ANNALES DE L'INSTITUT FOURIER

F. DUMORTIER
ROBERT ROUSSARIE

Smooth linearization of germs of R^2-actions and holomorphic vector fields

Annales de l'institut Fourier, tome 30, nº 1 (1980), p. 31-64
<http://www.numdam.org/item?id=AIF_1980__30_1_31_0>

Ann. Inst. Fourier, Grenoble
30, 1 (1980), 31-64

SMOOTH LINEARIZATION OF GERMS OF R^2-ACTIONS AND HOLOMORPHIC VECTOR FIELDS

by F. DUMORTIER (*) and R. ROUSSARIE

The paper contains a generic condition permitting the linearization in class C^k, $0 \leqslant k \leqslant \infty$ of germs of singular infinitesimal $\mathbf{R}^2$-actions on $\mathbf{R}^n$ $(n \geqslant 2)$ and of singular holomorphic vector fields on C^n $(n \geqslant 1)$. It generalizes a similar result of S. Sternberg for germs of singular (real) vector fields on $\mathbf{R}^n$. An announcement has been made in [5].

The problem was put to us by J. Palis during a stay in the Instituto de Matemática Pura e Aplicada in Rio de Janeiro. We like to thank that institution for its hospitality.

1. Definitions.

DEFINITION 1.1. – *A germ in* $0 \in \mathbf{R}^n$, $n \geqslant 2$, *of a singular infinitesimal* $\mathbf{R}^2$*-action of class* C^∞ *is a Lie algebra homomorphism* $\rho : \mathbf{R}^2 \longrightarrow \mathfrak{G}^n$, *where* $\mathfrak{G}^n$ *denotes the Lie algebra of germs in* $0 \in \mathbf{R}^n$ *of* C^∞ *vector fields vanishing in* 0. *We denote by* $\mathrm{Act}(\mathbf{R}^2, \mathbf{R}^n)$ *the set of germs of singular infinitesimal* $\mathbf{R}^2$*-actions of class* C^∞ *on* $\mathbf{R}^n$. *For more information about* $\mathfrak{G}^n$ *see* [10]. *Once we have chosen a basis in* $\mathbf{R}^2$, *such an infinitesimal action can be identified with a pair of germs in 0 of commuting vector fields i.e.* $(\rho(e_1), \rho(e_2))$, *where* (e_1, e_2) *denotes the chosen basis, and* $[\rho(e_1), \rho(e_2)] = 0$; $[\cdot, \cdot]$ *denotes the Lie bracket. To each* $\rho \in \mathrm{Act}(\mathbf{R}^2, \mathbf{R}^n)$, *we*

(*) "Aangesteld Navorser" of the "Nationaal Fonds voor Wetenschappelijk Onderzoek" of Belgium during part of the preparation of this paper.

can associate by integration a germ of an $\mathbf{R}^2$*-action on* $\mathbf{R}^n$: $\widetilde{\rho} : \mathbf{R}^2 \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$ *where (after taking representatives of the germs)* $\widetilde{\rho}(t, s ; x) = Y_t \circ X_s(x)$ *with* $X = \rho(1,0)$, $Y = \rho(0,1)$ *and* Z_t *denotes the flow of* Z.

DEFINITION 1.2. – *We say that* $\rho_1, \rho_2 \in \mathrm{Act}(\mathbf{R}^2, \mathbf{R}^n)$ *are* C^k*-conjugate,* $0 \leqslant k \leqslant \infty$ *if there exists a germ* g *in* $0 \in \mathbf{R}^n$ *of a* C^k*-diffeomorphism preserving the origin such that*

$$\widetilde{\rho}(u, g(x)) = g \circ \widetilde{\rho}_2(u, x) \ \forall \ (u, x) \in \mathbf{R}^2 \times \mathbf{R}^n$$

(in the neighbourhood of $(0,0)$ *for representatives of the germs). If* $k \geqslant 1$, *this condition is equivalent to* $g * \rho_2 = \rho_1$.

Restricting to ℓ-jets of vector fields we can define the ℓ-jet $j^\ell\rho(0)$, $1 \leqslant \ell \leqslant \infty$, of each $\rho \in \mathrm{Act}(\mathbf{R}^2, \mathbf{R}^n)$; we denote by $J^\ell \mathrm{Act}(\mathbf{R}^2, \mathbf{R}^n)$ the set of ℓ-jets of singular infinitesimal actions. These sets are naturally endowed with a structure of algebraic set.

DEFINITION 1.3. – *We say that* $\alpha \in J^\ell \mathrm{Act}(\mathbf{R}^2, \mathbf{R}^n)$ *is "determining in class* C^k*" if all* $\rho_1, \rho_2 \in \mathrm{Act}(\mathbf{R}^2, \mathbf{R}^n)$ *with*

$$j^\ell\rho_1(0) = j^\ell\rho_2(0) = \alpha$$

are C^k*-conjugate. Identifying* $\mathbf{C}$ *with* $\mathbf{R}^2$ *and* $\mathbf{C}^n$ *with* $\mathbf{R}^{2n}$, *we can consider each germ in* $0 \in \mathbf{C}^n$ *of a singular holomorphic vector field as a germ of a singular infinitesimal* $\mathbf{R}^2$*-action on* $\mathbf{R}^{2n}$. *We denote by* $\mathrm{Act}(\mathbf{C}, \mathbf{C}^n)$ *the set of germs in 0 of these singular holomorphic vector fields. The inclusion of* $\mathrm{Act}(\mathbf{C}, \mathbf{C}^n)$ *in* $\mathrm{Act}(\mathbf{R}^2, \mathbf{R}^{2n})$ *permits to restrict to germs of singular holomorphic vector fields the notions of conjugacy in class* k *and the notion of determining* ℓ*-jet. Let us remark that once chosen a coordinate system for* $\mathbf{R}^n$ *each 1-jet* $\alpha \in J^1 \mathrm{Act}(\mathbf{R}^2, \mathbf{R}^n)$ *can be identified with a linear action (meaning an action* $\alpha \in \mathrm{Act}(\mathbf{R}^2, \mathbf{R}^n)$ *such that* $\alpha(u)$ *is a linear vector field for each* $u \in \mathbf{R}^2$*).*

A linear action α is C^k-determining if and only if each $\rho \in \mathrm{Act}(\mathbf{R}^2, \mathbf{R}^n)$ with $j^1\rho(0) = j^1\alpha(0)$ is C^k-linearizable. The same remark can be made for holomorphic vector fields.

DEFINITION 1.4. – *A 1-jet* $\alpha \in J^1 \mathrm{Act}(\mathbf{R}^2, \mathbf{R}^n)$ *or a linear infinitesimal singular* $\mathbf{R}^2$*-action is called "hyperbolic" if there exists* *a basis* (e_1, e_2) *of* $\mathbf{R}^2$ *and a coordinate system* L *in* $\mathbf{R}^n$ *such that* $X = \alpha(e_1)$ *and* $Y = \alpha(e_2)$ *are hyperbolic, diagonalized over* $\mathbf{C}$ *in* L *and with respective eigenvalues*

$$\lambda_1, \ldots, \lambda_p, \lambda_{p+1} \pm i\beta_{p+1}, \ldots, \lambda_{p+\ell} \pm i\beta_{p+\ell} \quad \text{for } X$$

and

$$\lambda'_1, \ldots, \lambda'_p, \lambda'_{p+1} \pm i\beta'_{p+1}, \ldots, \lambda'_{p+\ell} \pm i\beta'_{p+\ell} \quad \text{for } Y$$

with all $\lambda, \lambda', \beta, \beta' \in \mathbf{R}$ *and satisfying the following conditions:*

$$\forall j \in \langle p+1, \ldots, p+\ell \rangle : \lambda'_j + i\beta'_j \notin \mathbf{R}(\lambda_j + i\beta_j) \ (\Longleftrightarrow \beta'_j/\beta_j \neq \lambda'_j/\lambda_j)$$

$$\forall j, k \in \langle 1, \ldots, p+\ell \rangle \quad \text{and} \quad j \neq k : \lambda'_j/\lambda_j \neq \lambda'_k/\lambda_k .$$

We recall that a linear vector field is called hyperbolic if all its eigenvalues have real part different from zero.

Remarks. – 1) The basis (e_1, e_2) in Definition 1.4 can be chosen so that all eigenvalues of X as well as of Y are pairwise different.

2) A linear infinitesimal $\mathbf{R}^2$-action α is hyperbolic according to Definition 1.4 if and only if the induced linear action $\widetilde{\alpha} : \mathbf{R}^2 \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is hyperbolic as in [2].

That paper contains a lot of information about hyperbolic $\mathbf{R}^k \times \mathbf{Z}^\ell$-actions. The non-hyperbolic 1-jets of infinitesimal actions form a semi-algebraic subset of codimension 1 in $J^1 \operatorname{Act}(\mathbf{R}^2, \mathbf{R}^n)$.

If we endow $J^1 \mathfrak{g}^n$ with the natural euclidean topology and $J^1 \operatorname{Act}(\mathbf{R}^2, \mathbf{R}^n)$ with the induced compact-open topology, then the set of hyperbolic 1-jets is open and dense in $J^1 \operatorname{Act}(\mathbf{R}^2, \mathbf{R}^n)$, [2].

A 1-jet of a singular holomorphic vector field Z is hyperbolic in the sense of Definition 1.4 if and only if no two eigenvalues of Z are dependent over the reals i.e. $\lambda_i \notin \mathbf{R}\lambda_j \ \forall i \neq j$. The 1-jets of non-hyperbolic holomorphic vector fields also form a semi-algebraic set of codimension 1 in $J^1 \operatorname{Act}(\mathbf{C}, \mathbf{C}^n)$, endowed with its structure of real algebraic set.

DEFINITION 1.5. – *Let* $(k_1, \ldots, k_n) \in \mathbf{N}^n$ *with* $k_1 + \ldots + k_n \geq 2$. *We say that* $\alpha \in J^1 \operatorname{Act}(\mathbf{R}^2, \mathbf{R}^n)$ *satisfies the resonance condition of order* $(k_1, \ldots, k_n)$ *if* $\forall u \in \mathbf{R}^2$ *there exists an arrangement* $\mu_1, \ldots, \mu_n$

of the eigenvalues of $\alpha(u)$, *counted with their multiplicity such that:*

$$\mu_1 - \sum_{i=1}^{n} k_i \mu_i = 0 .$$

Let us denote by $S(k_1, \ldots, k_n)$ the set of 1-jets of infinitesimal actions which are either non-hyperbolic or which satisfy the resonance condition of order $(k_1, \ldots, k_n)$. This set is a semi-algebraic set of codimension 1 in $J^1 \operatorname{Act}(\mathbf{R}^2, \mathbf{R}^n)$, as well as the set

$$S_k = \bigcup_{k_1+\ldots+k_n \leqslant k} S(k_1, \ldots, k_n)$$

defined for $k \geqslant 2$.

We for instance prove the semi-algebraicity on the 1-jets in Jordan form and then apply the theorem of Tarski-Seidenberg [10]. We denote $S_\infty = \bigcup_k S_k$.

The complement of S_∞ in $J^1 \operatorname{Act}(\mathbf{R}^2, \mathbf{R}^n)$ is a residual subset, a countable intersection of open and dense subsets (hence dense).

For $\alpha \in J^1 \operatorname{Act}(\mathbf{C}, \mathbf{C}^n)$ Definition 1.5 is equivalent to say that α satisfies the resonance condition of order $(k_1, \ldots, k_n)$ if there exists an arrangement $\mu_1, \ldots, \mu_n$ of the eigenvalues of α such that $\mu_1 - \sum_{i=1}^{n} k_i \cdot \mu_i = 0$.

We denote by $\Sigma(k_1, \ldots, k_n)$ the semi-algebraic set of codimension 1 of 1-jets of holomorphic vector fields which are not hyperbolic or satisfy the resonance condition of order $(k_1, \ldots, k_n)$. $\Sigma_k = \bigcup_{k_1+\ldots+k_n \leqslant k} \Sigma(k_1, \ldots, k_n)$, $k \geqslant 2$ and $\Sigma_\infty = \bigcup_k \Sigma_k$.

The complement of each Σ_k in $J^1 \operatorname{Act}(\mathbf{C}, \mathbf{C}^n)$ is open and dense, while the complement of Σ_∞ is residual.

2. Statement of the results and some comments.

THEOREM A. — *Let* $\alpha \in J^1 \operatorname{Act}(\mathbf{R}^2, \mathbf{R}^n) \setminus S_\infty$ *(respectively* $\alpha \in J^1 \operatorname{Act}(\mathbf{C}, \mathbf{C}^n) \setminus \Sigma_\infty$), *then* α *is determining in class* C^∞.

In other words: if a germ ρ of a singular infinitesimal $\mathbf{R}^2$ action on $\mathbf{R}^n$ or a germ ρ of a singular holomorphic vector field on $\mathbf{C}^n$ has a hyperbolic 1-jet whose eigenvalues do not satisfy any of the resonance conditions, then ρ is C^∞ linearizable.

We also obtain a theorem on C^k-linearization for hyperbolic $\mathbf{R}^2$ and $\mathbf{C}$-infinitesimal actions in the absence of certain resonances.

THEOREM B. – *Let* $\alpha \in J^1 \operatorname{Act}(\mathbf{R}^2, \mathbf{R}^n)$ *(respectively* $\alpha \in J^1 \operatorname{Act}(\mathbf{C}, \mathbf{C}^n)$). *Then there exists a neighbourhood* U *of* α *in* $J^1 \operatorname{Act}(\mathbf{R}^2, \mathbf{R}^n)$ *(resp. in* $J^1 \operatorname{Act}(\mathbf{C}, \mathbf{C}^n)$) *such that for all* ℓ, $0 \leqslant \ell < \infty$, *we can find* $k < \infty$ *(depending on* ℓ *and* U) *for which all* β, $\beta \in U \setminus S_k$ *(resp.* $\beta \in U \setminus \Sigma_k$) *are determining in class* C^ℓ.

Theorem B implies that a germ of a singular $\mathbf{R}^2$-action (resp. of a singular holomorphic vector field) is linearizable in class C^k for some $k \geqslant 0$ (hence topologically linearizable) as soon as its 1-jet belongs to some open and dense subset in $J^1 \operatorname{Act}(\mathbf{R}^2, \mathbf{R}^n)$ (resp. $J^1 \operatorname{Act}(\mathbf{C}, \mathbf{C}^n)$).

This open and dense subset is defined by the condition of hyperbolicity and locally near α by a finite number of non-resonance conditions between the eigenvalues depending on k and α. A precise description of this open and dense subset –which is smaller than the set of all hyperbolic actions– is in study.

This result is related with the results of Camacho, Kuiper and Palis in [3, 4] where they obtain among other results that each hyperbolic $\mathbf{C}$-action in $\mathbf{C}^3$ is topologically linearizable.

Camacho in [2] has proven topological linearizability for hyperbolic $\mathbf{R}^2$ actions on $\mathbf{R}^n$, $n \leqslant 3$.

On $\mathbf{R}^4$ he can topologically linearize using the weaker notion of equivalence instead of conjugacy. Topological equivalence means that the homeomorphism preserves the orbit structure without necessarily respecting the parameter. For the same notion of C^0-equivalence, Guckenheimer for $\mathbf{C}$-actions [6] and Camacho for $\mathbf{R}^2$ actions [2] have proven linearizability on any $\mathbf{R}^n$ when the action is hyperbolic and contains an attractor.

Theorem A permits to linearize (although in class C^∞) more singular holomorphic vector fields then the ones that for instance

Siegel [12] and Brjuno [1] proved to be holomorphically linearizable (using a holomorphic change of coordinates). A complete proof of Theorem A will occupy the rest of the paper. The proof of Theorem B is completely similar to that of Theorem A. At each step one has to pay attention on the loss of differentiability. It seems a nice calculation to find the exact expression of $k(\ell, U)$.

3. Reduction of Theorem A to a system of partial differential equations.

Suppose ρ is some germ of a singular infinitesimal $\mathbf{R}^2$ action of class C^∞ on $\mathbf{R}^n$ $(n \geqslant 2)$ whose 1-jet $j^1\rho(0)$ lies in $J^1\,\mathrm{Act}(\mathbf{R}^2, \mathbf{R}^n)\backslash S_\infty$.

(If ρ is a germ of a singular holomorphic vector field on $\mathbf{C}^n$, we treat it as an $\mathbf{R}^2$-action on $\mathbf{R}^{2n}$.)

We choose a basis (e_1, e_2) of $\mathbf{R}^2$ such that the two commuting vector fields $X = \rho(e_1)$ and $Y = \rho(e_2)$ which characterize ρ completely have 1-jets $X_0 = j_1X(0)$ and $Y_0 = j_1Y(0)$ each with all eigenvalues different. Moreover we take X_0 such that its eigenvalues do not satisfy any of the resonance conditions. Such generators X and Y can obviously be found and we can even assume X_0 and Y_0 as near as we want in $J^1\,\mathfrak{G}^n$.

Because of the theorem of Sternberg [14], [10], X can be C^∞ linearized: a germ $\exists\,\varphi \in \mathfrak{G}D^n$, of a C^∞ diffeomorphism: $\widetilde{\varphi} : (V \subset \mathbf{R}^n, 0) \hookleftarrow$ such that $\varphi * X = X_0$ i.e. $\varphi * X$ is the germ of the vector field $\widetilde{\varphi} * \widetilde{X}$, where $\widetilde{X}$ is a realization of X on V and

$$\widetilde{\varphi} * \widetilde{X}(x) = d\widetilde{\varphi}_{\widetilde{\varphi}^{-1}(x)}(\widetilde{X}(\widetilde{\varphi}^{-1}(x)) \quad \text{for} \quad x \in V,$$

since $j_1\varphi(0) = j_1(\mathrm{Id})\,(0) \Longrightarrow j_1(\varphi * Y)\,(0) = j_1(Y_0)\,(0)$ and of course $[X_0, \varphi * Y] = \varphi * [X, Y] = 0$.

As the eigenvalues of X_0 do not satisfy resonance conditions it follows that formally $\varphi * Y$ needs to be linear, meaning that $\varphi * Y = Y_0 + P$ with $j_\infty(P)\,(0) = 0$.

We are not going to repeat here these calculations, which can for instance be found in [10].

From now on we do no longer need to suppose that X_0 is resonance-free. The non-resonance conditions are only needed in the formal part of the proof.

We now want to find a germ of a diffeomorphism $\psi \in \mathfrak{G}D^n$ such that :

$$\begin{cases} \psi * X_0 = X_0 \\ \psi * Y_0 = Y_0 + P . \end{cases} \tag{1}$$

For this we use the "method of the path" as used in [10] to linearize a single vector field.

The only difference here is that we also need to X_0 invariant.

Let us consider the C^∞ 1-parameter family of vector fields $Y_\tau = Y_0 + \tau P$, $\tau \in [0,1]$, we see $[Y_\tau, X_0] = 0$.

Let us try to find a 1-parameter family of germs of diffeomorphisms g_τ with $g_0 = \mathrm{Id}$, $j_\infty(g_\tau)(0) = j_\infty(\mathrm{Id})(0)$ such that:

$$\begin{cases} (g_\tau) * X_0 = X_0 \\ (g_\tau) * Y_0 = Y_0 + \tau P . \end{cases} \tag{2}$$

It suffices to find a 1-parameter family of vector fields Z_τ with $j_\infty(Z_\tau)(0) = 0$ and satisfying

$$\begin{cases} [Y_\tau, Z_\tau] = -\dot{Y}_\tau = -P \\ [X_0, Z_\tau] = 0 . \end{cases} \tag{3}$$

Integrating Z_τ over τ we find the family of diffeomorphisms g_τ with the properties we need. g_1 solves then our initial problem. For more details see [10].

Some notations:

$\mathcal{E}(n)$: ring of germs in $0 \in \mathbf{R}^n$ of C^∞ functions on $\mathbf{R}^n$

$\mathfrak{M}(n) \subset \mathcal{E}(n)$: the only maximal ideal of $\mathcal{E}(n)$ consisting of germs of C^∞ functions vanishing in the origin.

$\mathfrak{M}^{r+1}(n)$: ideal of $f \in \mathcal{E}(n)$ with $j_r(f)(0) = 0$ $(1 \leqslant r \leqslant \infty)$.

$V(n)$: the $\mathcal{E}(n)$-module of germs in 0 of C^∞ vector fields on $\mathbf{R}^n$
(remark: $\mathfrak{M}(n)\, V(n) = \mathfrak{G}^n$).

$\mathcal{E}_\tau(n)$: the ring of C^∞ 1-parameter families (over $\tau \in [0,1]$) of elements of $\mathcal{E}(n)$.

$V_\tau(n) = \mathcal{E}_\tau(n) \underset{\mathcal{E}(n)}{\otimes} V(n)$: the $\mathcal{E}_\tau(n)$-module of C^∞ 1-parameter families of elements of $V(n)$.

$V_\tau(n, X_0)$: the $\mathbf{R}$-vector subspace of elements of $V_\tau(n)$ commuting with X_0.

Hence $\mathfrak{M}_\tau^\infty(n)\, V_\tau(n, X_0)$ is an $\mathbf{R}$-vector space consisting of C^∞ 1-parameter families of germs in 0 of ∞-flat vector fields commuting with X_0 ; for instance $\dot{Y}_\tau = P \in \mathfrak{M}_\tau^\infty(n)\, V_\tau(n, X_0)$.

Also Z_τ must be an element of this space.

Let us try to characterize this space in a better way. Therefore we first recall some well known properties of hyperbolic singularities of C^r vector fields and hyperbolic fixed points of C^r diffeomorphisms f.

($p \in \mathbf{R}^n$ with $f(p) = p$ is called a hyperbolic fixed point if Df_p, the differential of f in p has only eigenvalues with modulus different from 1).

In a sufficiently small neighbourhood V of such a fixed point one can find two embedded C^r manifolds $W_s(f)$ and $W_u(f)$ intersecting transversally in p, $W_s(f) \cap W_u(f) = \{p\}$, both invariant under f and

$$W_s(f) = \{y \in V \mid \lim_{n \to \infty} f^n(y) = p\}$$

$$W_u(f) = \{y \in V \mid \lim_{n \to \infty} f^{-n}(y) = p\} ;$$

$W_s(f)$ is called the stable manifold of f and $W_u(f)$ the unstable manifold.

If $T_p\mathbf{R}^n = E_s \oplus E_u$ represents the spectral decomposition of $T_p\mathbf{R}^n$ associated to Df_p, then $W_s(f)$ is tangent to E_s in p, $W_u(f)$ is tangent to E_u, $\dim W_s(f) = \dim E_s$; $\dim W_u(f) = \dim E_u$. Both manifolds are uniquely defined. See for instance [8, 13]. The same remains true for a hyperbolic singularity of a vector field: $W_s(X)$ and $W_u(X)$ are now invariant under the flow of X and have all the described properties with respect to each X_t instead of f.

PROPOSITION 3.1. – *Let* L *be a hyperbolic isomorphism of* $\mathbf{R}^n$ *with eigenvalues* $\lambda_1, \ldots, \lambda_n$ *satisfying*

$$|\lambda_1| \leqslant |\lambda_2| \leqslant \ldots \leqslant |\lambda_p| < 1 < |\lambda_{p+1}| \leqslant \ldots \leqslant |\lambda_n| .$$

Let $\ell, k \in \mathbf{N}$ *satisfy the relation* $|\lambda_p|^{k-\ell} < \dfrac{|\lambda_1|}{|\lambda_n|^\ell}$. *If* $g : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ *is a* C^k *mapping,* $Lg = gL$ *and* $j_k(g)(0) = 0$, then

$$j_\ell(g)(x) = 0 \quad \forall x \in W_s(L) .$$

Remarks. – 1) To obtain a similar result for $W_u(L)$ we need $|\lambda_{p+1}|^{k-\ell} > \dfrac{|\lambda_n|}{|\lambda_1|^\ell}$.

2) If L is the time 1-map of the flow of a vector field V with eigenvalues $\mu_1, \ldots, \mu_n$ such that

$$\operatorname{Re}\mu_1 \leqslant \ldots \leqslant \operatorname{Re}\mu_p < 0 < \operatorname{Re}\mu_{p+1} \leqslant \ldots \leqslant \operatorname{Re}\mu_n ,$$

then the conditions on k, ℓ in the proposition and remark 1 respectively are

$$k > \frac{\operatorname{Re}\mu_1 + \ell(\operatorname{Re}\mu_p - \operatorname{Re}\mu_n)}{\operatorname{Re}\mu_p} \quad \text{and} \quad k > \frac{\operatorname{Re}\mu_n + \ell(\operatorname{Re}\mu_{p+1} - \operatorname{Re}\mu_1)}{\operatorname{Re}\mu_{p+1}} .$$

Proof. – 1) $Lg = gL$ implies that $g(x) \in W_s(L)$ $\forall x \in W_s(L)$ since $\lim\limits_{n\to\infty} L^n g(x) = \lim\limits_{n\to\infty} gL^n(x) = g(\lim\limits_{n\to\infty} L^n(x)) = g(0) = 0$.

2) $\ell = 0$ $|\lambda_p|^k < |\lambda_1|$ is exactly the condition used by N. Kopell [9] to prove that $g|W_s(L) \equiv 0$. Next step contains the method to prove this.

3) by induction: take $\ell > 0$ and suppose the proposition has already been proven for $p-1$ with $1 \leqslant p \leqslant \ell$.

Then $j_p(g)(x)$ can be represented in each $x \in W_s(L)$, by $D^p g(x)$, the differential of order p of g in x. Since $g = L^{-1}gL$ and $D^q g(x) = 0$ $\forall 0 \leqslant q < p$, $\forall x \in W_s(L)$, we find $\forall x \in W_s(L)$:

$$\begin{aligned}
D^p g(x) &= DL^{-1}_{gL(x)} \circ D^p g_{L(x)} \circ [\underbrace{DL_x, \ldots, DL_x}_{p}] \\
&= L^{-1} \circ D^p g_{L(x)} \circ [L, \ldots, L] \\
&= L^{-2} \circ D^p g_{L^2(x)} \circ [L^2, \ldots, L^2] \\
&\quad \vdots \\
&= L^{-m} \circ D^p g_{L^m(x)} \circ [L^m, \ldots, L^m] \\
&= \lim_{m\to\infty} L^{-m} \circ D^p g_{L^m(x)} \circ [L^m, \ldots, L^m] .
\end{aligned}$$

So $\|D^p g(x)\| \leqslant \lim_{m\to\infty} \|L^{-m}\| \, \|D^p g_{L^m(x)}\| \, \|L^m\|^p$ (*).

By supposition $\|D^p g(y)\| = \|y\|^{k-p} . \alpha_p(y)$, with α_p bounded in a neighbourhood of $y = 0$, hence

$$(*) = \lim_{m\to\infty} \|L^{-m}\| \, \|L^m\|^p \, \|L^m(x)\|^{k-p} \, \alpha_p(L^m(x))$$

and $\lim_{m\to\infty} \alpha_p(L^m(x))$ is bounded since $\lim_{m\to\infty} L^m(x) = 0$; furthermore $\forall x \in W_s(L)$:

$$L^k x \in W_s(L)$$

$$L^m(x) = (L | W_s(L))^m(x)$$

and $\quad \|L^m(x)\| = \|(L|W_s(L))^m(x)\| \leqslant \|(L|W_s(L))^m\| \, \|x\|$.

Hence it suffices to prove that:

$$\lim_{m\to\infty} (\|L^{-m}\| \, \|L^m\|^p \, \|(L|W_s(L))^m\|^{k-p})^{1/m} < 1 \; (**)$$

but

$$\lim_{m\to\infty} \|L^{-m}\|^{1/m} = \frac{1}{|\lambda_1|}$$

$$\lim_{m\to\infty} \|L^m\|^{1/m} = |\lambda_n|$$

$$\lim_{m\to\infty} \|(L|W_s(L))^m\|^{1/m} = |\lambda_p|$$

(**) Is thus satisfied because of the conditions in the proposition.

Remark. – Suppose $h = A + g$ is a diffeomorphism with A linear, $j_k g(0) = 0$ for some $k \geqslant 1$, and $Lh = hL$. Then considering the 1-jet of this commutator we see that $LA = AL$ and also $Lg = gL$.

Proposition 3.1 implies that each $Y_\tau \in \mathfrak{M}_\tau^\infty(n) \, V_\tau(n, X_0)$ consists of germs of vector fields which are ∞-flat along $W_s(X_0) \cup W_u(X_0)$. Choosing a good coordinate system (linear in our case) we may suppose that $W_s(X_0) = \mathbf{R}^p \times \{0\}$ and $W_u(X_0) = \{0\} \times \mathbf{R}^{n-p}$. Y_τ can be extended naturally along the orbits of X_0 to a germ along $\mathbf{R}^n \times \{0\} \cup \{0\} \times \mathbf{R}^{n-p}$, commuting with X_0. By Proposition 3.1 it is clearly ∞-flat in all $x \in \mathbf{R}^n \times \{0\} \cup \{0\} \times \mathbf{R}^{n-p}$.

Let $S \subset \mathbf{R}^p \times \{0\}$ be a $(p-1)$-sphere centered in 0 and transverse to the orbits of X_0. Let $D \subset \{0\} \times \mathbf{R}^{n-p}$ be an $(n-p)$-ball centered in 0.

We denote by $V_\tau(S)$ the space of C^∞ 1-parameter families over $\tau \in [0,1]$ of germs along $S \times \{0\}$ of sections of $T\mathbf{R}^n$ over $S \times D \subset \mathbf{R}^p \times \mathbf{R}^{n-p} = \mathbf{R}^n$.

Let us also denote by $\mathcal{E}_\tau(S)$ the ring of 1-parameter families of germs of C^∞ functions on $S \times D$ along $S \times \{0\}$, and by $\mathfrak{M}_\tau^\infty(S)$ those which are ∞-flat along $S \times \{0\}$, i.e. ∞-flat in all points of $S \times \{0\}$.

We have a natural restriction mapping:

$$\rho : \mathfrak{M}_\tau^\infty(n)\, V_\tau(n, X_0) \longrightarrow \mathfrak{M}_\tau^\infty(S)\, V_\tau(S).$$

PROPOSITION 3.2. – $\rho : \mathfrak{M}_\tau^\infty(n)\, V_\tau(n, X_0) \longrightarrow \mathfrak{M}_\tau^\infty(S)\, V_\tau(S)$ *is an isomorphism of* $\mathbf{R}$-*vector space.*

The proof of this proposition can be found in [10] but is also an immediate consequence of our Theorem 2 (§ 5). The proposition provides a nice description of the ∞-flat vector fields commuting with a hyperbolic linear vector field X_0 and permits to construct non trivial examples (even when X_0 is resonance-free).

As a consequence of Proposition 3.2 equation (3) is equivalent to

$$[Y_\tau, Z_\tau]\Big|_{S \times D} = -\dot{Y}_\tau\Big|_{S \times D}. \tag{3'}$$

Let us now introduce a basis for $T\mathbf{R}^n$: $\dfrac{\partial}{\partial x_1}, \ldots, \dfrac{\partial}{\partial x_n}$.

Each $T_\tau \in \mathfrak{M}_\tau^\infty(S)\, V_\tau(S)$ can be expressed as:

$$T_\tau = \sum_{i=1}^n t_{i,\tau} \frac{\partial}{\partial x_i} \quad \text{with} \quad t_{i,\tau} \in \mathfrak{M}_\tau^\infty(S).$$

We denote by $U_\tau \in V_\tau(S)$ the projection along $S \times D$ of Y_τ on $T(S \times D)$ parallel to the direction of X_0 : i.e. $\forall x \in S \times D$ and $\forall \tau \in [0,1]$

$$U_\tau(x) = Y_\tau(x) - \frac{\langle N_x, Y_\tau(x)\rangle}{\langle N_x, X_0(x)\rangle} X_0(x)$$

where N_x is the inner pointing normal to $T_x(S \times D)$ in $T_x \mathbf{R}^n$. We denote by $\widetilde{U}_\tau$ the germ of the vector field commuting with X_0, defined in some neighbourhood W of $S \times D$ in $\mathbf{R}^n$ and such that $\widetilde{U}_\tau\Big|_{S\times D} = U_\tau$.

We take W to be a union of segments of X_0-trajectories of length ϵ centered in the points of $S \times D$.

If ϵ is sufficiently small $\widetilde{U}_\tau$ can be defined in W without difficulties.

In W we can decompose $Y_\tau = \alpha_\tau X_0 + \widetilde{U}_\tau$, where each α_τ is a germ of a function defined on W with $X_0 . \alpha_\tau = 0$.

We denote by $T_1, \ldots, T_n$ the vector fields on W commuting with X_0 and such that $T_i\Big|_{S\times D} = \dfrac{\partial}{\partial x_i}$.

We decompose

$$Z_\tau = \sum_{i=1}^{n} A_{i,\tau} T_i$$

$$-\dot{Y}_\tau = \sum_{i=1}^{n} B_{i,\tau} T_i \quad .$$

So that equation (3) gets:

$$\left[Y_\tau, \sum_{i=1}^{n} A_{i,\tau} T_i\right] = \sum_{i=1}^{n} B_{i,\tau} T_i$$

$$\sum_{i=1}^{n} (Y_\tau . A_{i,\tau}) T_i + \sum_{i=1}^{n} [Y_\tau, T_i] A_{i,\tau} = \sum_{i=1}^{n} B_{i,\tau} T_i .$$

As Z_τ commutes with X_0: $X_0 . A_{i,\tau} = 0$ and hence:

$$Y_\tau . A_{i,\tau} = \widetilde{U}_\tau . A_{i,\tau} .$$

We put $[Y_\tau, T_j] = \sum_{i=1}^{n} F_{ij,\tau} T_i$ then:

$$[Y_\tau, Z_\tau] = \sum_{i=1}^{n} \left(\widetilde{U}_\tau . A_{i,\tau} + \sum_{j=1}^{n} F_{ij,\tau} A_{j,\tau}\right) T_i ,$$

showing the equivalence between (3) and

$$\widetilde{U}_\tau . A_{i,\tau} + \sum_{j=1}^{n} F_{ij,\tau} A_{j,\tau} = B_{i,\tau} \quad i = 1, \ldots, n . \tag{4}$$

Equation (3′) is equivalent to:

$$U_\tau a_{i,\tau} + \sum_{j=1}^{n} f_{ij,\tau} a_{j,\tau} = b_{i,\tau} \qquad i = 1, \ldots, n \tag{4'}$$

where $a_{i,\tau}$ are the components of $Z_\tau\big|_{S\times D}$, $b_{i,\tau}$ those of $-\dot{Y}_\tau\big|_{S\times D}$ and $f_{ij,\tau} = F_{ij,\tau}\big|_{S\times D}$.

Since all these functions are X_0-invariant, in consequence of Proposition 3.2 our problem is reduced to prove following theorem:

THEOREM 1. – *Let* $U_\tau \in V_\tau(S)$ *be a 1-parameter family of germs of vector fields on* $S \times D$ *as defined here above and let* $f_{ij,\tau} \in \mathcal{E}_\tau(S)$ *be arbitrary;* $i, j = 1, \ldots, n$. *Then the system:*

$$U_\tau a_{i,\tau} + \sum_{j=1}^{n} f_{ij,\tau} a_{j,\tau} = b_{i,\tau} \qquad i = 1, \ldots, n$$

has solutions $a_{i,\tau} \in \mathfrak{M}_\tau^\infty(S)$ *for any second member* $b_{i,\tau} \in \mathfrak{M}_\tau^\infty(S)$.

Short summary of the remaining paragraphs:

In the remaining paragraphs we prove Theorem 1.

In § 4 we study the vector field U_0 associated to Y_0.

We prove that all singularities and limit cycles of U_0 are hyperbolic and detect some peculiar features about the position of their stable and unstable manifolds.

In § 5 we prove Theorem 1 to be exact in the neighbourhood of hyperbolic singularities and hyperbolic closed orbits.

In § 6 we use the results of § 4 to show that U_τ behaves similarly as U_0 and has especially following property near to S: if we denote $\forall x \in S$ by $W_s(x)$ the set of points on $\{x\} \times D$ whose positive U_τ-trajectory tends to $S \times \{0\}$ and by $W_u(x)$ the set of points on $\{x\} \times D$ whose negative U_τ-trajectory tends to $S \times \{0\}$, then these two subsets are contained in two cones of $\{x\} \times D$ intersecting in $\{x\} \times \{0\}$; this property depends continuously on x. We then use this in § 7 to show that the local solutions to our system of differential equations can be globally extended in the neighbourhood of $S \times \{0\}$, proving Theorem 1.

4. Properties of the vector field U_0.

We denote by U_0 the vector field on $S \times \mathbf{R}^{n-p}$ (see § 3) which we obtain by projecting the linear vector field Y_0 on $T(S \times \mathbf{R}^{n-p})$ parallel to X_0:

$$\text{i.e.} \qquad \forall x \in S \times \mathbf{R}^{n-p} : U_0(x) = Y_0(x) - \frac{\langle N_x , Y_0(x)\rangle}{\langle N_x , X_0(x)\rangle} X_0(x).$$

Let us remember that we take X_0 and Y_0 to be hyperbolic, diagonalized in $\mathbf{C}$, satisfying the conditions of Definition 1.4, and as near to each other that we can arrange their eigenvalues as:

$$\lambda_1 \pm i\beta_1 , \ldots , \lambda_k \pm i\beta_k , \lambda_{2k+1} , \ldots , \lambda_p , -\mu_1 \pm i\gamma_1 , \ldots , -\mu_\ell \pm i\gamma_\ell , -\mu_{2\ell+1} , \ldots , -\mu_{n-p} \quad \text{for } X_0$$

and:

$$\lambda_1' \pm i\beta_1' , \ldots , \lambda_k' \pm i\beta_k' , \lambda_{2k+1}' , \ldots , \lambda_p' , -\mu_1' \pm i\gamma_1' , \ldots , -\mu_\ell' \pm i\gamma_\ell' , -\mu_{2\ell+1}' , \ldots , -\mu_{n-p}' \quad \text{for } Y_0 .$$

All $\lambda , \lambda' , \beta , \beta' , \mu , \mu' , \gamma , \gamma'$ are strictly positive and the eigenspace E_ν corresponding to an eigenvalue ν of X_0 is also the eigenspace $E_{\nu'}$ corresponding to the eigenvalue ν' of Y_0 $(E_\nu = E_{\nu'})$.

Under these assumptions the following observations about U_0 can be verified by trivial calculations:

1) The singularites of U_0 are exactly the intersections of $S \times D$ with the 1-dimensional eigenspaces E_λ corresponding to the eigenvalues $\lambda_{2k+1} , \ldots , \lambda_p$ (resp. $\lambda_{2k+1}' , \ldots , \lambda_p'$). In these singularities U_0 is hyperbolic and the eigenvalues of $j_1(U_0)$ in the two singularities $E_{\lambda_{j_0}} \cap S$ have following expression:

$$\lambda_j' - \frac{\lambda_{j_0}'}{\lambda_{j_0}} \lambda_j , \; -\mu_j' + \frac{\lambda_{j_0}'}{\lambda_{j_0}} \mu_j , \; \left(\lambda_j' - \frac{\lambda_{j_0}'}{\lambda_{j_0}} \lambda_j\right) \pm i\left(\beta_j' - \frac{\lambda_{j_0}'}{\lambda_{j_0}} \beta_j\right)$$

$$\text{and } \left(-\mu_j' + \frac{\lambda_{j_0}'}{\lambda_{j_0}} \mu_j\right) \pm i\left(\gamma_j' - \frac{\lambda_{j_0}'}{\lambda_{j_0}} \gamma_j\right) .$$

The eigenspaces corresponding to these eigenvalues are resp. $E_{\lambda_j} , E_{-\mu_j} , E_{\lambda_j \pm i\beta_j} , E_{-\mu_j \pm i\gamma_j}$ with $j \neq j_0$.

2) The closed orbits of U_0 are exactly the intersections of $S \times D$ with the 2-dimensional eigenspaces E_λ corresponding to the eigenvalues

$$\lambda_1 \pm i\beta_1, \ldots, \lambda_k \pm i\beta_k \quad (\text{resp. } \lambda'_1 \pm i\beta'_1, \ldots, \lambda'_k \pm i\beta'_k)$$

U_0 is hyperbolic in these closed orbits. The multiplicators (eigenvalues of the differential of the associated Poincaré mapping) for the closed orbit $E_{\lambda_{j_0}} \cap S$ have as modulus:

$$\exp\left(\frac{2\pi\left(\lambda'_j - \frac{\lambda'_{j_0}}{\lambda_{j_0}}\lambda_j\right)}{\left|\beta'_{j_0} - \frac{\lambda'_{j_0}}{\lambda_{j_0}}\beta_{j_0}\right|}\right) \quad \text{or} \quad \exp\left(\frac{2\pi\left(-\mu'_j + \frac{\lambda'_{j_0}}{\lambda_{j_0}}\mu_j\right)}{\left|\beta'_{j_0} - \frac{\lambda'_{j_0}}{\lambda_{j_0}}\beta_{j_0}\right|}\right).$$

Their corresponding eigenspaces are again the eigenspaces E_j of X_0 for $j \neq j_0$.

3) U_0 is tangent to $S \times \{0\}$.

We denote by $\overline{U}_0$ the restriction of U_0 to $S \times \{0\}$. The singularities and closed orbits of U_0 are also those of $\overline{U}_0$.

4) Let us now range the singularities (meaning zeroes and limit cycles) of $\overline{U}_0$, denoting them by γ_i with $i \in \{1, \ldots, k + 2(p-2k)\}$, in such a way that the corresponding λ'/λ satisfy:

$$i < j \Longrightarrow \frac{\lambda'_i}{\lambda_i} \leqslant \frac{\lambda'_j}{\lambda_j}$$

(equality can only happen in the case γ_i and γ_j represent the two singularities associated to a same eigendirection E_ν of X_0).

By this convention, γ_1 (and possibly also γ_2) is a source for $\overline{U}_0$, while γ_{2p-3k} (and possibly $\gamma_{2p-3k-1}$) is a sink for $\overline{U}_0$. The ω-(and α)-limit set of each point of $S \times \{0\}$ can easily – using Lyapunov functions – be shown to be one of the singularities. As a matter of fact we can find two invariant filtrations by spheres:

$$S \times \{0\} = S^{n_1} \supset S^{n_2} \supset \ldots \supset S^{n_{p-k}} \text{ and } S \times \{0\}$$
$$= S^{m_1} \supset S^{m_2} \supset \ldots \supset S^{m_{p-k}} \text{ with}$$

$$S^{n_i} = \{(x_1, \ldots, x_p) \mid x_1 = \ldots = x_{n_i} = 0,\ x_{n_i+1}^2 + \ldots + x_p^2 = 1\} \text{ and}$$

$$S^{m_j} = \{(x_1, \ldots, x_p) \mid x_{m_j+1} = \ldots = x_p = 0 \,,\; x_1^2 + \ldots + x_{m_j}^2 = 1\}$$

such that:

1) if $\forall i \in \{1, \ldots, p-k-1\}$ we denote by A_i the non-wandering set of $\overline{U}_0$ in $S^{n_i} \backslash S^{n_{i+1}}$, then A_i consists either of one closed orbit $(n_i = n_{i+1} + 2)$ or of two zeroes $(n_i = n_{i+1} + 1)$ and in each case $W^u(A_i) = S^{n_i} \backslash S^{n_{i+1}}$;

2) if $\forall j \in \{1, \ldots, p-k-1\}$ we denote by B_j the non-wandering set of $\overline{U}_0$ in $S^{m_j} \backslash S^{m_{j+1}}$, then B_j consists either of one closed orbit $(m_j = m_{j+1} + 2)$ or of two zeroes $(m_j = m_{j+1} + 1)$ and in each case:

PROPOSITION 4.1. – *Let γ be a singularity of $\overline{U}_0$. There exists a $k(\gamma)$, $0 \leqslant k(\gamma) \leqslant n-p$ such that if $W_s(\gamma)$ and $W_u(\gamma)$ denote the stable and unstable manifold of γ for U_0, and $\overline{W}_s(\gamma)$ and $\overline{W}_u(\gamma)$ those of γ for $\overline{U}_0$, we have:*

$$W_u(\gamma) = \overline{W}_u(\gamma) \times \{y_1, \ldots, y_{k(\gamma)}\}$$

and

$$W_s(\gamma) = \overline{W}_s(\gamma) \times \{y_{k(\gamma)+1}, \ldots, y_{n-p}\}$$

where $\{y_\ell, \ldots, y_k\} \subset \mathbf{R}^{n-p}$ is the linear subspace generated by $y_\ell, \ldots, y_k$.

Proof. – The vector field U_0 is the sum of a vector field parallel to the $\mathbf{R}^{n-p}$-factor and a vector field parallel to S which is independent of y.

Because of this almost product structure it follows that $W_u(\gamma)$ is the product of $\overline{W}_u(\gamma)$ with the product of the directions E_i of $\mathbf{R}^{n-p}$ such that $\dfrac{\mu_i'}{\mu_i} < \dfrac{\lambda_j'}{\lambda_j'}$. This proves the result.

PROPOSITION 4.2. – *Let γ_1 be a singularity of U_0 such that $\gamma_1 \in \overline{M}_u(\gamma)$ (meaning that γ lies in the boundary of $\overline{W}_u(\gamma_1)$). Then $k(\gamma_1) \leqslant k(\gamma)$.*

Proof. – If $\gamma_1 \in \overline{M}_u(\gamma)$, *then* $\dfrac{\lambda_{\gamma_1}'}{\lambda_{\gamma_1}} < \dfrac{\lambda_\gamma'}{\lambda_\gamma}$, hence if $\dfrac{\mu_i'}{\mu_i} < \dfrac{\lambda_{\gamma_1}'}{\lambda_{\gamma_1}}$, also $\dfrac{\mu_i'}{\mu_i} < \dfrac{\lambda_\gamma'}{\lambda_\gamma}$.

The set $\overline{\mathrm{M}}(\gamma) = \overline{\mathrm{M}}_u(\gamma) \cap \overline{\mathrm{M}}_s(\gamma)$ is an open neighbourhood of γ and is invariant by the flow of $\overline{\mathrm{U}}_0$. Therefore $\{\overline{\mathrm{M}}(\gamma)\}_\gamma$ forms an open covering of $\mathrm{S} \times \{0\}$. Let us write $\mathrm{M}(\gamma) = \overline{\mathrm{M}}(\gamma) \times \mathbf{R}^{n-p}$.

Based on Propositions 4.1 and 4.2 and the almost-product structure of U_0 described in the proof of Proposition 4.1, we obtain:

PROPOSITION 4.3 – *If* $m \in \mathrm{M}(\gamma)$ *belongs to the unstable (resp. stable) manifold of some singularity of* U_0, *then*

$m \in \overline{\mathrm{M}}(\gamma) \times \{y_1, \dots, y_{k(\gamma)}\}$ *(resp.* $m \in \overline{\mathrm{M}}(\gamma) \times \{y_{k(\gamma)+1}, \dots, y_{n-p}\}$*)*.

In particular, the stable and unstable manifolds of the singularities of U_0 *have no points in common outside* $\mathrm{S} \times \{0\}$.

5. Solving a system of partial differential equations in the neighbourhood of a hyperbolic singularity or a hyperbolic closed orbit.

Let M be an oriented C^∞ manifold without boundary and let X be a C^∞ vector field on $\mathrm{M} \times [0,1]$ such that $\forall (x, \tau) \in \mathrm{M} \times [0,1]$: $\mathrm{X}(x, \tau) \in \mathrm{T}_x\mathrm{M} \times \{\tau\}$.

Such a vector field can be considered as a C^∞ 1-parameter family of vector fields on M.

We denote by X_τ the vector field $\mathrm{X} | \mathrm{M} \times \{\tau\}$.

Let $\gamma \subset \mathrm{M}$ be a subset diffeomorphic to S^1 or diffeomorphic to a point; γ is of codimension n in M.

Let us suppose that $\forall \tau \in [0,1]$, $\gamma \times \{\tau\}$ is a hyperbolic singularity of X_τ. In the case γ is a closed orbit we moreover ask the Poincaré mapping to be orientation preserving on the stable manifold, as well as on the unstable manifold.

The theorem of the existence of stable and unstable manifolds of a hyperbolic singularity (being a zero or a closed orbit) applied to the fields X_τ –[8]– implies that there exists a neighbourhood V of $\gamma \times [0,1]$ and a C^∞ diffeomorphism:

$$\varphi : \gamma \times [0,1] \times \mathbf{R}^n \longrightarrow \mathrm{V}, \text{ such that}$$

1) $\varphi(\gamma \times \{\tau\} \times \mathbf{R}^n) \subset \mathrm{M} \times \{\tau\} \quad \forall \tau \in [0,1]$.

2) The stable and unstable manifolds of the X_τ in V are the images by φ of resp. $\gamma \times \{\tau\} \times \mathbf{R}^p \times \{0\}$ and $\gamma \times \{\tau\} \times \{0\} \times \mathbf{R}^{n-p}$ where $\mathbf{R}^n = \mathbf{R}^p \times \mathbf{R}^{n-p}$ and p is the codimension of the unstable manifold of U_τ.

We can now modify φ in order to construct an "adapted neighbourhood" of γ in the sense of following definition:

DEFINITION. – *Let γ and X be as defined here above. A diffeomorphism φ from $\gamma \times [0,1] \times U$ onto a neighborhood V of $\gamma \times [0,1]$ in $M \times [0,1]$ determines an "adapted neighbourhood" for γ if:*

1) $U \subset \mathbf{R}^n = \mathbf{R}^p \times \mathbf{R}^{n-p}$ is a manifold with corners, compact neighbourhood of $0 \in \mathbf{R}^n$ such that $\partial U = t_s \cup t_u \cup t_i$ where:

$$t_s = S^{p-1} \times D^{n-p}_{1/2} \subset \mathbf{R}^p \times \mathbf{R}^{n-p},$$

$$t_u = D^p_{1/2} \times S^{n-p-1} \subset \mathbf{R}^p \times \mathbf{R}^{n-p},$$

t_i is diffeomorphic to $S^{p-1} \times S^{n-p-1} \times [0,1]$ and $\partial t_s \cup \partial t_u = \partial t_i$,

where S^i denotes the sphere of radius 1 in $\mathbf{R}^i$ and D^i denotes the disc of radius 1 in $\mathbf{R}^i (D^i_{1/2}$ is the disc of radius $1/2)$.

2) $\varphi(\gamma \times \{\tau\} \times U) \subset M \times \{\tau\}$.

3) $\varphi(\gamma \times [0,1] \times D^p \times \{0\})$ is the union W_s of the stable manifolds of the X_τ in V,

$\varphi(\gamma \times [0,1] \times \{0\} \times D^{n-p})$ is the union W_u of the unstable manifolds of the X_τ in V.

4) If $T_u = \varphi(\gamma \times [0,1] \times t_u)$

$T_s = \varphi(\gamma \times [0,1] \times t_s)$

$T_i = \varphi(\gamma \times [0,1] \times t_i)$

then $\partial V = T_u \cup T_s \cup T_i$.

The vector field X enters V along T_s, leaves V along T_u and is tangent to T_i.

On T_i the flow goes from ∂T_s to ∂T_u. (T_s and T_u could be called the stable resp. unstable fence.)

5) In $V \setminus (W_s \cup W_u)$ each orbit goes in a finite time from a point of T_s to a point of T_u ; $\gamma \times \{\tau\}$ is the only singularity of X_τ in $V \cap (M \times \{t\})$.

The purpose of this paragraph is to study partial differential equations associated to X in such an adapted neighbourhood V of a hyperbolic orbit γ. More precisely, if $(f_{ij})_{i,j=1,\ldots,k}$ are k^2 C^∞ functions on V, we consider the system of equations :

$$X a_i + \sum_{j=1}^{k} f_{ij} a_j = b_i \qquad i = 1, \ldots, k . \tag{E}$$

If $K \subset V$ is a closed subset we denote by $C^\infty(V, K)$ the space of C^∞ functions in V, ∞-flat on K i.e. the space of those C^∞ functions h such that $j_\infty(h)(y) = 0$, $\forall y \in K$.

Let us suppose that F is some closed subset of V, invariant by the flow of X.

We are going to consider system (E) in the space $C(V, F \cup W_u)$ with initial conditions in $C^\infty(T_u, T_u \cap (F \cup W_u))$, as well as in $C^\infty(V, F \cup W_s)$ with initial conditions in $C^\infty(T_s, T_s \cap (F \cup W_s))$. We have following result:

THEOREM 2. – *Let* $b_1, \ldots, b_k \in C^\infty(V, F \cup W_u)$ *and*

$$\alpha_1, \ldots, \alpha_k \in C^\infty(T_u, T_u \cap (F \cup W_u)) .$$

The system (E) *has a unique solution* $(a_i)_{i=1,\ldots,k}$ *with*

$$a_i \in C^\infty(V, F \cup W_u)$$

such that $a_i | T_u = \alpha_i$, $i = 1, \ldots, k$. *In the same way, if*

$$b_1, \ldots, b_k \in C^\infty(V, F \cup W_s)$$

and $\alpha_1, \ldots, \alpha_k \in C^\infty(T_s, T_s \cap (F \cup W_s))$, *system* (E) *has a unique solution* $(a_i)_{i=1,\ldots,k}$ *with* $a_i \in C^\infty(V, F \cup W_s)$ *and such that* $a_i | T_s = \alpha_i$, $i = 1, \ldots, k$.

Remarks. – This statement generalizes different results in [10] and [11] concerning the equations $Xf = h$ and $[X, Y] = Z$, when looking for first integrals or vector fields commuting with X. It can still be generalized to compact invariant manifolds on which X is normally hyperbolic (and some extra conditions) using the same kind of proof.

Proof. – We limit ourselves to the study of the system (E) in $C^\infty(V, F \cup W_u)$. The study in $C^\infty(V, F \cup W_s)$ can be done in an analogous way.

Let $\varphi_u(m)$, $m \in V$, $u \in \mathbf{R}$ denote the flow of X. Restricted to each trajectory $\{\varphi_u(m)\}_u$ the system (E) reduces to a system of ordinary differential equations in the variable u:

$$\frac{\partial}{\partial u} a_i \circ \varphi_u(m) + \sum_j f_{ij}(\varphi_u(m))\, a_j(\varphi_u(m)) = b_i(\varphi_u(m)). \qquad (\mathrm{E}')$$

We introduce for each $m \in V$ the fundamental matrix solution of this system (E′):

$$A(u, m) = (A_{ij}(u, m))_{i,j} \qquad i, j \in \{1, \dots, k\}$$

i.e.

$$\begin{cases} \dfrac{\partial}{\partial u} A_{ij}(u, m) + \sum\limits_k f_{ik}(\varphi_u(m))\, A_{kj}(u, m) = 0 \\ A_{ij}(0, m) = \delta_{ij}. \end{cases}$$

A being invertible we put $B(u, m) = A^{-1}(u, m)$ with: $B(u, m) = (B_{ij}(u, m))_{i,j}$.

We look for the solutions a_i, restricted to the orbit of m, in the form $a_i(\varphi_u(m)) = \sum\limits_j A_{ij}(u, m)\, \overline{a}_j(u, m)$.

Equation (E′) is equivalent to:

$$\frac{\partial \overline{a}_i}{\partial u}(u, m) = \sum_j B_{ij}(u, m)\, b_j(\varphi_u(m)). \qquad (\mathrm{E}'')$$

If $u(m)$ denotes the value of the parameter such that $\varphi_{u(m)}(m) \in T_u$ in case $m \notin W_s$ and $u(m) = +\infty$ if $m \in W_s$, then following functions (if they exist) are solutions of (E″)

$$\overline{a}_i(u, m) = \int_{u(m)}^{u} \sum_j B_{ij}(s, m)\, b_j(\varphi_s(m))\, ds + c_i(m)$$

where $c_i(m)$ is an arbitrary function only depending on the orbit of m.

Filling in the initial conditions and writing the expression for a_i we obtain:

$$a_i(m) = \int_{u(m)}^{0} \sum_j B_{ij}(s, m)\, b_j(\varphi_s(m))\, ds + \sum_j B_{ij}(u(m), m)\, \alpha_j(\varphi_{u(m)}(m)). \qquad (\mathrm{F})$$

We need to show that this expression still has a meaning for $m \in W_s$ and defines a function in $C^\infty(V, F \cup W_u)$. First we can observe that formula (F), for all i, surely defines a C^∞ function outside W_s.

Furthermore, if the b_j are ∞-flat on $F \cup W_u$ and if the α_j are ∞-flat on $T_u \cap (F \cup W_u)$ then the functions a_i defined by (F) are ∞-flat on $(F \cup W_u) \setminus W_s$.

To prove the theorem it hence suffices to show that a_i is defined and C^∞ in the points of W_s.

We identify V with the "sub-manifold" $\gamma \times [0,1] \times U$ of $\gamma \times [0,1] \times \mathbf{R}^n$ by means of the diffeomorphism φ trivializing V.

We introduce coordinates (z, x, y) in $\gamma \times [0,1] \times \mathbf{R}^n$ with:

$$\begin{cases} z \in \gamma \times [0,1] & \\ x \in \mathbf{R}^{n-p} & \text{(for the unstable manifold)} \\ y \in \mathbf{R}^p & \text{(for the stable manifold)}. \end{cases}$$

We denote the components of $\varphi_u(m)$ by

$$\varphi_u(m) = (x_u(m), y_u(m), z_u(m))$$

and if necessary by $\varphi_u(m) = (\varphi_u^1(m), \dots, \varphi_u^i(m), \dots)$ where the $\varphi_u^i(m)$ are the numerical components of the flow.

The result of whatsoever differentiation of the second member of the formula (F) is a sum of terms of the following form:

$$I(m) = \int_{u(m)}^{0} D^\alpha B_{ij}(s, m)\, D^\beta b_j(\varphi_s(m))\, D^{\gamma_1} \varphi_s^{k_1}(m) \dots D^{\gamma_\nu} \varphi_s^{k_\nu}(m)\, ds$$

and of terms of the form

$$J(m) = D^\alpha B_{ij}(u(m), m)\, D^\beta \begin{cases} b_j \\ \alpha_j \end{cases} (\varphi_{u(m)}(m)) \cdot D^{\gamma_1} \varphi_{u(m)}^{k_1}(m)$$
$$\dots D^{\gamma_\nu} \varphi_{u(m)}^{k_\nu}(m) \dots D^{\delta_1} u(m) \dots D^{\delta_n} u(m).$$

The symbols D^α represent partial derivatives on the set of variables of the respective functions.

In order that the a_i be C^∞ in the points of W_s it is sufficient that the integrals of the form $I(m)$ converge uniformly with respect to m when $s \longrightarrow \infty$, and that the functions $J(m)$ are ∞-flat along W_s:

i.e. $$|J(m)| = 0(\|x(m)\|^k) \qquad \forall k \in \mathbf{N}.$$

These properties follow from the majorations due to the hyperbolicity of X.

We are going to indicate these majorations (sometimes too rough) without comments.

More about the justification of these majorations – rather easy to establish – can be found in [10].

1) There exist constants c_1, d_1, $0 < c_1 < d_1$, such that:

$$\begin{aligned} &\|x(m)\| e^{c_1 u} \leqslant \|x_u(m)\| \leqslant \|x(m)\| e^{d_1 u} \\ &\|y(m)\| e^{-d_1 u} \leqslant \|y_u(m)\| \leqslant \|y(m)\| e^{-c_1 u} \end{aligned} \quad \text{in } V.$$

(For some well chosen metric in the case γ is a closed orbit.)

2) parameter $u(m)$ is defined by the condition $\|x_{u(m)}(m)\| = 1$. Using (1) we see that

$$u(m) \leqslant - c_2 \log \|x(m)\| \qquad \left(\text{with } c_2 = \frac{1}{c_1} > 0\right)$$

and also

$$\|y_{u(m)}(m)\| \leqslant \|y(m)\| \cdot \|x(m)\|^{c_2'} \qquad \left(\text{with } c_2' = \frac{c_1}{d_1} > 0\right).$$

3) For each component $\varphi_u^k(m)$ of the flow:

$$|D^\alpha \varphi_u^k(m)| \leqslant d_{3,\alpha}\, e^{c_{3,\alpha} u}$$

$\forall$ multi-index $\alpha \neq 0$; $d_{3,\alpha}$, $c_{3,\alpha} > 0$.

4) $$|D^\alpha B_{ij}(u, m)| \leqslant d_{4,\alpha}\, e^{c_{4,\alpha} u}$$

$\forall$ multi-index α ; $d_{4,\alpha}$, $c_{4,\alpha} > 0$.

5) Differentiating the relation $\|x_{u(m)}(m)\| = 1$ we obtain $\forall \alpha \neq 0$:

$$|D^\alpha u(m)| \leqslant d_{5,\alpha}\, \|x(m)\|^{-c_{5,\alpha}}$$

with $d_{5,\alpha}$, $c_{5,\alpha} > 0$.

6) Since the functions α_j and b_j are ∞-flat along $T_u \cap W_u$ and W_u resp., we have:

$$|D^\alpha \alpha_j(m)| \leqslant c_{6,\alpha,k}\, \|y(m)\|^k \quad m \in T_u$$

$$|D^\beta b_j(m)| \leqslant c_{6,\beta,k}\, \|y(m)\|^k \quad \forall k \in \mathbf{N}.$$

The constants $c_{6,\alpha,k}$ and $c_{6,\beta,k}$ are >0 and depend on the functions α_j and b_j.

Let us now show that $\mathrm{J}(m)$ is ∞-flat along W^s:

from 2) and 4): $\exists d_{7,\alpha}$, $c_{7,\alpha} > 0$

$$|\mathrm{D}^\alpha \mathrm{B}_{ij}(u(m), m| \leqslant d_{7,\alpha} \, \|x(m)\|^{-c_{7,\alpha}}$$

from 6) and 2)

$$|\mathrm{D}^\beta \left\{ \begin{matrix} b_j \\ \\ \alpha_j \end{matrix} \right. (\varphi_{u(m)}(m))| \leqslant d_{6,\beta,k} \, \|y_{u(m)}(m)\|^k \leqslant d_{8,\beta,k} \, \|x(m)\|^{kc_2'}$$

$$\forall k \in \mathbf{N} \quad \text{and} \quad d_{8,\beta,k} > 0 .$$

from 2) and 3)

$$|\mathrm{D}^{\gamma_i} \varphi_{u(m)}^{k_i}(m)| \leqslant d_{9,\gamma_i} \, \|x(m)\|^{-c_{9,\gamma_i}} \quad \text{with} \quad d_{9,\gamma_i},\ c_{9,\gamma_i} > 0$$

and from 5)

$$|\mathrm{D}^{\delta_j} u(m)| \leqslant d_{5,\delta_j} \, \|x(m)\|^{-c_{5,\delta_j}} .$$

Hence finally: $|\mathrm{J}(m)| \leqslant d_{10,k} \, \|x(m)\|^{c_{10,k}}$ with $d_{10,k} > 0$ and

$$c_{10,k} = -\left[c_{7,\alpha} + \sum_{i=1}^{\nu} c_{9,\gamma_i} + \sum_{j=1}^{\mu} c_{5,\delta_j} \right] + k\, c_2' \ ;$$

$c_{10,k}$ can be made arbitrarily big by taking k big.

For the integrands of the $\mathrm{I}(m)$ we have following majorations:

$$|\mathrm{D}^\alpha \mathrm{B}_{ij}(s, m)| \leqslant d_{4,\alpha} \, e^{c_{4,\alpha} s}$$

$$|\mathrm{D}^{\gamma_i} \varphi_s^{k_i}(m)| \leqslant d_{3,\gamma_i} e^{c_{3,\gamma_i} s}$$

$$|\mathrm{D}^\beta b_j(\varphi_s(m)| \leqslant c_{6,\beta,k} \, \|y_s(m)\|^k \leqslant c_{6,\beta,k} \, e^{-c_1 ks} .$$

The integrand is bounded in absolute value by the function:

$$d_{11} \cdot e^{[c_{4,\alpha} + \sum_i c_{3,\gamma_i} - c_1 k]s} \quad \text{with} \quad d_{11} > 0 .$$

The integral of this function is convergent if we choose k sufficiently large.

This proves the theorem.

6. Properties of the 1-parameter family of vector fields U_τ .

In the sense of § 5 we denote by U a vector field on $S \times \mathbf{R}^{n-p} \times [0,1]$, representative of the 1-parameter family of germs defined in § 3. For each $\tau \in [0,1]$ we will denote by the same symbol U_τ the restriction of U to $S \times \mathbf{R}^{n-p} \times \{\tau\}$ and we suppose that U_0 is the vector field described in § 4.

For each neighbourhood W of $S_0 = S \times \{0\} \times [0,1]$ in $S \times \mathbf{R}^{n-p} \times [0,1]$ and each closed subset $F \subset W$ we denote by $C^\infty(W, F)$ the space of C^∞ functions defined in W and ∞-flat along F.

Theorem 1 is a consequence of the following:

THEOREM 3. – *Let* U *be the vector field defined as above and let* $(f_{ij})_{i,j=1,\dots,n}$ *be any* C^∞ *functions on* $S \times \mathbf{R}^{n-p} \times [0,1]$. *Then there exist two neighbourhoods* W *and* W' *of* S_0, $W \subset W'$ *such that for each n-tuple of functions* $(b_i)_{i=1,\dots,n}$, $b_i \in C^\infty(W', S_0)$ *there exists a n-tuple of functions* $(a_i)_{i=1,\dots,n}$, $a_i \in C^\infty(W, S_0)$ *solution of the system*

$$Ua_i + \sum_{j=1}^{n} f_{ij} a_j = b_i \quad i = 1, \dots, n \text{ in } W. \tag{E}$$

In this paragraph we are going to show how to choose W and W' in an appropriate way. The theorem will only be proven in § 7.

Let γ_i be a singularity (a zero or a closed orbit) of U_0. As for all τ, γ_i is also a singularity for U_τ, we say that γ_i is a singularity of U. We denote by $W_s(\gamma_i, \delta)$ and $W_u(\gamma_i, \delta)$ the stable and unstable manifold of $U | W(\delta)$, relative to γ_i, which means the union of the stable resp. unstable manifolds of the $U_\tau | W_\tau(\delta)$.

$$W(\delta) = S \times D^{n-p}(\delta) \times [0,1], \quad W_\tau(\delta) = S \times D^{n-p}(\delta) \times \{\tau\}$$

and $$D^{n-p}(\delta) = \{y \in \mathbf{R}^{n-p} \mid \|y\| < \delta\}.$$

For $\delta' < \delta$ we have:

$$W_{s(\text{or } u)}(\gamma_i, \delta') \subset W_{s(\text{or } u)}(\gamma_i, \delta) \cap W(\delta')$$

(in general this inclusion is a strict one).

Let us also denote by

$$W_s(\delta) = \bigcup_{\gamma_i} W_s(\gamma_i, \delta)$$
$$W_u(\delta) = \bigcup_{\gamma_i} W_u(\gamma_i, \delta).$$

PROPOSITION 6.1. – *There exists a* $\delta > 0$ *such that*

$$C^\infty(W(\delta), S_0) = C^\infty(W(\delta), W_s(\delta)) + C^\infty(W(\delta), W_u(\delta)).$$

This means that each function defined in $W(\delta)$ and ∞-flat along S_0 can be expressed as a sum of a function ∞-flat on $W_s(\delta)$ and a function ∞-flat on $W_u(\delta)$; let us remark that

$$S_0 \subset W_u(\delta) \cap W_s(\delta), \ \forall \delta > 0.$$

The proposition is going to follow from the special position of the sets $W_u(\delta)$ and $W_s(\delta)$ associated to U. This position will be deduced from the properties of U_0 which we established in § 4 (prop. 4.3).

Let us recall that we had found for each γ_i a $k(\gamma_i) \in \mathbf{N}$ and an open neighbourhood $\overline{M}(\gamma_i)$ of γ_i in $S \times \{0\}$ such that the $\{\overline{M}(\gamma_i)\}_i$ form an open covering of $S \times \{0\}$ and such that:

$$W_u^0 \cap (\overline{M}(\gamma_i) \times \mathbf{R}^{n-p}) \subset \overline{M}(\gamma_i) \times \{y_1, \ldots, y_{k(\gamma_i)}\}$$
$$W_s^0 \cap (\overline{M}(\gamma_i) \times \mathbf{R}^{n-p}) \subset \overline{M}(\gamma_i) \times \{y_{k(\gamma_i)+1}, \ldots, y_{n-p}\}$$

where W_u^0 (resp. W_s^0) denotes the union of the unstable (resp. stable) manifolds of U_0.

We now choose for each γ_i an open neighbourhood $\overline{N}(\gamma_i)$ of γ_i in $S \times \{0\}$ such that $C\ell(\overline{N}(b_i)) \subset \overline{M}(\gamma_i)$ and such that the $\{\overline{N}(\gamma_i)\}_i$ still form an open covering of $S \times \{0\}$.

We also choose cones $C_u(\gamma_i)$ and $C_s(\gamma_i)$, cones in $0 \in \mathbf{R}^{n-p}$ and open outside $\{0\}$ such that:

(i) $[\text{Closure } C_u(\gamma_i)] \cap \{y_{k(\gamma_i)+1}, \ldots, y_{n-p}\} = \{0\}$

(ii) $[\text{Closure } C_s(\gamma_i)] \cap \{y_1, \ldots, y_{k(\gamma_i)}\} = \{0\}$.

(iii) $C_u(\gamma_i) \cup C_s(\gamma_i) = \mathbf{R}^{n-p}$.

We put:
$$C_u(\gamma_i, \delta) = \overline{N}(\gamma_i) \times (C_u(\gamma_i) \cap D^{n-p}(\delta)) \times [0,1]$$
$$C_s(\gamma_i, \delta) = \overline{N}(\gamma_i) \times (C_s(\gamma_i) \cap D^{n-p}(\delta)) \times [0,1]$$
$$N(\gamma_i, \delta) = C_u(\gamma_i, \delta) \cup C_s(\gamma_i, \delta) = \overline{N}(\gamma_i) \times D^{n-p}(\delta) \times [0,1].$$

Proposition 6.1 is a consequence of:

PROPOSITION 6.2. – *There exists a* $\delta > 0$ *such that for each singularity* γ_i *of* U *we have*

$$W_s(\delta) \cap [\text{Closure } C_u(\gamma_i, \delta)] \subset \overline{N}(\gamma_i) \times \{0\} \times [0,1] \subset S_0$$

$$W_u(\delta) \cap [\text{Closure } C_s(\gamma_i, \delta)] \subset \overline{N}(\gamma_i) \times \{0\} \times [0,1] \subset S_0 .$$

Let us first show how Proposition 6.2 implies Proposition 6.1. For that purpose we consider $f \in C^\infty(W(\delta), S_0)$ with δ as defined in Proposition 6.1.

Since $\{N(\gamma_i, \delta)\}_{\gamma_i}$ is an open covering of $W(\delta)$ we can write f as a sum of functions h such that each function h has a support contained in one of the $N(\gamma_i, \delta)$.

If h is such a function we can write $h = h_1 + h_2$ with:

$$\text{Support } h_1 \subset C_s(\gamma_i, \delta)$$

$$\text{Support } h_2 \subset C_u(\gamma_i, \delta).$$

This can easily been obtained by for instance blowing up $N(\gamma_i, \delta)$ along $\overline{N}(\gamma_i) \times \{0\} \times [0,1]$ in the $\mathbf{R}^{n-p}$-direction, since C^∞ functions which are ∞-flat along M are in $(1-1)$ correspondance with C^∞ functions which are ∞-flat along the blown-up of M [10]; the transversal planes $\{y_{k(\gamma_i)+1}, \ldots, y_{n-p}\}$ and $\{y_1, \ldots, y_{k(\gamma_i)}\}$ transform into two manifolds without intersection and with the blown-ups of $C_u(\gamma_i, \delta)$ resp. $C_s(\gamma_i, \delta)$ as open neighbourhoods covering the blown-up of $N(\gamma_i, \delta)$.

But the conditions in the statement of Proposition 6.2 make that h_1 is ∞-flat on $W_u(\delta) \cap N(\gamma_i, \delta)$ – hence $h_1 \in C^\infty(W(\delta), W_u(\delta))$ – while h_2 is ∞-flat on $W_s(\delta) \cap N(\gamma_i, \delta)$, hence $h_2 \in C^\infty(W(\delta), W_s(\delta))$.

(i) $\varphi_m|_{\{\tau=0\}}$ is the identical embedding

(ii) φ_m is ∞-tangent to the identical embedding along

$$\overline{V}_m \times \{0\} \times [0,1]$$

(iii) For all $\tau \in [0,1]$

$$\varphi_m(\overline{V}_m \times D^{n-p}(\epsilon(m)) \times \{\tau\} \subset S \times \mathbf{R}^{n-p} \times \{\tau\}$$

iv) $W_u(\gamma_i, \delta(m)) \cap \varphi_m(\overline{V}_m \times D^{n-p}(\epsilon(m)) \times [0,1])$
$$\subset \varphi_m(\overline{V}_m \times D^{k(\gamma_i)}(\epsilon(m)) \times [0,1])$$

where $D^{k(\gamma_i)}(\epsilon(m)) = D^{n-p}(\epsilon(m)) \cap \{y_1, \dots, y_{k(\gamma_i)}\}$.

We have a similar statement for the stable manifolds.

Remark. – In view of Proposition 6.2 it is clear that it suffices to prove this Proposition 6.3 for each singularity γ_i separately. The associated φ_m, which depend on i, do not need to satisfy certain compatibility conditions as is usually the case in constructing invariant fibrations.

Proof. – As in § 4 we denote by $\overline{W}_u(\gamma_j)$ the unstable manifold of $\overline{U}_0 = U_0 | S \times \{0\}$ relative to γ_j. $S \times \{0\} = \bigcup_j \overline{W}_u(\gamma_j)$ (disjoint union).

Let us denote by $\widetilde{U}_0$ the vector field on $S \times \mathbf{R}^{n-p} \times [0,1]$, equal to U_0 on each factor $S \times \mathbf{R}^{n-p} \times \{\tau\}$.

U and $\widetilde{U}_0$ are ∞-near to each other along $S_0 = S \times \{0\} \times [0,1]$.

1) First we suppose that $m \in \overline{W}_u(\gamma_j)$ with $j < i$.

We can find a neighbourhood $\overline{V}_m$ of m in S such that $\overline{V}_m \subset \bigcup_{k<i} \overline{W}_u(\gamma_k)$, implying that $\overline{V}_m \cap \overline{W}_u(\gamma_i) = \emptyset$. The existence of $(\varphi_m, \epsilon(m), \delta(m))$ as in the statement is now trivial, because for δ sufficiently small $W_u(\gamma_i, \delta)$ does not come in the neighbourhood of m.

2) We now suppose that $m \in \gamma_i$.

As U only differs from $\widetilde{U}_0$ by a term ∞-flat on S_0, there exists a diffeomorphism φ' from an adapted neighbourhood Z_0 (in the sense of § 5) of $\gamma_i \times \{0\} \times [0,1]$ in $S \times \mathbf{R}^{n-p} \times [0,1]$, respecting the parameter τ, ∞-near to the identity along $S \times \{0\} \times [0,1]$, to an adapted neighbourhood Z of $\gamma_i \times \{0\} \times [0,1]$ for U, sending $\widetilde{U}_0$ to U.

The existence of such a diffeomorphism can be shown using Theorem 2 of § 5 and the method of the path as explained in § 3 (see also [10]).

The required diffeomorphism can be obtained by restricting this φ' to a neighbourhood of the form $\overline{V}_m \times D^{n-p}(\epsilon(m)) \times [0,1]$ inside Z_0.

We take $\delta(m)$ so small that $W_u(\gamma_i, \delta(m))$ has no intersection with $T_s(Z)$ (stable fence), which is possible by point 1). Let us remark that we have obtained at once the result for all the points of $\overline{V}(\gamma_i) = \overline{V}_m$.

3) If we now consider $m \in \overline{W}_u(\gamma_i) \backslash \gamma_i$ we can take some $m_0 \in \overline{V}(\gamma_i)$ such that there exists an orbit of U_0 going from m_0 to m.

We can now easily transport the obtained properties in the neighbourhood of $\{m_0\} \times \{0\} \times [0,1]$ to similar properties in the neighbourhood of $\{m\} \times \{0\} \times [0,1]$ since the flow mappings of $\widetilde{U}_0$ and U are ∞-near to each other along this orbit. We hence obtain the required result for m.

4) The remaining part of the proof can be achieved by induction: given $j > i$, we suppose the result to be true for all $m \in \overline{W}_u(\gamma_k)$ with $k < j$, and we will try to prove it for $m \in \overline{W}_u(\gamma_j)$. Using the same argument as in point 3) we see that it really suffices to prove it for the points $m \in \gamma_j$.

So let $m \in \gamma_j$.

Using Theorem 2 we can find an adapted neighbourhood V_0 of $\gamma_j \times \{0\} \times [0,1]$ for $\widetilde{U}_0$ and a diffeomorphism ϕ_1 from V_0 onto V , sending the vector field $\widetilde{U}_0$ to U such that ϕ_1 is ∞-near to the identity along S_0 and such that the stable fences $T_s(V_0)$ and $T_s(V)$ are equal in a neighbourhood of $\overline{T}_s = T_s(V_0) \cap S_0 = T_s(V) \cap S_0$. We suppose $\overline{T}_s = \overline{T}_{s_0} \times [0,1]$ with $\overline{T}_{s_0} = \overline{T}_s \cap S \times \{0\} \times \{0\}$. (For the notation see § 5 – we will also identify S with $S \times \{0\}$ and with $S \times \{0\} \times \{0\}$ if no confusion is possible). The existence of ϕ_1 follows from Theorem 2; the extra property of T_s is permitted because of the choice on the intial conditions in this theorem.

We may also suppose that $\exists \mu > 0$ such that $\overline{T}_s \times D^{n-p}(\mu)$ is a neighbourhood of $\overline{T}_s$ in $T_s(V_0) \cap T_s(V)$.

We know that $\overline{T}_s \subset [\bigcup_{k<j} \overline{W}_u(\gamma_k)] \times \{0\} \times [0,1]$ and each of these submanifolds $\overline{W}_u(\gamma_k) \times \{0\} \times [0,1]$ is transverse to $\overline{T}_s$ in S_0.

We can choose μ so small that the unstable manifolds $W_u(\gamma_k)$ of U are transverse to $\overline{T}_s \times D^{n-p}(\mu)$.

We now consider $\overline{T}'_s = \overline{T}'_{s_0} \times \{0\} \times [0,1]$ with $\overline{T}'_{s_0} \subset \overline{T}_{s_0}$ defined in such a way that $\overline{T}'_{s_0}$ is a neighbourhood of $\overline{W}_s(\gamma_j) \cap \overline{T}_{s_0}$, the closure of $\overline{T}'_{s_0}$ lies in the interior of $\overline{T}'_{s_0}$, and $\overline{T}'_{s_0}$ is diffeomorphic to the product of a sphere with a disc.

By the induction hypothesis it is easy to show that to each point $q \in \overline{T}'_{s_0}$ we can associate a neighbourhood $\overline{O}_q$ of q in $\overline{T}_{s_0}$, numbers $\epsilon(q)$, $\delta(q) > 0$ and an embedding

$$\phi_q : \overline{O}_q \times D^{n-p}(\epsilon(q)) \times [0,1] \longrightarrow T_s$$

such that ϕ_q is C^∞, ∞-near to the identical embedding along $S \times \{0\} \times [0,1]$, respecting the parameter τ,

$$\phi_q(\overline{O}_q \times D^{n-p}(\epsilon(q)) \times [0,1]) \subset \overline{T}_s \times D^{n-p}(\mu), \quad \text{and}$$

$$W_u(\gamma_i, \delta(q)) \cap \phi_q(\overline{O}_q \times D^{n-p}(\epsilon(q)) \times [0,1]) \subset \phi_q(\overline{O}_q \times D^{k(\gamma_i)}(\epsilon(q)) \times [0,1]).$$

(ϕ_q can be obtained by restricting φ_q in the induction hypothesis to some $\overline{O}_q \times D^{n-p}(\epsilon(q)) \times [0,1]$, and projecting it along the orbits of U onto T_s – in small flow boxes for instance).

Taking a finite covering of $\overline{T}'_{s_0}$ we can find $\epsilon > 0$, $\delta > 0$ and a C^∞ embedding $\phi : \overline{T}'_s \times D^{n-p}(\epsilon) \longrightarrow T_s$, ∞-near to the identity along $\overline{T}'_s \times \{0\}$, respecting τ, and such that

$$W_u(\gamma_i, \delta) \cap \phi(\overline{T}'_s \times D^{n-p}(\epsilon)) \subset \phi(\overline{T}'_s \times D^{k(\gamma_i)}(\epsilon)) \subset \overline{T}_s \times D^{n-p}(\mu).$$

As ϕ is ∞-near to the identity along $\overline{T}'_s$ and $k(\gamma_i) \leqslant k(\gamma_j)$ (see § 4) we can, choosing ϵ sufficiently small and reparametrizing ϕ obtain that ϕ and ϕ_1 coincide on

$$(\overline{W}_s(\gamma_j) \cap \overline{T}_{s_0}) \times [0,1] \times (D^{n-p}(\epsilon)) \cap \{y_{k(\gamma_j)+1}, \ldots, y_{n-p}\},$$

and that they are even ∞-near to each other on that set. (Here we use the results of point 2).

We can now extend $\phi : \overline{T}'_s \times D^{n-p}(\epsilon) \longrightarrow \overline{T}_s \times D^{n-p}(\mu)$ to some diffeomorphism ϕ' from $T_s(V_0)$ onto $T_s(V)$ in such a way that ϕ' and $\phi_1 | T_s(V_0)$ are ∞-near to each other on $\overline{T}_s \cup W_s(\widetilde{U}_0, \gamma_j)$. Hence $\phi_1^{-1} \circ \phi' : T_s(V_0) \longrightarrow T_s(V_0)$ is a diffeomorphism which is ∞-near to the identity on $\overline{T}_s \cup W_s(\widetilde{U}_0, \gamma_i)$.

Because of Theorem 2 it is possible to find a diffeomorphism $\phi_2 : V_0 \longrightarrow V_0$ such that:

$$\phi_2\Big|_{T_s(V_0)} = \phi_1^{-1} \circ \phi' \quad \text{and} \quad (\phi_2)_* \widetilde{U}_0 = \widetilde{U}_0 .$$

Then $\psi = \phi_1 \circ \phi_2$ is a diffeomorphism from V_0 onto V such that $\psi_* \widetilde{U}_0 = U$ and $\psi \mid \overline{T}_s' \times D^{n-p}(\epsilon) = \phi$.

If now $K \subset T_s(V)$, we denote by Sat K the union of the trajectories of U in V issueing from points in K, then $\overline{V}_0' = \overline{\text{Sat}(\overline{T}_{s_0}')}$ is a neighbourhood of γ_j in S and $\overline{\text{Sat}(\overline{T}_s')} = \overline{V}_0' \times [0,1] = \overline{V}'$.

Let us by $W_0 \subset V_0$ denote an adapted neighbourhood for $\widetilde{U}_0$ whose stable fence $T_s(W_0)$ is such that $T_s(W_0) \cap S_0 = \overline{T}_s'$ and near $\overline{T}_s'$, $T_s(W_0)$ is exactly $\overline{T}_s' \times D^{n-p}(\epsilon)$. $\psi(W_0) = W$ is an adapted neighbourhood for U with $T_s(W) = \psi(T_s(W_0))$. We choose $0 < \delta' \leqslant \delta$ with

$$\delta' \leqslant \min \{\|y\| \mid (x, y, \tau) \in T_s(W) \setminus \psi(\overline{T}_s' \times D^{n-p}(\epsilon))\} .$$

We furthermore take a neighbourhood $\overline{W}_0'$ of γ_j inside $\overline{V}_0'$ such that $\overline{W}_0' \times D^{n-p}(\epsilon') \subset W_0$ (with $0 < \epsilon' \leqslant \epsilon$, $W' = \overline{W}_0' \times [0,1]$). We claim that $\psi : \overline{W}_0' \times D^{n-p}(\epsilon') \times [0,1] \longrightarrow S \times \mathbf{R}^{n-p} \times [0,1]$ and the numbers ϵ' and δ' satisfy the required properties.

As a matter of fact: suppose $x \in W_u(\gamma_j, \delta') \cap \psi(\overline{W}' \times D^{n-p}(\epsilon'))$. Since $\psi(\overline{W}' \times D^{n-p}(\epsilon)) \subset W$, $\exists x_0 \in T_s(W)$ and $\exists u \in R_+$ such that $U_u(x_0) = x$, where U_u denotes the flow of U.

By this choice of δ', we necessarily have $x_0 \in \psi(\overline{T}_s' \times D^{n-p}(\epsilon))$. But $x_0 \in \psi(\overline{T}_s' \times D^{n-p}(\epsilon')) \cap W_u(\gamma_i, \delta')$, with $\delta' \leqslant \delta$, implies that $x_0 \in \phi(\overline{T}_s' \times D^{k(\gamma_i)}(\epsilon'))$, hence $\exists \alpha \in D^{k(\gamma_i)}(\epsilon')$ and $q \in \overline{T}_s'$ such that $x_0 = \phi(q, \alpha) = \psi(q, \alpha)$. Now, since $\psi_* \widetilde{U}_0 = U$ and since we know that the flow of $\widetilde{U}_0$ keeps invariant the foliation by the planes $\{y_1, \dots, y_{k(\gamma_i)}\}$, we find that

$$x = U_u(\psi(q, \alpha)) \in \psi(U_u(q) \times D^{k(\gamma_i)}(\epsilon')) .$$

This ends the proof of the proposition.

Besides the result of Proposition 6.3, we are also going to use the following.

PROPOSITION 6.4. – *Let* $\delta > 0$ *be given. There exists a system* $V(\gamma_1), \ldots, V(\gamma_m)$ *of adapted neighbourhoods for the singularities* $\gamma_1, \ldots, \gamma_m$ *of* U *(ordered as in § 4) such that*

(i) $W = \bigcup_{\gamma_i} V(\gamma_i) \subset W(\delta)$

(ii) W is a neighbourhood of S_0

(iii) $\begin{cases} T_s(\gamma_i) \cap V(\gamma_j) = \phi & \text{if } j > i \\ T_u(\gamma_i) \cap V(\gamma_j) = \phi & \text{if } j < i . \end{cases}$

Proof. – Because of the properties of the field $\overline{U}_0$ it is easy to find a system of adapted neighbourhoods $V_1(\gamma_1), \ldots, V_1(\gamma_m)$ such that iii) holds in intersection with S_0, such that $\forall i : W_u(\gamma_i)$, $W_s(\gamma_i) \subset W(\delta)$ and such that $W_1 = \bigcup_{\gamma_i} V_1(\gamma_i)$ is a neighbourhood of S_0 ($W_u(\gamma_i)$ and $W_s(\gamma_i)$ are the unstable and stable manifold of γ_i in $V_1(\gamma_i)$).

We can now modify the $V_1(\gamma_i)$ in such a way (by retracting each $V_1(\gamma_i)$ along $W_s(\gamma_i) \cup W_u(\gamma_i)$) that the three conditions in the statement are verified.

7. Proof of Theorem 3.

Let $\delta > 0$ be as in Proposition 6.1 and let $V(\gamma_1), \ldots, V(\gamma_m)$ be a system of adapted neighbourhoods for the singularities of U as in Proposition 6.4.

We are going to prove Theorem 3 for the neighbourhoods $W' = W(\delta)$ and $W = \bigcup_{\gamma_i} V(\gamma_i)$.

According to Proposition 6.1, each function $b \in C^\infty(W', S_0)$ is a sum of a function in $C^\infty(W', W_s(\delta))$ and of a function in $C^\infty(W', W_u(\delta))$. It suffices to prove Theorem 3 for a second member (b_j) where $\forall j : b_j \in C^\infty(W', W_u(\delta))$ or $\forall j : b_j \in C^\infty(W', W_s(\delta))$. We are only going to treat the first case; the second is analogous. Let us hence suppose that all the given b_j belong to $C^\infty(W', W_u(\delta))$. We will show the existence of the a_j by induction on the index of the singularities.

1) *For* γ_m. As each b_j is ∞-flat on $V(\gamma_m) \cap W_u(\delta) \supset W_u(\gamma_m)$, unstable manifold of U in $V(\gamma_m)$, there exists a solution to system (E) in $V(\gamma_m)$ (using Theorem 2; we can for instance take the solution with initial conditions $a_j\big|_{T_u(\gamma_m)} = 0 \ \forall j = 1, \ldots, n)$. The solution (a_j) is such that each a_j is C^∞ in $V(\gamma_m)$ and ∞-flat along $V(\gamma_m) \cap W_u(\delta)$.

2) Suppose we have found (a_j) on $V(\gamma_m) \cup \ldots \cup V(\gamma_{m-k+1})$ such that each a_j is C^∞ and ∞-flat along

$$[V(\gamma_m) \cup \ldots \cup V(\gamma_{m-k+1})] \cap W_u(\delta).$$

We consider $V(\gamma_{m-k})$. The restriction of each a_j to $T_u(\gamma_{m-k})$ is C^∞ on $T_u(\gamma_{m-k}) \cap [V(\gamma_m) \cup \ldots \cup V(\gamma_{m-k+1})]$ and ∞-flat on $T_u(\gamma_{m-k}) \cap [V(\gamma_m) \cup \ldots \cup V(\gamma_{m-k+1})] \cap W_u(\delta)$. We choose an extension α_j of this restriction on $T_u(\gamma_{m-k})$ which is C^∞ and ∞-flat along $T_u(\gamma_{m-k}) \cap W_u(\delta)$ by the Whitney extension theorem. Then system (E) has a unique solution $(\overline{a_j})$ in $V(\gamma_{m-k})$, C^∞ and ∞-flat on $V(\gamma_{m-k}) \cap W_u(\delta)$, and such that $\overline{a_j} \mid T_u(\gamma_{m-k}) = \alpha_j$. We claim that $\overline{a_j} = a_j$ on $V(\gamma_{m-k}) \cap [V(\gamma_m) \cup \ldots \cup V(\gamma_{m-k+1})]$. Indeed, if $m \in V(\gamma_{m-k}) \cap V(\gamma_{m-\ell})$ with $\ell < k$, we consider the orbit $U_t(m)$ of m. As $m \in V(\gamma_{m-k})$ $\exists s > 0$ such that $U_s(m) \in T_u(\gamma_{m-k})$. Then the whole segment

$$U_{[0,s]}(m) \subset V(\gamma_{m-k}) \cap V(\gamma_{m-\ell}).$$

If not, there would exist an $s' \in]0,s]$ such that $U_{s'}(m) \in T_u(\gamma_{m-\ell})$. But then $U_{s'}(m) \in V(\gamma_{m-k}) \cap T_u(\gamma_{m-\ell})$, and hence

$$V(\gamma_{m-k}) \cap T_u(\gamma_{m-\ell}) \neq \emptyset,$$

unlike the conditions described in Proposition 6.4.

In the point $U_s(m) : \overline{a_j}(U_s(m)) = \alpha_j(U_s(m)) = a_j(U_s(m)) \ \forall j$. The two solutions (a_j) and $(\overline{a_j})$ of equation (E) coincide on $U_s(m)$ and hence coincide on the whole part of the orbit contained in the intersection of their domain of definition $V(\gamma_{m-\ell}) \cap V(\gamma_{m-k})$. In particular $q_j(m) = \overline{a_j}(m) \ \forall j = 1, \ldots, n$ and $\overline{a_j}$ is indeed an extension of a_j.

By induction on $k = 1, \ldots, m-1$ we find a solution (a_j) defined and C^∞ on $W = \bigcup_{\gamma_i} V(\gamma_i)$. Q.E.D.

Remark 1. – It is perhaps good to notice that our method permits to prove the following:

THEOREM C. – *If* $\rho \in \mathrm{Act}(\mathbf{R}^2, \mathbf{R}^n)$ (resp. $\rho \in \mathrm{Act}(\mathbf{C}, \mathbf{C}^n)$) *has a hyperbolic 1-jet and is formally linearizable, then* ρ *is* C^∞ *linearizable.*

Proof. – If ρ is formally linearizable, then (by a theorem of Borel) there exists a C^∞ diffeomorphism φ such that

$$j_\infty(\varphi * \rho)(0) = j_\infty(\alpha)(0)$$

where α is a linear inf. $\mathbf{R}^2$ action such that $j_1(\alpha)(0) = j_1(\rho)(0)$.

Since $j_1(\rho)(0)$ is hyperbolic we can choose two hyperbolic generators X and Y for $\varphi * \rho$: $X = X_0 + \widetilde{X}$ and $Y = Y_0 + \widetilde{Y}$ where X_0, Y_0 are linear, while $j_\infty(\widetilde{X})(0) = 0 = j_\infty(\widetilde{Y})(0)$. Now using a result of [10] we know the existence of a diffeomorphism $\psi : (\mathbf{R}^n, 0) \hookleftarrow$ such that $\psi * (X) = X_0$; ψ is ∞-near to the identity in 0, hence $j_\infty(\psi * Y)(0) = j_\infty(Y_0)(0)$. We are now exactly in the same situation as for Theorem A (in § 3) after using the theorem of Sternberg to linearize one of the generators of the action.

Remark 2. – The method is also appropriate to prove the linearization of certain singularities of $\mathbf{R} \times \mathbf{Z}$ and $\mathbf{Z} \times \mathbf{Z}$-actions.

BIBLIOGRAPHY

[1] A.D. BRJUNO, The analytic form of differential equations, *Trans. Moscow Math. Soc.*, Vol. 25 (1971), 131-288.

[2] C. CAMACHO, On $\mathbf{R}^k \times \mathbf{Z}^\ell$-actions, Dynamical Systems, *Proc. of the Salvador Symposium*, Ed. M.M. Peixoto, (1973), 23-70.

[3] C. CAMACHO, N.H. KUIPER, J. PALIS, La topologie du feuilletage d'un champ de vecteurs holomorphes près d'une singularité, *C.R. Acad. Sc.* Paris, t. 282 (1976), 959-961.

[4] C. CAMACHO, N.H. KUIPER, J. PALIS, The topology of holomorphic flows with singularity, Preprint of the I.H.E.S., Bures-sur-Yvette.

[5] F. DUMORTIER, R. ROUSSARIE, Linéarisation différentiable des germes singuliers d'actions de $\mathbf{R}^2$ et de champs de vecteurs holomorphes, *C.R. Acad. Sc.* Paris, t. 285 Série A (1977), 841.

[6] J. GUCKENHEIMER, Hartman's theorem for complex flows in the Poincaré domain, *Compositio Math.*, 24 (1972), 75-82.

[7] P. HARTMAN, On the local linearization of differential equations, *Proc. Amer. Math. Soc.*, 14 (1963), 568-573.

[8] M. HIRSCH, C.C. PUGH, M. SHUB, Invariant manifolds, *Lecture Notes in Math.*, 583 (1977).

[9] N. KOPELL, Commuting diffeomorphisms global analysis, *Proc. of Symp. in Pure Math.*, XIV, (1970), 165-184.

[10] R. ROUSSARIE, Modèles locaux de champs et de formes, *Astérisque*, Vol. 30 (1975).

[11] R. ROUSSARIE, Singularités des germes de champs de vecteurs, *Bol. Soc. Brasileira de Matemática* (to appear).

[12] G.L. SIEGEL, Nachr. Akad. Wiss Göttingen, *Math. Phys.*, I (1952), 21-30.

[13] S. SMALE, Differentiable dynamical systems, *Bull. Amer. Math. Soc.*, 73 (1967).

[14] S. STERNBERG, On the structure of local homeomorphisms of Euclidean n-space II, *J. of Math.*, Vol. 80 (1958), 623-631.

Manuscrit reçu le 22 juin 1979.

F. DUMORTIER,
Limburgs Universitair Centrum
Universitaire Campus
3610 Diepenbeek (Belgium)
&
Instituto de Matemática
Pura e Aplicada
Rua Luiz de Camoes, 68
20.000 Rio de Janeiro (Brasil).

R. ROUSSARIE,
Département de Mathématiques
Université de Dijon
214, rue de Mirande
21000 Dijon (France).