensional class because the fixed-point sets have codimension at least 2.

LEMMA 10.   *The link Σ has the homotopy type of a 4-dimensional Poincaré complex.*

Proof. By general position, the free stratum of Σ is a manifold, but there could be nonmanifold points in the singular set. Let $p \in \Sigma$ be an isolated fixed point, with link $L = \ell k(p)$ in Σ. Since p is isolated, the π action on $\ell k(p)$ is free and L is a connected 3-manifold by Lemma 9. However, L is dominated by a

degree-1 map from S^3, and so L is a homotopy 3-sphere and Σ is a homology manifold around p. A similar argument deals with neighbourhoods of points which lie on 2-dimensional fixed sets. ∎

COROLLARY 11. Σ *is homotopy equivalent to* $P^2(\mathbf{C})$.

Proof. The Poincaré duality space Σ is the quotient space of a free S^1 action on $\tilde{\Sigma}$. But by the same domination argument, there is a degree-1 map $S^5 \to \tilde{\Sigma}$. Therefore $\tilde{\Sigma}$ is homotopy equivalent to S^5 and $H_2(\Sigma) = \mathbf{Z}$. Now the existence of the degree-1 map $f\colon P^2(\mathbf{C}) \to \Sigma$ implies that Σ has the integral homology of $P^2(\mathbf{C})$, and $\pi_1(\Sigma) = 1$. ∎

4. The proof of Theorem A (1). We will prove that our equivariant moduli space $(\mathcal{M}, \pi)$ has nice neighbourhoods around each reducible connection. First we need some preparation.

LEMMA 12. *Let* $C \subseteq \mathcal{M}^*_{(\pi')}$ *be a noncompact connected component with* $\pi' \neq 1$. *If* $\dim C \geqslant 5$, *then* C *must be empty. If* $\dim C = 3$, *then the closure* $\overline{C} \subset \mathcal{M}$ *must intersect the Taubes boundary* $\partial \mathcal{M} = X \times 0 \subset \mathcal{M}$.

Proof. If $\dim C \geqslant 5$, by general position (or more specifically the existence of normal cone bundle neighbourhoods to the strata) the closure $\overline{C} \subset \mathcal{M}$ does not intersect the closure of the free stratum except at reducible connections. If $\dim C = 3$, we will assume that $\overline{C}$ also has only reducibles as its limit points and derive a contradiction.

Choose a nontrivial subgroup $\mathbf{Z}/\mathbf{p} \subseteq \pi'$, and put $(\mathcal{M}, \mathbf{Z}/\mathbf{p})$ in general position with respect to the action restricted to $\mathbf{Z}/\mathbf{p} \subset \pi$. Then the deformed stratum C' is a manifold stratum, fixed under $\mathbf{Z}/\mathbf{p}$, and $\overline{C'} = C' \cup \{\text{reducibles}\}$. Let $[D_1], \ldots, [D_r] \in \overline{C'}$ be the set of reducible limit points of $\overline{C'}$, and let W denote the compact manifold with boundary obtained from $\overline{C'}$ by removing small open neighbourhoods of the $[D_i]$. Then W has odd dimension, say $2d + 1$, $d \geqslant 1$, and has 1 boundary component $\partial_i W \approx P^d(\mathbf{C})$ for each reducible $[D_i]$ in the closure of C'.

Bierstone general position implies that the inclusion of each link $P^d(\mathbf{C}) \subset \mathcal{B}^*$ induces a weak homotopy equivalence with the link of $[D_i]$ in $\mathcal{B}^*$ through dimensions $\leqslant 2d$. If $\pm e_i \in H_2(X, \mathbf{Z})$ is the Poincaré dual of $c_1(L_i)$, where L_i is the line bundle associated to the reduction, then

$$\mu(e_i)|_{\partial_j W} = -\langle c_1(L_j), e_i \rangle \cdot h_j = \pm \delta_{ij}$$

where $h_j \in H^2(\partial_j W, \mathbf{Z}) \cong H^2(P^d(\mathbf{C}), \mathbf{Z})$ is the positive generator. For any cocycle representative $\tilde{\mu}(e_i)$ of $\mu(e_i)$ we have

$$\langle \tilde{\mu}(e_i)^d, [\partial W] \rangle = \langle \delta(\tilde{\mu}(e_i)^d), [W] \rangle \equiv 0 \qquad \text{mod } 2.$$

But since

$$\langle \mu(\widetilde{e_i})^d, [\partial W]\rangle = \sum_{j=1}^{r} \langle \mu(e_i)^d, [\partial_j W]\rangle = \langle h_i^d, [P^d(\mathbf{C})]\rangle \equiv 1 \qquad \mod 2,$$

we have a contradiction. ∎

Recall that $\Sigma \subseteq \partial \mathcal{U} \cap \mathcal{M}^*$ denotes the link of a reducible connection $[D]$ inside the closure of the free stratum. As in §4 (c), we identify $\mathcal{U} \approx H_D^1/S^1$, and $\mathcal{U} \cap \mathcal{M} \approx \phi^{-1}(0)/S^1$, using a local chart map $\phi: H_D^1 \to H_D^2$ in general position. Consider the decomposition of $V = H_D^1$ into isotypical π-subspaces $V = \bigoplus V(\chi_i)$, where each $V(\chi_i)$ is a direct sum of copies of the same irreducible complex representation of π. There is a similar decomposition for $W = H_D^2$, and we further write $V = V_0 \oplus V'$, $W = W_0 \oplus W'$, where $V_0 = \ker df_0$ and $W_0 = \operatorname{coker} df_0$ are determined by the differential (at zero) of a general position map $f: V \to W$. We may assume V_0 and W_0 have no irreducible representations in common [19, 1.11]. Let $\deg(\chi)$ denote the degree of a complex representation χ, and $m_0(\chi)$ its multiplicity as a constituent of V_0.

COROLLARY 13.   *If χ_j is an irreducible complex constituent of V_0, then $m_0(\chi_j) \cdot \deg(\chi_j) \leqslant 3$, with equality holding if and only if $\deg(\chi_j) = 3$, $m_0(\chi_j) = 1$, and χ_j is faithful as a projective representation of π into $PGL_3(\mathbf{C})$.*

Proof. If χ_j is an irreducible complex constituent of V_0, then by general position there is a nonempty stratum component in $\mathcal{M}^*$ of (real) dimension $\geqslant 2m_0(\chi_j) \deg(\chi_j) - 1$ with minimal isotropy subgroup equal to the minimal isotropy subgroup occurring in a complex projective space of (real) dimension $2d = 2 \deg(\chi_j) - 2$ on which π acts via the associated projective representation of χ_j into $PGL_d(\mathbf{C})$.

If χ_j is a faithful projective representation of π, then its minimal isotropy group is the trivial group. Since every component of the free stratum of the moduli space has dimension 5 (from the usual Atiyah-Hitchin-Singer index calculation [19, page 28]), we conclude that $2m_0(\chi_j) \deg(\chi_j) - 1 \leqslant 5$. If equality holds, then the solution $m_0(\chi_j) = 3$ and $\deg(\chi_j) = 1$ would produce a 3-dimensional projectively trivial subrepresentation of V_0, and hence an open subset of $\mathcal{M}^*$ on which π acted trivially. This is ruled out by the effectiveness of the π-action on $\mathcal{M}$, which is a consequence of our assumption that the original equivariant metric on X was chosen to be real analytic. (Recently [20], S. Illman has shown that every smooth compact group action admits an equivariant analytic structure.)

On the other hand, if χ_j is not faithful, its minimal isotropy group is the kernel $\pi' \neq 1$ of the representation. Then, by Lemma 12, any connected component of $\mathcal{M}^*_{(\pi')}$ has dimension $\leqslant 3$ and so $m_0(\chi_j) \cdot \deg(\chi_j) \leqslant 2$. ∎

It will also be important for us to estimate how many singular strata intersect Σ. For each irreducible complex constituent χ_j of V_0, let $\Sigma(\chi_j)$ denote the intersection of Σ with the complex projective subspace $\mathbb{P}(\chi_j) = S(V(\chi_j))/S^1 \subset \mathcal{U}$, and call this the singular stratum in Σ associated to $V(\chi_j)$. Note that points $x \in \operatorname{Fix}(\Sigma, \pi)$ correspond to π-invariant 1-dimensional complex linear subspaces of V_0, so

that these singular strata contain all the points of $\text{Fix}(\Sigma, \pi)$. Let $d_j = m_0(\chi_j) \cdot \deg(\chi_j) - 1$.

LEMMA 14. *There are at most 3 nonempty singular strata $\Sigma(\chi_j)$ of real dimension $2d_j$ in the link Σ, and $\sum (d_j + 1) \leqslant 3$ where we sum over all irreducible complex constituents χ_j of V_0.*

Proof. We note that each nonempty $\Sigma(\chi_j)$ has the integral homology of a projective space, of real dimension $2d_j$, where $0 \leqslant d_j \leqslant 2$ by Corollary 13. Moreover, if $d_j \geqslant 1$, then the inclusion induces an isomorphism $H_2(\Sigma(\chi_j), \mathbf{Z}) \to H_2(\Sigma, \mathbf{Z})$. The argument is similar to the proof of Corollary 11: each singular stratum is the orbit space of a simplicial complex homotopy equivalent to a sphere under a free S^1-action.

Since the different subspaces $\mathbb{P}(\chi_j)$ are all disjoint, so are the different $\Sigma(\chi_j)$. This implies that if $\Sigma(\chi_j) \neq \varnothing$ and $d_j = 2$, then $\Sigma = \Sigma(\chi_j)$ has just 1 singular stratum.

Next, $d_j = 1$ for some constituent χ_j of V_0 would imply the existence of a π-invariant S^2 in Σ. But the nontrivial self-intersection number of the generator of $H_2(\Sigma, \mathbf{Z})$ implies that all other singular strata $\Sigma(\chi_i)$, $i \neq j$ must have $d_i = 0$.

We will finish the proof by considering 2 cases.

Case (i). Suppose that the $k + 1$ nonempty singular strata all have $d_j = 0$. In other words, $\text{Fix}(\Sigma, \pi)$ consists of $k + 1$ distinct points, and we need to show that $k \leqslant 2$. Let $Y = \Sigma - \{x_1\}$, where $x_1 \in \Sigma$ denotes 1 of the fixed points.

Then Y is a finite simplicial π-complex homotopy equivalent to S^2, and the π-action has k fixed points corresponding to distinct (nontrivial) linear representations χ_j of π, each a complex constituent of V_0 with multiplicity 1. Since the free stratum of Σ is dense, the representation V_0 must be projectively faithful. For a sum of linear characters, this can only happen if π is abelian.

Pick an element $1 \neq g \in \pi$ of prime order p. Then by the P. A. Smith theory, $\text{Fix}(Y, g)$ is a $\mathbf{Z}/\mathbf{p}$-homology sphere containing $\text{Fix}(Y, \pi)$. It follows that $\text{Fix}(Y, g)$ is either an S^2 or consists of 2 distinct points. Since this fixed set is π-invariant, we obtain $k = 2$ by an easy inductive argument.

Case (ii). Suppose that one of the singular strata, $\Sigma(\chi_j)$, has $d_j = 1$, and the other $(k - 1)$ strata have $d_i = 0$ corresponding to distinct linear representations χ_j of π, each a complex constituent of V_0 with multiplicity 1. Let Y be obtained from Σ by collapsing the invariant 2-sphere $\Sigma(\chi_j)$ to a point; then Y is a finite simplicial π-complex homotopy equivalent to S^4, and the π action has $k \geqslant 1$ distinct points in $\text{Fix}(Y, \pi)$.

Consider a composition series $\pi_0 \lhd \pi_1 \lhd \cdots \lhd \pi_r = \pi$ for π, where each π_i/π_{i-1} is simple or solvable. Assume by induction that $\text{Fix}(Y, \pi_{i-1})$ is either a 2-sphere or $\leqslant 2$ distinct points. If π_i/π_{i-1} is solvable, then we show as in Case (i) that $\text{Fix}(Y, \pi_i)$ is either a 2-sphere or $\leqslant 2$ distinct points. If π_i/π_{i-1} is simple, the trivial representation is the only representation of π_i/π_{i-1} with degree 1, and so $\text{Fix}(Y, \pi_i)$ contains at most 2 points. ∎

The following result was proved for π a cyclic 2-group in [5], [6], using other techniques.

THEOREM 15. *Let π act smoothly on X, inducing the identity on homology. Then for each reducible connection $[D]$, there is a π-invariant neighbourhood of $[D]$ in $(\mathcal{M}, \pi)$ which is smooth away from the cone point and π-homeomorphic to the cone of a linear action on $P^2(\mathbf{C})$. These 2 stratified spaces are equivariantly diffeomorphic away from the cone point.*

Proof. By Lemma 12 we may now assume that $\dim \mathcal{M}^*_{(\pi')} < 5$ for each stratum with nontrivial isotropy group π'. Furthermore, by Lemma 10 within a local chart we can assume that there is a neighbourhood $\mathcal{U} \subset \mathcal{B}$ of each reducible $[D]$ such that $\mathcal{U}^* \cap \mathcal{M}^*$ is connected. Our local model for $\mathcal{M}^*$ near $[D]$ is determined by a $\pi \times S^1$ equivariant map $f \colon H^1_D \to H^2_D$ in general position. As above, we write $V = H^1_D$, $W = H^2_D$ and decompose $V = V_0 \oplus V'$, $W = W_0 \oplus W'$, where $V_0 = \ker df_0$ and $W_0 = \operatorname{coker} df_0$.

By Lemma 14 there are at most 3 singular strata in the link Σ of $[D]$ in $\mathcal{M}^*$. Moreover, $\sum (d_j + 1) \leqslant 3$ implies that $V_0 \subseteq V$ has at most 3 summands in its decomposition into isotypical components, and $\dim V_0 \leqslant 6$. However, by the index calculation, $\dim H^1_D - \dim H^2_D = 6$ so $W_0 \subseteq W$ must be trivial, and it follows that $\mathcal{M}$ is locally modelled on a transverse equivariant map near each reducible. ∎

Part 1 of Theorem A is now proved: the link of a reducible is the cone on some linear π-action on $P^2(\mathbf{C})$ and in particular, $\pi \subset PGL_3(\mathbf{C})$.

5. The proof of Theorem A (2), and Theorem B. Suppose now that $X = \#^n_1 P^2(\mathbf{C})$ with $n \geqslant 2$. Theorem B is completed by Theorem 15 and the following result. Note that in the case of involutions ($p = 2$), the equivariant link $(P^2(\mathbf{C}), \pi)$ of each reducible contains a fixed 2-sphere [18, 2.1]. This accounts for the last assertion in the statement of Theorem B.

THEOREM 16. *Let $\pi = \mathbf{Z}/\mathbf{p}$, p prime.*

(i) *Each 2-dimensional component $F_i \subset \operatorname{Fix}(X, \mathbf{Z}/\mathbf{p})$ has $[F_i] = \sum a_{ik} e_k$ where $a_{ik} = 0, 1$ and (for each i) not all the a_{ik} are 0.*

(ii) *No nonempty collection $\{F_i\} \subset X = \partial\overline{\mathcal{M}}$ bounds a fixed set in $\mathcal{M}^* \cup \partial\overline{\mathcal{M}}$.*

(iii) *No nonempty collection of 2-dimensional fixed sets in the links of the reducible connections bounds a compact fixed set in $\mathcal{M}^*$.*

Proof. Parts (i) and (ii) are proved by similar arguments to those above, using the μ-map. Note that each noncompact 3-dimensional component of $\operatorname{Fix}(\mathcal{M}^*, \mathbf{Z}/\mathbf{p})$ is canonically oriented since its closure in $\overline{\mathcal{M}}$ meets $\partial\overline{\mathcal{M}} = X$ and the set of reducibles. But the Donaldson orientation on $\mathcal{M}$ induces the standard orientation of $P^2(\mathbf{C})$ times the inward normal on the link $\Sigma = P^2(\mathbf{C})$ of each reducible, and the π-action on Σ is complex linear, so the 2-dimensional fixed set in the links repre-

sent the homology classes $P^1(\mathbf{C}) \subset P^2(\mathbf{C})$. The compatibility of this orientation with that of X specifies the sign in the expression for $[F_i] \in H_2(X, \mathbf{Z})$. Part (iii) follows from Lemma 12. ∎

The next result will be used in the proofs of Theorems C and D.

LEMMA 17. *If $\pi = \mathbf{Z}/\mathbf{p}$, p an odd prime, then each noncompact 1-dimensional fixed set in $\mathcal{M}^*$ has at least 1 reducible connection as a limit point in $\mathcal{M}$.*

Proof. Let γ denote the closure in $\overline{\mathcal{M}}$ of a 1-dimensional fixed stratum of $\mathcal{M}^*$. Then if γ contains no reducible connections, its endpoints $\{p_1, p_2\}$ lie in $X = \partial\overline{\mathcal{M}}$. Suppose that (a_1, b_1) and (a_2, b_2) denote the rotation numbers at p_1 and p_2 respectively. Then clearly (a_1, b_1) and (a_2, b_2) are a cancelling pair, or in other words we can assume that $(a_2, b_2) = (-a_1, b_1)$.

To rule out this pattern, we recall that a fixed-point component, such as the arc γ, corresponds to a π-equivariant structure on our original SU(2)-bundle P over X (compare [14]). Also recall that this bundle admits a concentrated connection at the point p_1 by means of Taubes construction. An equivariant interpretation of this last connection allows us to describe explicitly this π-bundle (π, P) structure as follows.

Consider a small invariant 4-ball $B(p_1)$ around p_1. By collapsing the complement $X - B(p_1)$ of this 4-ball to a point, we obtain a 4-sphere $S^4 = \mathbf{H} \cup \{\infty\}$ with a linear action and an equivariant, degree-1 map $g_1: X \to S^4$ sending p_1 to the north pole, $g_1(p_1) = 0$, and sending the complement $X - B(p_1)$ to the south pole, $g_1(X - B(p_1)) = \infty$. Locally near the north pole, the group action is given by the rotation numbers (a_1, b_1) and near the south pole by $(-a_1, b_1)$. As described in [21, Theorem 3], there exists an equivariant SU(2)-bundle Q over S^4 whose Chern number $c_2(Q) = -1$ and whose rotation number $(\pi, Q|_0)$ at the north pole is $\{(a_1 + b_1), -(a_1 + b_1)\}$ and at the south pole is $\{(-a_1 + b_1), -(-a_1 + b_1)\}$. Pulling back Q by means of g_1, we obtain the desired equivariant bundle $(\pi, g_1^*(Q))$ structure on P.

Note that at any point $p_3 \in \text{Fix}(X, \pi)$, with $p_3 \neq p_1, p_2$, the rotation numbers of the pull-back equivariant bundle $(\pi, g_1^* Q)$ are the same as at the south pole $\{(-a_1 + b_1), -(-a_1 + b_1)\}$. Now the same pull-back-bundle construction can be repeated using an invariant 4-ball $B(p_2)$ around p_2, a corresponding degree-1 map $g_2: X \to S^4$, and the pull-back $(g_2)^*(Q')$ of an equivariant bundle Q' over S^4. This time, however, the rotation numbers of $(\pi, g_2^*(Q'))$ at p_3 are $\{(-a_2 + b_2), -(-a_2 + b_2)\}$.

In order that there exists an arc γ of self-dual equivariant connections connecting the 2 concentrated connections at p_1 and p_2, these 2 equivariant bundles $(\pi, g_1^*(Q))$ and $(\pi, g_2^*(Q'))$ must be the same. Therefore the rotation numbers $\{(-a_1 + b_1), -(-a_1 + b_1)\}, \{(-a_2 + b_2), -(-a_2 + b_2)\}$ at p_3 must also agree. Since (a_1, b_1) and (a_2, b_2) represent a cancelling pair, it follows that either $2a_1 \equiv 0 \bmod |\pi|$ or $2b_1 \equiv 0 \bmod |\pi|$. This is impossible because $|\pi|$ is odd and the action (π, X) has isolated fixed points at p_1 and p_2. ∎

We can now complete the proof of Theorem A, part 2 by considering the symmetries of the singular set patterns in $\mathscr{M}$ under the action of π. By the first part of Theorem A, we can assume that $\pi \subset PGL_3(\mathbf{C})$ and that the action is a cone over some linear π-actions on $P^2(\mathbf{C})$ around each reducible connection.

To prove part 2 we refer to the list given in [18, §1] for the various families of finite subgroups of $PGL_3(\mathbf{C})$. We must eliminate (i) nonabelian subgroups of $U(2)$, (ii) subgroups of the Hessian group, and (iii) the simple groups A_5, A_6, and $PSL_2(\mathbb{F}_7)$. These cases are handled by the following 3 lemmas: Lemma 18 for types (B)–(D), Lemma 19 for types (E)–(G), and Lemma 20 for types (H)–(J). Note that A_5, A_6 contain A_4, and $PSL_2(\mathbb{F}_7)$ contains a dihedral group of order 14.

LEMMA 18.   *If π contains a normal cyclic subgroup π' on which the conjugation action of π is nontrivial and sends each element $g \mapsto g^{-1}$, for $g \in \pi'$, then π does not act smoothly and effectively on $X = \#_1^n P^2(\mathbf{C})$, inducing the identity on homology, for $n > 1$.*

Proof.   The linear actions of π on $P^2(\mathbf{C})$ restricted to the cyclic normal subgroup π' are of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta^a & 0 \\ 0 & 0 & \zeta^{-a} \end{bmatrix}$$

where a is relatively prime to the order of the subgroup. Therefore these actions have 3 isolated points for the π' action on the link of each reducible connection in $\mathscr{M}$. It follows easily that at least 2 distinct reducibles must be connected by a 1-dimensional stratum of the fixed point set of π' intersecting the 2 linking copies of $P^2(\mathbf{C})$ at points p and q, say. However, since this pattern has an extra $\mathbf{Z}/2$ symmetry, the rotation numbers at p and q must be of the form $(a, -a)$ and $(b, -b)$. Moreover, these rotation numbers at p and q must be identified under an *orientation-reversing* π'-equivariant homeomorphism, and this is impossible since $a \not\equiv 0$.   ∎

LEMMA 19.   *If $\pi = \mathbf{Z}/3 \times \mathbf{Z}/3$ acts smoothly on X inducing the identity on homology, then the π-action on $\mathscr{M}$ has 3 isolated fixed points on the link of each reducible connection.*

Proof.   The other possible pattern for the singular set of π on $P^2(\mathbf{C})$ has 3 isolated fixed points for each $\mathbf{Z}/3$ subgroup, but no fixed points for π [18, 2.2]. Again we observe that at least 2 distinct reducible connections must be joined by a 1-dimensional fixed set for the $\mathbf{Z}/3$-action. However, this pattern has no additional symmetry and so the π-action does not exist.   ∎

LEMMA 20.   *The group $\pi = A_4$ does not act smoothly on X inducing the identity on homology.*

Proof. We consider the singular set for the action of the subgroup $\mathbf{Z}/2 \times \mathbf{Z}/2 \subset \pi$. By Lemma 17 we can conclude that at least 2 reducibles are again connected by a 1-dimensional fixed set in $\mathscr{M}$, and this pattern has no extra $\mathbf{Z}/3$ symmetry. ∎

Proof of Theorem C. First we suppose that $\pi = \mathbf{Z}/\mathbf{p}$ and consider the fixed-point set $\mathrm{Fix}(\mathscr{M}, \pi)$, which contains $\mathrm{Fix}(X, \pi)$. By Theorem 15 we can assume that each reducible $[D]$ has an equivariant neighbourhood in $\mathscr{M}$ which is a cone over a linear π-action on $P^2(\mathbf{C})$. Each such cone has a fixed set consisting of either 3 distinct lines or a line and a cone over a 2-sphere. In addition, these components are disjoint except at the cone point $[D]$ and correspond to different isotropy subgroups of the $\pi \times S^1$ action in the local model.

Case (i). Assume that the $\mathbf{Z}/\mathbf{p}$-action has only isolated fixed points. Then by Theorem 16 the same is true for the linear actions linking the reducibles. We regard the reducibles $[D_i]$ and the distinct points $x_j \in \mathrm{Fix}(X, \pi)$ as the *vertices* of a graph. Two such vertices are connected by an *edge* if there is an equivalence class of paths in $\mathrm{Fix}(\mathscr{M}, \pi)$ connecting them. We consider such paths to be *equivalent* if they lie in the same component of $\mathrm{Fix}(\mathscr{M}, \pi)$ in every small open neighbourhood of the vertices in $\mathscr{M}$. The vertices $[D_i]$ have valence 3 (connected to exactly 3 others), and the vertices $\{x_j\}$ have valence 1, by Lemma 17. Since there are $n + 2$ fixed points in X and only n reducibles, at least 1 reducible must be joined to 2 vertices in X.

On the other hand, if $n > 1$ it is not possible for 1 reducible to be joined to 3 points in X, since then the remaining $n - 1$ reducibles would be connected by $n - 1$ edges. This would produce a closed path, say γ, of edges and γ would intersect 1 of the cones in 2 lines ℓ_1, ℓ_2 of fixed points with distinct isotropy groups $\pi(\ell_1) \neq \pi(\ell_2)$ in the $\pi \times S^1$ action of the local model. But an irreducible $[D] \in \mathscr{B}^\pi$ corresponds to a lifting of the action on X to a π-SU(2) bundle structure on P, and the S^1 determinant line bundles over the strata containing ℓ_1, ℓ_2 are both "positively oriented" by the complex structure on H_D^1. This contradicts the fact that the oriented determinant line bundle extends over γ, and shows that such a closed path γ does not exist.

We can now count the edges and conclude that the reducibles may be ordered $[D_1], \ldots, [D_n]$ so that there is an edge $e(i, j)$ connecting $[D_i]$ and $[D_j]$ if and only if $j = i + 1$, for $1 \leqslant i \leqslant (n - 1)$. The remaining edges connect $[D_1]$ and $[D_n]$ to 2 fixed points each in X, and $[D_i]$, $2 \leqslant i \leqslant (n - 1)$, to 1 fixed point each in X. This is exactly the pattern of the equivariant connected sum of linear actions. From this we can also conclude (from the normal bundles to $\mathrm{Fix}(\mathscr{M}^*, \pi)$ as usual) that the tangential representations at fixed points in X are the same as those in an equivariant connected sum of linear actions.

Case (ii). When the $\mathbf{Z}/\mathbf{p}$ action has 2-dimensional components in $\mathrm{Fix}(X, \pi)$ the analysis is similar, but the definition of the graph must be modified. The vertices as before are the reducible connections together wth the components of

Fix(X, π). The edges now have weights: $w(e) = 1$ if the path lies in a 1-dimensional fixed-point set and $w(e) = 2$ otherwise. Now the valence (the weighted sum of the edges) of each reducible is still 3. Consider one of the sets S_i of reducibles given by Theorem B. If k is the number of elements in S_i, and $n > k$, then 1 of the reducibles in S_i must be joined by an edge of weight 1 to a reducible not in S_i. Otherwise, the $n - k$ remaining reducibles are joined by edges with weighted sum $n - k$, and a loop γ would exist as before. The same contradiction eliminates this possibility, and we can conclude that the subgraph with vertices the reducibles is connected. An edge count shows that we again have the fixed-point pattern for an equivariant connected sum of linear actions. The tangential representations are determined by the normal bundles to the fixed sets as before.

Case (iii).   Suppose now that π is a cyclic group, and $\mathbf{Z}/\mathbf{p}$ is a subgroup of π. If the $\mathbf{Z}/\mathbf{p}$-action has isolated fixed points, then the π-action is pseudofree, and Theorem C follows in this case. If the $\mathbf{Z}/\mathbf{p}$-action has 2-dimensional fixed sets, the group π acts on Fix($\mathscr{M}$, $\mathbf{Z}/\mathbf{p}$) with fixed set Fix($\mathscr{M}$, π). Again, it follows that this fixed-point pattern is the same as that of an equivariant connected sum of linear actions, and the proof of Theorem C is complete.   ∎

6. A nonsmoothable action.   In [11] a semifree action of $\pi = \mathbf{Z}/5$ on $X = P^2(\mathbf{C}) \# P^2(\mathbf{C})$ was constructed. Start with the linear actions on $P^2(\mathbf{C})$ having rotation numbers $\{(2, 3), (3, 1), (-1, 2)\}$ and $\{(1, 2), (-1, 1), (-1, 3)\}$. After the equivariant connected sum at the cancelling pair $(-1, 2)$, $(1, 2)$, we obtain a smooth action on X with rotation numbers

$$\{(2, 3), (3, 1), (-1, 1), (-1, 3)\}.$$

In particular, these values satisfy the Atiyah-Singer formula. Now, replacing the cancelling pair $(3, 1)$, $(-1, 3)$ by the cancelling pair $(1, 1)$, $(1, -1)$, and applying Theorem 5, the result is a topological, locally linear action with rotation numbers

$$\{(2, 3), (1, 1), (-1, 1), (1, -1)\}$$

at the 4 isolated fixed points.

Such an action cannot arise as an equivariant connected sum of actions on $P^2(\mathbf{C})$ since the fixed-point data $(1, 1)$ would imply the existence of 2-spheres in the fixed-point set. Since the action induces the identity on homology, the Euler characteristic of the fixed point set is 4, and so the action is pseudofree.

THEOREM 21.   *There is no smooth action of $\pi = \mathbf{Z}/5$ on $P^2(\mathbf{C}) \# P^2(\mathbf{C})$ which induces the identity on homology and has isolated fixed points with tangential representations given by $\{(1, 1), (1, 4), (1, 4), (2, 3)\}$.*

Proof.   The equivariant moduli space $\mathscr{M}$ has 2 singular points $[D_1]$, $[D_2]$ arising from reducible connections. By Lemma 17, the 4 isolated fixed points in X

provide 4 1-dimensional fixed point sets in $\mathscr{M}$, and each of these has either $[D_1]$ or $[D_2]$ as its closure. By Theorem 15, the moduli space is locally just the cone on some linear actions on $P^2(\mathbf{C})$ near the $[D_i]$. Since the fixed arcs from X intersect the linking $P^2(\mathbf{C})$ in isolated fixed points, these linear actions have 3 isolated fixed points. However, the equivariant normal bundles to these fixed arcs in $\mathscr{M}$ are product bundles, and this implies that the slice representation $(1, 1)$ would appear in a linear action on $P^2(\mathbf{C})$, a contradiction. ∎

Remark 22. A. Edmonds has informed us that it is possible to produce a wide variety of rotation number configurations as the number of copies of $P^2(\mathbf{C})$ in the connected sum increases. The idea is to find "cancelling cyles" of rotation numbers, instead of just cancelling pairs. Combining this work with Theorem 5 and Theorem C would probably produce many other nonsmoothable locally linear actions of $\mathbf{Z}/\mathbf{p}$ on connected sums of $P^2(\mathbf{C})$.

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