SMOOTH MAPS TRANSVERSE TO A FOLIATION

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1. Introduction. This article presents a Smale-Hirsch-type classification theorem for smooth maps transverse to a foliation. Let M, W be smooth manifolds, with tangent bundles TM, TW, and let $\operatorname{Hom}(M, W)$, $\operatorname{Hom}(TM, TW)$ represent the spaces of smooth maps $M \to W$ and of fibrewise linear maps $TM \to TW$, where we give to $\operatorname{Hom}(TM, TW)$ the compact-open topology, and to $\operatorname{Hom}(M, W)$ the C^1-compact-open topology; thus the map $d:\operatorname{Hom}(M, W) \to \operatorname{Hom}(TM, TW)$, which associates to each smooth map its differential, is continuous.

Suppose W carries a foliation $\mathfrak{F}$, and let $T\mathfrak{F}$ denote the subbundle of TW tangent to $\mathfrak{F}$ (i.e. the embedding $T\mathfrak{F} \to TW$ is an integrable distribution). Let $\operatorname{Trans}(TM, T\mathfrak{F})$ be the subspace of $\operatorname{Hom}(TM, TW)$ consisting of those maps fibrewise transverse to $T\mathfrak{F}$, and let

$$\operatorname{Trans}(M, \mathfrak{F}) = d^{-1} \operatorname{Trans}(TM, T\mathfrak{F}) \subset \operatorname{Hom}(M, W).$$

THEOREM 1. *If M is open, then the differential map $d:\operatorname{Trans}(M, \mathfrak{F}) \to \operatorname{Trans}(TM, T\mathfrak{F})$ is a weak homotopy equivalence.*

Suppose now W has a Riemannian metric, so we can define $N\mathfrak{F}$, the normal bundle to $\mathfrak{F}$, to be the bundle whose fibre at $x \in W$ is the orthogonal complement to $T\mathfrak{F}_x$. Then the space $\operatorname{Epi}(TM, N\mathfrak{F})$ of fibrewise linear and surjective maps $TM \to N\mathfrak{F}$ is a subspace and, in fact, a deformation retract, of $\operatorname{Trans}(TM, T\mathfrak{F})$. If we let $p:\operatorname{Hom}(TM, TW) \to \operatorname{Hom}(TM, TW)$ be composition with fibrewise orthogonal projection of TW onto the sub-bundle $N\mathfrak{F}$ then Theorem 1 has the immediate corollary:

THEOREM 2. *If M is open, then the map $p \circ d:\operatorname{Trans}(M, \mathfrak{F}) \to \operatorname{Epi}(TM, N\mathfrak{F})$ is a weak homotopy equivalence.*

REMARKS. Theorem 1, which was proposed to the author by J. W. Milnor, has a special case (where $\mathfrak{F}$ = the foliation by points) the

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author's classification of submersions [6, Theorem A]. This theorem, as well as the rest of the Smale-Hirsch-type theorems for open manifolds, also follows from a general theorem proved by M. L. Gromov in his dissertation [2].

As an application, let us give a short proof of the following result from [7].

THEOREM 3. *Let σ be a q-plane field on an open manifold M. If the structural group of σ considered as a q-plane bundle can be reduced to a discrete group, then σ is homotopic to the q-plane field normal to a foliation.*

PROOF (see also [2]). Let S be the total space of the bundle σ; by a theorem of Ehresmann [1], [3], S has a foliation $\mathfrak{F}$ of codimension q of which the zero cross-section is a leaf. Orthogonal projection: $TM \to \sigma$ can be interpreted as an element H_0 of $\mathrm{Epi}(TM, N\mathfrak{F})$ via the usual identification of σ with the tangent space to the fibres of S along the zero cross-section. Theorem 2 implies that H_0 is homotopic in $\mathrm{Epi}(TM, N\mathfrak{F})$ to $H_1 = p \circ df$, where $f \in \mathrm{Trans}(M, \mathfrak{F})$. If $t \to H_t$ is the homotopy then $t \to$(orthogonal complement of ker H_t) gives a homotopy between σ and the q-plane field normal to the pulled-back foliation $f^* \mathfrak{F}$.

Theorem 1 has also been applied to the study of classifying spaces for foliations [4], [5].

2. **Outline of proof of Theorem 1.** The proof follows the lines of the proof of the submersion theorem of [6]. This method of proof can be summarized as follows. In order to compare the spaces $\mathrm{Trans}(M, \mathfrak{F})$ and $\mathrm{Trans}(TM, T\mathfrak{F})$, we consider the pair of spaces $\mathrm{Trans}(U, \mathfrak{F})$ and $\mathrm{Trans}(TU, T\mathfrak{F})$ for each closed submanifold-with-boundary[1] $U \subset M$, with $\dim U = n = \dim M$. The assignments $U \to \mathrm{Trans}(U, \mathfrak{F})$, $U \to \mathrm{Trans}(TU, T\mathfrak{F})$ can be thought of, following Gromov, as contravariant functors from the category $\mathcal{C}_M$ of closed n-dimensional submanifolds-with-boundary of M and inclusion maps, to the category $\mathfrak{I}$ of topological spaces.

DEFINITION. A functor $A: \mathcal{C}_M \to \mathfrak{I}$ will be called admissible if it has the following properties.

(a) A is locally defined, in that if $U_1 \cap U_2 \in \mathcal{C}_M$, and if $X \subset A(U_1) \times A(U_2)$ is defined by $X = \{(f_1, f_2), f_1 | U_1 \cap U_2 = f_2 | U_1 \cap U_2\}$, then the natural map $A(U_1 \cup U_2) \to X$ is a homeomorphism.

[1] If $U \subset M$ is a manifold with boundary, we define $f \in \mathrm{Trans}(U, W)$ to mean that f extends to a transverse map of an open neighborhood of U in M, and TU to be $TM|U$. These manifolds with boundary may have corners, as described in [6, §0].

(b) If $V \supset U$ is a collarlike neighborhood of U (see [6, p. 176] for a precise definition) then the restriction map $A(V) \to A(U)$ is a weak homotopy equivalence, and has the covering homotopy property.

(c) If $V = U$ with a handle of index λ attached, then the restriction map $A(V) \to A(U)$ has the covering homotopy property. If M is open and n-dimensional then this property need only be satisfied for handles of index $\lambda \leqq n-1$.

PROPOSITION (SMALE-THOM-HIRSCH-PALAIS-HAEFLIGER-POENARU THEOREM PROVING MACHINE). *Let $A, B: \mathcal{C}_M \to \mathfrak{I}$ be admissible functors, and let $\Phi: A \to B$ be a natural transformation. If $\Phi: A(D^n) \to B(D^n)$ is a weak homotopy equivalence for each embedded n-disc $D^n \subset M$, then so is $\Phi: A(M) \to B(M)$.*[2]

PROOF. See [6, §6].

Theorem 1 will follow from this Proposition once it is shown that, on an open manifold, Trans(, $\mathfrak{F}$) and Trans(T , $T\mathfrak{F}$) are admissible functors, and that d:Trans(D^n, $\mathfrak{F}$)$\to$Trans(TD^n, $T\mathfrak{F}$) is a homotopy equivalence. Most of this is a straightforward generalization of the corresponding lemmas for submersions. The only point that seems to require new analysis is showing that Trans(, $\mathfrak{F}$) has property (c). This is treated in the next two sections.

3. **The covering homotopy property.** The proof of this property in the submersion case involves a long, geometric argument [6, §4]; examination of this argument shows that it uses only the following facts about submersions:

(a) submersions are stable in the sense of [6, Lemma 3.1];

(b) submersions form an open and locally defined subspace of Hom(M, W);

(c) if $f: M \to W$ is a submersion and h is a diffeomorphism of M, then $f \circ h$ is a submersion.

Facts (b) and (c) are clearly also true of maps transverse to a foliation. (It turns out that facts (b) and (c) alone are sufficient, and that if M is open the appropriate Smale-Hirsch type theorem holds for any subspace of Hom(M, W) satisfying these two conditions. This observation is due to Gromov [2].) In order to use the "good position" method of proof, it remains to establish an analogue to the stability lemma; the statement is below.

Let $U \subset M$ be a compact manifold-with-boundary, and suppose

[2] If M is not compact, let $A(M)$ be the inverse limit of $A(U_i)$ where $U_i \in \mathcal{C}_M$, $U_i \subset U_{i+1}$ and $\bigcup U_i = M$.

given $f \in \text{Trans}(U, \mathfrak{F})$. By definition, f extends to a transversal map of an open neighborhood of U. In particular, we may suppose that $f = \bar{f} | U$, where $\bar{f} \in \text{Trans}(L, \mathfrak{F})$ and $L \subset M$ is a compact manifold-with-boundary, $U \subset \text{Int } L$. Let E be the total space of $\bar{f}^* T\mathfrak{F}$, let $\beta: E \to T\mathfrak{F}$, $\pi: E \to L$ be the canonical maps, and let $\mathcal{G}$ be the foliation of E by fibres.

LOCAL FACTORING LEMMA. *With data as above, there exist*

(1) *an open tubular neighborhood N of the zero cross-section in E (we will consider N as an open manifold with boundary $\partial N = \pi^{-1}\partial L$)*;

(2) *a submersion $\phi: N \to W$ with $d\phi(T\mathcal{G}) \subset T\mathfrak{F}$*;

(3) *a neighborhood η of f in* $\text{Hom}(U, W)$;

(4) *a continuous map $\nu: \eta \to \text{Aut}(N, \mathcal{G})$ (the space of foliation-preserving diffeomorphisms of N which are the identity near ∂N) such that $\nu_f = \text{id}$ and such that $g = \phi \circ \nu_g | U$ for $g \in \eta$.*

REMARKS. Roughly speaking, this lemma means that f can be extended to a submersion ϕ of a larger manifold N in such a way that maps nearby to f can be obtained by composing ϕ with leaf-preserving diffeomorphisms of N nearby to the identity, and which are equal to the identity near ∂N. Condition (2) implies that if a map h is transversal to $\mathcal{G}$, then $\phi \circ h$ will be transversal to $\mathfrak{F}$.

This lemma is proved in the next section. Let us now see how it is used to lift an arc of maps from $\text{Trans}(U, \mathfrak{F})$ to $\text{Trans}(V, \mathfrak{F})$. The technical details involved in lifting a homotopy of a cube of dimension >0 are completely analogous to those for the submersion case. The pictures in [6, §4], which illustrate the special case of this argument for $\mathfrak{F}$ the foliation by points, should be consulted.

Lifting an arc. Suppose $F_0 \in \text{Trans}(V, \mathfrak{F})$ and that f_t, $0 \leq t \leq 1$, is a homotopy of $f_0 = F_0 | U$. Each f_t has a neighborhood η_t as described above; clearly, we may suppose that $f([0, 1]) \subset \eta_0$. Let $\phi: N \to W$ be the submersion corresponding to η_0, and let $\nu: \eta_0 \to \text{Aut}(N\mathcal{G})$ be as in the local factoring lemma.

We define a *collar neighborhood* C of U in V to be a neighborhood diffeomorphic to $U \cup \dot{U} \times [0, 1]$, where $\dot{U} \cong S^{\lambda-1} \times D^{n-\lambda}$ is the attaching surface of the handle. Let $\dot{C}$ be the boundary of C in V (see [6, Figure 4.5].)

We say that F_0 is *in good position with respect to* ϕ if we can find a collar neighborhood C of U in V and an embedding $\beta: C \to N$ such that

(1) $\beta | U$ is the zero cross-section;

(2) $\phi \circ \beta = F_0 | C$;

(3) $\beta(\dot{C}) \subset \partial N$;

(4) β is transverse to $\mathcal{G}$.

If F_0 is in good position with respect to ϕ, then the arc f_t can be lifted to Trans(V, $\mathfrak{F}$) by defining

$$F_t(x) = F_0(x), \qquad x \in V - C,$$
$$= \phi \circ \nu_t \circ \beta(x), \qquad x \in C.$$

Otherwise we remark that F_0 is in good position with respect to $\phi | S$, where $S \subset N$ is some smaller tubular neighborhood. This follows from comparing the maps $F_0 | L \cap V$ and $\phi | L \cap V$ (where $L \subset N$ as the zero cross-section). These maps agree on U, so they are close near U, so since ϕ is a submersion there exists, by [6, Lemma 3.1], an embedding $\kappa: \hat{U} \to N$, where $\hat{U} \subset L \cap V$, $\hat{U} \cong U \cup \dot{U} \times [0, 1]$ is a collar neighborhood of U in V, such that $\phi \circ \kappa = F_0 | \hat{U}$, and $\kappa | U$ is the zero cross-section. If $\hat{U}$ is chosen small enough κ will be transverse to $\mathcal{G}$. Then pick L' such that $U \subset \text{Int } L'$ and $\pi \circ \kappa(\hat{U}) \cap \partial L' = \pi \circ \kappa(\dot{U} \times \{\frac{1}{2}\})$. Let $S = \pi^{-1} L' \cap N$. Then taking $C = U \cup \dot{U} \times [0, \frac{1}{2}]$ and $\kappa | C: C \to S$ shows that F_0 is in good position with respect to $\phi | S$.

Now pick an $\epsilon > 0$ such that $\nu_t(U) \subset S$ for $t \leqq \epsilon$ and such that the arc of embeddings $\nu_t | U: U \to S$ can be realized by composing the zero cross-section with an arc σ_t in Aut(S, $\mathcal{G}$). Then the argument above shows how to lift $f([0, \epsilon])$ to an arc $\hat{F}: [0, \epsilon] \to$ Trans(V, $\mathfrak{F}$) starting at F_0.

In order to continue past ϵ we change $\hat{F}$ to a new lifting $\tilde{F}$ such that $\tilde{F}_\epsilon$ is in good position with respect to $\phi \circ \nu_\epsilon$. Briefly, this is done by a $\mathcal{G}$-transverse isotopy of $\kappa | \dot{U} \times [\frac{1}{2}, 1]$, keeping ends fixed, in a larger tubular neighborhood $\hat{N}$. The isotopy described in [6, Sublemma 4.6] may be performed on $\pi \circ \kappa$ and lifted under π, starting at κ, to give the desired arc of maps.

4. Proof of local factoring lemma.

PROOF. (With notation from §3). Pick a connection in TW such that if $v \in T\mathfrak{F}$, the arc $t \to \exp(tv)$ lies in a leaf, and define $\phi: E \to W$ by

$$\phi(v) = \exp_{\bar{f}(\pi(v))} \beta(v).$$

It follows from transversality of $\bar{f}$ that this ϕ is a submersion along L (which we identify with the zero cross-section in E) and therefore on some tubular neighborhood N of L in E, e.g. $\{|v| < \epsilon\}$ for some $\epsilon > 0$. This is essentially a "foliated tubular neighborhood," as described in [8, Proposition 3.1]. Observe that by the choice of connection, $d\phi(T\mathcal{G}) \subset T\mathfrak{F}$, as required.

Suppose $g \in$ Hom(U, W) is C^1-close to f. Then since ϕ is a submersion there is an embedding $\mu_g: U \to N$ such that $\phi \circ \mu_g = g$; in fact the argument of [6, Lemma 3.1] gives a continuous map $\mu: \eta$

$\to \mathrm{Emb}(U, N)$ (where η is a neighborhood of f in $\mathrm{Hom}(U, W)$ and $\mathrm{Emb}(U, N)$ is the space of embeddings) such that $\mu_f =$ the zero cross-section and $\phi \circ \mu_g = g$ for $g \in \eta$. If g is close enough to f, μ_g will also be transverse to the fibres of $\mathcal{G}$, and will extend to an embedding: $L \to N$ transverse to the fibres and equal to the zero cross-section near ∂L. This embedding in turn will extend to a fibre-preserving diffeomorphism ν_g of N, which leaves a neighborhood of $\partial N = \pi^{-1}\partial L$ fixed; it is easy to check that these extensions can be defined so as to depend continuously on g.

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