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AND n-SIDED PATCHES

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ABSTRACT. The theory of 'geometric continuity' within the subject of CAGD is reviewed. In particular, we are concerned with how parametric surface patches for CAGD can be pieced together to form a smooth C^k surface. The theory is applied to the problem of filling an n-sided hole occurring within a smooth rectangular patch complex. A number of solutions to this problem are surveyed.

1. Introduction

The topics addressed by these tutorial lectures are those of polygonal patches and the theory of geometrically smooth parametric surfaces. In particular, we will consider the problem of filling a polygonal hole occurring within a smooth rectangular patch complex. This problem, which is frequently encountered by current 'free-form' or 'sculptured' surface modellers, illustrates the need for the theory of 'geometric continuity' within the subject of CAGD (Computer Aided Geometric Design). This theory is concerned with how parametric surface patches can be pieced together to give a smooth C^k surface.

A C^k surface is one which locally admits a C^k parameterization. Thus the surface can be considered as a collection of overlapping patches, each defined as a C^k map from an open domain in $\mathbb{R}^2$ into $\mathbb{R}^3$. In CAGD, however, the surface is composed of a number of non-overlapping patches, each defined on a closed domain in $\mathbb{R}^2$. We are thus concerned with how to join together such closed patches in order to obtain a c^k surface.

The simplest type of join between two patches is that of a 'parametric continuous' C^k join, where the parameter domains of two patches abut along a common edge and the surface is C^k on the composite domain. In effect, the surface is one composite patch, the domain being the union of the individual patch domains. This is the situation most frequently encountered when composing a surface of rectangular patches, where the patch complex can be considered as a mapping from a parameteric domain subdivided by a regular rectangular mesh. More generally, two patches can have a 'geometric continuous' GC^k join. Here we will see that, locally, there is a C^k reparameterization in which the composite surface is parametrically C^k. A parametric continuous join is then the

special case of where the reparameterization is defined by the identity map.

Whilst current surface modellers use almost entirely rectangular patches, it is impossible to model a complex surface as a single map from a regular rectangular meshed domain. More complex surface topologies will require n-sided polygonal holes within the rectangular patch complex to be filled in. This situation typically arises where two or more rectangularly meshed surfaces are to be blended together. One solution might be to allow n rectangular patches to meet at a common vertex in R^3, whilst another solution might be the construction of a special polygonal patch. In either case the concept of a geometrically continuous join of the patches is necessary.

The lectures are organised as follows. Section 2 develops the geometric continuity tools needed for the study of smooth parametric surface patch complexes. In section 3, rectangular patch representations are briefly reviewed and the n-sided polygonal hole problem is described. Solutions to the polygonal hole problem are then discussed in sections 4 and 5.

2. Geometrically Smooth Parametric Surfaces

The geometric continuous GC^k join of two surface patches is a terminology that is now well established in CAGD. However, the differential geometer might well prefer to state that two such patches meet with ' contact of order k' . The idea will be explained through that of a reparameterization of the surface, see, for example, DeRose [21], Hollig [52], Gregory and Hahn [39],and Hahn [48]. Here we will summarize some of the material contained in Gregory [37] which is based on the exposition of Hahn [48]. We first introduce the reparameterization approach by considering the simpler case of planar curves.

2.1 Geometric continuous curves

Consider the two planar curve segments

$$
\left.
\begin{aligned}
p(t) &= (t, t^2 + t^3) \ , \ -2 \leq t \leq 0 \ , \\
q(t) &= (2t, 4t^2) \ , \ \ 0 \leq t \leq 1 \ .
\end{aligned}
\right\}
\tag{2.1}
$$

Clearly p(2t), -1 < t < 0, describes the same curve segment as p(t) but with different parameterization. Moreover, p(2t) and q(t) are (parametrically) C^2 at t = 0. We thus say that the orginal parameterizations p(t) and q(t) have a geometric continuous GC^2 join. This idea will be formalized in the definition below but first the need for the use of 'regular' parametric representations is explained.

A univariate parameterization $p : [a,b] \to \mathbb{R}^2$ is said to be a regular parametric representation of class C^k, k ≥ 1, if the first derivative tangent vector $p^{(1)}(t)$ does not vanish and if the component functions of p are k times continuously differentiable on [a,b]. Regularity, that is

a non-vanishing first derivative, is in general essential for the smooth-
ness of the curve. For example,consider the parametric curve

$$p(t) = (t^3, t^2) \ , \ -1 \leq t \leq 1 \ , \tag{2.2}$$

as shown in figure 2.1. Here p(t) is infinitely differentiable on [-1,1]
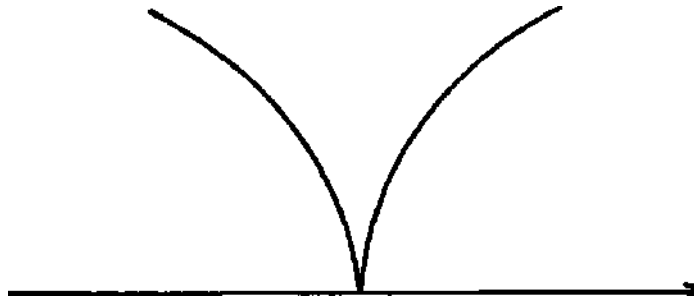
Figure 2.1. A non-regular parametric curve

but the curve does not have a continuous unit tangent vector at (0,0),
where t = 0. This behaviour is possible because the first derivative
vanishes at t = 0, where the parameterization is said to be singular.
Thus we restrict the discussion to parameterizations that are regular.
    We now define and give appropriate conditions for the geometric
continuous join of two regular parametric curves, see Barsky and DeRose
[8] and Goodman [31].

Definition 2.1. Let $\mathbf{p}$: $[a,b] \rightarrow \mathbb{R}^2$ and $\mathbf{q}$: $[c,d] \rightarrow \mathbb{R}^2$ be two regular
parametric representations of class C^k, $k \geq 1$. Then p and q join with
geometric continuity GC^k, at b and c respectively, if there exists a C^k
mapping φ : $\mathbb{R} \rightarrow \mathbb{R}$, defined in a neighbourhood of c, which is such that
(see figure 2.2)
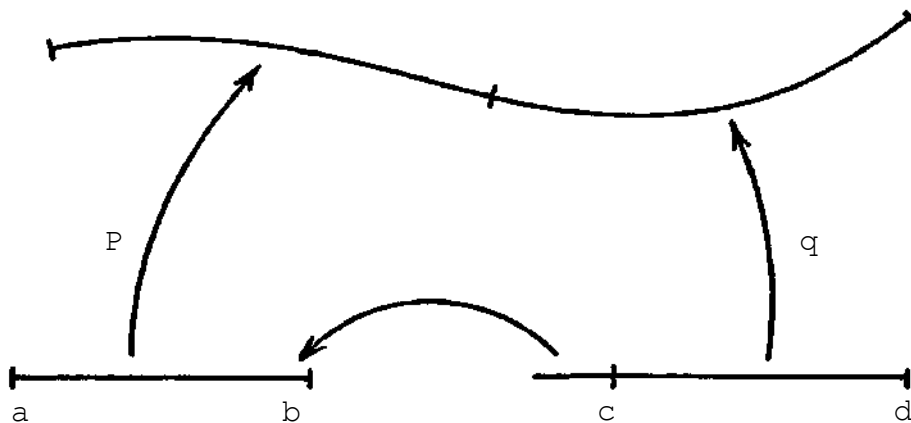
Figure 2.2.   Reparameterization for a C^k curve.

$$\varphi(c) = b, \varphi^{(1)}(c) > 0,$$

$$\frac{d^i}{dt^i} q(t) \Big|_{t=c} = \frac{d^i}{dt^i} p(\varphi(t)) \Big|_{t=c} \quad, i = 0, \ldots, k. \tag{2.3}$$

Here $\varphi(t)$ defines the reparameterization function and the condition (c) > 0 means that φ is monotonic increasing in a neighbourhood of c. This ensures that the reparameterization is orientation preserving and hence that the composite curve will not have a cusp at the join. For our planar curves, conditions (2.3) mean that the composite curve is a C^k function in a neighbourhood of the join, with respect to the projection onto the tangent line. The conditions are also equivalent to the composite curve being of class C^k with respect to the arc length parameterization. Thus the terminology of a 'geometric continuous arc length join' is sometimes used to distinguish this case from another type of geometric join, see Hagen (45] and Dyn and Micchelli [26], which is referred to in [37] as a 'geometric continuous Frenet frame join'.

Conditions (2.3) can be expressed in a form which avoids explicit reference to the mapping φ. For example, the curves join with GC^2 continuity if

$$\begin{aligned} q(c) &= p(b) \\ q^{(1)}(c) &= \beta_1 p^{(1)}(b) \quad, \quad \beta_1 > 0 \;, \\ q^{(2)}(c) &= \beta_2 p^{(1)}(b) \quad, \quad \beta_1^2 p^{(2)}(b) \;. \end{aligned} \tag{2.4}$$

These conditions are obtained using the chain and product rules in (2.3), giving the identifications

$$\beta_1 = \varphi^{(1)}(c) \quad, \quad \beta_2 = \varphi^{(2)}(c) \;.$$

They are given as k'th order contact of curve conditions for $k \leq 6$ in Geise [30] and their general form, Goodman [31], is given in the following proposition.

Proposition 2.2. The curves p and q join with geometric continuity GC^k if and only if there exist constants $\beta_1 > 0$ and $\beta_2, \ldots, \beta k$ such that

$$q^{(i)}(c) = \sum_{j=0}^{i} a_{ij} p^{(j)}(b) \;, \quad j = 0, \ldots k \;, \tag{2.5}$$

where

$$a_{i0} = \delta_{i0}, i = 0, \ldots, k,$$

$$a_{ij} = \sum_{i_1+\ldots+i_j=i} \begin{bmatrix} i \\ i_1 \cdot \ldots i_j \end{bmatrix} \beta_{i_1} \ldots \beta_{i_j}, 1 \le j \le i \le k.$$

(2.6)

Here the summation in (2.6) is over positive integers, where if $\{i_1, \ldots ij\}$ comprise r distinct elements with multiplicities $m_1, --., m_r$, then

$$\begin{bmatrix} i \\ i_1 \cdot \ldots i_j \end{bmatrix} = \frac{i!}{i_1! \ldots i_j! m_1! \ldots m_r!}.$$

(2.7)

The construction of geometric continuous parametric curves has been considered by a number of authors, see for example Manning [60], Nielson [61], and Barsky and Beatty [7]. Much of this work has motivated the interest in geometric continuity within the subject of CAGD. Our interest here is in generalizing the reparameterization approach to that of considering the join of two surface patches.

2.2 Geometric continuous surfaces

Given a parametric representation $p : \Omega \to R^3$ of class c^k on the closed domain $\Omega \subset R^2$, we use the notation of the derivative multi-linear map $\partial^i p|x$ at $X = (u,v)$. Directional derivatives are then denoted by

$$\partial^i P \mid_X (U_1, \ldots, U_i) = \frac{\partial^i P(X)}{\partial U_1 \ldots \partial U_i}$$

$$= \frac{\partial^i}{\partial s_1 \ldots \partial s_i} P(X + s_1 U_1 + \ldots s_i U_i) \Big|_{s_1 = \ldots = s_i = 0}$$

(2.8)

where u_j, $j = 1, \ldots, i$, are i vectors (directions) in R^2. In particular, first partial derivatives of p(X), X=(u,v), will be denoted variously by

$$\partial p \mid x^{(1,0)} = \partial_{1,0} p(x) = \partial_u p(x) = p_u(x),$$

$$\partial p \mid x^{(0,1)} = \partial_{0,1} p(x) = \partial_v p(x) = p_v(x).$$

(2.9)

A derivative along $U(X) = (\alpha(X), \beta(X))$ is then

$$\partial P \mid X^{(\alpha\ X),\beta(X))} = \frac{\partial P(X)}{\partial U}$$
$$= \alpha(x)\partial_{1,0}p(x) + \beta(x)\partial_{0,1}p(x). \tag{2.10}$$

Given $\varphi : \Omega_2 \to \Omega_1$ and $p : \Omega_1 \to R^3$, which are of class C^1, then the chain rule

$$\partial(p \circ \varphi) \mid_x = \partial p \mid_{\varphi(x)} \circ \partial\varphi \mid_x \tag{2.11}$$

applies. This can be expressed as a multiplication of the Jacobian matrix representations

$$\partial\varphi \mid x = [\partial_{1,0}\varphi\partial_{0,1}\varphi]] = \begin{bmatrix} \partial_{1,0} & \varphi_1 & \partial_{0,1} & \varphi1 \\ \partial_{1,0} & \varphi_2 & \partial_{0,2} & \varphi_2 \end{bmatrix} \tag{2.12}$$

$$\partial p \mid x = [\partial_{1,0}\varphi\partial_{0,1}p] = \begin{bmatrix} \partial_{1,0} & p_1 & \partial_{0,1} & p_1 \\ \partial_{1,0} & p_2 & \partial_{0,1} & p_2 \\ \partial_{1,0} & p_3 & \partial_{0,1} & p_3 \end{bmatrix} \tag{2.13}$$

Here we have assumed a column vector representation $\varphi = (\varphi_1, \varphi_2)^T$, $p = (P_1, P_2, P_3)'T$ of the maps and then, for example, U in (2.10) would be expressed in column vector form $U(X) = (\alpha(X), \beta(X))^T$.

The parameterization p(X) is said to be 'regular' if ∂_1, $op(^x)$ and $\partial_0 l^p(^x)$ are linearly independent, that is if the Jacobian matrix (2.13) has rank 2. This regularity assumption corresponds to that for the case of curves and guarantees that a well defined tangent plane exists at each point X of the surface. This implies that cones, cusp ridges and other types of singularity are avoided.

Following Hahn [48], we now define a C^k patch and consider how two C^k patches should join to give a   surface. For our purposes, we restrict the discussion to patches defined on polygonal domains which hence have straight line edges.

Definition 2.3 (C^k patch). A C^k patch, $k \geq 1$, is a mapping $p : \Omega \to R^3$ of a closed polygonal domain $\Omega \subset R^2$ into R^3 which is a regular parametric
representation of class C^k. Thus p is k-times continuously differentiable on Ω and $\partial p|x$ has rank 2 for all $X \in \Omega$.

Definition 2.4 (GC^k join). Let $p : \Omega_1 \to R3$ and $q : \Omega_2 \to R3$ be two patches defined on closed polygonal domains Ω_1 and Ω_2 Also, let $E_2 : [0,1] \to R^2$ be a regular C^k parameterization of an edge of Ω_2 Then p joins q with geometric continuity GC^k (along the edge E_2) if there

exists a C^k diffeomorphism $\varphi : R^2 \to R^2$, defined in a neighbourhood of E_2, such that

(i) (Domain continuation)

$$E_1(s) = \varphi(E_2(S)) \ , \ s \in [0,1] \ , \tag{2.14}$$

is an edge of Ω_1 and interior points of Ω_1 in a neighbourhood of E_1 are mapped from exterior points of Ω_2

(ii) (Patch continuation) For $s \in [0,1\}$

$$\partial^i q\big|_{E_2}(S) = \partial^i(p \circ \varphi)\big|_{E_2}(S) \ , \ i = 0,\ldots,k \ . \tag{2.15}$$
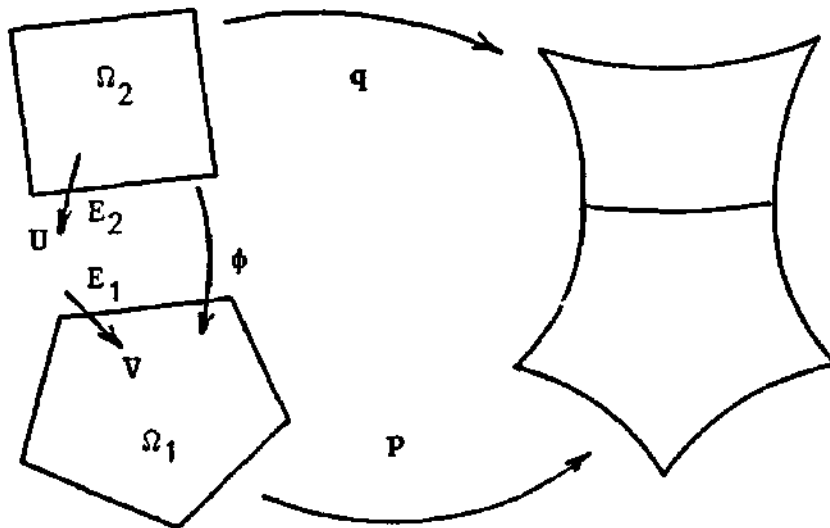

Figure 2.3. Reparameterization for a C^k surface.

The mapping φ defines a reparameterization of p for which the composite surface is parametrically C^k. The conditions on φ in (i) ensure that the reparameterization is orientation preserving. The continuity conditions (2.15) can be replaced by the requirement that for $s \in [o,l]$

$$\frac{\partial^i q}{\partial U^i}(E_2(s)) = \frac{\partial^i(p \circ \varphi)}{\partial U^i}(E_2(s)) \ , \ i = 0,\ldots k \ , \tag{2.16}$$

where $U = U(s)$ is any non-zero C^{k-1} transversal vector field defined along the edge $E_2(s)$ in an outward direction, say. Applying the chain and product rules then gives, for example, the GC^2 surface analogue of (2.4), namely

$$q(E_2(S)) = P(E_1(S)),$$
$$\frac{\partial q}{\partial U}(E_2(S)) = \frac{\partial P}{\partial V_1}(E_1(s)), \quad V_1(s) = \frac{\partial \varphi}{\partial U}(E_2(S)),$$
$$\frac{\partial^2 q}{\partial U^2}(E_2(S)) = \frac{\partial P}{\partial V2}(E_1(s)) + \frac{\partial^2 P}{\partial V_1^2}(E_1(s)), \quad V_2(s) = \frac{\partial^2 \varphi}{\partial U^2}(E_2(S)).$$

$$(2.17)$$

Here $V_1(s)$ will be a C^1 non-zero vector field transversal to $E_1(s)$ in an inward direction and $V_2(s)$ will be a C° vector field. More generally we have (cf. Proposition 2.2):

Proposition 2.5. Let $U : [0,1] \to R^2$ be an (outward) transversal vector field to the edge E_2. Then the patches p and q join with geometric continuity GC^k if and only if there exist C^{k-1} vector fields $V_i : \{0,1\} \to R^2$ such that

$$\frac{\partial^i q}{\partial U^i}(E_2(s)) = \sum_{j=0}^{i} \sum_{i_1+\ldots+i_j=i} \begin{bmatrix} i \\ i_1 \cdots i_j \end{bmatrix} \frac{\partial^j p(E_1(s))}{\partial Vi_1 \ldots \partial Vi_j}, \quad i = 0, \ldots, k, \qquad (2.18)$$

where V_1 is an (inward) transversal vector field to the edge E_1.

Given the C^k diffeomorphism φ such that (2.16) holds, the necessity of this proposition is demonstrated by application of the chain and product rules. This gives (2.18) where

$$E_1(s) = \varphi(E_2(s)) \quad \text{and} \quad V_i(s) = \frac{\partial^i \varphi}{\partial U^i}(E_2(s)), \quad i = 1, \ldots, k. \qquad (2.19)$$

Conversely, given C^K vector fields satisfying (2.18) then a C^K diffeomorphism (p, satisfying (2.19), is defined by the Taylor interpolant.

$$\varphi(E_2(s)) + tU(s)) = E_1(s) + \sum_{i=1}^{k} \frac{t^i}{i!} v_i(s). \qquad (2.20)$$

A more general construction for the case of the c^{k-i} vector fields of the proposition is given in Hahn [48].

2.3 A particular parameterization

Consider the case where $p : \Omega_1 \to R^3$ and $q : \Omega_1 + R^3$ are parameterized such that they have the C° join

$$q(s,0) = p(0,s), \quad s \in [0,1].$$

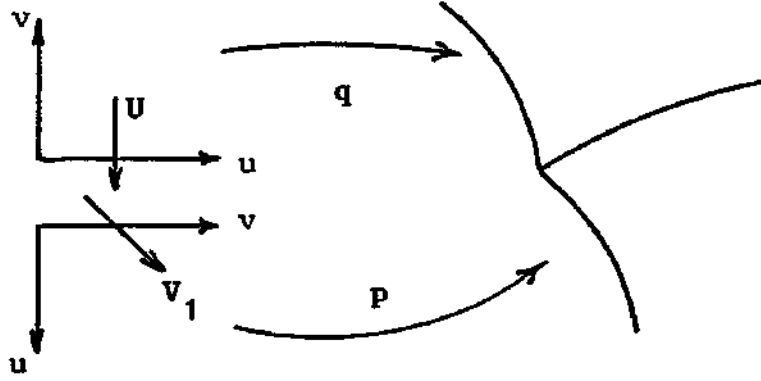Thus $E_1(s) = (0,s)$ and $E_2(s) = (s,0)$, see Figure 2.4.

Figure 2.4. A particular parameterization

Furthermore, let

$$U(s) = -\gamma(s) \ (0,1) \ , \ \gamma(s) > 0,$$

$$V_1(s) = \alpha(s) \ (1,0) + \beta(s) \ (0,1) \ , \ \alpha(s) > 0,$$

$$V_2(s) = \sigma(s) \ (1,0) + \upsilon(s) \ (0,1) \ , \tag{2.21}$$

where the positivity of $\alpha(s)$ and $\gamma(s)$ ensures the orientation condition on $U(s)$ and $V_1(s)$ (more generally $\alpha(s) \ \gamma(s) > 0$). Then given $\gamma(s)$, the GC^2 conditions are that there exist $\alpha(s), \beta(s), \sigma(s)$ and $\upsilon(s)$ such that

$$q(s,0) = (s,0) = P(0,s),$$

$$-\gamma(s)\partial_{0,1} \ q(s,0) = \alpha(s)\partial_{1,0} P(0,s) + \beta(s)\partial_{0,1}P(0,s) \ ,$$

$$\gamma^2(s)\partial_{0,2}q(s,0) = \sigma(s)\partial_{1,0} P(0,s) + v(s)\partial_{0,1}P(0,s)$$

$$+ \alpha^2(s)\partial_{2,0}P(0,s) + 2\alpha(s)\partial_{1,1}P(0,s) + \beta^2(s)\partial_{0,2}P(0,s) \ , \tag{2.22}$$

The diffeomorphism φ, given by (2.20), is then defined by

$$\varphi(u,v) = (0,u) - \frac{v}{\gamma(u)} \ (\alpha(u),\beta(u)) + \frac{v^2}{2\gamma^2(u)} \ (\sigma(u),v(u)) \ . \tag{2.23}$$

We may let $\gamma(s) \equiv 1$ without loss of generality and hence, for example, may replace the GC^1 condition by

$$\bar{\partial}_{1,0}q(s,0) = \hat{\alpha}(s)\,\partial_{1,0}\,p(0,s) + \hat{\beta}(s)\,\partial_{0,1}(0,s) \ . \tag{2.24}$$

In the case of polynomial patches, however, (2.24) can lead to the mistaken assumption that $\hat{\alpha}(s)$ and $\hat{\beta}(s)$ must be polynomial, whereas taking the cross product with $\partial_{1,0}P(0,s)$ and $\partial_{0,1}P(0's)$ shows these to be rational with identical denominators. Hence the form (2.22) is more appropriate. This was observed in Liu [57], Liu and Hoschek [59] and more recently Peters [66].

The GC^1 condition in (2.22) is sometimes replaced by the necessary determinant condition

$$\det\ [q_{1,0}\ p_{1,0}\ p_{0,1}\ ]\ =0\ . \tag{2.25}$$

For example, DeRose [22] uses this condition to develop a method for verifying the GC^1 join between two Bernstein-Bézier patches. The determinant condition does net preclude cusp like joins, however, since the orientation requirement $\alpha(s)\ \gamma(s) > 0$ is not included.

2.4 Continuity at a vertex

In the two previous subsections, conditions for the smooth join of two patches were developed, either via the existence of a diffeomorphism φ defining a reparameterization as in (2.15) or, equivalently, via explicit consistency conditions of the form (2.17)/(2.22). In order to build surfaces, however, we must consider the situation where a number of patches join subsequently and surround a common vertex in R^3. In this situation, the GC^k joins around the vertex must be consistent. More concretely, consider n patches $pj:\Omega j \to R^3, j=0,\dots,n-1$, parameterized such that they have the C^0 joins

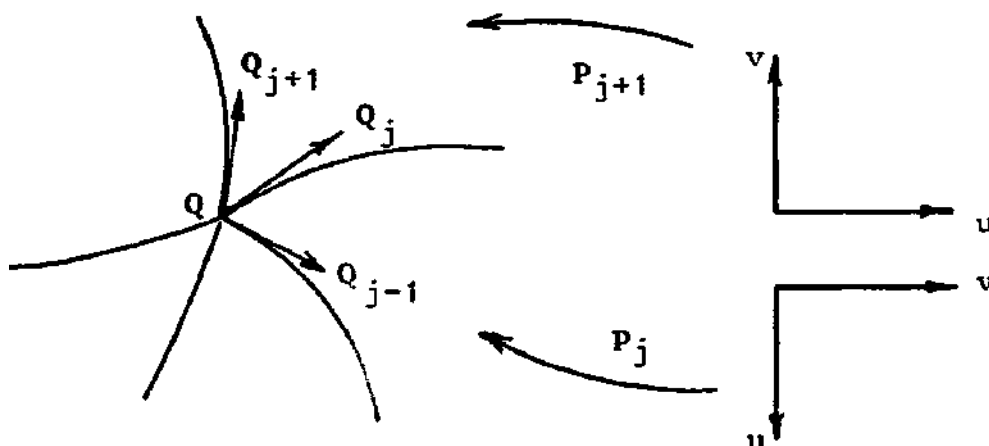$$P_{j+1}(s,0) = p_j(0,s)\ ,\ s \in\ [0,1]\ ,\ j = 0,\ \dots,l\ , \tag{2.26}$$

Figure 2.5 Patches surrounding a vertex

as in subsection 3.3, see figure 2.5. Thus the n patches meet at the common vertex

$$P_j(0,0) = Q , \quad j = 0,\ldots,n-1 , \qquad (2.27)$$

say, and we call such a vertex an n-vertex.

The GC^k condition between p_j and p_{j+i} has the representation in terms of diffeomorphisms φ_j as

$$\partial^i P_{j+1}\Big|_{(s,0)} = \partial^i (p_j \circ \varphi_j)\Big|_{(s,0)} , \quad i = 0,\,.0\ldots,k , \quad \text{and} \quad j = 0,\ldots, n-1 , \quad (2.28)$$

where $\varphi_j(s,0) = (0,s)$. The requirement that on marching around the vertex one gets back to the starting point then leads to the vertex condition (Hahn [47] and [48])

$$\partial^i p_0\big|_{(0,0)} = \partial^i (p_0 \circ \varphi_0 \circ \varphi_1 \circ \cdots \circ \varphi_{n-1})\big|_{(0,0)} , i = 0,\ldots\ldots,k , \qquad (2.29)$$

Hahn shows that this is equivalent to

$$\partial^i (\varphi_0 \circ \varphi_1 \circ \ldots \circ \varphi_{n-1})\Big|_{(0,0)} = \partial^i \text{ id}\Big|_{(0,0)} , \quad i = 0,\ldots,k . \qquad (2.30)$$

In particular, with the matrix representation (2.12) we obtain the GC^1 vertex condition

$$\partial\varphi_0\Big|_{(0,0)} \circ \partial\varphi_1\Big|_{(0,0)} \circ \ldots \circ \partial\varphi_{n-1}\Big|_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \qquad (2.31)$$

An explicit representation of these matrices is given in Hahn [47]. Suppose that the n patches surround the vertex Q with GC^1 continuity and let

$$\partial_{0,1}p_j(0,0) = \partial_{1,0}p_{j+1}(0,0) = Q_j, j = 0,\ldots, n-1, \qquad (2.32)$$

denote the first partial derivative vectors at the vertex. These vectors will form a star in the tangent plane at Q. The GC^1 condition for i = 1 in (2.28) at the vertex can be written, using the chain rule, as

$$\partial p_{j+1}\Big|_{(0,0)} = \partial p_j\Big|_{(0,0)} \circ \partial\varphi_j\Big|_{(0,0)}$$

that is

$$[Q_j \ Q_{j+1}] = [Q_{j-1} \ Q_j] \circ \partial\varphi_j\big|_{(0,0)} \tag{2.33}$$

Thus

$$\partial\varphi_j\big|_{(0,0)} = \begin{bmatrix} 0 & \mu_j \\ 1 & \lambda_j \end{bmatrix} \tag{2.34}$$

say, represents the matrix of the coordinate change from the basis $[Q_j \ Q_{j+1}]$ to $[Q_{j-1} \ Q_j]$ and the GC^1 vertex condition takes the form

$$Q_{j+1} = \mu_j Q_{j-i} + \lambda_j Q_j, = 0, \ldots.n-1 \tag{2.35}$$

A simple calculation gives

$$\mu_j = -\frac{r_{j+1} \sin\theta_j}{r_{j-1} \sin\theta_{j-1}}, \ \lambda_j = \frac{r_{j+1} \sin(\theta_j + \theta_{j-1})}{r_j \sin\theta_{j-1}}, \tag{2.36}$$

where $r_j = \|Q_i\|$ and θ_j is the angle between Q_j and Q_{j+i} Conversely, if we wish to construct λ_i and μ_j satisfying the GC^1- vertex constraint (2.35), then the general solution takes the form (2.36) where $\Sigma \ \theta_j = 2\pi$. A particular symmetric solution is given by $r_j = 1$ and $\theta_j = 2\pi/n$ so that

$$\mu_j = -1 \ , \ \lambda_j. = 2\cos(2\pi/n) \ , \tag{2.37}$$

see Hahn [48] and van Wijk [76].
   The GC^1 vertex condition (2.35) can clearly also be obtained via the explicit GC^1 condition between the patches of the form

$$\alpha_j(s) \ \partial_{1,0}P_j(0,s) + \beta_j(s)\partial_{0,1}P_j(0,s) + \gamma_j(s) \ \partial_{0,1}P_{j+1}(s,0) = 0 \ ,$$

$$j = 0,\ldots,n-1 \ , \tag{2.38}$$

where $\alpha_i(s) \gamma_i(s) > o$. letting $\gamma_j(0) = 1$ without loss of generality then gives, at $s = 0$,

$$\alpha_j(0)Q_{j-1} + \beta_j(0)Q_j + Q_{j+1} = 0 \ , \ j = 0 \ldots.n-1 \tag{2.39}$$

Thus $\alpha_j(0) = -\mu_j$ and $\beta_j(0) = -\lambda_j$ in terms of our previous notation.

2.5 Some general problems in constructing smooth surfaces

The construction of geometric continuous patch complexes involves the development of patch methods which either explicitly or implicitly satisfy the conditions for geometric continuous joins. In the previous subsections we have tri d to present some of the tools which are required in the study of such j ins. Before considering the particular case of

the polygonal hole problem, we conclude this section by describing three general difficulties that can arise in the construction of geometrically smooth patch complexes. The first two of these involves the vertex conditions.

2.5.1 Twist constraint: Suppose the individual patches $p_j : \Omega_j \to \mathbb{R}^3$ surrounding a vertex Q, are of class C^2 and let

$$\partial_{1.1} P_j(0,0) = Q_{j-1.j} \quad , \; j = 0,\ldots,n-1 , \qquad (2.40)$$

denote the 'twist vectors' at the vertex Q. Differentiating (2.38) with respect to s and evaluating at s = 0 then gives the constraints

$$\alpha_j(0)Q_{j-1,j} + Q_{j,j+1} = R_j, \; j = 0,\ldots,n-1 , \qquad (2.41)$$

where

$$-R_j = \alpha_j^{(1)}(0)Q_{j-1} + \beta_j^{(1)}(0)Q_j + \gamma_j^{(1)}(0)Q_{j+1} + \beta_j(0)\partial_{0,2}p_j(0,0) .$$
$$(2.42)$$

Equations(2.41), considered as a system in the twist variables $Q_{j-1,j}$, generates a cyclic matrix with determinant

$$1 + (-1)^{n-1}\prod_{j=0}^{n-1}\alpha_j(0), \qquad (2.43)$$

where

$$\prod_{j=0}^{n-1}\alpha_j(0) = \prod_{j=0}^{n-1}(-\mu_j) = 1 . \qquad (2.44)$$

Thus the system is singular if n is even and non-singular when n is odd. The observation that an even number of vertices can give rise to difficulties in the analysis has been noted by van Wijk [76], Watkins [81], Jones [54] and Peters [65]. It does not imply that analysis in the even vertex case is impossible but that this rank deficient case must be treated with care. An alternative approach which avoids the twist condition is to consider patches which are of class C^1 but not C^2. For example, the rationally corrected patches of section 3 have such a property, as do patches whose domain is split into subdomains at a vertex (for example, the 'Clough Tocher' triangle split). It should be noted that the case of parametric $C^{1,1}$ continuity on a regular rectanglular grid patch network, where n = 4 at the vertices, is a special one. Here the constraint (2.41) simplifies and is identically satisfied.

2.5.2 A non-regular GC^1 constraint for rectangular patches

Consider the GC^1 constraint (2.38), where s = 0 corresponds to an

n-vertex and s = 1 corresponds to an m-vertex. The theory of subsection 2.4 suggests the particular symmetric conditions

$$\alpha_j(0) = \gamma_j(0) = 1, \beta_j(0) = 2\cos(2\pi/n),$$
$$\alpha_j(1) = \gamma_j(1) - 1, \beta_j(1) = 2\cos(2\pi/m). \qquad (2.45)$$

Hence, linear interpolation on $0 \leq s \leq 1$ gives a sufficient condition for a GC^1 join as

$$\partial_{1,0}P_j(0,S) - {}^2\{(1-s)\cos(2T\pi/n) - s\cos(2\pi/m)\}\partial_{0,1}p_j(0,s) + \partial_{0,1}\ p_{j+1}\ (s\ ,0) = 0 \qquad (2.46)$$

This is a particularly simple and attractive condition but it leads to a non-regular parameterization if one (but not both) of the vertices is a 4-vertex and the rectangular patch is C^2, for example, a polynomial patch. Thus suppose n. $\neq$ 4 and m = 4 so that $\cos(2\pi/m) = 0$ in (2.46). Differentiating(2.46) with respect to s then gives, at s = 1,

$$\partial_{1,1}\ P_j\ (0,1) + 2\cos\ (2\pi/n)\,\partial_{0,1}\ P_j\ (0,1) + \partial_{1,1}\ P_{j+1}(1,0) = 0\ . \qquad (2.47)$$

Since $\partial_{1,1}Pj(0,1) = -\partial_{1,1}Pj+1(1,0)$ at the 4-vertex, we thus have
$\partial_{1,0}P_j(0,1) = 0$ (unless n = 4 when both vertices are 4-vertices).

We will see later that this problem can be avoided by use of higher degree polynomial coefficients in (2.47). This approach is also used by Goodman [32] in the construction of spline spaces for closed surfaces of rectangular patches. Alternatively, the use of C^1 rationally corrected patches avoids this problem, see subsection 4.2.

2.5.3 Blending geometric smooth surfaces

Suppose that a patch p has GC^k joins k $\geq$ 1, with patches $q_1 : \Omega \rightarrow \mathbb{R}^3$ a nd q2 : $\Omega \rightarrow \mathbb{R}^3$ along an edge E(s) of ft in the sense that, for j = 1,2,

$$\partial^i q_j\Big|_{E(s)} = \partial^i(p\circ\varphi_j)\Big|_{E(s)}\ ,\quad i = 0,\ \ldots\ ,k, \qquad (2.48)$$

where the diffeomorphisns are such that $\varphi_1(E(s)) = \varphi_2(E(s))$. Then the blended surface

$$q(X) = \omega_1(X)q_1(X, + \omega_2(X)q_2(X)\ ,\ ,\ \omega_1(X) + \omega_2(X) = 1\ , \qquad (2.49)$$

where $\omega_j : \Omega \rightarrow \mathbb{R}$, j = 1., are C^k functions, does not, in general, have a GC^k join with p for a   k > 1. This occurs because of the different connecting diffeomorp s φ_j, j = 1,2, in (2.48) or equivalently different vector fields   the right hand sides of derivative conditions

of the form (2.18). A simple example is provided by

$$p(u,v) = (u,u^2,v) \ , \ u \le 0 \ ,$$

$$q(u,v) = \frac{1}{2}(u,u^2,v) + \frac{1}{2}(2u,4u^2,v) \ , \ v \ge 0 \ , \qquad (2.50)$$

which only join GC^1 along $(0,v)$, whilst the individual components of the average q have GC^∞ joins with p.

The example does provide an illustration of the following result, however. Suppose the connecting diffeomorphisms agree up to order $\hat{k} < k$ along $E(s)$, that is the vector fields in the equivalent derivative conditions agree up to order k. Then the blended surface (2.49) will have a $\hat{k} + 1$' st order continuous join with p. Thus, since $\varphi_1(E(s)) = \varphi_2(E(s))$, the blended surface will always have at least a GC^1 join with p.

The fact that blending GC^k surfaces does not automatically result in GC^k surface for $k > 1$, causes difficulty in the construction of high order continuous blended parametric surfaces. We will consider the blending function technique as one method of treating the polygonal hole problem. The difficulty of using blends for the specific polygonal hole problem is discussed in Gregory and Hahn [39].

3. Rectangular Patches and the Polygonal Hole Problem

We briefly review some standard rectangular patch methods from Approximation Theory and CAGD and then introduce the polygonal hole problem. In particular, we will need to consider some representations of bicubic tensor product patches and their associated 'rationally corrected' forms. For further details see, for example, the survey papers of Barnhill [5], Boehm et al [14], Pratt 168], or the book of Farin [29]. For simplicity of presentation, most of the patch functions $p(u,v)$ are defined on the unit square $0 \le u,v \le 1$ so that $p : [0,1]^2 \to \mathbb{R}^3$. For $(s,t) \in [a,b] \times [c,d]$ the transformation

$$u = (s-a)/(b-a) \ , \ v = (t-c)/(d-c) \qquad (3.1)$$

can be made.

3.1 Bicubic Hermite patch

The bicubic Hermite patch $p : [0,1]^2 \to \mathbb{R}^3$ is defined by

$$p\{u,v) = [H_0^3(u),H_1^3(u),H_2^3(u),H_3^3(u)]Q[H_0^3(v),H_1^3(v),H_2^3(v),H_3^3(v)]T \ ,$$

$$(u,v) \in [0,1]^2 \ . \qquad (3.2)$$

where

$$Q= \begin{bmatrix} P(0,0) & P(0,1) & P_V(0,0) & P_V(0,1) \\ p(1,0) & p(1,1) & p_V(1,0) & p_V(1,1) \\ p_u(0,0) & p_u(0,1) & p_{uv}(0,0) & p_{uv}(0,1) \\ p_u(1,0) & p_u(1,1) & p_{uv}(1,0) & p_{uv}(1,1) \end{bmatrix} \tag{3.3}$$

Here, the cubic Hermite basis on [0,1] is defined by

$$[H_0^3(u), H_1^3(u), H_2^3(u), H_3^3(u)] = [1, u, u^2, u^3]M, \tag{3.4}$$

where

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \tag{3.5}$$

The coefficients of the matrix Q define the interpolation data. The boundary curves p(u,0), p(u,1), p(0,v), p(1,v) and the cross bouydar tangent vector derivatives $P_v(u,0)$, $P_v(u,1)$ $P_u(0,v)$, $P_u(1,v)$ ar univariate cubic Hermite functions. Thus, for example.
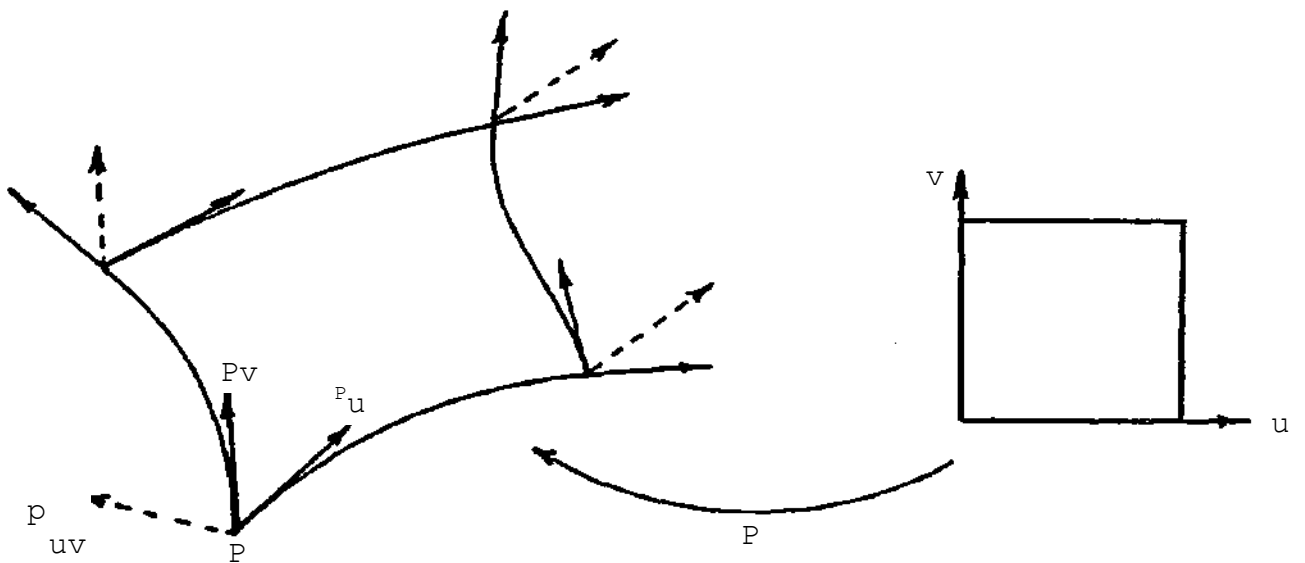

Figure 3.1 Bicubic Hermite patch

$$P(u,0) = H_0^3(u)\,p(0,0) + H_1^3(u)\,p(1,0) + H_2^3(u)\,p_u(0,0) + H_3^3(u)\,p_u(1,0)$$

$$P_v(u,0) = H_0^3(u)\,P_v(0,0) + H_1^3(u)\,P_v(1,0) + H_2^3(u)\,P_{uv}(0,0) + H_3^3(u)\,P_{uv}(1,0)\ .$$

$$(3.6)$$

3.2 Bicubic Bernstein-Bézier patch

The bicubic Bernstein-Bezier patch is defined by

$$P(u,v) = \sum_{i=0}^{3} \sum_{j=0}^{3} B_j^3(u) B_i^3(u) B_j^3(v) Q_{i,j}$$

$$= \left[B_0^3(u),\, B_1^3(u),\, B_2^3(u),\, B_3^3(u) \right] Q \left[B_0^3(V),\, B_1^3(u),\, B_2^3(u),\, B_3^3(u) \right]^T \quad (u,v) \in [0,1]^2$$

$$(3.7)$$

where

$$B_i^3(u) = \binom{3}{i} u^i (1-u)^{3-i} \tag{3.8}$$

Thus

$$\left[B_0^3(u),\, B_1^3(u),\, B_2^3(u),\, B_3^3(u) \right] = \left[1, u, u^2, u^3 \right] M\ , \tag{3.9}$$

where

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

$$(3.10)$$

The coefficient matrix $Q = [Q_{i,j}]$ is the control point matrix of the Bernstein-Bezier patch and the boundary curves and the cross boundary derivatives are univariate Bernstein-Bézier cubics. For example

$$P(u,0) = \sum_{i=0}^{3} B_I^3(u) Q_{i,0}$$

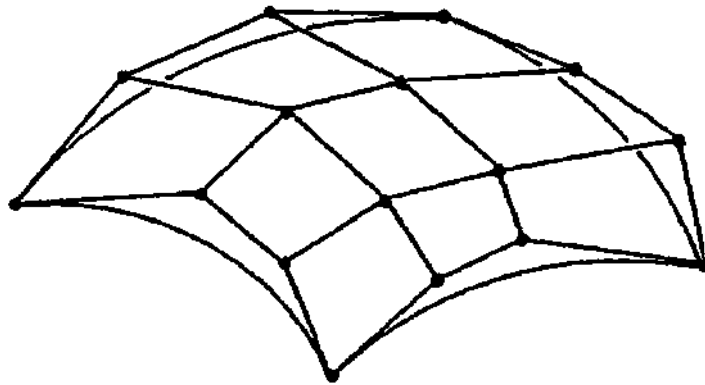$$p_v(u,0) = 3 \sum_{i=0}^{3} B_i^3(u)\, (Q_{i,1} - Q_{i,0})$$

$$(3.11)$$

Figure 3.2   cubic Bernstein-Bezier patch

3.3 Bicubic   -spline patch

The bicubi   spline surface $p : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by

$$p(s,t) = \sum_{\ell \in z} \sum_{m \in z} N_\ell(s) \hat{N}_m(t) Q_{\ell,m} ,$$

(3.12)

where N,ℓ(s $\hat{N}m(t)$ denote the normalized cubic B- splines with the
local suppc $(s\ell-2'{}^S\ell+2)$ and (t_{m-2}, t_{m+2}) respectively, *on* the non-
decreasing   partitions $\{s\ell\}$, .$\ell \in$ Z and $\{t_m\}$, m $\in$ z. The B-spline
surface ha;:   metric continuity $C^{2'2}$, that is continuity up to and
including c   in each of the independent variables. The restriction
(s,t) $\in$ [s_i x[t_j, t_{j+2}] gives the bicubic B-spline patch

$$p(s,t) = [N \quad N_i(s), N_{i+1}(s), N_{i+2}(s)] Q [N_{j-1}(t), N_j(t), N_{j+1}(t), N_{j+2}(t)]^T ,$$

(3.13)

where Q = [(,   = i-1,...,i+2, m = j-1,...,j+2, is the control point
matrix of t   tch. A calculation in terms of the Bernstein-Bezier
form, follow,   construction of Boehm [10], gives the explicit B-spline
basis repress   ion

$$[N_{i-1}(s), \quad N_{i+1}(s), N_{i+2}(s)] = \left[B_0^3(u), B_1^3(u), B_2^3(u), B_3^3(u) \right] W$$

$$= [1, u, u^2, u^3] MW ,$$
(3.14)

where M is c   by (3.10) and

$$u = (s - s_i.)/h_i. , \quad h_i = s_{i+1} - s_i.$$
(3.1.5)

Here W is th   isformation matrix

$$W = \begin{bmatrix} \lambda_i & 1-\lambda_i-\mu_i & \mu_i & 0 \\ 0 & 1-\alpha_i & \alpha_i & 0 \\ 0 & \beta_i & 1-\beta_i & 0 \\ 0 & \lambda_{i+1} & 1-\lambda_{i+1}-\mu_{i+1} & \mu_{i+1} \end{bmatrix} \qquad (3.16)$$

with

$$\left. \begin{aligned} \lambda_i &= h_i^2 / \big[(h_{i-1}+h_i)(h_{i-2}+h_{i-1}+h_i)\big], \\ \mu_i &= h_{i-1}^2 / \big(\big[(h_{i-1}+h_i)(h_{i-1}+h_i+h_{i+1})\big]\big), \\ \alpha_i &= h_{i-1} / (h_{i-1}+h_i+h_{i+1}), \\ \beta_i &= h_i+1 / (h_{i-1}+h_i+h_{i+1}). \end{aligned} \right\} \qquad (3.17)$$

The basis functions $\hat{N}m\,(t)$, $m = j-1, \ldots, j+2$, are defined similarly in terms of the variable

$$v = (t-t_j)/\hat{h}_j \ , \quad \hat{h}_j. = t_{j+1}. \ . \ t_j. \ . \qquad (3.18)$$

For equal intervals, the familiar basis matrix

$$MW \quad = \frac{1}{6} \begin{bmatrix} 1 & 4 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \qquad (3.19)$$

is obtained in (3.12) for the case of uniform B-splines.
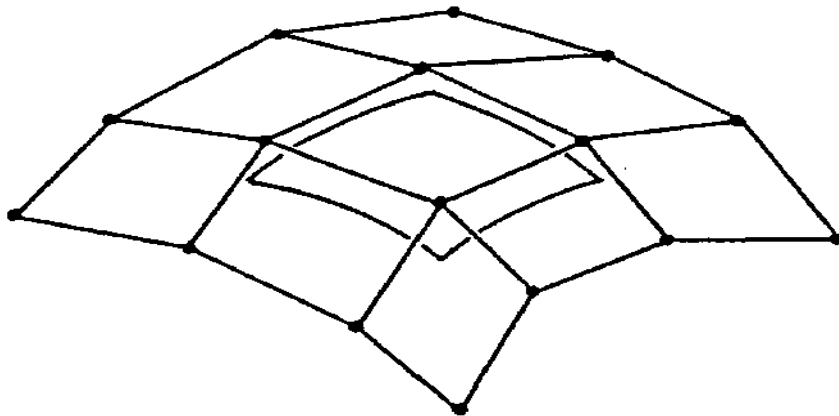
Figure 3.3 Bicubic B-spline patch

3.4 Coons' bicubically blended patch

The Coons' bicubically blended patch is defined by

$$p(u,v) = [H_0^3\{u\}, H_1^3(u), H_2^3(u), H_3^3(u)][p(0/v)/p(1,v), p_u(0,v), p_u(1/v)]^T$$

$$+ [p\{u,0\}, p(u,1), p_v(u,0)/_{Pv}(u,1)][H_0^3(v), H_1^3(v), H_2^3(v), H_3^3(v)]^T$$

$$- [H_0^3(u), H_1^3(u)/H_2^3(u), H_3^3(u)]Q[H_0^3(v), H_1^3(v), H_2^3(v), H_3^3(v)]^T \qquad (3.20)$$

where Q is defined as in (3.3) and the cubic Hermite basis functions are
defined in (3.4) and (3.5). The Coons' patch is sometimes referred to as
a 'Boolean sum' blend of two 'lofted' surface patches and the bicubic
Hermite 'tensor product[1] patch (3.2).   The boundary curves p(u,0),
p(u,l), p(0,v), p(1,v) and the cross boundary derivatives Pv(u,0),
Pv(u'1)' Pu(0'v)' Pu(1'v) are not   restricted to be cubics. However for
(3.20) to be meaningful, we must   assume that these boundary terms are
consistent with a $C^{1,1}$ surface.   In particular, we have the necessary
twist condition that

$$P_{uv} \equiv \partial^2 p/\partial u \partial v = \partial^2 p/\partial v \partial u = pvu \qquad (3.21)$$

at the four corners of the patch. Otherwise (3.20) is not well defined.
In particular, Pv(u,0) and Pv(u,l) are not interpolated by the patch.

3.5 Rationally corrected patches

The rationally corrected form of the Coons' bicubically blended patch
allows for different twist terms P_{uv} and p_{vu} at the corners of the patch.
Thus, for example

$$P_{uv}(0,0) = \partial/\partial u \, P_v(u,0)|_{u=0}$$
$$P_{vu}(0,0) = \partial/\partial v P_u(0,v)|_{v=0} \qquad (3.22)$$

are not necessarily the same. The rationally corrected form of the
Coons' patch, see Gregory [34,35] and Barnhill [5}, is defined as in
(3.20) but with the lower right 2 x 2 'twist partition' block in the
definition of Q, equation (3.3), replaced by

$$
\left[
\begin{array}{cc}
\dfrac{v p_{uv}(0,0) + u p_{vu}(0,0)}{u + v} & \dfrac{(1-v)p_{uv}(0,1) + u p_{vu}(0,1)}{u + 1 - v} \\[2em]
\dfrac{v p_{uv}(1,0) + (1-u) p_{vu}(1,0)}{1 - u + v} & \dfrac{(1-v)p_{uv}(1,1) + (1-u)p_{vu}(1,1)}{1 - u + 1 - v}
\end{array}
\right] \qquad (3.23)
$$

It should be noted that the rational forms introduce singularities in order to allow for the possibility that $p_{uv} \neq p_{vu}$ at the corners. However, these singularities are removable up to first order (the patch is C^1). For numerical stability, the patch should not be evaluated at the corners. Instead, the given corner values should be used directly.

Replacement of the boundary functions and derivatives in the rationally corrected Coons' patch, with univariate cubic Hermite forms, gives the rationally corrected bicubic Hermite patch. This is defined as (3.2) but with the 'twist partition' in (3.3) replaced by

$$
\left[
\begin{array}{cc}
\dfrac{u p_{uv}(0,0) + v p_{vu}(0,0)}{u + v} & \dfrac{u p_{uv}(0,1) + (1-v) P_{vu}(0,1)}{u + 1 - v} \\[2em]
\dfrac{(1-u)p_{uv}(1,0) + v p_{vu}(1,0)}{1 - u + v} & \dfrac{(1-u)p_{uv}(1,1) + (1-v)p_{vu}}{1 - u + 1 - v}
\end{array}
\right]
$$
$$\qquad (3.24)$$

The cubic Hermite boundary derivatives $P_v(^{u,0})$, $P_v(^{u,1})$ and the derivatives $P_u(0,v)$ and $P_u(1,v)$ now have the different twist parameters P_{uv} and P_{vu} respectively in their definitions. For example $P_v(u,0)$ is defined as in (3.6) but $p_u(0,v)$ is defined by

$$
p_u(0,v) = H_0^3(v)\, p_u(0,0) + H_1^3(v)\, P_u(0,1) + H_2^3(V)\, P_{vu}(0,0) + H_3^3(V)\, p_{vu}(0,1) \; .
$$

$$\qquad (3.25)$$

A rationally corrected form of the bicubic Bernstein-Bézier patch can also be derived, by introducing two different control points for each of the four interior control points of the patch, as shown in Figure 3.4, see Chiyokura and Kimura [19] and Chiyokura [18]. Cross boundary derivatives are then defined in terms of the appropriate control points.
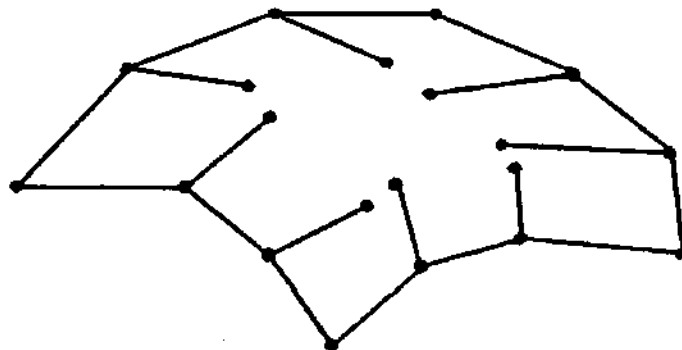

Figure 3.4 Control points of rationally corrected B-B patch

For example,

$$p_{v(u,0)} = 3[B_1^3(u)(Q_{0,1} - Q_{0,0}) + B_1^3(u)(Q_{1,1}^V - Q_{1,0})$$

$$+ B_2^3(u)(Q_{2,1}^V - Q_{2,0}) + B_3^3(u)(Q_{3,1} - Q_{3,0})] \tag{3.26}$$

where $Q_{1,1}^V$ and $Q_{2,1}^V$ denote the interior control points to be associated with the boundary' derivative Pv(u,0). The rationally corrected form of the bicubic Bernstein-Bëzier patch is then derived of the form (3.7) with the control point matrix

$$
\begin{bmatrix}
Q_{0,0} & Q_{0,1} & Q_{0,2} & Q_{0,3} \\
Q_{1,0} & \dfrac{uQ_{1,J}^V + VQ_{1,1}^u}{u+v} & \dfrac{uQ_{1,2}^V + (1-V)Q_{2,2}^u}{u+1+v} & Q_{1,3} \\
Q_{2,0} & \dfrac{(1-u)Q_{2,1}^V + VQ_{2,1}^u}{1-u+v} & \dfrac{(1-u)Q_{2,2}^V + VQ_{2,2}^u}{1-u+1-v} & Q_{2,3} \\
Q_{3,0} & Q_{3,1} & Q_{3,2} & Q_{3,3}
\end{bmatrix} \tag{3.27}
$$

Other types of rational schemes are described in the survey paper of Barnhill [5] and Worsey [82] has derived the appropriate rational corrections for the C^2 biquintic blended patch.

3.7 The polygonal hole problem

The rectangular patches described above are typically used to form surface patch complexes defined on regular, rectangular meshed domains. Thus the bicubic Hermite and Bernstein-Bézier patch representations are usually joined to give C^1 surface patch complexes, whilst the bicubic B-spline gives C^2 surface patch complexes. In the following sections we will consider the situation where a rectangular patch complex surrounds an n-sided hole described in the following manner.

Let q_j = : $\delta \to \mathbb{R}^3$, $j = 0,..,n-1$, = $[0,2] \times [-1,0]$, denote a rectangular patch complex surrounding an n-sided hole, where $q_j(s,0)$, $0 \le s \le 1$, defines the j' th edge of the hole, see Figure 3.5. The patches are assumed to form a C^k patch complex in the sense that the composite maps

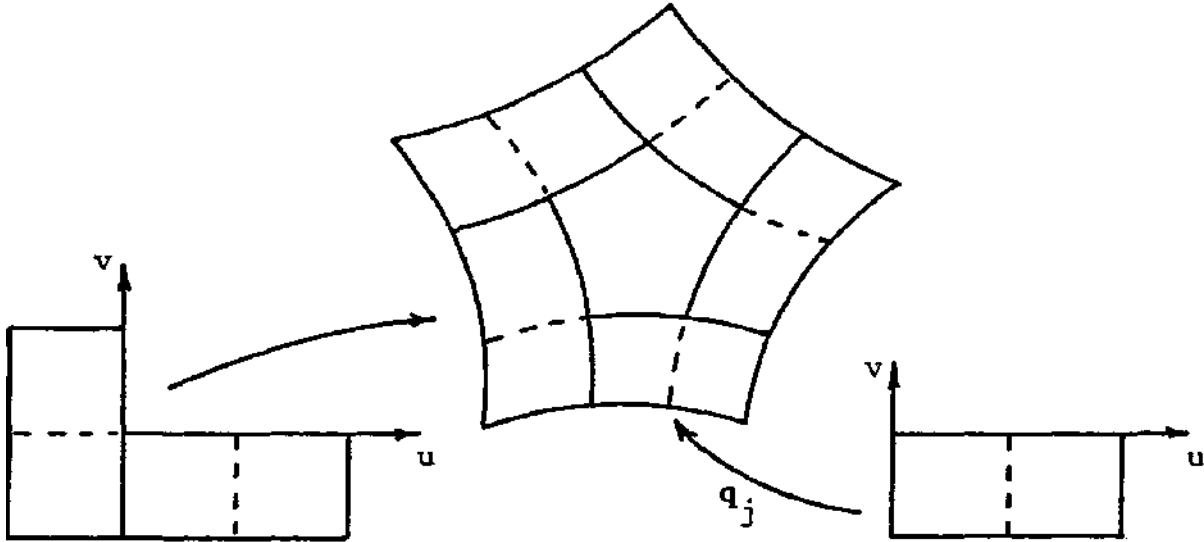$$(u,v) \to \begin{cases} q_{j-1}(1-v,u) \\ q_j(u,v) \end{cases} \tag{3.28}$$

Figure 3.5 Polygonal hole problem

are C^k continuous on $[-1,0] \times [-1,1] \cup [0,2] \times [-1,0]$ for $j = 0,\ldots,n-1$, see also figure 3.5. In practice, for tensor product patches, this composite map will usually also be $C^{k,k}$ continuous, particularly at the corner $(0,0)$, and this will be assumed in sections 4 and 5.

The q_j will normally consist of a sub-complex of two or more patches, although for mathematical purposes it has been denoted by one parametric surface here. For example, q_j could consist of two bicubic patches defined on $[0,1] \times [-1,0]$ and $[1,2] \times [-1,0]$, respectively, where $q_j(u,v)$, $(u,v) \in [0,1] \times [-1,0]$ defines the patch adjacent to the hole. The patch surface can then be used to give a C^1 surface about the hole. Alternatively, q_j could consist of a sub-complex of eight patches, where $q_j\{u,v\}$, $(u,v) \in [0,\frac{1}{2}] \times [-\frac{1}{2},0]$ and $(u,v) \in [\frac{1}{2},1] \times [-\frac{1}{2},0]$ define the two patches adjacent to the hole. It is then possible to surround the hole with a C^2 surface by appropriate choice of a bicubic B-spline representation.

We now consider how such polygonal holes can be filled in, making use of the geometric continuity tools developed in section 2.

4. Filling Polygonal Holes with Rectangular Patches

In subsections 4.1 - 4.3 we consider methods of filling an n-sided hole with n rectangular patches p_j, $j = 0,\ldots,n-1$, parameterized as in subsection 2.4. Thus the patches have the C^0 joins $p_{j+1}(s,0) = Pj(0,s)$ and meet at the common vertex $Pj(0,0) = Q$, $j = 0,\ldots,n-1$. Along the j ' th edge of the hole we assume that

$$\left.\begin{array}{l} p_j(s,1) = q_j(1/2-s/2,0), \\ \\ p_{j+i}(1,s) = q_j(1/2+s/2,0) \end{array}\right\} 0 \le s \le 1, \qquad (4.1)$$

where q_j is parameterized as in subsection 3.7. This situation is illustrated by figure 4.1. In subsection 4.4, a recursive subdivision technique for filling the hole is discussed.
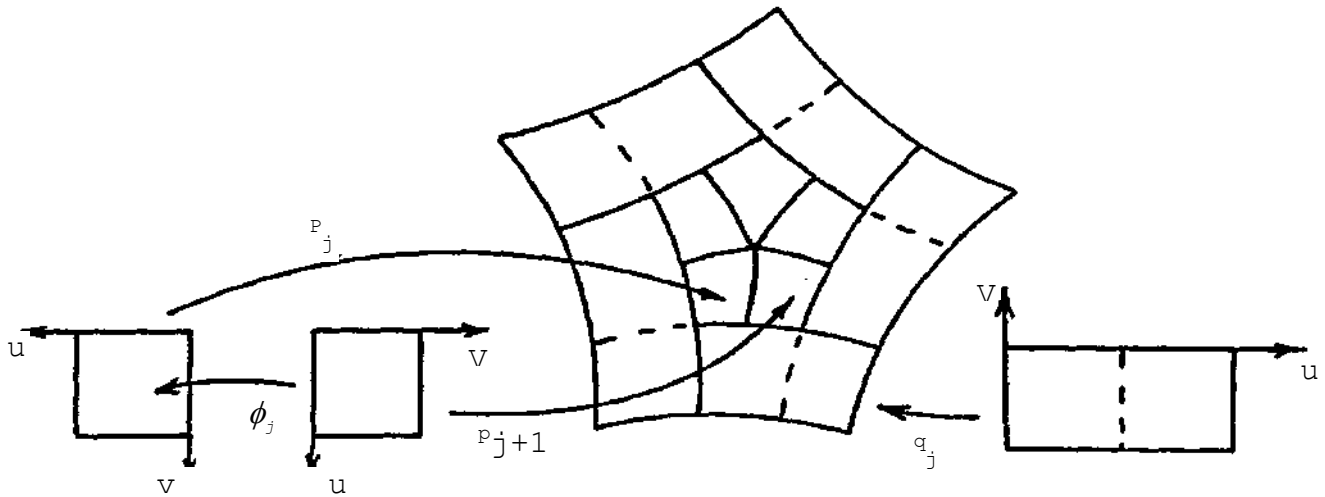

Figure 4.1 Filling an n-sided hole with rectangular patches

4.1 A bicubic Hermite scheme

Consider the situation where either $q_j(u,v)$, $(u,v) \in [0,1] \times [-1,0]$ is a bicubic patch or $q_j(u,v)$, $(u,v) \in [0,\frac{1}{2}] \times [-\frac{1}{2},0]$ and (u,v) $[\frac{1}{2},1) \times [-\frac{1}{2},0]$ are two bicubic patches, adjacent to the j'th edge of the hole. Then we consider the p_j, $j = 0,\ldots,n-1$ as bicubic patches, which along the j'th edge of the hole satisfy the C^0 conditions (4.1) and the GC^1 conditions

$$
\left. \begin{array}{l}
\partial_{0,1} p_j(s,1) = -\dfrac{1}{2}\ \partial_{0,1}\ q_j\ (1/2 - s/2, 0)\ , \\[2mm]
\partial_{1,0}\ p_{j+1}(s,1) = -\dfrac{1}{2}\ \partial_{0,1}\ q_j\ (1/2 + s/2, 0)\ ,
\end{array} \right\} \tag{4.2}
$$

(Here there is a 1/2 scaling factor due to the choice of the parameterizations of the q_j and p j.) These conditions will be satisfied by an appropriate identification of the Hermite data along the edge, see, for example, (3.6). In particular, at the 'mid-point' of the edge we define

$$
\left. \begin{array}{l}
B_j\ =\ q_j(1/2, 0)\ =\ p_j(0,1)\ =\ p_{j+1}(1,0) \\[2mm]
B_j^{u}\ =\ \dfrac{1}{2}\ \partial_{1,0}\ q_j\ (1/2, 0)\ =\ -\partial_{1,0}\ p_j(0,1)\ =\ \partial_{0,1}\ p_{j+1}(1,0) \\[2mm]
B_j^{v}\ =\ \dfrac{1}{2}\ \partial_{0,1}\ q_j\ (1/2, 0)\ =\ -\partial_{0,1}\ p_j(0,1)\ =\ \partial_{1,0}\ p_{j+1}(1,0) \\[2mm]
B_j^{u,v}\ =\ \dfrac{1}{2}\ \partial_{1,1}\ q_j\ (1/2, 0)\ =\ -\partial_{1,1}\ p_j(0,1)\ =\ \partial_{1,1}\ p_{j+1}(1,0)\ .
\end{array} \right\} \tag{4.3}
$$

At the n-vertex we define

$$
\left.
\begin{aligned}
Q &= P_j(0,0) = P_{j+1}(0,0) \\
Q_j &= \partial_{0,1} p_j(0,0) = \partial_{1,0} P_{j+1}(0,0) \\
Q_{j-1,j} &= \partial_{1,1} P_j(0,0)
\end{aligned}
\right\}
\tag{4.4}
$$

as in subsection 2.4.

It follows that $p_{j+1}(s,0) = p-i(0,s)$, since these two univariate functions share common Hermite data. Thus, for a C^1 surface patch complex, it remains to satisfy the GC^1 constraints

$$
\phi_j(s) = \alpha_j(s)\,\partial_{1,0}P_j(0,s) + \beta_j(s)\,\partial_{0,1}P_j(0,s) + \gamma_j(s)\partial_{0,1}P_{j+1}(s,0) = 0
$$

$$
j = 0,\ldots,n-1 , \tag{4.5}
$$

where $\alpha_j(s)\,\gamma_j(s) > 0$, see subsection 2.4. Some difficulties in satisfying such a condition have been discussed in subsection 2.5.2. In particular the simple choice of coefficients (see (2.46) with m=4)

$$
\alpha(s) = 1 = \gamma_j(s) , \quad \beta_j.(s) = -2(1-s)\cos(2\pi/n) , \tag{4.6}
$$

was observed to lead to a non-regular parameterization. An alternative quadratic choice which avoids this problem is

$$
\alpha_j(s) = 1 = \gamma_j(s) , \quad \beta_j(s) = -2(1-s)^2\cos(2\pi/n) . \tag{4.7}
$$

Gregory and Zhou [43] show that equating coefficients in (4.5) then leads to the following solutions for the polygonal hole problem:

(i) For n = 3, a triangular hole,

$$
\left.
\begin{aligned}
Q &= \frac{1}{3}V + \frac{1}{6}W \\
Q_j &= 2B_j + B_j^v - \tfrac{2}{3}V - \frac{1}{3}W , \\
Q_{j-1,j+1} &= -6B_j - 4B_j^v + 2V + \frac{3}{2}
\end{aligned}
\right\}
\tag{4.8}
$$

where

$$
V = B_1 + B_2 + B_3 , \quad W = B_1^V + B_2^V + B_3^V \tag{4.9}
$$

(ii)   For n > 4, a general polygonal hole.

$$
\left.
\begin{aligned}
B_j &= Q + 2Q_j + \frac{1}{2}(Q_{j-1,j} + Q_{j,j+1}) / \cos(2\pi/n) \ . \\[2mm]
B_j^v &= -3Q - \frac{1}{2}(Q_{j-1,j} + Q_{j,j+1}) / \cos(2\pi/n) \ .
\end{aligned}
\right\}
$$

(4.10)

Where

$$
Q_{j+1} - 2\cos(2\pi/n)Q_j + Q_{j-1} = 0, j = 0,\dots,n-1 \ .
$$

(4.11)

In the triangular hole case (4.8), the 3-vertex values are defined in terms of the boundary edge information. For n > 4, there are not enough degrees of freedom at the n-vertex to give such a solution in terms of the surrounding edge information. Thus in (4.10) we propose a solution whereby the Hermite edge values B_j and Bv_j are not given a priori but are determined in terms of the degrees of freedom at the n-vertex. In this case Q and the twists $Q_{j-i,j}$ can be chosen arbitrarily. The first derivative vectors must be chosen such that (4.11) holds. This approach of treating Bj and BY as the unknowns, with the twists being given, also avoids the twist constraint problem of having to distinguish between odd and even n-vertices as described in subsection 2.5.1.

Goodman [32] uses an identical constraint to (4.5), (4.7) in the development of local support bases for GC^1 spline spaces over closed surfaces of rectangular patches. Van Wijk [75] has also considered the geometric join of bicubic patches. He either assumes that all vertices are odd or assumes a mixture of three and four-vertices. Jones [54] uses higher degree patches to fill n-sided holes, again restricting the discussion to the case of n being odd so that the twist constraints (2.38) are not rank deficient. Biquintic patches are used for the GC^1 case and Jones also presents a GC^2 solution using biseptic patches. A biquartic solution to the GC^1- problem is provided by a special case of a result of Peters [65].

4.2 A rationally corrected scheme

Consider the situation of subsection 4.1, where the surrounding rectangular patch network consists of bicubic patches.   The p_j, j = 0,...,n-l, which are to fill the hole, are chosen as rationally corrected bicubic patches. The rationally corrected forms introduce additional degrees of freedom. These are used in the solution of the GC^1 constraint equations for the edges around the n-vertex. The rationally corrected forms avoid both the twist constraint problem of subsection 2.5.1 and the non-regular GC^1 constraint problem of subsection 2.5.2.

The use of rationally corrected forms for filling a polygonal hole is suggested by Chiyokura [17,18], using the Bernstein-Bézier representation. For simplicity and uniformity of presentation, we assume here that the p_j are represented as rationally corrected bicubic Hermite patches, see subsection 3.9. Thus we have the possibility of two twist

parameters $\partial_u\partial_v p_j$ and $\partial_v\partial_u p_j$ at each vertex.

The GC^1 conditions (4.2) along the j'th edge of the hole are satisfied by appropriate identification of the Hermite data along the edge, in a similar way to subsection 4.1. In particular, at the 'mid-point' of the j'th edge we define B_j. B_j^u B_j^v. as in (4.3) and the twists across the j'th edge of the hole as

$$B_j^{u,v} = \frac{1}{2}\partial_{1,1} q_j(1/2,0) = \partial_u\partial_v p_j(0,1) = -\partial_v\partial_u P_{j+1}(1,0) \qquad (4.12)$$

The twists $\partial_v\partial_u p_j(0,1)$ and $\partial_u\partial_v P_{j+i}(1,0)$ are yet to be determined. At the n-vertex we define Q and Q_j as in (4.4) and the two twists

$$Q_{j-1,j} = \partial_u\partial_v P_j(0,0) \;,\; Q_{j,j-1} = \partial_v\partial_u P_j(0,0) \;. \qquad (4.13)$$

The problem is now to satisfy GC^1 constraint equations j(s) = 0, of the form (4.5), for j = 0,...,n-1. The use of rationally corrected patches allows the simple choice of coefficients (4.6) in (4.5), which for polynomial patches would lead to a non-regular parameterization. With this choice we proceed in the following asymmetric way: For the patch p_j, assume the equal twists

$$\partial_v\partial_u p_j(0,1) = \partial_v\partial_u p_j(0,1) = B_j^{u,v} \qquad (4.14)$$

and for the patch p. define the twist parameter

$$B_j^{v,u} = -\partial_u\partial_v P_{j+1}(1,0) \qquad (4.15)$$

which differs from the other twist parameter $B_j^{u,v} = \partial_v\partial_u p_{i+1}.,(1,0)$. The constraint $\varphi_j(s) = 0$ is a cubic equation and hence is uniquely determined by the conditions

$$\varphi_j.(0) = \varphi_j(1) = \varphi'_j.(0) = \varphi'_j.(1) = 0 \qquad (4.16)$$

The condition $\varphi(1) = 0$ is identically satisfied and the other conditions give

$$\left.\begin{array}{l} Q_{j-1} + c_n Q_j + Q_{j+1} = 0 \,, \\ Q_{j,j+1} = -Q_{j,j-1} - c_n(6B_j - 6Q - 5Q_j + 2B_j^v \,, \\ B_j^{v,u} = B_j^{u,v} + c_n B_j^v \,, \end{array}\right\} \qquad (4.17)$$

Where

$$c_n = -2\cos(2\pi 2\pi/ \qquad (4.18)$$

These conditions can be satisfied for j = 0,...,n-1, giving a rationally corrected rectangular patch solution to the polygonal hole problem. Note that for polynomial patches $B_j^{u,v} .= B_j^{v,u}$ .and then $B_j^v .= 0$ which results in the non-regular parameterization described in subsection 2.5.2 (compare (2.47) with the third equation of (4.17)).

4.3 A GC^k scheme

For arbitrary k, it is clearly impossible to construct a rectangular patch solution to the polygonal hole problem based on equating coefficients in GC^k edge consistency constraints of the form (2.18). Hahn [47] thus tackles the problem by constructing explicit diffeomorphisms φ_j for the constraint equations

$$\partial^i P_{j+1} \Big|_{(s,0)} = \partial^i (p_j \ o \ \varphi_j) \Big|_{(s,0)} \ , i=0,...,k, and j=0,...,n-1, \tag{4.19}$$

(see (2.28) and figure 4.1). The surrounding rectangular patch network is assumed to be   In particular, the composite map (3.28) is assumed to be $C^{k'k}$.

From subsection 2.4 we have, at (0,0),

$$\varphi_j(0,0) = (0,0) \ , \ \partial\varphi_j\Big|_{(0,0)} = \begin{bmatrix} 0 & \mu_j \\ 1 & \lambda_j \end{bmatrix} \tag{4.20}$$

Similarly, at (1,0) we obtain

$$\varphi_j(1,0) = (0,1) \ , \ \partial\varphi_j\Big|_{(1,0)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{4.21}$$

The properties (4.20) and (4.21) are respectively satisfied by the affine maps

$$(u,v) \rightarrow (u.v, u + A \ v) \ , \tag{4.22}$$

$$(u,v) \rightarrow (-v,u) \ . \tag{4.23}$$

Hence Hahn defines φ_j as the blended map

$$\varphi_j(u,v) = a(u)(\mu_j \ v, u+\lambda_j \ v) + \beta(u)(-v,u) \ , \tag{4.24}$$

where

$$\alpha(u) + \beta(u) = 1 \ , \ \alpha(0) = 1 \ , \ \beta(1) = 1 \ . \tag{4.25}$$

In order to generate consistent C^k data, the α and β are constrained to have zero derivatives up to appropriate orders at 0 and 1, so that, up to order k, φ_j behaves like (4.22) and (4.23) at (0,0) and (1,0)

respectively. This means that 1'st derivatives of φj at (0,0) and (1,0)
satisfy the derivative equations in (4.20) and (4.21), whilst higher
derivatives are zero. Thus a special solution of the vertex constraints
(2.30) has been constructed. (Hahn also imposes further constraints on
the α and ß in order to generate consistent C^k data at the n-vertex so
that the patches can be constructed using Boolean sum interpolation.)

Given the data for one patch, say P_0, and consistent vertex values Q,
Q_j, j = 0,...,n-l the procedure is to march around the vertex. The data
for the other patches is thus generated via the connecting diffeo-
morphisms φ_j and from the surrounding rectangular patch network.

4.4 A recursive subdivision scheme

The final rectangular patch method we consider for filling an n-sided
hole has a very different structure to the previously proposed methods.
It is based on a recursive subdivision technique suggested by Catmull and
Clark [15] and Doo and Sabin [24]. The method generalizes the sub-
division technique for a bicubic B-spline surface over a regular
rectangular grid. Such subdivision techniques should be covered else-
where in these proceedings and we will only give a brief description of
the method here, following the notation of Ball and Storey [1,2,3,4],

Consider the bicubic B-spline patch of figure 3.3 with its 16 control
points. This patch can be subdivided into 4 sub-patches, each defined by
a subset of 16 control points from the 25 points shown in figure 4.2.
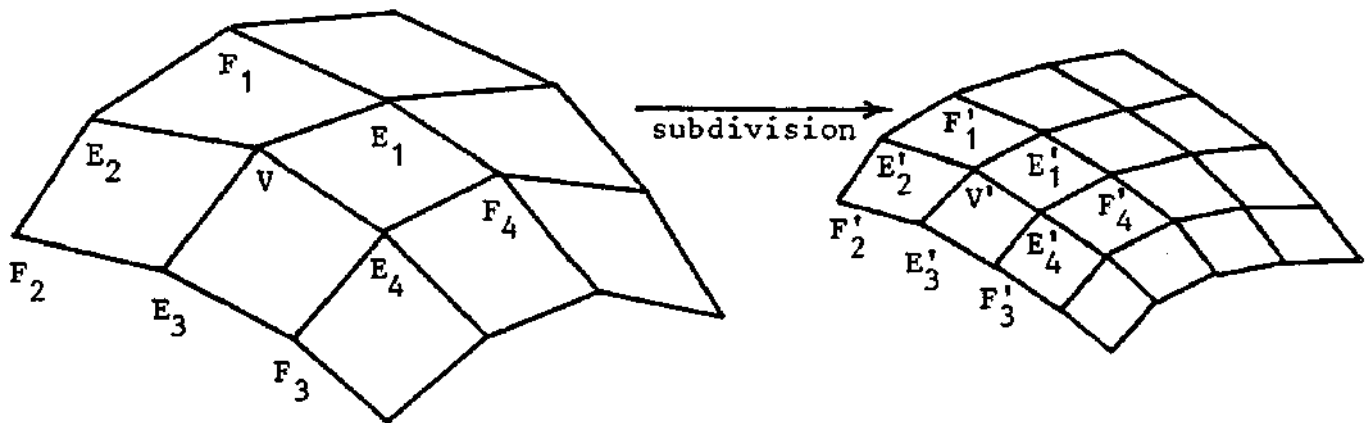The new control points are defined by formulae which are special cases of

Figure 4.2 Subdivision of bicubic B-spline

those given below. Repeated application of this procedure gives a
sequence of control point networks which converges to the B-spline patch.
We wish to generalize this subdivision process for control points
which are arranged in a network about an n-vertex control point V as
shown in figure 4.3. This structure is equivalent to a polygonal hole
problem since, away from the n-vertex, the control points define a
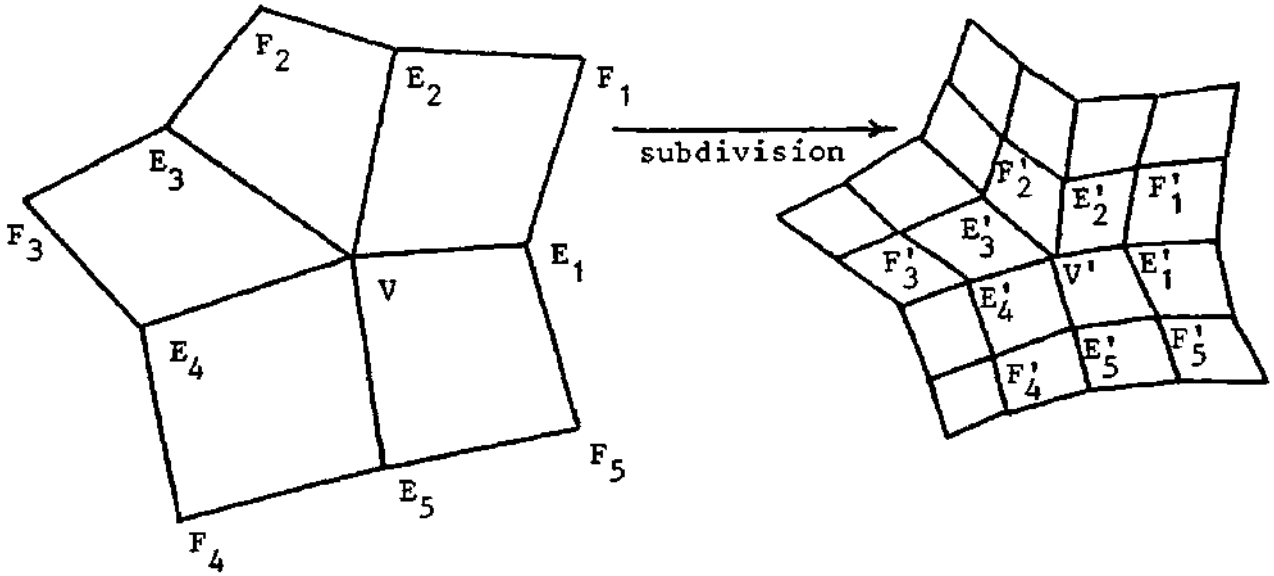bicubic B-spline surface around an n-sided hole.

Figure 4.3 Subdivision at an n-vertex

The formulae for the subdivision process are given by

$$F_j' = \frac{1}{2}(V + E_j + F_j + E_{j+1}),$$

Where

$$E_j' = \frac{1}{2}(V + E_j + F_{j-1}' + F_j'),$$

$$V' = \frac{1}{2}\alpha_n V + \beta_n \sum E_j / n + \gamma_n \sum F_j' / n ,$$

(4.26)

.

$$\alpha_n + \beta_n + \gamma_n = 1$$ (4.27)

The formulae for the subdivision process are given by

In [15] and [24] the simple choice of coefficient weights

$$\alpha_n = \beta_n = 1/n , = \gamma_n (n-2)/n$$ (4.28)

is suggested, which for n = 4 accords with the values required for the bicubic B-spline subdivision.

The procedure is equivalent to generating a sequence of rectangular B-spline surface patch complexes about an 'extraordinary point' defined by the limit of the sequence of V's. Away from the extraordinary point the surface is C^2. At the extraordinary point geometric considerations apply. Doo and Sabin [24] suggest a matrix eigenproperty analysis of the behaviour at the extraordinary point and this behaviour has been studied in the series of papers by Ball and Storey [1,2,3,4]. In particular, a method of optimizing the coefficient weights in (4.26) is suggested. (It should be noted that the coefficients in (4.26) are different from those used by Ball and Storey due to a rearrangement of the last equation in

(4.26).) Finally, in Storey and Ball [75] it is shown that the above recursive subdivision technique can be applied to fill a polygonal hole within a C^1 bicubic Hermite patch complex.

5. Filling Polygonal Holes with Polygonal Patches

In this section we consider methods which have been devised to fill an n-sided hole with a single patch. We thus refer to such a patch as a polygonal patch although a polygonal domain will not always be explicitly defined. For example, subsection 5.3 considers a technique for defining **a** patch on a non-planar domain. The first technique described is that of blending function interpolation.

5.1 Blending function methods

Given the n-sided hole problem, a blending function method of filling the hole takes the general form

$$p(v) = \sum_{j=0}^{n-1} \alpha_j(V)P_j(V), \quad V \in \Omega \ , \tag{5.1}$$

where Ω is a polygonal domain. The p_j;, $pj : \Omega \to R^3$, are interpolants which match the surrounding rectangular patch network along certain of edges of the domain. The $\alpha j(v)$, $\alpha j : \Omega \to R$, are the 'blending functions' which are chosen such that the patch (5.1) matches the rectangular patch network around the entire boundary. In practice, the blend will be a convex combination, that is

$$\sum_{j=0}^{n-1} \alpha_j(V) = 1, \alpha_j(V) \geq 0, \ \forall \ V \in \Omega \ . \tag{5.2}$$

Blending function methods for defining polygonal patches have been considered by Charrot and Gregory [-16,38], Gregory and Hahn [39,40,41] and Hagen [45]. We illustrate the technique by describing a GC^1 method, where the surrounding patch complex is a C^1 surface.
   Let the polygonal domain ft be a regular n-gon in R^2 with sides of length unity and centre the origin 0. Its vertices are denoted by Vj and its edges are Ej, parameterized as

$$E_j(S) = (1-s)V_j + sV_{j+1}. \ , \tag{5.3}$$

$j = 0,...,n-1$. The perpendicular distance d_j of a point V Ω from the side E_j is given by

$$d_j(V) = <V_j - V, Z_j - 0> / <Z_j - 0, Z_j - 0>^{\frac{1}{2}} \ , \tag{5.4}$$

where Z_j is the point of intersection of E_{j-1} and E_{j+1}, and $<.,.>$ is the Euclidean scalar product in $\mathbb{R}^2$. We then make use of either of the following local Cartesian coordinate systems ('coordinate charts') (u_j,v_j) about each vertex V_j, $j = 0,\ldots,n-1$, see figures 5.1.
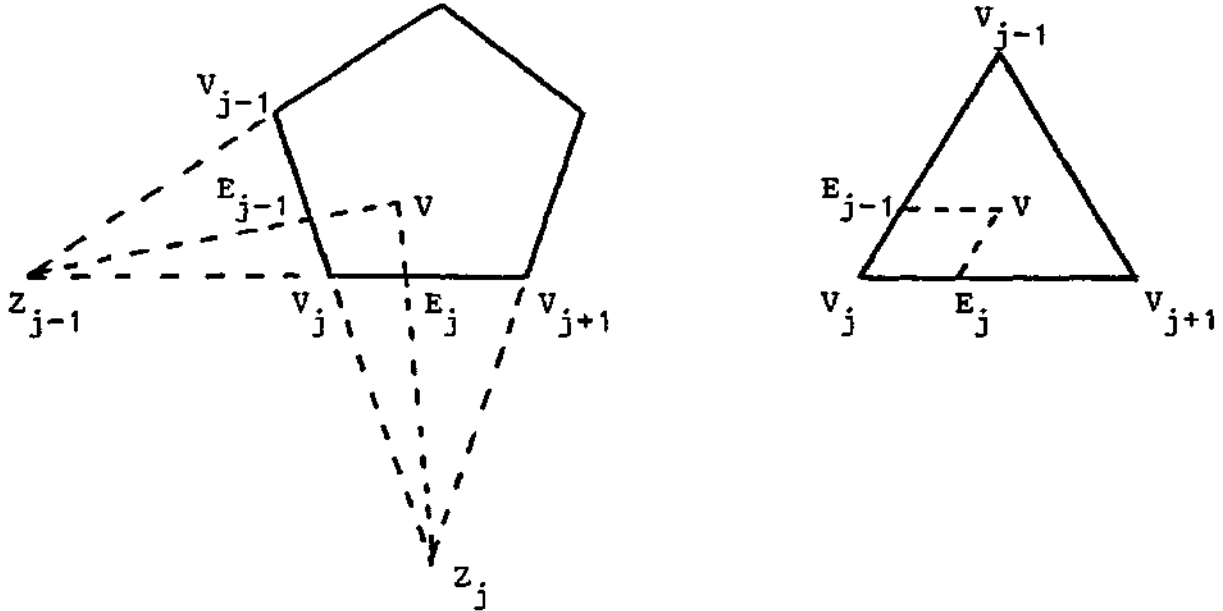
Figures 5.1 Construction of the coordinate charts

(i) Central (radial) projection coordinate charts. Let

$$(u_j(V), V_j(v)) = \left[\frac{d_{j-1}(V)}{d_{j+1}(V)+d_{j-1}(V)} , \frac{d_j(v)}{d_{j-2}(V)+d_j(V)} \right] . \tag{5.5}$$

Then $E_j(u_j) = (1-u_j)v_j + u_jv_{j+1}$ and $E_{j-1}(1-v_j) = (1-v_j)V_j+v_jV_{j-1}$. are the projections onto the sides of the rays from V to Z_j and V to Z_{j-i} respectively.

(ii) Parallel projections. Let

$$(u_j(v),v_j(v))=(\hat{d}_{j\_1}(v), \hat{d}_j(v)) , \hat{d}_j(v) = d_j(V)/\sin(2\pi/n) . \tag{5.6}$$

Then $E_j(u_j)$ and $E-_{j-1}(1-v_j)$ are the projections onto the sides of the rays from V parallel to the sides $E_{j\ i}$ and E_j respectively.

   For n > 4 we prefer the use of the central projections, whilst for n = 3 (when the central projections have singularities at the vertices) we prefer the use of the parallel projections. In the latter case, dj, j = 0,1,2, are the barycentric coordinates of the triangle with an unconventional index notation, due to the labelling used for the general polygon. Thus, for n = 3,

$$V = \sum_{j=0}^{2} \hat{d}_{j+1} V_j , \sum_{j=0}^{2} \hat{d}_{j+1} = 1 . \tag{5.7}$$

Consider now the rectangular patch complex q_j, j = 0,...,n-1, about the n-sided hole and, in particular, the C^1 composite map (3.28). Let $P_j(u,v)$, u ≥ 0, v ≥ 0, denote a C^1 extension of this map into the positive quadrant. (In the papers [16], [38], [40] and [41] this is achieved by Boolean sum Taylor interpolation.) Then

$$P_j(V) = P_j(u_j(V),v_j(V)) \qquad (5.8)$$

defines a function which has GC^1 joins with q_{j-i} and q_j along the edges E_{j-1} and E_j of the polygonal domain. The joins are geometric continuous ones since p_j matches the rectangular patches under a reparameterization defined by the coordinate chart map.

The polygonal patch is now defined by (5.1), where the blending functions are given by

$$\alpha_j(V) = \prod_{i \neq j,j-1} d_i^2 \Bigg/ \sum_{\ell=0}^{n-1} \prod_{i \neq \ell,\ell-1} d_i^2 \qquad (5.9)$$

Thus the α_j satisfy (5.2) and vanish up to first order along the edges E_i i ≠j-1,j. Thus p has a GC^1 join with q_j along the edge E_j since along this edge p is a convex combination of p_j and P_{j+1} both of which have GC^1 joins with q_j.

Whilst the above method will work to first order, a blend of k'th order components, k > 1, does not guarantee a GC^k result, see subsection 2.5.3. This difficulty is resolved in Gregory and Hahn [40] by constructing a reparameterization of the surface around the hole which is parametrically C^k around the hole. Also, a specific GC^2 construction is proposed in Gregory and Hahn [41]. Alternatively, a scheme will be proposed in [42], whereby p_j is defined as a reparameterization of a C^k extension of the composite map (3.28) into the positive quadrant. The mapping defining the reparameterization will be a blend of the vertex coordinate charts and by this means p_j and p_{j+i} will have identical GC^k joins with q_j.

Another technique, not described here, is that of blending one-sided interpolants, see, for example, Gregory [36]. Hagen [45] describes such a GC^2 scheme for the triangle and the method is readily generalized to polygonal domains.

5.2 Overlap patches

The following approach, due to Varady [77,78], assumes the situation where $q_j(u,v)$, (u,v) ∈ [0,1] × [-1,0], is one bicubic patch adjacent to the j'th edge of the hole. The bicubic patch is represented in Hermite form and Varady observes that this patch can be written as a sum of four components, each defined with respect to a coordinate chart about a vertex. Polygonal patches are then defined which have the same structure of a simple linear combination of vertex maps. The method is thus of a similar form to the blending function technique of the previous subsection. However, it differs from it in that the components of the blend only match the adjoining data at one vertex, rather than along

edges, and the blending functions are all unity. It is the choice of the coordinate charts at each vertex which play the important role of producing geometric continuous joins with the adjoining rectangular patches.

The bicubic Hermite patch (3.2) can be written as

$$P(u,v) = \sum_{j=0}^{3} P_j(u_j, v_j) \ , \tag{5.10}$$

where

$$P_j(u_j, v_j) = \left[H_0^3(u_j) \ H_2^3(u_j) \ H_2^3(u_j) \right] \begin{bmatrix} P_j & P_j^{v_j} \\ P_j^{u_j} & P_j^{u_j v_j} \end{bmatrix} \begin{bmatrix} H_0^3(V_j) \\ H_2^3(V_j) \end{bmatrix}. \tag{5.11}$$

Here we have the coordinate charts $(u_0, v_0) = (u,v)$ about $V_0 = (0,0)$; $(u_1 V_1) = (v, 1-u)$ about $V_1 = (1,0)$; $(u_2, v_2) = (1-u, 1-v)$ about $V_2 = (1,1)$; $(u_3, v_3) = (1-v, u)$ about $V_3 = (0,1)$. Also P_j, $P_j^{u_j}$, $p_j^{v_j}$, $p_j^{u_j v_j}$ represent the function and derivative data at the vertex V_j, with respect to the coordinate chart at V_j. The boundary behaviour is then given, for example, by

$$\left. \begin{aligned} p\,(u,o) &= P_0(u,0) + P_1(0,1,u) \ , \\ P_u(u,0) &= \partial_{1,0} P_0(u,0) - \partial_{0,1} P_1(0, 1-u) \ , \\ P_v(u,0) &= \partial_{1,0} P_0(u,0) + \partial_{0,1} P_1(0, 1-u) \ . \end{aligned} \right\} \tag{5.12}$$

Let Ω be the regular n-gon with vertices V_j and edges $E_j(s) = (1-s)V_{j+} \ sV_{j+1}$, $j = 0,\ldots,n-1$, defined as in subsection 5.1. Then the overlap patch takes the form

$$P(V) = \theta + \sum_{j=0}^{n-1} P_j(u_j, v_j) \ , \ V \in \Omega \ , \tag{5.13}$$

where

$$p_j(u_j, v_j) = \left[H_0^3(u_j) \ \ H_2^3(u_j) \right] \begin{bmatrix} P^{j-\theta} & P_j^{vj} \\ P_j^{uj} & P_j^{uj vj} \end{bmatrix} \begin{bmatrix} H_0^3(v_j) \\ H_2^3(v_j) \end{bmatrix} \tag{5.14}$$

Here θ is the 'origin' of the patch, which in the case n = 4 cancels from

(5.13) and (5.14) to give (5.10) and (5.11). Varady calls (5.13) the 'overlap patch' since it is a summation of individual patches which have an overlapping domain of influence within the polygon.

The choice of the coordinate charts $(u_j(V), v_j(V))$, $V \in \Omega$, about each vertex V_j is crucial to the success of the method. The choice must be made such that the overlap patch has a GC^1 join with its bicubic Hermite patch neighbours. Assume that $E_j(U_j)$ maps to $(u_j, 0)$ and $E_{j-1}(1-v_j)$ maps to $(0, vj)$ in the coordinate chart (u_j, v_j). Consider without loss of generality the behaviour of p on the side E_0. Assume that $p_j(U_j, V_j)$, $j \neq 0,1$, are zero to first order on E_0. (This is achieved either by constraining $u_j = 1$ or $V_j = 1$ on E_0 for $j \neq 0,1$, or by a zero extension of $P_j(u_j, v_j)$ outside $[0,1]^2$.) Then
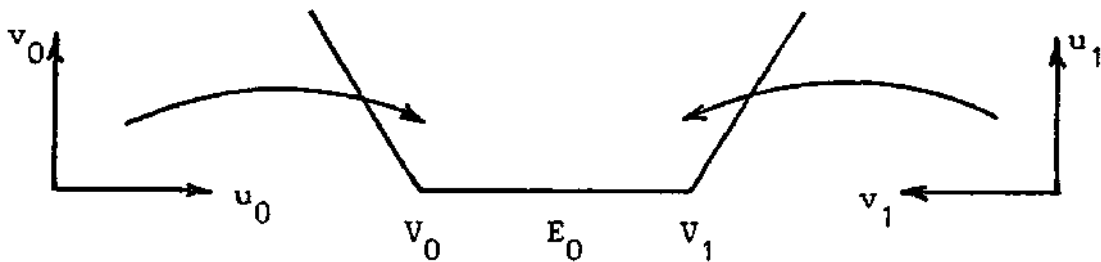

Figure 5.2 The coordinate charts at V_0 and V_1

$$p(E_0(s)) = P_0(s,0) + P_1(0, 1-s)$$

$$\partial_s P_0(E_0(S)) = \partial_{1,0} P_0(S,0) + \partial_{0,1} P_1(0, 1-s)$$

(5.15)

where $\partial_s p = \partial p/\partial s$ denotes differentiation along the side. Furthermore, differentiating along the inward normal to the side gives

$$\partial_n P(E_0(S)) = \alpha(S) \left[\partial_{1,0} P_0(S,0) + \partial_{0,1} P_1(0, 1-s) \right]$$

$$+ \beta(s) \left[\partial_{0,1} P_0(S,0) + \partial_{1,0} P_1(0, 1-s) \right] ,$$

(5.16)

where we assume the coordinate charts are such that

$$\alpha(s) = \left. \frac{\partial u_0}{\partial n} \right|_{E_0(S)} = - \left. \frac{\partial v_1}{\partial n} \right|_{E_0(S)},$$

$$\beta(s) = \left. \frac{\partial v_0}{\partial n} \right|_{E_0(S)} = - \left. \frac{\partial u_1}{\partial n} \right|_{E_0(S)}.$$

(5.17)

Thus p has a GC^1 join with its neighbour q_o (compare (5.16) with (5.12)).

Varady proposes two constructions of the coordinate charts which satisfy conditions of the above form and which hence produce GC^1 polygonal patch schemes. (The central and parallel projection coordinate charts of the previous subsection are not appropriate.)

5.3 Non-planax domains

Hosaka and Kimura [51] and Sabin [69,70,71] have approached the polygonal patch problem by defining surfaces on non-planar domains. The use of non-planar domains introduces degrees of freedom which can be used in the construction of appropriate geometric continuity constraints between patches. We will consider, in particular, the example of a triangular patch derived both by Sabin [71] and Hosaka and Kimura [51]. This patch fills a three-sided hole within a biquadratic Bernstein-Bezier patch complex.

Given an n-sided hole, a "multi-linear' domain is defined by the n-2 constraint equations

$$\sum_{j=0}^{n-1} \prod_{\ell=0}^{k} u_{j+\ell} = n - 2 - k + A_k \prod_{\ell=0}^{n-1} u_\ell, k = 0, \ldots, n-3, \qquad (5.18)$$

where the $u\ell \geq 0$ are the parameters of the patch and the A_k are constants. The domain edges are defined by $u_j = 0$, $j = 0,\ldots,n-1$. The u_j vary linearly along the adjacent edges $u_{j-1} = 0$ and $u_{j+1} = 0$. On all other edges $u_j = 1$.

For the triangular patch the domain is defined by the one constraint equation

$$u_0 + u_1 + u_2 = 1 + A_0 u_0 u_1 u_2 , u_i \geq 0 , i = 0, 1, 2. \qquad (5.19)$$

If $A_0 = 0$, the domain is a planar triangle. Sabin sets $A_0 = 2$ and defines a 'tricubic' patch by

$$p(u_0,u_1,u_2) = u_0^2 \; (1-2u_1u_2)A + 2u_0u_1(1-u_2)B$$

$$+ u_1^2 \; (1-2u_2u_0)C + 2u_1u_2(1-u_0)D$$

$$+ u_2^2 \; (1-2u_0u_1)E + 2u_0u_2(1-u_1)F$$

$$+ u_3^2 \; 0u_1u_2G \; ' \qquad (5.20)$$

where A - G define the   atrol points of the patch and (u_o,u_1,u_2) satisfies (5.19).

Along any edge $u_j = 0$ the patch is a univariate Bernstein-Bezier quadratic and hence can   used to fill a triangular hole within a biquadratic rectangular Bernstein-Bezier patch complex. We now show that

it is appropriate for a C^1 patch complex by considering the behaviour across the edge $u_2 = 0$ say. Let

$$\hat{P}(u_1,u_2) = p(u_0,u_1,u_2) \ , \quad u_0 = (1-u_1-u_2)/(1-2u_1u_2) \ , \tag{5.21}$$

be the representation of p as a rational patch $\hat{p}$ in the coordinate chart (u_1,u_2). Then along the edge $u_2 = 0$, where $u_0 = 1-u_1$, it can be shown that

$$\hat{p}(u_1,0) = u_0^2 A + 2u_0u_1 B + u_1^2 C \tag{5.22}$$

$$\hat{P}_{0,1}(u_1,0) - u_1^2\,\hat{p}_{1,0}(u_1,0) = 2u_0^2(F-A) + 4u_0u_1(G-B) + 2u_1^2(D-C) \ , \tag{5.23}$$

see figure 5.3. The first of these equations is the Bernstein-Bezier
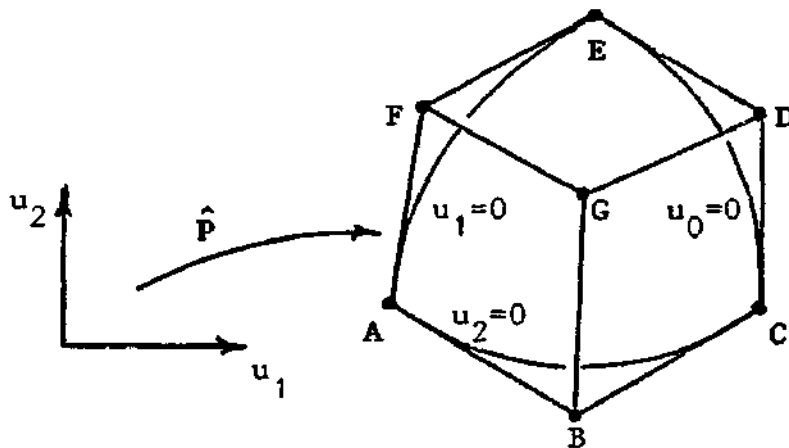
Figure 5.3 The tricubic patch

quadratic along the edge. The second equation is a cross boundary derivative which is compatible with the cross boundary derivative of a biquadratic Bernstein-Bezier patch. Thus the control points of the patches can be chosen such that GC^1 constraints are satisfied across the edges.

Pentagonal patch constructions are given in [5] and [71] but Sabin [72,73] has observed difficulties in using 'multi-linear' domains defined by (5.18) for $n \geq 6$. Thus a generalization of the domain may be appropriate.

6. Concluding Remarks

In this survey of geometric continuity methods, we have concentrated on the specific problem of filling an n-sided hole within a 'regular' rectangular patch complex. A recent polygonal patch method of DeRose and

Loop [23] has not been included. This method constructs a Bernstein-Bezier polynomial defined on an n-1 dimensional simplex. A projection of the simplex into $\mathbb{R}^2$ then defines the polygonal patch. However, geometric continuous joins, particularly around a vertex, have not yet been developed for this method.

We have also omitted the discussion of the creation of parametric surfaces over more general patch complexes, for example, complexes of triangular patches and 'irregular' complexes of rectangular patches. A number of methods for such complexes having GC^1 joins have been proposed, see for example Herror [49], Jensen [53], Nielson 162], Peters [66], Piper [67] and Saraga. [74].   An extended list of papers involving geometric continuity of surfaces is given in the references [1 - 82]. Recursive subdivision methods for defining surfaces over general mesh networks should be covered elsewhere in these proceedings.

Finally, a comparative study of the methods discussed here has not been made. Some methods are clearly easier than others to implement and the bias of current surface modelling systems towards rectangular patches might seem to favour rectangular patch methods for filling polygonal holes. However, it is a simple matter to reparameterize a given polygonal patch as a complex of rectangular patch mappings. This is achieved by quadrilateral subdivision of the polygonal domain about its centre.

Acknowledgements

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