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"What a curious attitude scientists have: '*We still don't know that; but it is knowable and it is only a question of time till we know it*'! As if that went without saying..."—[423]

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"Physics does not *explain* anything. It simply *describes* cases of concomitance."—[427]

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"People who are constantly asking '*why*' are like tourists, who stand in front of a building, reading Bädeker, and through reading about the history of the building's construction *etc etc* are prevented from *seeing* it."—[423]

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"Tolstoy: the meaning (importance) of something lies in its being something that everyone can understand. That is both true and false. What makes the object hard to understand—if it's significant, important—is not that you have to be instructed in abstruse matters in order to understand it, but the antithesis between understanding the object and what people *want* to see. Because of this precisely what is more obvious may be what is most difficult to understand. It is is not a difficulty for the intellect but one for the will that has to be overcome."—[423]

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"How hard it is for me to *see what is in front of my eyes*."—[422]

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Dedication

The present work is wholeheartedly dedicated to all theoretical/mathematical physics' researchers who "*risk their own lives, so that they may never be heard of again*" and, against all odds and with wax-plugged ears *contra* the Sirens' song of their contemporary research trends, take the trip into Feynman's "*wild blue yonder to see if they can figure it out*"...[1]

[1] Here we have partially and loosely quoted Feynman's words in [137] about the importance of a 'non-fashionable', perhaps 'iconoclastic', research pursuit of Quantum Gravity: "*...It is very important that we do not all follow the same fashion...It's necessary to increase the amount of variety...and the only way to do this is to implore you few guys to take a risk with your lives that you will not be heard of again, and go off in the wild blue yonder to see if you can figure it out...*".

$\mathcal{C}^{\infty}$-Smooth Singularities Exposed: Chimeras of the Differential Spacetime Manifold

Anastasios Mallios[†] and Ioannis Raptis[‡*§]

Abstract

The glaringly serious conflict between the principle of general covariance of General Relativity (GR) and the existence of $\mathcal{C}^{\infty}$-smooth singularities assailing the differential spacetime manifold on which the classical relativistic field theory of gravity vitally depends, is resolved by using the basic manifold independent and, *in extenso*, Calculus-free concepts, techniques and results of Abstract Differential Geometry (ADG). As a physical toy model to illustrate these ideas, the ADG-theoretic resolution of both the exterior, but more importantly, of the inner, Schwarzschild singularities of the gravitational field of a point particle is presented, with the resolution of the latter being carried out entirely by finitistic-algebraic and sheaf-theoretic means, and in two different ways. First, by regarding it as a localized, 'static' point-singularity, we apply Sorkin's finitary topological poset discretization scheme in its Gel'fand dual representation in terms of 'discrete' differential incidence algebras [318, 319] and the finitary spacetime sheaves thereof [310]. Then we exercise the ADG machinery on those sheaves in the manner of [270, 271, 272] to show that the vacuum Einstein equations still hold over the classically offensive *locus* occupied by the point-mass both at the 'discrete' level of the finitary sheaves and at the 'classical' continuum (inverse and direct) limit of infinite topological refinement of (a projective and inductive system of) the said sheaves. On these grounds alone we infer that the essentially algebraic differential geometric mechanism of ADG is in no way impeded by the presence of singularities on a geometrical base spacetime, be it a 'discretum' or a 'continuum'—a result which goes to show that our ADG theoresis of gravity and its singularities is genuinely background spacetime ('continuous' or 'discrete') independent. The second way in which we resolve the interior Schwarzschild singularity is more straightforward, but closely akin to the first. We carry it out by regarding the inner singularity as a non-localized, time-extended, distributional spacetime foam dense singularity in the sense of [274, 275]—essentially, by smearing the original point-singularity to a family of dense singularities extending along the 'wrist-watch' coordinate time-axis $\mathbb{R}$ of the point-particle. Then, again we show that the vacuum Einstein equations hold over the uncountable, densely singular *loci* in the point-mass' time-line $\mathbb{R}$ when sheaves of Rosinger's differential spacetime foam algebras of generalized functions (distributions) are used as structure sheaves of generalized coefficients or 'coordinates' and the ADG-theoretic mechanism is applied to them—as it were, to 'engulf' or 'absorb' them, but still retain the said essentially algebraic

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differential geometric mechanism in their very presence—in the manner of [262]. *In toto*, we intuit that Nature—*ie*, the laws of physics—have no singularities, let alone that they 'break down' in any (differential geometric) sense at their *loci*, or that the infinities that are normally associated with them have any physical significance or are necessarily unavoidable as they misleadingly appear to be from the manifold perspective, but that it is precisely our conventional, $\mathcal{C}^\infty$-smooth manifold based way of doing differential geometry, and in terms of which we have hitherto formulated those physical laws as differential equations relating $\mathcal{C}^\infty$-smooth—in the usual Calculus-theoretic, manifold based notion of differentiability or smoothness—fields, that 'stumbles and falters' on singularities and their unphysical infinities. It goes without saying that the traditional Calculus or Analysis-based technology of dealing with (*ie*, try to define and systematically study) $\mathcal{C}^\infty$-singularities always within the confines of the mathematical framework of Classical Differential Geometry (CDG), such as analytic inextensibility and the associated notion of (smooth) geodesic incompleteness, as well as the construction of various 'topological' boundaries to an otherwise 'regular' spacetime manifold on which singularities are then destined to be 'asymptotically' or 'marginally' situated, is completely evaded since ADG does not employ at all a base geometrical differential manifold to support its concepts and constructions. In the process of this total singularity evasion we provide a new, purely algebraic, Leibnizian-Kleinian expression of the principle of general covariance, which is manifestly smooth background spacetime manifold free as it does not involve at all the latter's diffeomorphism 'structure' group as in the classical, manifold based theory. Rather, it concerns solely the automorphism group of the vector and algebra sheaves involved, which, in turn, from a geometric (pre)quantization vantage, are the state spaces of the particles ('quanta') of the fields (*viz.* connections) acting categorically, as sheaf morphisms, on those (associated) vector sheaves' sections. On the one hand, this is a functorial (with respect to the structure sheaf $\mathbf{A}$ of generalized coordinates or arithmetics) expression of the principle of general covariance of GR since the differential equation of Einstein representing the law of gravity in ADG involves the (Ricci) curvature ('field strength') of the gravitational field (:connection), which curvature is an $\mathbf{A}$-sheaf morphism (an $\otimes_{\mathbf{A}}$-tensor). At the same time, we argue that this is an autonomous, 'self-referential' conception of general covariance which concerns only the fields and their particle quanta 'in themselves'—what we call here 'field-particle solipsism', without reference at all to an external, underlying spacetime manifold, and which we here coin 'synvariance'. At the heart of synvariance lies a radical revision—in fact, an inversion—of the notions of (gravitational) kinematics and dynamics to the effect that it enables us to argue, in striking contrast to the traditional process of the construction of GR (in point of fact, of any physical theory constructed so far), that 'dynamics comes before kinematics'. From a categorical perspective, as befits the aforesaid functorial expression of gravitational dynamics that ADG enables us to maintain, we argue that a natural transformation-type of principle underlies the notion of synvariance, which we coin the Principle of Algebraic Relativity of Differentiability (PARD). In turn, PARD may be philosophically interpreted as an abstract, generalized version of Einstein's principle of physical reality, here coined the Principle of Field Realism (PFR)—a principle which goes hand in hand with the field solipsism mentioned earlier. Subsequently, various implications that such a total ADG-theoretic evasion or bypass of singularities and of the differential spacetime manifold carrying them could have for our seemingly never ending attempts to arrive at a genuinely quantum and inherently finite theory of gravity are discussed in some detail. For instance, apart from completely circumventing various caustic issues that are supposed to trouble the (persistently differential manifold based) theory at the quantum level of description of gravity (*eg*, in the canonical or the path integral approach to quantum

or 'quantized' GR on a smooth spacetime continuum) such as the inner product/functional integral measure problem as well as the so-called problem of time, by the notion of synvariance and the autonomous conception of field dynamics that ADG enables us to posit and actually practice algebraically (:sheaf-theoretically), we can, already at the classical level of description of gravity, evade, unscathed, the whole of Einstein's hole argument—a 'no-go' argument in GR originally proposed in order to put to the test and ultimately 'shoot down' the principle of general covariance, when the latter is implemented via the diffeomorphism group of the underlying smooth spacetime manifold. Based on a generalized interpretation of the hole argument by Stachel, we argue that its total ADG-assisted bypass is virtually equivalent to the aforesaid priority of dynamics over kinematics that ADG allows us to maintain. Concerning quantum gravity in particular, we argue that the purely algebraic notion of field (*viz.* $\mathbf{A}$-connection $\mathcal{D}$, for a suitably chosen $\mathbf{A}$) in ADG, and the gravitational dynamics (the differential equations of Einstein) that it defines, is in a subtle sense 'already quantum' or 'quantized-by-itself' hence in no need of either a process of quantization, or conversely, of a correspondence principle (or classical limit theory). We coin this feature of ADG-based gravity, 'third', or a more mouthful, 'field (without an external spacetime, whether a continuum or a discretum) self-quantization'. Third quantization, as opposed to second quantization—let alone first quantization, which is non-existent in ADG—makes us question altogether the physical existence of a fundamental space-time scale in Nature, like the Planck length-time is supposed to be in the conventional (and persistently spacetime continuum based in one way or another) approaches to quantum gravity. In the end, we present a physico-philosophical critique of the quite unsuccessful way we try to apply differential geometric ideas in the quantum deep, as well as numerous potent arguments we have gathered so far for the possibility of doing ADG-theoretically field theory entirely by categorico-algebraic and finitistic means, by referring directly and solely to the (algebraic relations between the) fields themselves without at all the 'mediation' or interference in our constructions, or even intervention in our calculations (*ie*, in our Calculus) of a pointed background differential spacetime manifold in the guise of $\mathcal{C}^\infty$-smooth coordinates. In the light of this critique, the idea is entertained of potentially 'marrying' Stachel's 'two Einsteins' [368]—one, the nowadays more popular facet of Einstein's post GR work, advocating a unitary, 'continuous' field theory on the spacetime continuum (while at the same time apparently maintaining a polemic stance against the quantum, which he thought that a suitably completed and potentially singularity-free field theory could actually 'explain away'), the other, arguably a less popular and currently much overlooked aspect of Einstein's ideas, propounding a purely algebraic and finitistic-combinatorial physics on a 'fundamental discretum'. All in all, the present 'paper-book' may be viewed as a significant extension of the trilogy [270, 271, 272] to a tetralogy so as to include ADG's promising physical application towards evading the $\mathcal{C}^\infty$-smooth gravitational singularities of the differential spacetime manifold of GR, as well as to explore the potential technical and conceptual consequences that such an evasion has for both classical and quantum gravity research. This work's recent forerunner, of a more modest size, is the second author's paper [317], which the reader might like to have a look at first as a 'warmup reading'.

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Key words: general relativity, smooth singularities, differential algebras of generalized functions, spacetime foam dense singularities, abstract differential geometry, sheaf theory, category theory, causal sets, discrete differential incidence algebras of locally finite posets, discrete Lorentzian quantum gravity

Contents

1 Foreword

The present work encapsulates (admittedly, in quite a large 'capsule' which, paying tribute to Laurent Schwartz, may as well be coined '*paper-book*') more than 3-years' efforts of exploring the potential application of ideas, concepts, techniques and results of *Abstract Differential Geometry* (ADG) to *Classical* and *Quantum Gravity* (QG), as well as to (quantum) Yang-Mills (gauge) theories of matter. Primarily, it focuses on showing how ADG may be employed to evade completely the *singularities* and their associated unphysical infinities assailing the *smooth background spacetime manifold* and, *in extenso*, the *Classical Differential Geometry* (CDG) based *General Relativity* (GR), as well as to anticipate various conceptual (philosophical) and technical implications that such a total singularity-bypass may have for a plethora of currently 'caustic' topics in QG research. The present treatment of gravity by ADG-theoretic means is suitably called '*ADG-gravity*'.

On the physics side. Much in the same way that, in the pure mathematical subject of *Differential Geometry* (DG), ADG has shown us how to do DG *purely algebraically* (:sheaf-theoretically), based solely on the notion of an *algebraic connection* $\mathcal{D}$ (*viz.* generalized differential operator) and *manifestly without depending ourselves on a base differential* (:$\mathcal{C}^\infty$-smooth) *locally Euclidean space* (:manifold), in its application to gravity ADG enables us to formulate the latter as a *pure gauge theory* (*ie*, a theoresis of gravity based only on the gravitational connection field $\mathcal{D}$), in a *genuinely background spacetime manifold independent* fashion, and with algebraic 'quantum traits' built into the formalism from the very start (*ie*, without any need of a formal procedure of quantization of the classical relativistic field theory—GR—let alone of the base spacetime continuum underlying it). '*Background independence*' in particular, is a 'hot' issue on which various current approaches to QG—most notably, the *Loop Quantum Gravity* (LQG) approach to (canonical) *Quantum General Relativity* (QGR)—hinge. However, in contradistinction to these approaches where 'background independence' means '*background metric independence*' while a base differential manifold is still present in the theory galore, in 'ADG-gravity' not only *a metric is not employed at all* (as the sole dynamical variable in the theory is the gravitational gauge connection field $\mathcal{D}$), but also *no background spacetime manifold is present at all.* In fact, a background geometrical spacetime, whether a 'classical continuum' or a 'quantal discretum', plays absolutely no role whatsoever in ADG-gravity—*ie*, it plays no role in formulating the gravitational dynamics, as *differential equations* proper, in terms of the gravitational connection field $\mathcal{D}$ 'in-itself'. *In summa, ADG-gravity is a fundamentally background spacetimeless, 'pure gauge'*[1] *and 'innately quantum'*[2] *field theoresis*

[1]The (dynamical) formalism underlying ADG-gravity, in contradistinction to the original metric-based formulation due to Einstein (second order formalism) and to the more recent Palatini-Ashtekar one which is based on both the (smooth) connection and tetrad variables on a smooth manifold (first order formalism), is coined *half-order formalism*', since the only dynamical variable involved is the gravitational gauge connection field $\mathcal{D}$ and *a fortiori* no base differential manifold is present in the theory. Especially due to the latter, ADG-gravity is called a '*gauge theory of the third kind*', to distinguish it from the usual gauge theories of the first (global) and second (local) kind which, since they are based on an external (background) spacetime continuum, necessarily draw a distinction between external (spacetime) and internal (gauge) symmetries. All 'symmetries' in ADG-gravity are 'internal'—*ie*, of the ADG-gravitational field 'in-itself'. (See remarks below about the novel terminology that comes along with ADG-gravity.)

[2]Due to the quantum concepts and associated formalism which are built from the very start into (*ie*, almost 'by definition' of) the ADG-gravitational field, ADG-gravity is called '*third quantized*' or '*third quantum*' (field) theory, if anything to distinguish it from the usual first and second quantized (non-relativistic quantum particle/finite degrees of freedom and relativistic quantum field/infinite degrees of freedom, respectively) theories (of matter) in

of gravity.

In turn, this fundamental base spacetimelessness of ADG-gravity and its focusing solely on the gravitational field itself, helps us shed light on, or even 'resolve' and 'bypass' completely and directly (*ie*, in a 'cutting of the Gordian knot' sort of way), certain crucial problems encountered in both classical and quantum gravity research: from the problem of the $\mathcal{C}^\infty$-smooth spacetime singularities and Einstein's hole argument in the manifold based GR, to the so-called *inner product/quantum measure problem* in both the canonical and the covariant (path integral) approaches to QG, as well as the *problem of time* in (canonical) QGR and *the general problem of viewing (vacuum) Einstein gravity as a gauge theory proper*, especially in the quantum domain (QG) as a quantum gauge theory. And it must be emphasized at this point that in all of the aforesaid problems the *spacetime diffeomorphism 'symmetry' group* $\mathrm{Diff}(M)$ of the underlying differential manifold M—the automorphism group of the smooth spacetime background traditionally used to implement mathematically the *Principle of General Covariance* (PGC) of GR, is involved in one way or another. Thus, since ADG-gravity is background spacetime manifold independent (hence *a fortiori* no $\mathrm{Diff}(M)$ is involved in the theory), none of these problems are encountered; as it were, they *ab initio* become '*non-problems*' in formulating gravity ('classically' or 'quantally')[3] in the ADG-framework.

Concerning $\mathcal{C}^\infty$-smooth spacetime singularities in particular, which are arguably 'innate' in the spacetime manifold—*ie*, they are 'inherent' in the *structure sheaf* $\mathcal{C}^\infty_M$ of algebras of $\mathcal{C}^\infty$-smooth ('coordinate') functions on M (which sheaf, in turn, *defines* M as a differential manifold proper in the first place by Gel'fand duality), their total evasion in ADG-gravity brings to mind a tetrad of verses from Constantinos Cavafis' poem '*Ithaca*' [77]:

> "*...Laestrygonians and Cyclopes, angry Poseidon,*
> *Such obstacles you will never encounter in your way,*
> *As long as you do not carry them in your soul,*
> *As long as your soul does not raise them before you...*"

the basic 'moral' being here (metaphorically speaking of course) that *one will not encounter gravitational singularities*, at least as insuperable differential geometric obstacles (or even more formidably, as '*breakdown points*' of Einstein's field equations—the way singularities are commonly viewed by physicists nowadays) in the *aufbau* of one's theory of gravity, *as long as one does not model* (in one's theory) *spacetime after a differential manifold.* Accordingly, *mutatis mutandis* for the other background M and, *in extenso*, $\mathrm{Diff}(M)$-related classical and quantum gravity problems mentioned above.

On the mathematics side. If anything, by ADG-gravity we can reinstate the status of DG in QG research, a status and import of the manifold based CDG that has been questioned in the past by many workers in the field, with the most severe and direct critic being (to our knowledge)

which again a background space-time continuum is invariably involved in one way or another. (Again, see remarks below about the novel terminology used in ADG-gravity.)

[3] This signals that, as a matter of fact, the traditional epithet-distinctions 'classical' and 'quantum' (usually put in front of the noun 'gravity') lose their significance in ADG-gravity.

Chris Isham, who claimed fairly recently in [203]:[4]

> "*...at the Planck-length scale, differential geometry is simply incompatible with quantum theory*...[so that] *one will not be able to use differential geometry in the true quantum-gravity theory...*"

On the philosophy side. Since ADG, as a mathematical framework, enables us to do DG *entirely algebraically* (in a way reminiscent of the *relational* fashion in which Leibniz envisioned the development of his 'Geometric Calculus'—'*Ars Combinatoria*' and '*Calculus Ratiocinator*') by referring directly and solely to the 'geometrical objects' (:the physical fields) that 'live' on 'space(time)' without depending at all on that external (to the fields themselves), 'ambient' geometrical space(time manifold) for its concepts and constructions, philosophically speaking ADG-gravity may be viewed as a '*purely realist*' theory—supported by what we coin the '*Principle of Field Realism*' (PFR); or even for more effect, the '*Principle of Field Solipsism*'. The PFR can be accommodated by the ADG-framework due to the fact that the ADG-gravitational field $\mathcal{D}$ is a '*dynamically autonomous*' entity, in no need of an external spacetime (manifold) for its 'dynamical subsistence' (*ie*, the dynamical Einstein equations that $\mathcal{D}$ defines via its curvature $R(\mathcal{D})$, although still modelled after differential equations proper, do not depend at all on a geometrical background M, as *they represent the dynamics of the field 'in-itself'*). More noteworthy is our maintaining this realistic 'attitude' even in the quantum domain, in spite of the usual operationalistic (algebraic), instrumentalist, 'external (to the quantum system) observer dependent' conception of physical reality that quantum theory is traditionally supposed to entail.

This is also in contrast to the usual manifold based GR, which is supported by an 'operationalist' or 'instrumentalist' philosophy (and associated interpretation) according to which the components of the smooth metric tensor field $g_{\mu\nu}$ on the differential manifold M not only represent the gravitational field potentials, but also the (local) *chronogeometry* of M, as they engage into the (infinitesimal proper time) line element ds^2.[5] As a result of this interpretation, the gravitational field is supposed to encode all the information about our tampering with ('probing' or 'measuring') it with our (local) geometrical 'spacetime gauges' (:equicalibrated rods and synchronized clocks), with the results of these (local) measurements being organized into the aforesaid coordinate structure sheaf $\mathcal{C}^\infty_M$. Of course, GR is able to sustain an analogue of the PFR via the PGC, yet its dependence on a background geometrical differential manifold is vital: *how else can one represent the gravitational Einstein equations as differential equations proper in the first place?*

Unfortunately, a 'self-referential' *vicious circle* is lurking here: the standard perception nowadays of 'genuine' or 'real' (*ie*, not coordinate) gravitational singularities is as *loci* in the spacetime continuum where the Einstein field law, viewed as a (partial) differential equation, *breaks down.* In other words, the very structure sheaf $\mathcal{C}^\infty_M$ (or equivalently, the base differential manifold M) that enables one to set up (the dynamical law of) GR differential geometrically (*ie*, as a differential

[4]This quotation appears as (Q8.?) in the main text and has also been used in the past, in the introduction to [272]. Here, emphasis is ours.

[5]In the usual CDG-based (pseudo-)Riemannian geometry supporting GR, the affine gravito-inertial connection 'inherits' the chronogeometrical interpretation by being required to be compatible with the metric (metric or torsionless connection)—a condition which is only optional in ADG-gravity, and in a way it is *reversed* since now *the metric* (which is an optional, externally prescribed structure in ADG-gravity, imposed by the external to the gravitational field 'observer' or 'measurer') *is made to be compatible with the fundamental connection field, not the other way round.*

equation proper), is pregnant to GR's own 'destruction' in the form of singularities. In this subtle (differential geometric) sense we view the Wheeler-Bergmann 'Popperian virtue' of GR, according to which the latter '*carries in its belly the seeds of its own destruction*', its 'self-falsification' so to speak. From an ADG-theoretic vantage, it is not that the gravitational field (and the law that it defines/obeys) breaks down at a singularity, but simply that GR is formulated within the manifold based CDG-theoretic framework, which is out of its depth on the face of singularities (let alone *contra* deeper and conceptually/interpretationally more complicated QG issues). Plainly then, from the ADG-vantage singularities are a 'fault' of the mathematics—a glaring 'proof' that the mathematics (*ie*, the CDG used in GR) is inadequate or 'inappropriate', *not* of the physics (dynamical gravitational field law). Alas, in the manifold based GR the mathematics is so intimately enmeshed and entwined with the physics that it misleads one into thinking that '*physical spacetime*' is a (differential) manifold, when in contradistinction, from the viewpoint of ADG, if there is any 'spacetime' at all it is 'inherent' in the dynamical fields that comprise 'it'. This then is the main 'aphorism' in the present work: *in ADG-gravity, all is field* and no externally prescribed spacetime is involved at all, the epitome of the aforesaid '*field solipsism*'.

Moreover, when it comes to the quantum domain, where an operational, 'observer dependent realism' reigns supreme, it is plain that the geometrical base spacetime continuum glaringly 'miscarries'. For one thing, there is supposed to be a minimal space-time scale (the so-called Planck length-time $\ell_P \approx 10^{-35}m$–$t_P \approx 10^{-40}s$) below which one cannot localize ('measure') the gravitational field with infinite accuracy without creating a singularity (concealed beyond the horizon of a so-called 'black hole'). This then seems to suggest that below ℓ_P the smooth spacetime continuum should give way to a 'reticular-quantal and inherently cut-off' or regularized structure, with the inevitable loss of one's differential geometric privileges in the QG regime (see Isham quotation above). From the background spacetimeless (whether a continuum or a discretum) ADG-theoretic vantage, this is hard to swallow. Similarly to Einstein's explicit dissatisfaction with spacetime singularities in GR,[6] we cannot accept that there is a spacetime scale above which the field law holds, but below which it apparently breaks down or that it should be radically modified—especially *vis-à-vis* ADG-gravity where no external (to the fields), background spacetime (whether 'continuous' or 'discrete') is involved at all. This is a '*paraphysical antinomy*' of the very term '*physical law*' (pun intended!) and its supposed universality.

All in all, it is evident by the foregoing that in the present work we do not shy away from addressing conceptual (philosophical) issues, especially *vis-à-vis* ADG-gravity's potential QG import, for after all, as 't Hooft recently put it in [392]:

> "*...The problems of quantum gravity are much more than purely technical ones. They touch upon very essential philosophical issues*[7]*...*"

The gist of this 'paper-book'. *Vis-à-vis* so-called 'applied mathematics', 'mathematical physics', or even 'mathematical methods in physics', it is fair to say that twentieth century ('modern') theoretical physics was largely dominated by applications of *differential geometry*. The great interest of theoretical physicists in differential geometry may be attributed to their idea of modelling physical laws after *differential equations* if anything in order to implement their primitive notion of

[6]See the first quotation (Q2.1) in section 2.

[7]Our emphasis.

'*infinitesimal locality*' or '*differential local causality*',[8] and arguably *differential geometry was developed for (having a comprehensive theory of as well as for solving) differential equations.*[9] This of course is the 'classical' theory of differential geometry (CDG) which vitally relies in one way or another on the notion of a *smooth background space*: from the finite-dimensional (locally) Euclidean spacetime (manifold) of the special, but more importantly, the general theory of relativity in which the (pseudo-)Riemannian geometry employed is grounded, to the infinite-dimensional complete Euclidean (Hilbert) spaces modelling the quantum configuration/phase (state) spaces used in quantum mechanics and handled by functional analytic/operator-theoretic means.[10] *A fortiori*, the subsequent unison of quantum theory with SR to a QFT of matter and the concomitant realization that matter forces are in fact *gauge* forces, gave impetus for further development of differential geometric ideas, concepts and techniques in theoretical physics. To appreciate this, one has simply to recall the boom in applications of the mathematics of fiber bundles to gauge theory, to the extent that Michael Atiyah, the celebrated mathematician, coined in [23] gauges theories as:

"...*Physical* theories of a *geometrical* character[11]..."

while one is justifiably tempted to add the epithet '*differential*' to the word '*geometrical*' in the quotation above.

Yet there still comes Isham's quotation above to 'haunt' any approach to QG that uses CDG-ideas. That is to say, there is this 'nagging paradox' that while CDG has enjoyed enormous success in being applied to classical mechanics and field theory (*eg*, electrodynamics and GR), as well as to QM and QFT (*eg*, quantum gauge theories of matter), when one sets out to marry GR with quantum theory to a quantum theory of gravity, the CDG of smooth manifolds appears to be of little help (if any at all!) and out of its depth. We understand this simply on the fact that the main culprit for all the 'pathologies' encountered in either GR (in the form of singularities) or QFT (field-theoretic infinities)—arguably, *the* 'anomalies' that make us question in the first place the CDG of smooth manifolds in the QG realm—is *our* assumption of (physical) spacetime as a differential manifold. Here we have *the* example *par excellence* of the proverb '*throw away the baby together with the bath-water*' in the sense that if the manifold, which causes all these 'unpleasant unphysicalities', will have to go in the realm where GR is envisioned to be united with quantum theory (the Planck regime), so will *differential geometry* as a whole. For after all, *so far the only way we know how to do differential geometry is by basing ourselves in one way or another on a smooth manifold, which in physics is interpreted either as the spacetime continuum or as the configuration space of a physical system—classical or quantum.*[12] It is plain that if we have such

[8]The primitive intuition that events causally affect others in their 'infinitesimal neighborhood'.

[9]Much in the same way that *algebraic* geometry was originally developed for dealing comprehensively with *algebraic* equations.

[10]Let it be noted here that 19th century physics too was essentially dominated by Newton's Differential Calculus—the spry grandparent of nowadays modern differential geometry (CDG): from the classical particle mechanics of Lagrange and Hamilton, to the classical field theory of Faraday and Maxwell. It is fair to say that the iconoclastic *physical* theories of the last century—namely, GR and QM—introduced new, ground-breaking *physical* concepts, but essentially relied on Newton's background continuous space(time) dependent Differential Calculus (albeit, in the more sophisticated language and technotropy of the manifold based CDG) for their mathematical concepts, techniques, and more importantly, for their *calculations*.

[11]Our emphasis.

[12]And let it be noted here that fields are normally thought of as physical systems with an uncountably infinite number of degrees of freedom, thus a mathematical continuum such as a manifold appears to be tailor-cut for

a 'CDG-monopoly', if the manifold goes 'bankrupt' in the quantum deep, inevitably so does CDG which is vitally dependent on it.

It is precisely this issue we wish to challenge in the present work, as it were to counter and ultimately evade Isham's 'no-go' of differential geometry in QG:[13] we are going to show and argue that since ADG is manifestly base manifold-free, hence also ADG-gravity genuinely background spacetime manifold independent, *one can still do differential geometry in the realm of QG*. For one thing, as noted above, bypassing directly the manifold with its 'inherent' singularities and associated unphysical infinities, while at the same time retaining most (if not *all*!) the differential geometric panoply (mechanism) of CDG in its manifest absence, as well as formulating gravity as a 'pure gauge theory', could prove to benefit tremendously both classical and QG research.

In the present work we shed the weighty burden of the smooth manifold and the inertia that its physical interpretation as 'spacetime' presents to the theoretical/mathematical physicist, and we wish to travel light in Feynman's 'wild blue yonder' QG regime.[14] However, in order to appreciate how difficult it may turn out to be for one to overcome the said '*background spacetime manifold inertia*' and associated '*CDG-conservatism and monopoly*',[15] one may recall some recent remarks of Isham and Butterfield from [213], where the issue is raised of what structures (other than the CDG-supporting, smooth, and as 'spacetime' interpreted continuum) to consider in QG research, as well as how can the familiar structure and notion of the spacetime continuum (of GR) emerge (or be recovered) from those 'deeper' structures:[16]

> "*...The usual tools of mathematical physics depend so strongly on the real-number continuum, and its generalizations (from elementary calculus 'upwards' to manifolds and beyond), that it is probably even harder to guess what non-continuum structure is needed by such radical approaches, than to guess what novel structures of dimension, metric etc. are needed by the more conservative approaches that retain manifolds. Indeed, there is a more general point: space and time are such crucial categories for thinking about, and describing, the empirical world, that it is bound to be ferociously difficult to understand their emerging, or even some aspects of them emerging, from 'something else'...*"

On the other hand, Einstein's words from [120] immediately spring to mind here:[17]

> "Time and space are modes by which *we*[18] think, not conditions in which we live.",

modelling either the spacetime on which these fields dynamically propagate and interact, or their corresponding state spaces.

[13] And let it be stressed here that the *DG* that Isham was referring to was the CDG on smooth manifolds. A recent exchange of the second author with Chris about this quotation received back the counter-remark that "*I may have been wrong*". This we understand as implying *not* that CDG on manifolds could be of import to QG research after all (for this would be a sort of regress to old concepts and dated technology so to speak), but that other new *theoretical frameworks* for doing DG, such as Connes' Noncommutative Differential Geometry (NDG) [91] which has enjoyed numerous applications in the Standard Model and quantum spacetime and gravity in the past decade [92, 79, 80], as well as the category (topos) based Synthetic Differential Geometry (SDG) of Kock and Lawvere [232, 243], could be of great value to the QG quest. We are aware that Chris is particularly interested in the possibility of applying SDG-ideas to the quantum structure of spacetime and gravity [72, 208]. We too are very keen on exploring in the immediate future various close affinities between ADG, NDG and SDG, as well as, hopefully, to unite forces at the QG research front (see 8.3.1).

[14] Hopefully not at the cost that we will never be heard of again...

[15] Thus also how easy and likely it is for one to go off to the QG 'blue yonder' and never be heard of again indeed!

[16] In the excerpt below, all emphasis is ours due to its importance.

[17] This quotation also occurs in the main text (Q2.10).

[18] Our emphasis.

but more importantly, his urging us to question and scrutinize familiar, well established concepts (like for instance that of the spacetime continuum) in our physics (re)searches:[19]

> "*...Concepts* [like the spacetime manifold][20] *which have proved useful for ordering things easily assume so great an authority over us, that we forget their terrestrial origin and accept them as unalterable facts. They then become labelled as 'conceptual necessities', 'a priori situations', etc. The road of scientific progress is frequently blocked for long periods by such errors. It is therefore not just an idle game to exercise our ability to analyse familiar concepts, and to demonstrate the conditions on which their justification and usefulness depend, and the way in which these developed, little by little*[21]*...*" [127]

come to accompany, comfort and inspire our endeavors here. All in all, let the present paper-book mark the beginning of the end of the epopee of the smooth background (spacetime) manifold based (and thus largely CDG-dominated) theoretical/mathematical physics research, and QG in particular.[22]

However, in spite of all that 'background geometrical spacetime manifold inertia, habit and indolence' as well as our principal aim to overcome them herein, and apart from the fact that quantum gauge (*ie*, electrodynamics and non-abelian Yang-Mills) theories and QG is the focal issue in the present treatise *vis-à-vis physical applications* of ADG, one should not lose touch with *the* philosophical essence of ADG as a mathematical framework for doing differential geometry in a wider, broader and more thorough sense. The (philosophical) gist of ADG is primarily an '*epoptic*' one. In this respect, let us first recall Wittgenstein's opening remarks in his posthumously published book "*Culture and Value*" [422]:

> "...Our civilization typically constructs. Its activity is to construct a more and more complicated structure. And even clarity is only a means to this end and not an end in itself. *For me on the contrary clarity, transparency, is an end in itself. I am not interested in erecting a building but in having the foundations of possible buildings transparently before me*[23]*...*"

[19]What we find truly remarkable in this quotation is that Einstein's words came only one year after the development of GR, in which the concept of the base geometrical spacetime continuum (and the CDG-based pseudo-Riemannian geometry on it) triumphed. This quotation also occurs in the main text (Q7.27), as well as in the conclusion of our last joint paper [272].

[20]Our addition for making our point here clearer.

[21]Our emphasis throughout.

[22]We are pleasurably indebted to John Stachel for timely pointing out to us his belief that QG research in the new millennium will focus primarily on formulating the theory in a background independent fashion, to the extent that any current or future approach to QG shall be ultimately 'judged' on the degree that it has achieved a genuinely background independent formulation (Stachel in private communication with the second author at Imperial College, Fall 2004). And let us further add here our opinion that background independence should go all the way—*ie*, that we should not content ourselves only with background *metric* independence as it is 'fashionable' nowadays, but also look for background *smooth spacetime manifold* independent scenaria, like the one ADG-gravity will offer herein.

[23]Our emphasis. From another translation of "*Culture and Value*" [423] a year later than [422], we encounter another nice version of the same excerpt: "*...Our civilization is characterized by the word 'progress'. Progress is its form rather than making progress is one of its features. Typically it constructs. It is occupied with building an ever more complicated structure. And even clarity is sought only as a means to this end, not as an end in itself. For me on the contrary clarity, perspicuity are valuable in themselves. I am not interested in constructing a building, so much as in having a perspicuous view of the foundations of possible buildings...*". The reader can choose the one (s)he prefers.

and slightly modify them to suit the general spirit of the present work regarding, under the prism of ADG, the theory and (physical) applications of differential geometry 'at large':

> In this work we are not as much interested in tackling this or that particular problem nowadays encountered in classical and quantum gravity research as well as in quantum Yang-Mills theories, let alone to come up with a grand structural scheme—a *panacea* so to speak—for dealing with such problems, as to attain a clear, '*epoptic*', bird's-eye-view as it were, of possible applications of fundamental differential geometric ideas—ones that are *ab initio* free from any commitment to an *a priori* posited 'background space(time manifold) structure—to modern theoretical physics, as well as to anticipate and explore the potent physical implications of such a fundamental non-commitment ('background independence'). *How far can we go in modern theoretical physics with a base (spacetime) manifoldless differential geometry?*—that's *the* basic question we would like to ask in the light of ADG the modern theoretical physicist, and in particular the QG worker, who is still willing, persistently in spite of Isham's 'differential geometric pessimism' in the quantum deep quoted above, to use differential geometric ideas and technotropy in her QG research. Let's hope it's a long way...

All in all, in many, closely intertwined, cross-fertilizing and mutually affecting levels—physical, mathematical and philosophical—our principal aim in this paper-book is, once again emulating in a metaphorical way the latter, post-Tractarian Wittgenstein's [424, 425, 426] remark that:

> "...[The principal aim of my work][24] *is to show the fly out of the fly-bottle*[25]...",

the main aim of our work is

> to 'free' the mathematician, physicist and philosopher of physics (who wishes to apply differential geometry to theoretical physics) from the confines and shackles of the background manifold that anyway she assumed up-front (and she trapped herself into!) in order to do (and interpret!) field theory differential geometrically, which 'pseudo-physical' spacetime continuum in turn creates all the problems, in the guise of singularities and unphysical infinities, that she encounters in both the classical and the quantum (field-theoretic) domain.[26] Our venture here is kind of 'therapeutic': as it were it aims to dispel the chimerical 'surrounding spacetime nimbus' and the mesmerizing magic that a background geometrical (spacetime) manifold exercises on the

[24] Or of philosophy in general. Our addition.

[25] Our emphasis. In a nutshell, the meaning here is that philosophers trap themselves into philosophical (pseudo-)problems by wrong 'use' and associated misinterpretations of (everyday colloquial) language, so that by clarifying language—as it were, by casting the said problems in their 'natural language habitat' (by the way, as such 'habitats', Wittgenstein had introduced the relational and autonomous notion of 'language games')—the philosopher is led out of the trap that he himself set up in the first place to tackle those problems by means of language.

[26] The idea here is that our standard (*ie*, CDG-based) use of differential geometric concepts and constructions via a base differential manifold is 'wrong' ('unnatural' or 'unphysical')—the 'wrongness' being exemplified by the physically inadmissible singularities and infinities one encounters in applying the manifold-based CDG-ideas to both classical and quantum field theory. ADG studies differential geometry in its 'natural habitat'—that delimited by the 'geometrical objects' (:physical fields) (in-)themselves, without the mediation (in the guise of smooth coordinates) of a smooth background manifold, which is thus of no physical significance in (physical applications of) the theory.

> modern theoretical physicist, which in turn misleads her into thinking that problems such as singularities and infinities that she encounters when she models physical laws differential geometrically via a base M are actually *physical* problems, when in fact they are only shortcomings and anomalies of the *mathematical* framework (*ie*, CDG) that she employs in the first place. The 'ADG-therapy' prescribed here involves *algebra* (:'relational differential geometry'), and, arguably, *there is no infinity in algebra*, for infinities arise only in our geometrical base space(time) manifold-mediated Analysis (*ie*, CDG or Differential Calculus).

In the same line of thought we are tempted to recall Evariste Galois' words in [153]:

"*Les calcules sont impracticables*",[27]

and to modify them in our ADG-context to the following:

> The usual Differential Calculus (CDG), insofar as it is effectuated—and so far, let it be stressed, it has actually been effectuated(!)—via a base differential (spacetime) manifold, is of little import in the QG deep.[28]

On the terminology side. The novel perspective on gravity that ADG enables us to entertain is inevitably accompanied by *new terminology*. We have thus not refrained from engaging into vigorous '*lexiplastic activity*', so that the present paper-book abounds with new terms for novel concepts hitherto not encountered in the standard theoretical physics' jargon and literature, such as '*gauge theory of the third kind*', '*third quantization*', '*synvariance*' and '*autodynamics*', to name a few. In this respect, we align ourselves with Wallace Stevens' words in [376]:[29]

> "*...Progress in any aspect is a movement through changes in terminology...*"

with the 'changes in terminology' in our case being not just superficial (formal) 'nominal' ones introduced as it were for 'flash, effect and decor', but necessary ones coming from *a significant change in basic theoretical framework for viewing and actually doing DG*: from the usual geometrical manifold based one (CDG), to the background manifoldless and purely algebraic (:sheaf-theoretic) one of ADG. All the new terms and concepts are defined and explained in a '*Glossary for ADG-gravity*' appended at the end.

On the textual side. The reader will have already noticed that we are also not frugal in providing a plethora of *footnotes* and *quotations*, as well as a long list of *references* at the end. The purpose of footnotes is principally explanatory, when we do not want to go into lengthy digressions within the main text. We do hope that they prove to be more helpful to the reader than distracting. The purpose of quotations is sometimes to give a historical background and a 'motivational alibi'

[27] "*Calculations are impractical*".

[28] Actually, it is of *great* import, but of things of the 'wrong' kind, such as singularities, infinities and a host of other differential geometric anomalies and pathologies, which cumulatively give one the (false, in our opinion) impression that 'DG miscarries with quantum theory, and especially, in the realm of QG' (see again the Isham quotation above).

[29] This quote also appears in the main text, in (Q3.?).

for the ideas presented—namely, that what is being said has also been anticipated (*ie*, it is in accord with) similar thoughts that have been expressed in the past by great thinkers or specialist workers in the research field of QG. That is to say, quotations essentially come to remind the reader that '*we are standing on the shoulders of giants*', or at least that '*we are not alone in this venture*', and enhance the 'historical continuity' of what is being said. However, other times quotations are given as 'counterpoints' to (*ie*, they provide a contrasting platform against) the points being raised and discussed. The ultimate hope of the present authors is that this dialectical '*thesis-antithesis*' service of quotations will prepare the ground for a fruitful '*synthesis*' of the new theoretical paradigm for gravity (classical and quantum) that ADG-gravity is in our opinion pregnant to. Finally, although to the best of our ability and knowledge the long list of references at the end is (intended to be) complete, we are certain that important (even classic!) works of various people on a host of subjects have been overlooked and unfortunately omitted. This was done inadvertently and we apologize in advance.

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2 Introductory Remarks: General Covariance Versus Singularities

That General Relativity (GR) predicts the existence of singularities—*loci* in the smooth spacetime continuum where Einstein's gravitational field equations do not hold or even 'break down'—is by now a virtually undisputed fact. By some physicists this has been regarded as a positive feature of the classical relativistic field theory of gravity—as if, in a 'Popperian falsifiability' sense, GR is a sound theory because in a way it carries, as it were from within its conceptual and technical edifice, its own limitations and, ultimately, it foreshadows its own downfall, its self-destruction so to say. For others, including Einstein, singularities were clearly unphysical, pathological, at least anomalous, and certainly problematic, features of the theory that should be somehow excluded from it; in [125] he notes for example:[30]

(Q2.1) "...A field theory is not yet completely determined by the system of field equations. Should one admit the appearance of singularities?...*It is my opinion that singularities must be excluded. It does not seem reasonable to me to introduce into a continuum theory points (or lines etc.) for which the field equations do not hold*[a]..."

[a]Our emphasis.

Bergmann, for instance, described GR's 'autocatastrophic' prediction of singularities as follows:

(Q2.2) "...What about shortcomings of general relativity?...perfectly reasonable conditions at one time may lead to field singularities at another. *That looks as if general relativity carries within its conceptual belly the seeds of its own destruction*[a]..." [41]

[a]Our emphasis.

while, more recently, Ashtekar too described singularities and their associated infinities as an 'intrinsic fault' of GR:

(Q2.3) "...Although general relativity has been accurately verified on macroscopic scales, it has an internal blemish: it permits, and even predicts, the occurrence of singular configurations in which physical quantities become infinite..." [8]

The philosophical debate about whether singularities are 'positive' or 'negative' traits of GR aside, we believe that very few physicists would actually doubt that the principal pathogenic gene in GR's conceptual genome—one that is the main culprit for the existence of singularities—is the primitive conception, and as a result, the basic assumption, of *modelling spacetime after a differential* (*ie*, a $\mathcal{C}^\infty$-smooth) *manifold*. Bergmann again, for instance, immediately after he expressed his negative view about singularities above, added:

[30]Throughout the present paper, quotations will be left-labelled by 'Q*.⋆' (the first ordinal '*' corresponding to the section number, and the second, '⋆', to the number of the quotation in that section), while numerous remarks of ours that we would like to highlight and 'retro-refer' to at later parts of the paper, will be similarly marked by 'R*.⋆'.

(Q2.4) "...Thus, let me begin by saying that *all unitary field theories that I know have been based on one topological model, that of a manifold.* They differ on the kind of structures that they superimpose on that basic framework..."

while Joshi admitted rather categorimatically right at the very beginning of [219] that:

(Q2.5) "It is generally accepted now that any reasonable classical theory of gravitation must admit singularities where the curvatures grow unbounded and the usual laws of physics breakdown. *Within the classical framework, the physical universe is modelled by a spacetime manifold...*"

Indeed, granted that a smooth manifold supplies one with the usual ('classical' or 'standard') differential geometric structure and mechanism one needs in order to represent the laws of physics by *differential equations* relating the various relevant smooth physical quantities (fields), the coarse and intuitive picture one has of singularities is as 'regions' or 'locations' in the spacetime continuum where the 'differentiability properties' (smoothness) of physical quantities and, as a result, the dynamical relations—the laws of Nature modelled after differential equations—in which the latter participate,[31] *break down* in one way or another, while at the same time the very (smooth) fields that partake in those laws become *unphysically* and mathematically *unmanageably* infinite.

In fact, that singularities in GR (which, as it is well known, represents its sole dynamical variable, the gravitational field, by a smooth metric on a differential spacetime manifold) signal a breakdown of differentiability, that is to say, that they essentially mark the ineffectiveness of the entire edifice on which Classical Differential Geometry (CDG), the so-called Calculus on Manifolds, rests—*ie*, the $\mathcal{C}^\infty$-smooth manifold—is built into their very definition. Clarke, for example, remarks in [86]:[32]

[31]Thus, in a broad sense, by 'differentiability' we understand here the mathematical representation of 'dynamical variability' in physics—that physical quantities (can) change. Equivalently, for us, 'measurable dynamical attributes' (commonly known as '*observables*') and 'differentiable quantities' (or '*differentiables*' [272]) are effectively synonymous terms standing for *dynamically variable properties of physical systems*.

[32]In this quotation the words in in square brackets are our own additions for continuity, completeness and clarity. We will return to comment in more detail about this definition of smooth gravitational singularities in the next section.

(Q2.6) "...Thus the definition of a singularity depends on the definition of an [analytic] extension of [the] space-time [manifold], and so the question of what counts as a singularity depends on what sort of extension is allowed. We call a boundary point [of a smooth manifold] a class C^k geometrical singularity if there is no [analytic] extension with a C^k metric that removes it; *i.e. if it is associated with a breakdown of differentiability of the metric at the C^k level*[a]..."

[a]Our emphasis.

This prompts us to emphasize here that it is the basic contention of the present paper that

(R2.1) behind both the *unphysicality* ('physical inadmissibility') and the *unmanageability* ('mathematical inadequacy and ineffectiveness in the handling') of singularities and the infinities that they are associated with, lies the main culprit for it all: *the*—in fact, *our*—*representation of spacetime as a differential manifold and, as a result, the CDG-framework and the various smooth constructions (structures) within that framework that the manifold supports.*

2.1 About Unphysicality of Singularities

One of the main reasons for singularities and the infinities that they are associated with is that the manifold picture of spacetime allows one, even if just in theory, to pack an uncountable infinity of events into a finite spacetime volume. We have no actual physical experience of an infinity of events since we invariably record a finite number of them ('field values') during experiments of finite duration ('temporal extension') conducted in laboratories of finite size ('spatial extension'). Einstein himself, especially in view of the discrete or finitistic actions of quanta, was sceptical about the infinities assailing the geometrical spacetime continuum and, as a result, his continuous field theory of gravity based on it, for as he remarked upon concluding [125]:[33]

[33]Einstein's quotation below is the last paragraph of the last appendix D of "*The Meaning of Relativity*". The complete paragraph (quotation) is given subsequently in (Q2.?).

(Q2.7)

> "...One can give good reasons why reality cannot at all be represented by a continuous field. *From the quantum phenomena it appears to follow with certainty that a finite system of finite energy can be completely described by a finite set of numbers.*[a] This does not seem to be in accordance with a continuum theory, and must lead to an attempt to *find a purely algebraic theory for the description of reality*[b]..."
>
> [a]Our emphasis.
> [b]Our emphasis.

Thus, simply on pragmatic or 'experientially realistic' grounds, this alone should suffice as a motivation to try to somehow 'discretize', 'algebraicize' and, as a result, 'quantize' the geometrical spacetime manifold [318, 319] and, *in extenso*, gravity [270, 271, 272]. Indeed, as one can witness in [270], Finkelstein's telling words below were originally our principal (physical) motivation for initiating the application of the algebraico-categorical and sheaf theory based ADG to a finitistic, causal and quantal description of spacetime and (vacuum Einstein-Lorentzian) gravity, which was the subject matter of the aforementioned trilogy [270, 271, 272]:

(Q2.8)

> "...The locality principle seems to catch something fundamental about nature... Having learned that the world need not be Euclidean in the large, the next tenable position is that it must at least be Euclidean in the small, a manifold. The idea of infinitesimal locality presupposes that the world is a manifold. *But the infinities of the manifold (the number of events per unit volume, for example) give rise to the terrible infinities of classical field theory and to the weaker but still pestilential ones of quantum field theory.*[a] The manifold postulate freezes local topological degrees of freedom which are numerous enough to account for all the degrees of freedom we actually observe.
> The next bridgehead is a dynamical topology, in which *even the local topological structure is not constant but variable.*[b] The problem of enumerating all topologies of infinitely many points is so absurdly unmanageable and unphysical that dynamical topology virtually forces us to a more atomistic conception of causality and space-time than the continuous manifold..." [147]
>
> [a]Our emphasis.
> [b]Our emphasis.

2.1.1 The general covariance–versus–smooth singularities clash

'Experiential realism' or 'pragmatism' aside, we maintain that *prima facie* there is an even stronger discord between singularities and one of the conceptual pillars on which GR is founded: the *Principle of General Covariance* (PGC). Here, to express this fundamental disagreement, and also in order to prepare the reader for our ADG-theoretic musings in the sequel, we shall first abide by the following 'heuristic', albeit intuitively clear, version of the PGC which, in turn, is cast as a generalized statement of the very Principle of Relativity (PR) [125, 128]:[34]

(R2.2) the laws of Physics—here in particular, the law of gravity—are independent of our measurements, *ie*, of (the numerical results of) our observations of the physical fields that take part in them.

This, together with the following identification that we would like to emphasize in the present paper, namely that

(R2.3) *a differential manifold* M *is nothing but the algebra* $\mathbb{A} = \mathcal{C}^\infty(M)$ *of differentiable* (*ie*, smooth coordinate) *functions on*/(of) *it*(s points),

which vital interdependence (or equivalence) we may loosely cast as

$$M \rightleftarrows \mathcal{C}^\infty(M), \tag{1}$$

enables us to arrive swiftly at the usual (mathematical) expression for the PGC by the following syllogism:

1. 'Operationally' (algebraically) speaking, (the results of) our acts of measurement and localization of smooth fields on the points of the differential spacetime manifold M are organized into the algebra of (real-valued) smooth coordinates ${}^{(\mathbb{R})}\mathcal{C}^\infty(M)$[35] of those points [272, 266],[36]

[34] The generalized expression of the PR in (R2.?), the fundamental non-objectivity of quantum theory aside for the time being, is intimately related to Einstein's conception of (objective) "*physical reality*", as follows: "...Physics is an attempt conceptually to *grasp reality as something that is considered to be independent of its being observed*. In this sense one speaks of '*physical reality*'..." [128] (our emphasis).

[35] From now on we will omit the pre-superscript '$\mathbb{R}$' from $\mathcal{C}^\infty(M)$ as we will implicitly assume that our measurements 'yield' (or simply that we record) real numbers, unless of course we explicitly declare up-front in the sequel another set of numbers as our recorded (generalized) 'measurement' or 'coordinatization values'.

[36] More precisely, such measurement and point-event localization acts are conveniently grouped into *sheaves* $\mathbf{A} = \mathcal{C}^\infty_M$ of abelian topological algebras $\mathbb{A}$ (over the manifold M) such as $\mathcal{C}^\infty(M)$ [266], something that foreshadows our sheaf-theoretic, ADG based endeavors in the sequel (in this respect, see also the trilogy [270, 271, 272]).

2. But the smooth fields involved in the physical laws—for gravity in particular, as originally formulated in GR, the ten in principle arbitrarily (*ie*, to any order) differentiable gravitational potentials comprising the smooth metric tensor $g_{\mu\nu}$, which engages into Einstein's equations—are nothing else but $\otimes_{\mathbf{A}}$-*tensors*[37] [259, 260]; *ergo*,

3. Categorically speaking, the differential equations (physical laws) in which these fields partake are independent of ('invariant' so to speak, and the relevant fields are 'covariant', with respect to)[38] the group $\mathrm{Aut}(M)$ of automorphisms of M—the differential structure 'symmetry group' of 'active point-transformations' of the pointed base differential spacetime manifold on which these fields are soldered (localized) and from which they actually derive their 'differentiability properties' (*ie*, they actually qualify as being *smooth* in the classical sense of this word, so that in the first place they can participate into those laws, which are differential equations proper).[39]

4. Since, again categorically speaking,[40] $\mathrm{Aut}(M) \equiv \mathrm{Diff}(M)$, we have effectively recovered the standard (mathematical) expression of the PGC of GR. In other words, the usual expression of the PGC maintaining that the gravitational law of physics is a differential equation involving tensorial quantities relative to arbitrary coordinate transformations (which thus makes these quantities independent of the local reference frames to which they are referred to and being 'measured'),[41] assumes here a precise *functorial form with respect to our smooth point-localizations or 'measurements' (of the gravitational field) in* $\mathbb{A} \equiv \mathcal{C}^\infty(M)$: the law of gravity is a differential equation[42] involving smooth quantities obtained from our basic mea-

[37] That is, they are tensors whose components, expressed in local open coordinate patches U of M ($U \subset M$), are elements of $\mathcal{C}^\infty(U)$—smooth functions on U (or equivalently, in sheaf-theoretic terms, they are elements of $\Gamma(U, \mathbf{A} = \mathcal{C}^\infty_M)$—local sections of the structure sheaf $\mathcal{C}^\infty_M$ of smooth coordinates of the points of M).

[38] Technically speaking, 'invariance' usually refers to the Lagrangian (action) from which the laws derive from the variation of the field variables, while 'covariance' pertains directly to the dynamical laws themselves and to the field quantities involved in them [25].

[39] In this sense, $\mathrm{Aut}(M)$ may be thought of as the manifold's 'auto-transformation group' respecting the various fields' (classical) differentiability ($\mathcal{C}^\infty$-smoothness) property.

[40] That is to say, implicitly working in the category $\mathcal{M}an$ of (finite-dimensional) $\mathcal{C}^\infty$-smooth manifolds.

[41] And one should recall that, since M is by definition locally Euclidean (*ie*, locally its model is the Cartesian space $\mathbb{R}^4$ coming equipped with the usual differential structure), one identifies the 'structure group' of GR with $GL(4, \mathbb{R})$—the group of general (smooth) coordinate transformations.

[42] As we will see in the sequel, in ADG-theoretic terms even differential equations are of an essentially categorical character as they are equations between *sheaf morphisms*, the main sheaf morphism being the gravitational connection $\mathcal{D}$—the generalized ('localized' or 'gauged') differential operator ∂ [262, 270, 264, 271, 272]. It must also be noted that what appears as the basic variable in the equations themselves is the curvature of the connection which is an $\mathbf{A}$-morphism (tensor), thus our (classical) smooth 'observations'/localizations of the gravitational field (strength) in $\mathbf{A} \equiv \mathcal{C}^\infty_M$ respect it.

surements ('coordinatizations') in $\mathcal{C}^\infty(M)$ by the action of the *homological tensor product functor* $\otimes_{\mathbf{A}}$[43] [259, 260] on them.

A concise categorical distillation of the syllogism 1–4 above, thus effectively also of the PGC of the manifold based GR, is the following:

(R2.4) The gravitational law (Einstein's equations) is *functorial* relative to our smooth 'measurements' and 'localizations' of the relevant fields—processes which are represented algebraically by $\mathbb{A}$, or in a sheaf-theoretic (local) sense, by (local) sections of the sheaf $\mathbf{A}$.[a]

[a] In the sequel, where we will observe in ADG-theoretic terms that the basic gravitational variable (field) is the connection, rather than the metric, and in view of the fact that the actual 'observable' quantity—*ie*, the 'measurable' (geometrical) dynamical quantity engaging into the Einstein equations—is actually the curvature of the connection, which is an $\mathbf{A}$-morphism, the 'identification' above of the PGC with '*functoriality of dynamics relative to our measurements in* $\mathbf{A}$' will become even more transparent and quite natural.

An immediate 'corollary' of the above expression of the PGC is that *the law of gravity 'permeates' or 'sees through' the base smooth spacetime manifold insofar as the latter is, in an operational sense, identified with our operations of coordinatization of its point events, which in turn are conveniently grouped into the algebra* $\mathcal{C}^\infty(M)$ (1) (or equivalently and more 'locally speaking', into the sheaf $\mathcal{C}^\infty_M$ thereof over M).[44] Parenthetically we would like to point out that, of course, Einstein himself had clearly intuited that the laws of Nature should be unaffected by *our* (arguably subjective and arbitrary) choices of coordinates. Characteristically, he remarks in the '*Time, Space, and Gravitation*' article 12 in [121] about the essential motivation for his transition from SR (inertial frames) to GR (arbitrary coordinate frames) according to the PR:

[43] Again, it is tacitly assumed that we are working within the category $\mathcal{M}an$ of (finite-dimensional) differential manifolds M—and more particularly, in the category of sheaves $\mathbf{A} \equiv \mathcal{C}^\infty_M$ of abelian topological algebras $\mathbb{A} \equiv \mathcal{C}^\infty(M)$ of (real or complex-valued) smooth functions over them.

[44] So, in the following sense the aforesaid expression of the PGC is closely akin to a generalized sort of the PR [125]: *the laws of Physis are independent of the coordinates we lay out to chart (or in a Cartesian sense, to 'label'—ie, ascribe numerical values to or 'arithmetize') and localize the relevant fields on spacetime points.*

(Q2.9) "...Must the independence of physical laws with regard to a system of coordinates be limited to systems of coordinates in uniform movement of translation with regard to one another? *What has nature to do with the coordinate systems that we propose and with their motions?*[a] Although it may be necessary for our descriptions of nature to employ systems of coordinates that we have selected arbitrarily, the choice should not be limited in any way so far as their state of motion is concerned[b]..."

[a]Our emphasis.

[b]Or perhaps better expressed, (the said arbitrary choice of any particular system of) coordinates should not affect in any way the dynamical equations (laws) of motion of the fields in focus.

Thus, in this sense, *GR may be regarded as a manifestly background M-independent theory, as its dynamical laws involving the gravitational field (Einstein equations) 'see through' the $\mathcal{C}^\infty$-charted spacetime and its smooth 'self-transmutations' in* $\mathrm{Aut}(M) \equiv \mathrm{Diff}(M)$ [262, 264]. If in turn we abide to a Wheelerian-type of principle holding that

(R2.5) no theory is a *physical* theory unless it is a *dynamical* theory,

the above expression of the PGC, always in the classical (CDG) framework of our (differential geometric) conception and primitive assumption of spacetime as a smooth continuum, vindicates Einstein's

(Q2.10) "*Time and space are modes by which we think, not conditions in which we live*" [120],

since the differential spacetime manifold does not actively participate in the gravitational dynamics, thus it is not a physical, 'organic', 'living' condition on which, so to speak, the laws of Nature vitally depend.

Parenthetically, it must be noted here that, already back in the 1920s, Einstein had conceived of the spacetime continuum as an unphysical, ether-like substance [118, 272], "*acting but not being acted upon*" [125]. To be sure, however, the smooth spacetime continuum, although it does not participate itself into the dynamical equations for gravity, it provides us with the *vital 'differential conditions' or 'properties'*, or even better, the *essential differential geometric mechanism*, for representing these equations as *differential equations*: the structural and calculational apparatus of CDG (Calculus). This was again noted by Einstein in [118] in the form of '*differential or infinitesimal locality*' (Q2.?)—the basic assumption that field actions connect (*ie*, dynamically evolve and interact between) 'infinitesimally contiguous' or 'infinitely proximate' events in the smooth spacetime continuum, something that in turn qualifies (defines!) fields as being smooth (differentiable to an arbitrary order) and, concomitantly, the dynamical equations in which they participate as being *differential equations* proper (albeit in the classical sense 'afforded' and 'furnished' by the geometrical base differential spacetime manifold M). In connection with these remarks, we would also like to quote Finkelstein from [148]:

(Q2.11)

"...Soon after the physics of light led Einstein and others to special relativity, the physics of gravity led Einstein to general relativity. The heart of relativity is Einstein's locality principle:
The law of nature is local and causal.[a]
Here 'local' means that the dynamical law relates any event e only to events in its infinitesimal neighborhood. 'Causal' means that these other events are inside the light cone of e.
That is, *Einstein assumed that the law of nature was a differential equation*,[b] and that its characteristics (the surfaces across which there could be jumps in a solution) were null cones..."

[a] Finkelstein's emphasis.
[b] Emphasis is ours.

To return to the 'illusive' nature of spacetime, even more succinctly, Eddington pointed out, remarkably as early as 1916(!) [105], that the PR essentially entails the physical meaninglessness and artificiality of (the) base spacetime (continuum of GR), relegating it merely to *our own mental fiction*—or equivalently, Einstein's "*mode by which we think*" in (Q2.?) above—a virtual background scaffolding, a surrogate host (employed for the localization) of the relevant fields, that does not play any role in the actual physical dynamics:

(Q2.12) "According to the principle of relativity in its most extended sense, *the space and time of physics are merely a scaffolding in which for our own convenience we locate the observable phenomena of Nature. Phenomena are conditioned by other phenomena according to certain laws, but not by the space-time scaffolding, which does not exist outside our brains.*[a]"

[a]Our emphasis. Especially in the case of our sheaf-theoretic ADG-endeavors in the sequel, we will see how pertinent and prophetic(!) these remarks by Eddington are. Indeed, if there is any value at all to a background spacetime—be it a continuum or even a discretum as we shall see—it is as an external 'parameter' space, a virtual scaffolding assumed for our convenience, for the (sheaf-theoretic) localization (gauging) and concomitant dynamical variation of the actual 'geometrical objects' of Nature—the physical fields, but itself playing absolutely no (structural) role in that dynamics. The dynamics—*ie*, the laws of physics—'in itself' is essentially algebraic—*ie*, it is defined by the algebraic relations between the said fields (*viz.* connections) which are themselves '*inherently algebraic and autonomous entities*' and in no need of an external, background space(time) for their subsistence, efficacy, or operativeness.

In the light of these remarks about the 'virtual', 'imaginary' or 'mental' reality of the spacetime continuum, that is to say, if one abides by the aforesaid version of the PGC, the existence of singularities—*ie*, *à la* Einstein (Q2.?), *the existence in a theory* which, like GR, is based on the smooth spacetime manifold, *of points (or lines etc.) for which the field equations do not hold*, or even more graphically *à la* Joshi (Q2.?), *the existence of loci in the spacetime continuum where the laws of physics break down*—comes straight into conflict with it (*ie*, the PGC), for singularities entail a 'negative dependence' (even more impressively, a *breakdown*!) of the law of gravity on *our own* coordinatizations or measurements of the smooth physical fields localized on M's point events, or, what amounts to the same via the identification (1), on *our own*, and physically arbitrary(!), mathematical modelling of physical 'spacetime' by a differential manifold M.[45]

[45]For recall again Einstein's words in (Q2.?): "*What has nature* [*ie*, the dynamical laws of physics] *to do with the coordinate systems that we propose?*".

(R2.6) All in all, *the law of gravity appears to break down at singularities which are due to our modelling of spacetime after a differential manifold M, which in turn is tautosemous with our smooth coordinatizations of its points in $\mathcal{C}^\infty(M)$* (1): stated thus, how *contingent and inextricably dependent* on our mathematical model of spacetime as a $\mathcal{C}^\infty$-smooth continuum—ultimately, on our own 'smooth measurements or chartings' of fields represented by $\mathbf{A} = \mathcal{C}^\infty_M$—does the physical law of gravity appear to be?[a]

[a]This apparent contingency of the gravitational law of Nature, especially in the classical domain of GR (*ie*, where the 'observer dependence' and concomitant 'non-objectivity' of physical reality that quantum theory brought about is not supposed to play any role at all), on our (mathematical) model of spacetime, we find hard to accept in the present paper. For one thing, it comes straight into conflict with Einstein's PR in footnote 5, as well as with our version of the PGC of GR in (R2.?) and its categorical (functorial) expression in (R2.?). In fact, this (theoretically) unacceptable feature of singularities and *in extenso* of our $\mathcal{C}^\infty$-smooth manifold picture of spacetime hosting them was our principal motivation for writing this paper in the first place.

To our knowledge, it was Einstein first in [125], and subsequently Geroch in [155], who—the first implicitly as in (Q2.?), the second more explicitly—found the PGC of GR (much in the way it was expressed above) and singularities in glaringly serious 'physical disagreement'. In fact, in the very first sentence of the abstract of [155], the author notes:

(Q2.13) "The general covariance of relativity theory creates serious difficulties in formulating a suitable definition of a singularity in this theory."

To recapitulate, general covariance and singularities do not seem to go hand in hand; moreover, this discord is so strong that it apparently makes a clear-cut definition of singularities in the manifold based GR a very subtle, elusive and difficult task both technically (formally or 'mathematically') [87], but most importantly, conceptually (semantically or 'interpretationally').

In a nutshell, it is fair to say that the singularities-vs-PGC tension is due to the following (mathematical) 'vicious circle' [317]: on the one hand the $\mathcal{C}^\infty$-smooth structure of the spacetime manifold M appears to be a necessary assumption in order to represent the dynamical law of gravity by a *differential equation* proper, as well as to implement the PGC of GR via $\mathrm{Diff}(M)$, while on the other, and as we will argue extensively in the sequel, singularities are essentially built into $\mathcal{C}^\infty(M)$—the algebra that *defines* M as a differential manifold in the first place (1), and are (differential) geometrically perceived precisely as *loci* in M where that smooth structure 'breaks down', 'becomes anomalous', or at best 'misbehaves' in one way or another.

2.1.2 Begging the question: 'real' or 'virtual' singularities?

Having mentioned the conceptual and technical dissonance between the PGC and singularities in the theoretical framework of the differential manifold based CDG, it must be noted at this point that it appears to be unanimously accepted now that there are two kinds of singularities in GR, which might be coined '*virtual*' and '*real*'. As virtual are characterized the so-called *coordinate singularities*, the 'canonical' example being the exterior Schwarzschild one at a distance $r = 2m$ (horizon) from the gravitating point particle of mass m [109, 141]. Coordinate singularities are not regarded as being 'real' (*ie*, physically significant), since they are simply due to the fact that the physicist has laid down inappropriate coordinate patches—she has chosen an inappropriate coordinate frame of reference—in order to chart the spacetime continuum and are not intrinsic physical anomalies of that spacetime structure supporting the gravitational field.[46] Geroch, for instance, declares up-front in [155]:

(Q2.14) "...We shall not be concerned with so-called "coordinate singularities". This term refers to a spacetime which has been expressed in an improper coordinate system... *The presence or absence of a coordinate singularity is not a property of the spacetime itself, but rather of the physicist who has chosen the coordinates by which the spacetime is described*[a]..."

[a]Our emphasis.

Parenthetically, we could mention here, in contrast to Geroch's remarks in (Q2.?) above, but in a similar manner of expression in order to emphasize this contradistinction, ADG's basic thesis regarding singularities, which will be amply supported subsequently by the results of the present paper:

[46]In the case of the exterior Schwarzschild singularity for example, passing from spherical Schwarzschild coordinates to the 'logarithmic' ones of the so-called Eddington-Finkelstein frame allows one to convert it to a unidirectional membrane, thus show that it is not really a singularity, but just a '*coordinate artifact*' [109, 141]. We will return to it in more detail in section 5.

In ADG, we are not troubled at all, as far as the essentially algebraico-categorical differential geometric mechanism is concerned, by any kind of 'geometrical spacetime singularity', and *the distinction of the latter into 'real' and 'coordinate' ones loses its meaning in the theory.* Singularities refer to a field which has been coordinatized by, 'measured' in, or simply referred to, an 'improper' coordinate system—the term 'improper' pertaining to a structure sheaf **A** of coefficient functions or coordinates that host 'singularities' of some kind[a]—yet, singularities that do not affect or impede the said 'innately algebraic' mechanism, let alone cause it to 'break down' and, concomitantly, the law—*ie*, the differential equation which the field obeys (in fact, *defines*), to stop being in force (*ie*, not to hold over the 'singular *loci*')...*The presence or absence of a singularity is not a property of the field itself, which anyway physically exists independently of what we perceive and measure (ie, coordinatize) as being 'spacetime',*[b] *but rather of the physicist who has chosen the coordinates by which this field is described (ie, 'measured', 'coordinatized' or 'numerically perceived', thus soldered on or coordinated to, and placed within, the said ambient 'differential geometric spacetime framework'*[c]*) that, after all* <u>*we*</u> *impose on the field in order to represent mathematically—as a differential*
(R2.7) *equation—and study the law that it obeys*[d]*).*

[a]Indeed, in the classical case, $\mathbf{A} \equiv \mathcal{C}^{\infty}_{M}$, and it is the smooth functions that carry within them the seeds of the singularities and the differential geometric anomalies of the spacetime manifold and of the CDG (Calculus) based on it.

[b]See Einstein's PR in footnotes 5 and 15, Eddington's remarks in (Q2.?), as well as our categorical (functorial) expressions of the PGC of GR in (R2.?) and (R2.?).

[c]Indeed again, the singular spacetime manifold of the classical theory corresponding to $\mathcal{C}^{\infty}_{M}$ that one assumes for coordinate structure functions of its point-events (1)—'events' here being understood as the (numerical) results of our field-probings and measurements, our 'Cartesian arithmetizations' of (our numerical ascriptions to) the physical fields, in accordance with Einstein's concluding sentence in the quotation of footnote 15. To be sure, as we are going to argue extensively in 7.5.5 motivated by Stachel's penetrating analysis into the deeper meaning of Einstein's hole argument and the significance of the PGC of GR, the points of M do not have *a priori* the right to be called '*spacetime events*'—'*a priori*' here meaning '*kinematically fixed—before the dynamical field of gravity is determined*' (as a solution of the field equations). We wish to thank John Stachel for bringing this subtle and (as we shall see later) important interpretational matter to our attention in a timely e-communication. However, keeping in mind the pending remarks in 7.5.5 about 'physically interpreting the points of a manifold as spacetime events proper', we may abuse language below and refer to M's points as events.

[d]As we will see in the context of ADG below, it is more accurate to say 'the law that the field *defines*', not 'the law that it *obeys*', although we may use the two verbs interchangeably in the future.

while Clarke in [86] further highlights the supposedly fundamental distinction between virtual and true singularities:[47]

(Q2.15)

> "...[Based on the definition of singularities as boundary points beyond which the smooth causal geodesics in the spacetime manifold cannot be analytically extended or continued,] for the ideal endpoint to a curve to be called a singularity, as opposed to a regular boundary-point, it must be the case that there is no extension of the space time in which the curve in question can be continued: *if there were such an extension the purported singularity would be regarded as analogous to a 'coordinate singularity' such as the Schwarzschild horizon written in Schwarzschild coordinates*[a]..."
>
> [a]Our emphasis.

It is plain from the above that 'real' singularities are supposed to represent true spacetime pathologies, uncircumventable simply by (smooth) coordinate transformations and insuperable in general by the technical $\mathcal{C}^\infty$-means of CDG obstructions to the smooth spacetime structure. Such singularities, as Clarke defined in (Q2.?) before, are *loci* where the differential structure of the spacetime manifold and, concomitantly, the physical laws that it supports as a smooth base space, manifestly break down.[48]

The last remarks bring us to the mathematical unmanageability of $\mathcal{C}^\infty$-smooth manifolds and the CDG based on them in coping with gravitational singularities, which in turn makes the latter be viewed (always of course from within the CDG-framework) as *incurable classical differential geometric diseases*. We will argue, based on some glaring both conceptual (physical) and technical (mathematical) 'oxymorons' and in view of our identification in (1), that the notion of singularity creates various 'chimeras', various 'false impressions' within the framework of the CDG-based GR. One of these chimeras is the distinction between virtual and real singularities, and we shall argue that this distinction is in effect 'begging the question' of, and to a large extent obscures, what 'truly' is a singularity in GR [155] as well as of what is its actual mathematical 'origin'. The points

[47]Again, in the quotation below, the words in square brackets are our own additions for completeness and clarity of exposition.

[48]In the case of the Schwarzschild solution, the fact is that the inner $r = 0$ singularity cannot be, similarly to the exterior one, 'transformed away' by a coordinate change, thus it shows us that *the classical theory—based on the spacetime manifold—is out of its depth when trying to calculate the gravitational field right at its point mass source* [109, 141]. It is widely accepted nowadays that only a quantum theory of gravity may be able to describe the gravitational field on its source, right at its 'true singularity' [303, 305, 279, 200]. We will return to such 'real' singularities and to the associated question whether the 'true' quantum gravity will be able to 'heal' or ultimately to remove singularities in sections 3 and 6, respectively.

to be raised will allow us then to introduce rather naturally ADG's view on singularities, as well as to present its basic concepts and methods for evading them altogether [273, 262, 274, 264, 275, 265, 267].

2.2 About Mathematical Unmanageability

On the face of it, the discussion above is pregnant to a physico-mathematical oxymoron. On the one hand we have our basic contention that the differential manifold *is* the algebra of smooth 'coordinate' functions on it (1), and on the other the nowadays general consensus that *real singularities are not just coordinate ones*. One way out of this impasse is to declare, by 'geometrical fiat' as it were, (real) singularities as being *ideal boundary points to the smooth spacetime manifold which, 'in themselves', are not parts of the set M of spacetime events proper, loci not lying in the 'bulk interior manifold' of the actual physical point-events where the gravitational field equations hold, which in turn are deemed to be regarded as being regular (non-singular).* Clarke, for example, makes it clear from the beginning that:[49]

(Q2.16) "...With each incomplete inextendible curve we associate a[n ideal] boundary point: a point added on to [the] space-time [manifold M] mathematically, but not forming part of the physical space-time... The aim of the construction of boundary points is to build a mathematical model based on a set $\overline{M} = M \cup \partial M$, where ∂M is the set of boundary points and M is the set of real physical points in space-time, with $\overline{M}$ given a topology which makes it the closure of M (so that it is sensible to speak of ∂M as a boundary)..." [86]

A synthesis of the quotes (Q2.?), (Q2.?) and (Q2.?) lead us to infer that, according to the definitions above, as *physical* (regular) spacetime events are regarded only the *loci* in the bulk (interior) M of $\overline{M}$ where the gravitational potentials are *differentiable* (smooth) functions and the law of gravity *holds* (*ie*, it can be expressed) as a differential equation in the realm of the smooth M and its CDG. This is well in line with our regarding smooth singularities as '*obstructions to (classical) differentiability*', always in the context of the CDG-theoretical framework.

On the other hand, such an admittedly forced or *ad hoc* geometrical declaration of singularities as 'marginal points at the edge of physical spacetime', having only mathematical purpose and

[49] Again, in the quotation below, the words in square brackets are our own additions for completeness and clarity. The idea of viewing singularities as *ideal* boundary points in spacetime—sites that are inaccessible (in 'finite time') by smooth (causal) paths followed by physical particles—was explicitly pitched in [159] although it was intuitively implicit already four years earlier in [155] in the guise of the idea of '*(causal) geodesic incompleteness*' (for more details, the reader should wait until the next section).

utility, but being almost devoid of physical significance, appears to come straight into conflict with Geroch's distinction between real and coordinate singularities in (Q2.?), because for him, *true singularities, unlike virtual ones, are purported to be properties of the (physical) spacetime itself and not merely the result of improper spacetime descriptions (coordinates) used by the physicist.*

2.2.1 Enter ADG

We contend that there is no need for such elaborate, artificial physico-mathematical distinctions (and apparently deep conceptual asymphonies!) between coordinate (virtual) and real (true) singularities as long as one seriously abides by (1), namely, by the mathematical fact (in fact, *definition*!) that the $\mathcal{C}^\infty$-smooth spacetime manifold M is nothing else but the algebra of its differentiable (coordinate) functions. The latter, in turn, provides the foundations for the *aufbau* of the entire theory of CDG which fares miserably upon trying to cope with singularities—singularities of the very smooth functions (and, *in extenso*, of the higher order smooth tensor fields built out of them by (anti/symmetric) iterations of the homological functor $\otimes_{\mathcal{C}^\infty_M}$ [260]) that define M as a differential manifold. Stated in a positive way, (R2.8) all the singularities of the smooth spacetime manifold M are 'coordinate' ones in the sense that they are anomalies of certain elements in the function algebra $\mathcal{C}^\infty(M)$ that defines it in the first place as a *differential space*[a] proper. At the same time, it is *our assumption* of modelling spacetime after a $\mathcal{C}^\infty$-manifold (which automatically forces us to adopt the usual concepts and technical tools of the CDG based on $\mathcal{C}^\infty(M)$), that 'stumbles and falters' (thus also the differential equations for which CDG was created in the first place[b] break down) on singularities. Hence, singularities are shortcomings of our own theory (model) of spacetime (the differential manifold) and of the way that we mathematically represent the laws of Nature, as differential equations, within the differential geometric framework (CDG) supported by that model. *In summa, one cannot think of singularities independently of the differential spacetime manifold M; they are differential geometric diseases built into $\mathcal{C}^\infty(M)$ and expressing the 'inherent limitations' of CDG, since the latter vitally depends on M, which in turn is tautosemous with $\mathcal{C}^\infty(M)$ or, sheaf-theoretically, with the structure sheaf $\mathcal{C}^\infty_M$ thereof.*

[a]That is, a space endowed with a differential structure.
[b]See subsection 7.6.

From this perspective it appears quite inappropriate—not to say '*hubristic*'—to ascribe singularities to Nature Herself (*ie*, give them a physical significance), since they are problems inherent in *our* spacetime model (M) and theory (CDG) of Her (*ie*, our regarding of physical space(time) as being smooth or, differential geometrically speaking, 'regular') (R2.?). In other words, as already briefly alluded to in (R2.?), we maintain that

(R2.9) *Nature* (*ie*, the physical laws and the physical fields participating in them) *has no singularities, but it is our smooth model of Her* (*ie*, of what we 'axiomatically' accept *a priori* as being 'physical space(time)') *that is of limited applicability and validity.*

Of course, we tacitly abide by the 'principle' that *whenever there seems to be an asymphony between physics and the mathematics one uses to describe and represent the physical processes, situations and phenomena involved in that physics, one should always question and change the maths* and never interpret the dissonance as a 'problem' of Physis, or even attempt to uncover some supposedly hidden physical essence and significance in those mathematical shortcomings and 'anomalies'—in our own 'misrepresentations' of physical phenomena so to speak.[50] For our main example above about the conflict between the PGC and smooth singularities, we find it unreasonable, to say the least, to doubt the first (which, after all, is more or less a physical principle![51]) instead of trying to question, find the main culprits for its disagreement with the physics and, hopefully, change the second (which, anyway, is merely a consequence of our mathematical, $\mathcal{C}^\infty$-smooth manifold model of spacetime, and by no means Nature's 'sacrosanct quintessence').[52]

To summarize things:

[50]To be sure, singularities (and their associated infinities) may indeed be thought of as trying to tell us that there is something fundamentally wrong with the (mathematical) means that we employ to describe Nature (here, the manifold based CDG and the description of physical laws—in particular, the law of gravity—as differential equations within the theoretical framework of Calculus), but surely not that there is a limitation of the physical fields and of the laws that they obey/define at their *loci* (Q2.?). We could summarize this subtle distinction in the following: *it is not that the gravitational field (law) breaks down at a singularity, but rather that the manifold and, in extenso, the CDG-based GR does so.*

[51]See the generalized Principle of Relativity in (R2.?) above and its associated 'definition' of (objective) physical reality by Einstein in footnote 5.

[52]This of course implies that if M, with its intrinsic pathologies (singularities), has to go, so will the standard mathematical representation by $\mathrm{Diff}(M)$ of the PGC in the M-based GR. However, the PGC, as a basic *physical* principle, should remain intact in the *physical* theory; only its mathematical representation should change. This is in line with what we said above, namely, that upon encountering a problem, blame it on the mathematics, not the physics, and consequently, try to change the former, not the latter.

(R2.10) One cannot think of singularities apart from our $\mathcal{C}^{\infty}$-smooth manifold model M for spacetime, for they are 'intrinsic' to $\mathcal{C}^{\infty}(M)$, which defines M in the first place (1). In this sense, all singularities are 'coordinate', 'virtual' ones physically speaking, and at the same time *mathematically 'real'* (*ie*, they are mathematical (arti)facts!), insofar as they express 'innate' pathologies of M and 'inherent' limitations of the CDG that is based on it. *Singularities are not physically real* (*ie*, they are not Nature's own),[a] but they simply express that it is our Calculus—our mathematical Analysis based on the smooth spacetime continuum—that is out of its depth when dealing with certain physical situations, like for example when trying to calculate the smooth gravitational field right at its point source mass as in the Schwarzschild black hole scenario [109, 141] while at the same time maintain that this offensive locus is part of the physical smooth spacetime manifold.

[a]One could actually say that this is *the* fundamental '*physical axiom*' (or intuition) of ours to which the epithet '*chimeras*' in the title of the present paper pertains. In connection with this basic intuition of ours, see the remarks about our principal motivation for writing this paper in the footnote of (R2.?).

The contents of (R2.?) and (R2.?) may be thought of as being 'post-anticipations' of Ashtekar's telling remarks following (Q2.?) above in which he comments on the singularities and their associated infinities which result, in quite generic physical situations in a smooth spacetime manifold, from the gravitational collapse or accretion of a cloud of 'cool, non-interacting matter' (dust):

(Q2.17) "...Now, one believes that such infinities do not actually occur in Nature, and their occurrence in a theory is a signal that one is applying the theory beyond its domain of validity..." [8]

But which *theory* is Ashtekar referring to?—the mathematical framework within which GR is cast (*ie*, CDG), or to the actual dynamical law for the gravitational field (*ie*, the Einstein equations) which defines GR as a *physical* theory? Our basic thesis here is that the theory referred to above is the manifold-based CDG, not the field laws of GR.[53] Indeed, at so-called 'real' singularities, CDG has reached the limit (in fact, a 'dead end' as far as physically meaningful constructions and calculations by means of Calculus are concerned) of its applicability and validity.[54]

[53]As noted earlier, in our opinion the 'confusion' that is reflected in the physicist's claim that GR, as a physical theory, breaks down at singularities, is due to her identification of physical spacetime with (the mathematical artifact corresponding to) a locally Euclidean space M.

[54]See 2.1 in the sequel.

2.2.2 ADG's kernel and leitmotif: 'differentiability' is independent of smoothness

Thus, confronted with CDG's unmanageability and ineffectiveness in coping with singularities, and abiding by the aforesaid heuristic 'working principle' of '*changing the maths instead of blaming it on the physics*', we would like to look for an alternative way of doing differential geometry, a way that is not vitally dependent—in fact, *not at all!*—on differential manifolds with their 'innate differential geometric diseases' in the guise of the $\mathcal{C}^\infty$-smooth singularities.

Such a yearning has been already expressed by Einstein in two ways, both of which have to do with his dissatisfaction with the classical geometric continuum (manifold) picture of spacetime *vis-à-vis* on the one hand the singularities that plague it, and on the other, the 'discrete' (finitistic) and algebraic (relational) quantum paradigm.

The first way is rather 'apologetic' and 'confessional':

(Q2.18) "...Adhering to the continuum originates with me not in a prejudice, but arises out of the fact that I have been unable to think up anything *organic*[a] to take its place..." [120],

[a]Our emphasis. We will come back to comment further on this remark in the last section (7.5).

while the second is more suggestive and 'wishful' (for the mathematical structure and theory based on it that would replace the classical one—CDG—on the geometrical spacetime continuum):[55]

(Q2.19) "...The problem seems to me how one can formulate statements about a discontinuum without calling upon a continuum spacetime as an aid; the latter should be banned from theory as a supplementary construction not justified by the essence of the problem—a construction which corresponds to nothing real. *But we still lack the mathematical structure unfortunately*[a]..."

[a]Our emphasis.

or even more suggestively:[56]

[55]Einstein's quotation below can be found in [365]. Again, the last sentence is written in *emphasis script*.

[56]This is a one-sentence completion of the quotation (Q2.?) taken from [125]. The last sentence is written in *emphasis script*.

(Q2.20) "...One can give good reasons why reality cannot at all be represented by a continuous field. From the quantum phenomena it appears to follow with certainty that a finite system of finite energy can be completely described by a finite set of numbers (quantum numbers). This does not seem to be in accordance with a continuum theory, and must lead to an attempt to find a purely algebraic theory for the description of reality. *But nobody knows how to obtain the basis of such a theory.*"

while, even his view above that a continuous field theory founded on the geometrical spacetime continuum has many shortcomings *contra* the algebraic quantum was essentially based on the then (and even still today!) prominent lack of having a (mathematical) theory—"*a method*" in his own words—that deals and handles effectively singularities (while, optimally/preferably, still retaining the field-theoretic picture and its differential geometric apparatus—*ie*, the laws in which these fields participate can still be modelled after differential equations [368]); as follows:

(Q2.21) "...Is it conceivable that a field theory[a] permits one to understand the atomistic and quantum structure of reality? Almost everybody will answer this question with 'no'. But I believe that at the present time nobody knows anything reliable about it. *This is so because we cannot judge in what manner and how strongly the exclusion of singularities reduces the manifold of solutions. We do not possess any method at all to derive systematically solutions that are free of singularities*[b]..." [125]

[a]Of course, Einstein was implicitly alluding to his unitary field theory, which, according to his vision, could hopefully 'explain away' quantum phenomena. (Again, the reader should refer to the last section for more discussion on this point.)

[b]Our emphasis.

On the other hand, there is a recently developed theory, ADG [259, 260, 269], that time and again has proven to be suitable for evading completely the geometrical $\mathcal{C}^\infty$-smooth differential manifold and its singularities [273, 274, 262, 275, 264, 265, 267]; moreover, by its very algebraic (in fact, sheaf-theoretic and categorical) character, it appears to be able to implement differential geometrically quantum spacetime and gravity ideas in a directly non-manifold based—*ie*, in a *manifestly background spacetime manifold independent*—way[57] [261, 270, 271, 272, 263, 266], while

[57]In this sense we mean that one of the core ideas and central results of ADG, its *leitmotif* so to speak, is that "*differentiability is independent of smoothness*" (see concluding sentence-slogan in [271]).

still being able to do field theory by differential geometric means, albeit, glaringly without M.[58] Indeed, *ADG offers us an entirely algebraic way of doing differential geometry without at all the use of any Calculus*,

(R2.10) a way which, in a Leibnizian-Machian sense, deals directly with the geometrical objects representing the physical fields themselves and derives from their algebraic interrelations—their 'dynamical propagations and interactions' so to speak—without its essential differential geometric mechanism being dependent at all on or influenced by the intervention or mediation of (smooth) coordinates, and *in extenso* by (1), on the base spacetime manifold M [265].

And we call this relational-algebraic (and finitistic! [270, 271, 272]) way of doing differential geometry independently of a background space(time) '*Leibnizian*', because Leibniz, in contradistinction to Newton, was searching for *a combinatory-algebraic way of doing Calculus* (an '*ars combinatoria cum calculus ratiocinator*'—a logico-combinatorial 'Geometric Calculus') *dealing directly with the geometrical elements or objects of 'space(time)'* without the presence, let alone the intervention, of an ambient space(time) as such, especially in the Cartesian guise of coordinates [384].[59] To further support this point, we quote the Bourbakis [61] maintaining that Leibniz wished to

(Q2.22) "Fonder un 'calcul géométrique' *opérant directement sur les éléments géométriques, sans l'intermédiaire des coordonnées.*"[a]

[a] "Found a 'geometric calculus' *which operates directly on the geometrical elements, without the mediation of coordinates.*" Our emphasis.

In connection with the Leibnizian, relational (*ie*, algebraic) character of ADG, we read from the prologue to the Russian edition of [259] for instance:

[58] See concluding section.

[59] We wish to highlight here the epithet '*Geometric*' in front of 'Calculus' in order to emphasize its striking contrast to the conventional (Cartesian-Newtonian) '*Analytic*' Calculus, thus anticipating our philosophical remarks in 7.3 about the Euclidean *versus* the Cartesian conception of (differential) geometry.

(Q2.23) "...This special [unexpected] help [from ADG] now, when the necessity has grown to study manifolds with singularities and even to remove the underlying space (for example, spacetime) *and proceed to a direct description of the structures on this manifold*, may be important for many branches of contemporary mathematical and theoretical physics..."[a]

[a]Our emphasis.

In the same Leibnizian gist, such an essential background spacetime manifold independence has been anticipated by Penrose, albeit not in a differential geometric context proper like ours, but in the infant developmental steps of his celebrated combinatorial (relational-finitistic) approach to quantum space(time and gravity) originally coined *spin-networks* [297],[60] much as follows:[61]

(Q2.24) "A reformulation is suggested in which quantities normally requiring continuous coordinates for their description are eliminated from primary consideration. In particular, *since space and time have therefore to be eliminated*, what might be called a form of Mach's principle must be invoked: *a relationship of an object to some background space should not be considered—only the relationships of objects to each other can have significance.*[a]" [295]

[a]The reader should refer to 7.5.8 for our ADG-theoretic generalization of Mach's principle in the context of GR.

Thus, in the same vain, having already applied ADG-theoretic ideas to a finitistic, causal and quantal model for spacetime structure and gravity [318, 270, 319, 271, 272], we focus here on ADG's 'resolution' of $\mathcal{C}^\infty$-smooth singularities.[62]

In particular, the present paper may be regarded as a slightly more technical and concrete

[60]Spin-networks and their *spin-foam* descendants are nowadays considered to be sound and promising 'discrete' approaches to non-perturbative (Lorentzian) quantum gravity [332, 59, 32, 36, 304, 328].

[61]Again, the part in the quotation below written in emphasis script highlights the point we wish to make.

[62]We will see in the sequel that it is not so accurate to say that ADG '*evades*' or '*resolves*' singularities, as that it '*integrates*' or '*engulfs*', or even '*absorbs*', them in the algebra (sheaf) $\mathbf{A}$ of 'generalized coefficients' (coordinates) [262, 317, 265, 267]. In fact, as we shall see later, none of the aforesaid about singularities, the general CDG-anomalies, their clash with the PGC of GR *etc*, could stand on its own two feet had not we had the full fledged ADG theory at our disposal. Thus, based on ADG, it is perhaps more precise, considering what we actually do to the $\mathcal{C}^\infty$-singularities of the classical theory (CDG), to say '*dissolution*' (in $\mathbf{A}$) rather than '*resolution*' of singularities. In any case, '*singularity-resolution*' is a term already preempted by methods of Algebraic Geometry (*eg*, singularity blow-up procedures) [178]. We do not wish to confuse the established methods and practices of Algebraic Geometry with the new, both technically and conceptually different ones of ADG.

physical applications oriented version of [265, 267] while, in turn, it may also be thought of as an extension of the trilogy [270, 271, 272] to a tetralogy so as to include ADG's promising prospects concerning $\mathcal{C}^\infty$-smooth spacetime singularities [273, 262, 274, 275, 267]. Thus, below we will investigate the most general possible scenario for smooth singularities and we wish to make plain precisely in what way ADG, especially after its successful application in discretizing the curved differential spacetime manifold of GR and in formulating the vacuum Einstein equations, entirely algebraico-categorically, on the finitary spacetime sheaves (finsheaves) of quantum causal sets (qausets) in the aforesaid trilogy, manages to cope (in fact, absorb, thus enable us to do explicit differential geometric constructions and calculations) with them. This quest emanates from the intriguing fact, repeatedly highlighted in our past papers, that

(R2.11) by ADG-theoretic means it has been shown that *the laws of physics* (*eg*, vacuum Einstein gravity and free Yang-Mills or gauge theories) *hold on spaces either hosting the most general and unmanageable by $\mathcal{C}^\infty$-means singularities* [273, 274, 262, 275, 264], *or spaces looking 'discrete' and structurally as remote as a space can be from the featureless spacetime continuum of macroscopic physics* and, moreover, these laws are expressed completely by algebraic, categorical in essence, means [272] (though still manifestly remaining *differential equations* proper!), without depending on a background $\mathcal{C}^\infty$-smooth spacetime manifold.

At this point it must be stressed that although the ADG-based form of the (differential) equations (modelling the laws of physics) remains effectively the same as in the classical theory, their *physical interpretation* changes significantly. In fact, while ADG uses only 'conventionally' or 'formally' the terms and concepts (*eg*, connection, curvature *etc*) of CDG, its essential abolition of $\mathcal{C}^\infty$-manifolds amounts to a radically different *use* and *meaning* of those very same terms. In a pragmatist sense, in ADG these differential geometric concepts and techniques are *used* in a drastically different way than in the M-dependent CDG, hence their *meaning* (physical interpretation) is also significantly different.

In toto, it is our contention that ADG provides us with a well geared and fine tuned, both conceptually and technically, candidate for the "*purely algebraic*" [125], "*organic*" [120] theoretical framework that Einstein was searching for motivated by the incurable headaches that singularities and the quantum brought him when viewed (as he viewed them indeed!) from the classical perspective of the smooth spacetime continuum [272].[63]

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[63]The reader should go to 8.? for a thorough discussion of our claims that, from a differential geometric point

Below, after we provide a short summary of section 1,[64] we give a brief outline of the contents of the paper.

2.3 Section's Résumé

The second paragraph of a recent paper, [400], happens to be tailor-cut for summarizing, by means of juxtaposition, this first, introductory section, as well as for preparing the reader for various important issues in both classical and quantum gravity that we raise, tackle and discuss from an ADG-theoretic perspective in the sections to follow:

(Q2.25)

> "...'*On the Planck scale there is a precise, rich, and discrete structure*', says Ashtekar...The Planck scale is the smallest possible length scale with units of the order of 10^{-23} centimeters...At this scale, Einstein's theory of general relativity fails. Its subject is the connection between space, time, matter and energy. But on the Planck scale it gives unreasonable values–absurd infinities and singularities. It carries therefore–as the American physicist John Wheeler, who knew Einstein personally, used to say–the seeds of its own destruction.[a] That means the theory indicates the limitations of its own applicability. This is a restriction, but at the same time also an advantage: physicists cannot avoid looking for a better and more complete theory for the laws of nature at this fundamental level. In other words: they need a theory of quantum gravity in order to explain the behavior of nature at all levels, from quarks to quasars..."[b]
>
> [a]Together with the Bergmann quote (Q2.2) from [41] about 'singularities as auto-destruction *loci* of GR', let us also mention here the Wheeler references [278, 418].
>
> [b]Our emphasis.

Let is itemize the points raised in the quotation above in order of importance for us as we will encounter them subsequently in the present work:

1. First, it is maintained that *GR* fails—*ie*, it reaches the limit of its own applicability, while its dynamical law (Einstein equations), which traditionally is interpreted as linking space-time geometry with matter-energy-momentum and is represented as a *differential* equation

of view (Q?.?), ADG could prove to be the 'right' (mathematical) framework in which to implement and perhaps complete Einstein's unfulfilled unitary field theory project.

[64]In this paper, each section closes with a summary of the section's basic contents.

by CDG-means, reaches the threshold of its own validity—*vis-à-vis* singularities and other unphysical infinities. In what follows in the present paper-book we shall elaborate in detail and repeatedly on the following subtle distinction: it is not exactly that GR, as a *physical* theory, breaks down at singularities and it is plagued by infinities, but simply that *our* background differential manifold model for spacetime, and the CDG-technology that comes hand in hand with it, misbehave and miscarry in the quantum deep. In other words, one should blame it on the mathematics,[65] not on the physics, so that one should search for a 'better' Calculus (or Analysis)—one that is not vitally dependent on a smooth base manifold M with its 'inherent' singularities and infinities, while also one that is essentially algebraic in character in order to accord with the quantum paradigm.

2. Second comes the supposition that there is a fundamental space-time scale in Nature—the so-called Planck scale—below which the spacetime continuum (:manifold) of the classical theory (GR) gives way to something 'discrete' (finitistic) and 'quantal', with the concomitant loss of one's differential geometric privileges in the quantum deep. As we shall argue extensively in the sequel in the light of ADG-gravity, ADG is quite indifferent to the character of the background spacetime structure—whether it is a classical continuum or a quantal discretum. ADG-gravity is concerned exclusively and solely with the gravitational field (*viz.* connection) and the dynamics (Einstein equations, still differential geometrically modelled after differential equations) that it obeys (or better, *defines*), without caring about the (mathematical) nature of the background, external (to the gravitational field itself) spacetime. ADG-gravity is fundamentally (:'by definition') spacetimeless, thus it is begging the question in the theory on the one hand to posit (and impose!) a fundamental spacetime length in Nature—a cut-off scale put in by theoretical fiat in order to regularize and control the infinite integrals arising in attempts to quantize gravity, and on the other, to expect that a conceptually cogent and calculationally finite QG will fundamentally hinge on (or even result in) a quantization of spacetime itself.

3. And third, it follows from the above, GR is not a complete theory exactly because its dynamical law (Einstein equations) that define it as a *physical* theory proper stumbles and falters on the singularities due to the underlying geometrical spacetime manifold and, as a result, exactly because it is not universal—*ie*, because of the presence of the minimum spacetime length of Planck which is introduced and utilized precisely in order to make sense of the Analytical nonsense (*eg*, infinite expressions for physical quantities) that we obtain when we formally apply the rules of QFT to the quantization of gravity by explicitly (and forcedly!)

[65]Philologically speaking, the seeds of GR's destruction were planted in M, and 'flowered' with the CDG and GR, which are based on it.

retaining a background continuum. The operative word here is 'Analytical nonsense', and the crux of the argument is that the 'fault' lies with our mathematical model and technical tools (manifold and CDG, respectively) that we employ to describe the gravitational field quantally, not with the gravitational field—regarded as a *physical* entity—itself. All in all, our maths (CDG, Calculus or Analysis) needs rectifying, completing and, ultimately, fundamental revising, not the physical fields and the laws that they define.

Paper-book overview. *Ex altis* viewed, and very briefly, the present paper unfolds as follows: in the next section we revisit $\mathcal{C}^\infty$-smooth singularities, making precise and plain the sense in which they cannot be thought of independently of our differential manifold M model for spacetime and of the CDG that goes hand in hand with it. Since singularities are 'innate' in M, the basic observation is that it is quite understandable that the manifold based CDG (Analysis) cannot cope with them, thus we infer that it is high-time we found another method of doing differential geometry—one that is manifestly background smooth manifold independent, hence prepare the ground for ADG. In section 4 we present GR under the prism of ADG by basing ourselves on [259, 260, 262, 270, 271, 272, 269]. We highlight the sense in which the dynamical laws (Einstein equations) for the gravitational field, which in our scheme is represented by an *algebraic connection* and not by a smooth metric as in the original formulation of GR by Einstein, are manifestly smooth background spacetime M independent, thus further supporting our generalized version of the PGC in (R2.?). In section 5 we explain the way in which ADG is purported to 'absorb' or 'engulf', thus 'dissolve', singularities in the structure algebra sheaf $\mathbf{A}$ of generalized coordinates or 'arithmetics', while leaving intact the essentially algebraic differential geometrical mechanism according to which the physical laws are expressed as differential equations proper. We thus infer that smooth singularities are mathematical artifacts ('physically virtual', 'coordinate' singularities in the usual sense) and not real physical 'objects', let alone 'incurable differential geometric diseases' (*eg*, insuperable obstacles to differentiability) in any sense, for *differentiability is independent of smoothness*.[66] Even more powerfully, we contend that from an ADG-theoretic perspective, the so-called 'real' singularities are as regular as the regular points of the smooth spacetime manifold, with the latter locally Euclidean realm being totally absent from our theory. In the subsequent section, we present a concrete toy physical model in which the (abstract) ideas of section 5 are put to work and come to fruition, namely, the ADG-theoretic 'absorption' or 'dissolution' of both the exterior ($r = 2m$), but more importantly, of the inner ($r = 0$), Schwarzschild singularities, and in two different ways. At the same time, and to illustrate the power and effectiveness of ADG, we

[66] To be sure, singularities are indeed incorrigible differential geometric anomalies, but only when viewed from the classical perspective and tackled by the $\mathcal{C}^\infty$-technical means of the usual Calculus on Manifolds (CDG) which, ultimately, is due to our particular choice $\mathbf{A} \equiv \mathcal{C}^\infty_M$ for generalized arithmetics.

recall situations where ADG applies to GR over (Euclidean and locally Euclidean) space(time)s much more pathological and anomalous than Schwarzschild's—ones containing an uncountable infinity of the *most non-linear and robust singularities possible*—Rosinger's differential algebras of generalized functions (non-linear distributions), the so-called *spacetime foam dense singularities* [273, 274, 262, 275]. Even more remarkably, these uncountable singularities are seen to lie densely in the spacetime manifold M's 'bulk', not just 'sitting marginally', like virtual mathematical points as it were, at its boundary, as the usual smooth singularities are supposed to (in fact, *defined as*!) [84, 86, 87]. In the penultimate section, mainly based on our trilogy [270, 271, 272], but also on the numerous applications of ADG so far to field quantization [261, 263, 265, 266], we give various theoretical reasons why ADG may be of significant import not only to classical (GR), but also to *quantum gravity* (QG) proper. In particular, the essentially base spacetimeless—whether this background realm is assumed to be a continuum or a discretum—formalism of ADG-gravity prompts us to question the physical viability and significance of a fundamental spacetime length in Nature, like the Planck length, which is usually evoked in various essentially continuum based quantization schemes (covariant or canonical) for gravity in order to 'regularize' and render finite various (Calculus based) expressions of physically important quantities. We also question whether a genuinely quantum theoresis of the gravitational field—like, we contend, ADG-gravity is—should be concerned with a quantization of spacetime itself, something that in Loop QG for example has very recently been used to 'resolve' the interior Schwarzschild singularity [279] and the (initial) cosmological ones [51, 200]. The paper, based on the general didactics of ADG and their contrast against the glaring shortcomings of CDG in dealing with gravitational singularities already at the classical level, concludes with a general philosophical discussion of some of the issues raised and treated in sections 2–7, being particularly interested in discussing the extent to which differential geometric ideas can fruitfully be applied to QG research. The key point in this discussion is the subtle distinction between the *Euclidean* (relational) way of doing (differential) geometry—which involves directly and solely the geometrical objects (in our context, the physical fields) *per se* (*ie*, 'in-themselves') without reference to an external, ambient space(time), and the usual *Cartesian* (analytic) way involving coordinates and, inevitably, the mediation of a background geometrical space(time) (manifold). This distinction, we contend guided by the central didagma of ADG,[67] lies at the heart of the contrast between Leibniz's algebraic (relational) conception of Calculus (Q2.?) and the more popular analytic-geometrical (background space—intervening coordinates dependent) Newtonian way [265]. Based on this subtle distinction, we draw a boundary between '*physical geometry*' and '*geometric physics*' further inspired by some telling remarks of

[67] Which, we recall again, is that *one can do differential geometry directly with the geometrical objects (fields) themselves without the intervention of coordinates or, in extenso, of space*—especially of the spacetime manifold employed by the CDG to support the smooth gravitational field, which is identified with the metric, in GR (R2.??, Q2.??).

Peter Bergmann in [41]. We also suggest, again based on ADG, a way of doing 'continuous' field theory entirely by categorico-algebraic and finitistic means manifestly independent of a background spacetime continuum and its $\mathcal{C}^\infty$-smooth singularities, thus potentially marrying the two seemingly incompatible facets of Einstein [118]. As a result, we continue some points made in [272] and argue that ADG may be a promising candidate for the "*organic*" [120], non-continuum based, "*purely algebraic theory for the description of reality*" [125] that "*the other Einstein*" [368] was looking for in view of the pathologies (singularities) of the geometric spacetime manifold of GR and the 'discontinuous' or 'discrete' algebraic actions of quanta.

3 General Relativity: $\mathcal{C}^\infty$-Gravitational Singularities Revisited

Our principal aim in this section is to lay bare and 'beyond any doubt' so to speak *the inextricable dependence of singularities on the differential spacetime manifold*—essentially, how one cannot think of the former apart from the latter and in what way smooth singularities represent 'innate' shortcomings of CDG—anomalies that are built into $\mathcal{C}^\infty(M)$—which therefore are uncircumventable by its $\mathcal{C}^\infty$-manifold based concepts and analytic techniques. For arguably, the anomalous and problematic character of singularities in the manifold and CDG-based GR, and quite apart from the physical meaninglessness of the infinities that are usually associated with them, is reflected on the fact that so far there has not been given any precise, 'unambiguous' and unanimously accepted definition of gravitational singularities, not least because of the aforesaid clash between the PGC of GR (which is mathematically implemented via the diffeomorphism group $\mathrm{Diff}(M)$ of the smooth spacetime manifold M) and the very existence of $\mathcal{C}^\infty$-singularities in GR (Q2.?) [155, 87, 184]. This will firmly support our preliminary remarks in (R2.?) and prepare the reader for our ADG-theoretic perspective on singularities in the sequel. To this end, we first wish to review briefly and scrutinize to a certain extent all kinds of (attempts at definitions of) singularities in GR that have been proposed so far by different people in various places like for example [84, 85, 86, 87, 155, 156, 159, 184, 185, 186, 333].[68]

[68]This we by no means claim to be a complete list of references to the various works on gravitational singularities, but it is sufficient for capturing pretty much all the basic 'definitions' of smooth singularities in GR that have been attempted hitherto. Especially [184] and the more recent book [87] (and references therein), provide one with a rather complete picture of classical, $\mathcal{C}^\infty$-smooth spacetime singularities and their study by Analytic (CDG) means.

3.1 All the Perceptions and 'Definitions' (so Far) of $\mathcal{C}^\infty$-Smooth Singularities

From the very early attempts at defining precisely what is a singularity in GR, it has become clear that two central notions in those definitions are:[69]

- the notion of *(analytic) inextensibility of the spacetime manifold* M past the offensive *loci*—differential spacetime manifold extensions employed, as it were, to 'remove' singularities,[70]

  and

- the notion of *(causal) geodesic incompleteness*—the inability so to speak of particles to 'reach' singularities by following smooth (causal) paths (geodesics) under the influence of the gravitational field, something which in turn defines, topologically speaking, the anomalous *loci* as points at the boundary or 'edge' ∂M of the 'physical, regular spacetime bulk interior' M (Q?.?), sites not properly belonging to the latter.[71]

In fact, an explicit combination of these two notions in a concise, albeit informal and tentative definition of singularities by Clarke in [86], essentially reads as follows:

(R3.1) a singularity may be thought of as an ideal, boundary or 'end-point' of M—an unphysical *locus* not belonging to the physical and otherwise regular spacetime bulk (interior of) M—assigned to an incomplete inextensible curve.

while Hawking, based on these two notions, gives a concise definition of a singular spacetime (manifold) in [186], as follows:

[69] Remarkably, Einstein, as early as 1918 and in the context of de Sitter's solution to GR's field equations [115], had more or less figured out—even if he did not express it in the technical terms we do today—the two 'properties' below that a *locus* in the spacetime continuum must possess in order to qualify as a singularity proper.

[70] The importance of (analytic) inextensibility as such was implicitly noted first by Finkelstein (and then explicitly by Kruskal [237]) upon encountering the interior Schwarzschild singularity in [141].

[71] The importance of (causal) geodesic incompleteness was explicitly noted first in [155]. Subsequently, null geodesic incompleteness was the central prediction of the celebrated singularity theorems of Hawking and Penrose in [185]. However, as Clarke points out in [87], one need not consider only 'free falling' observers following causal *geodesics*, since other physically admissible frames—ones with bounded acceleration for example—may be able to reach the point-*loci* in question in finite proper time, even though geodesic observers cannot. In order to include the world-lines of such in principle arbitrarily accelerated observers, curves more general than geodesics—ones parameterized not by proper time, but by an arbitrary so-called *(general) affine parameter* (see below)—must also be included in the definition of incompleteness.

(Q3.1) "*A spacetime* [manifold] *is singular if it is timelike or null geodesically incomplete but cannot be embedded in a larger spacetime* [manifold]."

Leaving the notion of ideal, boundary points of M for the more 'rigorous' definition of singularities below,[72] we first make more precise the two notions of inextensibility and incompleteness above.

3.1.1 Analytic (or smooth) inextensibility of the (analytic or smooth) spacetime manifold M

We initially discuss continuous extensibility of (curves in) M, then we strengthen the notion of extensibility of M so as to include smooth and analytic extensions.[73] In what follows, we make it precise and clear what the term '*spacetime extension*'—when M is taken to be a $\mathcal{C}^0$-, $\mathcal{C}^\infty$-, or $\mathcal{C}^\omega$-manifold—actually refers to.

By a curve α, with *(general) affine parameter* t, in the manifold M, we mean, in general, a map $\alpha(t)$ from the 'clopen' interval $I_\tau = [t = 0, t = \tau) \subset \mathbb{R}$ into M, that is to say,

$$\alpha : \ t \in I_\tau \hookrightarrow M \ni \alpha(t) \tag{2}$$

[72]To the knowledge of these authors, such ideal 'end-points' to causal curves were first explicitly defined in [159].

[73]Recalling of course that differentiability (smoothness) is stronger than continuity, and analyticity still a bit stronger than smoothness. In a familiar Calculus-theoretic or (Real) Analytic sense, and for the ($\mathbb{R}$-valued) functions that are defined on a manifold M, these two 'strength relationships' are commonly expressed as follows: 'differentiability (of the functions involved) *implies* continuity (of those functions)' (write: 'smoothness>continuity'), and 'power (Taylor) series expansibility (of the functions involved) *implies* differentiability to any order (of those functions)' (write: 'analyticity>smoothness'). For the manifold M itself, this 'order of strength' reads: 'a topological manifold is weaker than a smooth one, and a smooth one still weaker than a (real) analytic one' (symbolically, write: ${}^{(\mathbb{R})}\mathcal{C}^0 < {}^{(\mathbb{R})}\mathcal{C}^\infty < {}^{(\mathbb{R})}\mathcal{C}^\omega$). Plainly then, an analytic manifold is automatically smooth, hence continuous as well. These last remarks hint at something that will prove to be of great import in the sequel, namely that, *structure-wise, whether a manifold M is regarded as being topological, smooth or (real) analytic one essentially depends on (in fact, it is defined by!) whether the 'structure coordinate' functions on it are (defined or assumed to be) continuous, smooth, or (real) analytic ones, respectively.* Ultimately, in a manifold-type of jargon: beginning with M as a structureless point-set, the epithets '*topological*', '*smooth*' (or, anyway, of finite order k of differentiability) or '(real) analytic' that may be given to it depend on whether we employ $\mathcal{C}^0$-, $\mathcal{C}^\infty$- (or, anyway, C^k-), or $\mathcal{C}^\omega$-charts to cover or 'coordinatize' it(s points). In turn, these charts *define* structurally M as a *topological*, *smooth* or *analytic* manifold, respectively. (**Note:** from now on, by an *analytic manifold* we mean a *real analytic* one, which is usually denoted by ${}^{\mathbb{R}}\mathcal{C}^\omega$, or simply, omitting the pre-superscript '$\mathbb{R}$', by $\mathcal{C}^\omega$. In fact, and for all practical intents and purposes, in the sequel we will not distinguish between a $\mathcal{C}^\infty$- and a $\mathcal{C}^\omega$-manifold, in spite of their slight technical difference mentioned above. That is, in our ADG-theoretic musings below, we do not distinguish between the terms 'Calculus/Differential Geometry on Smooth Manifolds' and 'Analysis'. Indeed, here we regard the terms *Differential Calculus* and *Analysis* as being synonymous.)

α may be regarded as a continuous, smooth or analytic injection (embedding) of I_τ into M depending on whether the coordinates of its points in M are $\mathcal{C}^0$-, $\mathcal{C}^\infty$-, or $\mathcal{C}^\omega$-functions of t, which in turn is conditional on whether M itself is assumed to be a topological ($\mathcal{C}^0$), smooth ($\mathcal{C}^\infty$) or analytic ($\mathcal{C}^\omega$) manifold in the first place.[74]

We first note that, by definition, the curve α has no end-point (in M), since $t = \tau \notin [0, \tau)$. Then, in the case of continuous extension, α is said to be (continuously) extensible if it is possible to *extend the map continuously to an end-point* $\alpha(\tau)$ *in* M. Otherwise, α is called *(continuously) inextensible*.

This definition of (continuous) inextensibility of (a curve in) M prompts us to make the following remarks which, in a way, foreshadow our 'global' stance against smooth and, *in extenso*, analytic extensibility of a space(time) to be presented in the sequel:

(R3.2) since continuity is a question of topology, and since M, regarded as a topological manifold, *is* the algebra $\mathcal{C}^0(M)$,[a] one could say that the continuous inextensibility of a curve α to a point $\alpha(\tau)$ in M, means essentially that we have reached the limit of applicability of the algebra $\mathcal{C}^0(M)$ and, ultimately, the representation of M as a topological manifold. This however does not mean that by changing the algebra of 'continuous' functions on M—thus, in effect, by changing the topology of spacetime[b]—we cannot convert a previously continuously inextensible curve in M regarded as a topological manifold, to one that is perfectly continuous at its end-point. Plainly, it is *we* that assign a topology on space(time) and, in the classical continuum case, *we* that assume that spacetime is modelled after a $\mathcal{C}^0$-manifold precisely by choosing $\mathcal{C}^0$-functions to label ('coordinatize') its points.

[a]That is to say, M, initially taken to be a structureless point-set, is covered by $\mathcal{C}^0$-charts—*ie*, its points are 'coordinatized' by $\mathcal{C}^0$-functions (see penultimate footnote).

[b]For instance, one can change the algebra from $\mathcal{C}^0(M)$ to some other functional algebra which one may assume up-front as being 'continuous', now of course with respect to a topology on M (a topology that one is free to *define* on M!) different from the usual (Euclidean) $\mathcal{C}^0$-one that defines it as a topological manifold.

[74]See last footnote. Of course, $\mathbb{R}$ is the archetypal (1-dimensional) $\mathcal{C}^\omega$-manifold, carrying the usual Euclidean differential (smooth) and topological structure.

The remarks in (R2.2) above apply then *mutatis mutandis* to smooth and analytic extensions. For $\mathcal{C}^\infty$- and $\mathcal{C}^\omega$-extensions of M, we may convert them to the following positive statement:

(R3.3) The smooth or analytic inextensibility of a manifold M past a certain *locus* (which is thus deemed to be identified with a singularity subsequently) marks *our* inability to cover ('coordinatize') the latter by $\mathcal{C}^\infty$- or $\mathcal{C}^\omega$-charts (*ie*, label them with smooth or analytic 'coordinates') thus, concomitantly, apply the usual differential geometric (analytic) concepts and techniques to it in the same way we do to the other so-called 'regular' (smoothly or analytically 'chartable') *loci* of M.

3.1.2 $\mathcal{C}^\infty$-smooth singularities: theoretical 'end-points' of both Calculus and of the Calculus based GR

In view of (R2.2) and (R2.3), the following 'heuristic' remark appears to be suitable here:

(R3.4) *Vis-à-vis* smooth or analytic extensions, *singularities represent* 'mathematical limits'*, 'impasses' or* 'end-points' *to the applicability of CDG (Analysis)*,[a] or simply, *they are loci where Calculus 'breaks down' or at best it is 'ineffective'*, so to speak.

[a]Or, since GR depends essentially on CDG for its mathematical formulation, which Calculus is in turn supported by the smooth M, $\mathcal{C}^\infty$-singularities (with their associated unphysical infinities) represent '*physical limit points*' to the applicability of the classical spacetime continuum based relativistic field theory of gravity at and beyond which the *physical theory* itself (GR)—precisely because of the inadequacy of the mathematical model for spacetime (M) and of the differential geometric framework on which it rests (CDG)—is well out of its depth (R2.?, Q2.??)—see also (R3.?) below. Plainly, there apparently is an inevitable conflation (or even 'confusion') of the notions '*physical theory*' (GR) and '*mathematical theory*' or '*mathematical framework*', since in GR we tend to identify *a priori physical spacetime* with the mathematical artifact *smooth manifold*, and, concomitantly, identify the physical (dynamical) structures involved in the relativistic field theory of gravity with those 'afforded' by that $\mathcal{C}^\infty$-background—the smooth fields on M (in the case of GR, $g_{\mu\nu}$). (The reader is advised to refer to section 7—and more in particular, to 7.4 and 7.5.5—for an ADG-theoretic critique of this '*physical spacetime*$\equiv M$' identification and *a priori* assumption in GR.)

A preliminary physical corollary of (R2.3) and (R2.4) is that

(R3.5) we tend to identify '*physical*' spacetime events with *regular points* (in M) on which the Einstein equations (and their solutions) hold. Another way to say this, 'physical' spacetime events are precisely those points in the bulk (interior) of M—those to which CDG concepts and techniques apply readily (*ie*, the differential equations of Einstein hold), without a problem.[a] Formally, we write:

[a] We thus put the word 'physical' in single quotation marks and write it in *emphasis script* just in order to preliminarily catch the reader's attention about the point made here, namely, that there is nothing really physically real about the point-events of the smooth manifold (pun intended), for as we will argue shortly, the bulk of M (consisting of regular points) is identified as being physical simply because we can actually do CDG on it, and CDG on manifolds (*ie*, Calculus) is actually the only way we know how to do differential geometry!—see (R2.6) next.

$$\text{`}\textit{physical}' \text{ spacetime events}^{75} \equiv \text{regular points (in the interior) of } M \qquad (3)$$

The reader should note here the 'negative' or 'by exclusion' sense in which singularities are defined. They are 'negative' or '$\mathcal{C}^\infty$-manifold excluding' and their definition is expressed, in a way, by 'negation' or 'exclusion' as follows: it is as if 'real' singularities, in contradistinction to 'virtual' or 'coordinate' ones (and, of course, 'regular' spacetime points), are precisely the ones to which we cannot further apply CDG.[76] In fact, this eliminative or negative 'definition-by-exclusion' of singularities is the very essence of current CDG-based attitudes and approaches to them, since it appears to be both technically and conceptually very hard, if not impossible, to give a direct, concise and 'positive' definition of them within the confines of Calculus (Analysis). For instance, below we quote the very first paragraph of the preface to [87]:

[75] Under the proviso that, physically speaking, the points of M should not be *a priori* interpreted as spacetime events proper—'*a priori*' meaning here *before the gravitational field obeying (or in ADG, defining) the gravitational dynamics (Einstein equations) is specified* (again, see 7.5.5). Thus, after this clarification and warning, from now on whenever we refer to the points of the manifold M as spacetime events without direct allusion to the field equations, we will put the designation '*sic!*' to remind the reader of this subtle interpretational point.

[76] See 2.1.3 below.

(Q3.2) "The central aim of this book is the development of results and techniques needed to determine when it is possible to extend a space-time through an 'apparent singularity' (meaning, a boundary-point associated with some sort of incompleteness in the space-time). Having achieved this, we shall obtain a characterisation of a 'genuine singularity' as a place where such an extension is not possible. *Thus we are proceeding by elimination: rather than embarking on a direct study of genuine singularities, we study extensions in order to rule out all apparent singularities that are not genuine.* It will turn out, roughly speaking, that the genuine singularities which then remain are associated either with some sort of topological obstruction of an extension, or with the unboundedness of the Riemann tensor when its size is measured in a suitable norm."

Implicit here is the following '$\mathcal{C}^\infty$*-manifold and CDG-conservative attitude*' (which, in fact, permeates throughout the whole of the spacetime continuum based field physics, whether classical or quantum!):

$\mathcal{C}^\infty$-conservatism and monopoly. Since Calculus on manifolds is the only way we know how to do differential geometry and it has undoubtedly served us well in the past for modelling mathematically relativistic gravitational physics (GR) for example,[a] $\mathcal{C}^\infty$-smooth gravitational singularities, which mark the ultimate inapplicability or 'breakdown' of classical differential geometric concepts and techniques to GR, are distinguished from the '*physical*' spacetime events (*sic*!) in the smooth M (*ie*, regular points or coordinate singularities) by being pushed to the boundary of the regular M (on which CDG still applies!). Metaphorically speaking (R2.4), this 'end of Calculus' at singularities signifies that we cannot 'calculate' (*ie*, apply differential geometric ideas and methods, and obtain 'sensible' numerical results—finite numbers) in the presence of (*ie*, in the vicinity of, let alone right at the spacetime *loci* of) singularities. At the same time, the corresponding CDG-based relativistic gravitational field physics (GR) is assailed by physically unacceptable, because nonsensical, infinities for many important $\mathcal{C}^\infty$-smooth fields[b] (Q?.?, Q?.?, Q?.?). (R3.6)

[a]Not to mention the manifold successful applications of the Minkowski spacetime continuum—itself carrying the technical and conceptual panoply of CDG—to the flat relativistic quantum theories of matter fields (QFT); while, in the same context, one should not to forget of course the numerous applications of CDG, in the guise of smooth fiber bundle theory, to (quantum) Yang-Mills (gauge) theories. Of course, these theories too are plagued by infinities coming from the assumption of a base spacetime continuum M, infinities which are not that different from the singularities of GR, although admittedly the quantization procedure alleviates a bit their robustness and strength (see section 6).

[b]That is, fields that are represented by $\otimes_{\mathcal{C}^\infty(M)}$-tensors, such as the spacetime metric or the Riemann curvature tensor.

The said '*smooth spacetime continuum conservatism*'—albeit, above Planck scale[77]—is nicely wrapped up in the following quote by Hawking in the preamble to the first chapter of [186], just before he discusses singularities in GR and the global (causal) structure of the classical spacetime continuum:

[77]That is, still in the domain of applicability of classical gravity (GR)—see 7.2 for an ADG-theoretic critique of the Planck length scale and its apparent physical importance (Q?.?).

(Q3.3) "...Although there have been suggestions that spacetime may have a discrete structure, *I see no reason to abandon the continuum theories that have been so successful.* General relativity is a beautiful theory that agrees with every observation that has been made. *It may require modifications on the Planck scale, but I don't think that will affect many of the predictions that can be obtained from it*[a]..."

[a]Our emphasis throughout.

with the principal 'negative prediction' of GR being, of course, its own 'autocatastrophe': the existence of $\mathcal{C}^\infty$-smooth spacetime singularities. We may thus discern in (Q?.?) a rather positive view of continuum singularities quite in contrast with Einstein's remarks in (Q?.?); furthermore, one may infer a question (and possibly a doubt?) by Hawking whether QG will ultimately remove singularities.[78]

However, going against the grain (or trend!) of '$\mathcal{C}^\infty$-conservatism', let us fuse into the following 'philological', albeit concise, statement the mathematical with the physical spacetime continuum and CDG inapplicability remarks in (R3.?–R3.?):

(R3.7) Physically nonsensical infinities come from our persistent trying to calculate (physically important quantities modelled by $\mathcal{C}^\infty$-smooth fields) at—that is, our insistence on applying Calculus to—singularities, which anyway come from *our* assuming up-front the 'structure algebra' $\mathcal{C}^\infty(M)$ (or its associated structure sheaf $\mathcal{C}^\infty_M$) to coordinatize spacetime events (*sic*!) in the first place.

3.1.3 (Causal geodesic) incompleteness

Let us return now to our affinely parametrized curve $\alpha(t)$ in (2). α is said to be incomplete if it has finite (general) affine parameter t-length, by which we mean that

$$\ell_E(\alpha) = \int_0^\tau \sqrt{\sum_{i=0}^{3}(x_i^2)}\; dt < \infty, \tag{4}$$

where the subscript 'E' denotes a *vierbein* that is parallelly translated along α and with respect to which the components x_i of the tangent vector $\dot{\alpha} := \frac{d\alpha}{dt}$ of α are measured.

[78]In anticipation of (Q?.?).

For *causal* (geodesic) curves α,[79] incompleteness may be also thought of as expressing that it takes finite (now proper) 'length of time' (duration) for free falling material particles to reach the end-points of their evolution,[80] end-points which, as we shall see subsequently, are deemed to be identified with singularities. All in all, and in a physical sense, incompleteness is supposed to capture the apparently contradictory quality of singularities as being '*finitely asymptotic (gravito-focal) loci*'[81] marking, in the case of free falling matter, the end of the dynamical histories of material particles propagating and accreting under the focusing (attractive) action of the gravitational field.

3.1.4 $\mathcal{C}^\infty$-Singularities: Mathematical artifact-points at the edge of the 'physical' (regular) spacetime manifold

Having the notions of inextensibility and incompleteness at our disposal, and following Clarke [86, 87], we can then proceed in two steps to define singularities more precisely:

- i) Such incomplete (causal) curves (ICs)[82] define boundary points of the spacetime manifold.
- ii) Some of these 'marginal' spacetime points are identified with 'true' singularities.

Starting formally, one may initially assume that the differential spacetime manifold M is, topologically speaking, *open and bounded* (in the usual $\mathcal{C}^0$-topology), with closure $\overline{M} := M \cup \partial M$, where ∂M is the set of boundary points of M. *En passant*, and in accordance with (Q?.?) and i) above, we note that ICs define points on ∂M. In particular, *boundary points represent equivalence classes of ICs in* M, with the defining equivalence relation being "*ending at the same point*" [86].[83] In other words, with each point p on ∂M we associate a collection of ICs in M, namely, all those that have p as (common) terminus. Of course, this definition of boundary points as $\mathfrak{t}$-classes of ICs in M does not give one an explicit expression for the relation '$\mathfrak{t}$'; consequently, it does not tell one how to actually construct ∂M as a boundary proper (at least in the topological sense of the term 'boundary').

[79] By which we mean curves whose tangent vector $\dot{\alpha}$ is timelike or lightlike (null) everywhere on them.

[80] A tetrad field 'E' like the one in (2) is now supposed to be comoving with each and every one of such particles, thus defining a geodesic frame adapted to (*ie*, with origin at) the particle.

[81] 'Finite asymptoticness' pertaining to the fact that it takes finite proper time (for geodesic frames adapted to free falling particles) or affine parameter length (for general, arbitrary frames with bounded acceleration) to reach (or 'converge' to) the purported singular *locus* which, however, itself does not belong to the 'physical', regular spacetime M (see 2.1.4 next).

[82] Correspondingly, incomplete and inextensible curves will be abbreviated by 'IICs'.

[83] Formally, for incomplete curves c_1 and c_2 ($c_1, c_2 \in$IC), we write '$c_1 \mathfrak{t} c_2$' for the aforesaid '*having the same terminal point*' equivalence relation.

Without going into any technical detail,[84] we mention two approaches for deciding when two ICs terminate at the same point on ∂M, thus effectively defining '$\mathfrak{t}$' and pointing to a method for actually constructing the boundary of M:

- α) The first, which is the physically more intuitive of the two—here to be called the '*c-method*',[85] essentially rests on $\mathfrak{t}$-identifying two *future directed causal curves* $\overrightarrow{c}_1, \overrightarrow{c}_2$[86] when every point to the past of a point in, say, $\overrightarrow{c}_1$, is also to the past of a point in $\overrightarrow{c}_2$; write formally:

$$\overrightarrow{c}_1 \mathfrak{t} \overrightarrow{c}_2 \Leftrightarrow \text{“}Past(\overrightarrow{c}_1) =: I^-(\overrightarrow{c}_1) \equiv I^-(\overrightarrow{c}_2) := Past(\overrightarrow{c}_2)\text{”}^{87} \tag{5}$$

Then, this particular definition of '$\mathfrak{t}$' leads to the well known definition of *indecomposable past sets* which figure prominently in the construction of the *conformal boundary* (c-boundary) and associated '*null horizon*' in [159]. The c-method of defining singular boundary points—in particular, one that uses causal *geodesics*—appears to be more 'natural' for defining singularities which form either from the gravitational accretion of free falling non-interacting 'cool matter' (dust) like for example in the theoretical scenario for stellar collapse in [185, 334], or in a cosmological setting for the radiative focusing of black body emission [185, 184]. In both cases, a geometrical configuration called '*trapped surface*' forms—trapping as it were the aggregated matter and confining the gravitationally focused radiation—which in turn points, *vis-à-vis* the singularity theorems of Hawking and Penrose [184, 185], either to the existence of a boundary point of M or to a violation of causality within the physical spacetime M.[88]
And since causality is to be 'protected' in any (intuitively) physically reasonable model of

[84] For a detailed presentation, analysis and discussion of how to construct the boundary of M, the reader is referred to [87].

[85] 'c' for '*causal*'.

[86] In keeping with our previous notation in the trilogy [270, 271, 272], a right-pointing arrow over a symbol corresponds to the adjective '*causal*' (which, of course, may vary in meaning from context/model to context/model).

[87] The right hand side expression reading: "*the past of* $\overrightarrow{c}_1$ *coincides with the past of* $\overrightarrow{c}_2$" [86]. Similarly for past directed causal curves (write $\overleftarrow{c}_1$ and $\overleftarrow{c}_2$) and the identification of their futures $I^+(\overleftarrow{c}_1)$ and $I^+(\overleftarrow{c}_2)$.

[88] The reader may wish to recall that the singularity theorems of Hawking and Penrose were essentially based on smooth causal (predominantly null) geodesics arguments as well as on the following four rather generic assumptions about them, M and their relation to the smooth gravitational field strength (smooth spacetime curvature): i) '*null geodesic focusing*': the null geodesics from any point of M are eventually focused (or alternatively, the null geodesics from some closed 2-surface are converging), ii) '*Ricci curvature positivity*': for every causal vector V, $V^r \mathcal{R}_{rs} V^s > 0$ holds, iii) '*Riemann curvature non-degeneracy*': every causal geodesic $\overrightarrow{\gamma}$ has a point where $\dot{\overrightarrow{\gamma}}_{[r} R_{s]tu[v} \dot{\overrightarrow{\gamma}}_{w]} \dot{\overrightarrow{\gamma}}^t \dot{\overrightarrow{\gamma}}^u \neq 0$ holds, and finally, iv) '*causality protection condition*': M contains no closed timelike curves. For more technical details and comments about the physical significance of the general assumptions i)–iv), the reader is again advised to refer to [87].

relativistic spacetime structure, (singular) boundary points to M appear to be inevitable consequences of the theory.[89]

- β) The second method, which is the analytically (mathematically) the more robust, elegant and fruitful of the two—here to be called the '*b-method*' as it eventually leads to the definition of the celebrated '*singularity b-boundary*' of Schmidt [333, 184, 87]—involves the construction of ∂M not by working directly with ICs in M, but rather with the frame bundle FM over M—essentially, by (parallelly) propagating frames along ICs in M.[90] The b-method appears to be analytically more versatile and of greater calculational import than the c-one, because it endows $\overline{M}$ with a 'natural' topology, inherited from FM,[91] relative to which it is (eventually) meaningful to talk about a neighborhood of a singular point at ∂M, or calculate how the (Riemann) curvature tensor (in a particular local frame) behaves (grows) 'asymptotically' as the frame ('local observer/particle') approaches that singular *locus*.

On the other hand, leaving aside these admittedly ingenuous technical attempts at defining ∂M—all of which are manifestly based on the (analytic) concepts and tools of CDG, which is in turn vitally dependent on the smooth spacetime continuum M in one way or another—we wish to emphasize here that on the one hand,

(R3.8) the points of ∂M are *ab initio* declared to be only mathematically adjoined to M, but devoid of any physical significance in the sense that only the points of the smooth bulk interior M of $\overline{M}$ are regarded as being *physical spacetime events*[a] (Q?.?, R2.5) [86].

[a] *Sic!*

while on the other, that

(R3.9) (smooth or analytic) inextensibility of M past points of ∂M is precisely the criterion or 'attribute' that separates the latter into 'regular' ('virtual' or 'coordinate' singularities included in M) and 'singular' ('actual', 'true' or 'real' singularities, excluded from M) (Q?.?, Q?.?, R2.5).

[89] Roughly in this sense one says that under quite general assumptions and conditions GR predicts singularities.

[90] For a brief account of the construction of the b-boundary, the reader is referred to [86], but for more details, either the original paper [333], or again the book [87], should suffice.

[91] That is to say, a topology 'projected down' from $\overline{FM}$—the Cauchy completion of FM with respect to Cauchy convergence (in the metric topology of a suitably and rather naturally defined metric) of sequences of frames tending to certain ones at the end-points of ICs in M [333, 86, 87].

In 2.1.5 next, we make it clear, following Clarke [86, 87], which points on ∂M qualify as 'actual' or 'real', as opposed to 'virtual' or 'coordinate', singularities, and we highlight for the reader that, so far, there have been proposed three kinds of the former all of which depend vitally on that 'quintessential' property that qualifies them as 'true' singularities in the first place: *the (analytic or smooth) inextensibility of M past them.*

3.1.5 Three types of 'real' $\mathcal{C}^{\infty}$-smooth singularities: what underlies them all

There are three kinds of boundary points that are usually regarded as being *'real' singularities*. As we will see below, 'common denominator' to all three types of singularities is the notion of (analytic or smooth) inextensibility, while what distinguishes them from one another is an auxiliary condition that must be satisfied on top of our inability to extend M analytically beyond them.[92]

Thus, first returning to ii) of 3.1.4, we recall Clarke's words from (Q?.?) maintaining that what distinguishes between 'regular' boundary points from 'real' singularities is that while there is an (analytic) extension of M past the former, there is no such 'continuation' so as to include the latter into the 'physical', regular spacetime continuum (R2.8, R2.9). In a nutshell, and in accord with our preliminary rationale in (R2.2, R2.3) and the concomitant 'heuristic remark' in (R2.4),

(R3.10) 'true' (and physically interesting or important) singularities, as opposed to 'virtual' or 'coordinate' ones (*ie*, regular boundary points of M), cannot be (smoothly or analytically) coordinatized so as to be regarded as being themselves 'regular' differential geomerically or analytically speaking; consequently, CDG (Analysis) cannot be applied to them: the latter has simply reached its limit of applicability and validity (Q?.?, Q?.?), or even more graphically and in a brute sense, it (and the physical laws that it models differential geometrically as differential equations) 'breaks down'.

On the other hand, as briefly alluded to above, so far there have been proposed three types of 'truly' singular boundary points of M, all of which have of course (analytic) inextensibility at their basis, but at the same time are distinguished from one another by an extra physico-mathematical condition that must be satisfied in the vicinity of their *locus*. In order, they are:

- α) *(Differential) geometric singularities* (DGS): boundary points for which there is no C^k-

[92] Indeed, not actually of M, but of its '**A**'; when one says 'M', one invariably (albeit, more often implicitly) refers to the '**A**' one employs in one's differential geometric constructions and calculations!

differential extension of (the metric on) M so as to remove them.[93] DGSs are precisely the singularities alluded to in (Q?.?) and, as Clarke points out there, they explicitly mark a breakdown of differentiability (of the metric) at the C^k-level. DGSs are direct (mathematical) evidence of the ineffectiveness of CDG and the $\mathcal{C}^\infty$-smooth manifolds supporting it in coping with singular points already at a finite order of differentiability—and it must be noted that the lower the order 'k' of a C^k-DGS, the more robust the latter is supposed to be[94] [184, 86, 87].

- β) *(Various) energy singularities* (VES): boundary points for which there is no (analytic) extension of M that removes them satisfying at the same time various energy conditions, the most prominent one being the *weak energy condition* in the celebrated singularity theorems of Hawking and Penrose [185, 184, 87].

- γ) *(Solution) field singularities* (SFS): boundary points for which there is no (analytic) extension of (the metric on) M that removes them and is a solution of the Einstein field equations in question.[95] The important thing to mention here is that the term *solution* to the field equations means *generalized* or *smeared*, what is commonly known as *distributional*, solution [86, 87].[96]

Loosely comparing the three species of 'real' singularities. The order in which the three species of 'true' singularities were presented above was intended to depict their 'conceptual and structural/mathematical (differential geometric) strength'. To be more precise, and from a mathematical standpoint, DGSs may be regarded as being more 'fundamental' than either VESs or SFSs as they express a direct inapplicability of CDG, since 'smoothness' or 'differentiability' (of the metric, whose components, as noted earlier, represent the gravitational potentials in GR) is supposed to break down at a finite level of differentiation in their vicinity. Apart from the smooth or analytic inextensibility of M past them, DGSs are manifest examples of the ineffectiveness of

[93] Following footnote ??, a C^k-differential inextensibility of M past one of its boundary points means that there is no C^k-chart (*ie*, coordinates of finite order 'k' of differentiability) to cover it. Equivalently, there is no (isometric) embedding of M, of (finite) differential order k ($1 \leq k \leq \infty$), into a 'larger' manifold $M^{'}$ (with $g_M = g_{M'}$) [184].

[94] With continuous (*ie*, $\mathcal{C}^0$-) DGSs representing the most singular, topological obstructions to (even defining!) fields on the spacetime continuum. However, since the Einstein equations are second order PDEs, the lack of even C^2-extensions, let alone $\mathcal{C}^0$-ones, past a *locus* in M, indicates the existence of a 'true' gravitational DGS [184].

[95] As Clarke mentions in [86], the Einstein equations referred to above may be taken for example to be the Einstein-scalar or the Einstein-fluid equations.

[96] As we will see in more detail in section 5, in [141] Finkelstein implicitly characterized the interior Schwarzschild singularity as an SFS, since he up-front assumed that the vacuum Einstein equations hold on an analytic spacetime manifold M (even though he did not explicitly state his assumption of a possibly distributional curvature solution to them).

the basic 'structural pillars' supporting the entire edifice of CDG[97] in dealing with them. A DGS may be thought of as being more basic than a singularity belonging to the other two 'species' in the sense that it is not just that a solution of Einstein's field equations does not hold (or even uncontrollably blows up!) at its *locus* (Q?.?), but that *differentiability itself breaks down at it*, so that one cannot even in principle set up a differential equation (to represent the gravitational law) over it (Q?.?).

On the other hand, and now from a more physical perspective, SFSs do not challenge directly the basic differential properties of the spacetime manifold *per se*, since one is supposed to be able to write 'valid' differential (Einstein) equations on the smooth manifold in focus; rather, the auxiliary physical condition imposed in addition to the analytic inextensibility of M is that the Einstein equations do not admit even generalized (distributional) gravitational field strength (curvature) solutions.[98] Finally, in both conceptual and structural strength, VESs lie 'in between' DGSs and SFSs, for as Clarke remarks in [86],

(Q3.3)

> "...In the 'singularity theorems' of Hawking and Penrose an intermediate case [*ie*, in between what we called DGSs and SFSs above][a] arises in which one does not impose full field equations but only energy conditions. Correspondingly, we shall call a boundary point where there is no extension satisfying the weak energy condition (for example[b]) *energy singularity*. One could also class this situation with the field singularities,[c] taking the view that the energy conditions stand in for the field equations in those felicitous situations where one can get away without using the full content of the field equations..."
>
> [a]Our addition for continuity with the preceding text, as well as for clarity.
>
> [b]For an analytical treatment of the role played by the various energy conditions in the theory of and theorems about spacetime singularities, the reader is once again recommended to refer to either [184], or to [87].
>
> [c]The ones we called SFSs above.

Formally, we depict this 'conceptual and structural strength' order of 'genuine' singularities as follows:

[97]That is to say, in view of our remarks in section 1—and in particular, of the identification (1)—of the 'structure coordinate algebra (sheaf)' $\mathcal{C}^\infty(M)$ ($\mathcal{C}^\infty_M$)!

[98]For example, in ?? we will present the ADG-theoretic resolution of the interior Schwarzschild singularity using sheaves of Rosinger's algebras of generalized functions (distributions) as coefficient structure sheaves—generalized (smooth) functions hosting singularities on everywhere dense subsets of finite-dimensional (locally) Euclidean spaces (manifolds).

$$\boxed{DGSs} > \boxed{VESs} > \boxed{SFSs} \tag{6}$$

and further note that our ADG-theoretic musings in the sequel will concentrate mainly on the (mathematically) most basic and strongest kind of the three: the DGSs. Nevertheless, we will also tackle SFSs ADG-theoretically, being especially motivated by ADG's successful application in formulating Einstein's (vacuum) gravitational equations over a space(time) M 'coordinatized' by algebras of generalized functions (differential algebras of non-linear 'multifoam spacetime' distributions) having singularities (of the most pathological and classically unmanageable kind) everywhere densely in M, and not just on its boundary [273, 274, 262, 275, 264, 265].

It must be stressed however that from a physical perspective, but always in the context of CDG, SFSs—in particular those associated with specific differential equations of physical interest, like for example the Einstein equations—may be viewed as being physically more 'relevant' and of greater importance (to the physicist) than the 'purely geometrical' DGSs. One way of expressing this is how Clarke opens chapter 4 of [87]:

(Q3.4) "Although many of our considerations will be geometrical, treating space-time as a pseudo-Riemannian manifold and asking whether or not this geometrical structure is breaking down, it must always be remembered that we are working with a physical theory, governed by particular physical equations for fields and particles, and that it is the breakdown of the physics that is primarily of interest. *The breakdown of geometry is simply one possible manifestation of the breakdown of the physics...*"

On the other hand, and as we shall see subsequently from an ADG-theoretic perspective, we will expressly disagree with certain points in (Q?.?) above. For one thing, and this goes glaringly against the grain of (R2.?) and (R2.?) which capture the very essence of the present paper,[99] as well as with Ashtekar's remarks on the unphysical infinities associated with $\mathcal{C}^\infty$-singularities in (Q?.?) and (Q?.?), "*the breakdown of* (differential) *geometry*"—the one based on "*treating space-time as a* (smooth) *pseudo-Riemannian manifold*" (*ie*, CDG)—cannot possibly mislead us into thinking that the physics itself (*ie*, the laws of Nature, represented by differential equations) breaks down at singularities. Of course, this is so as long as one does not confuse the actual *physical 'spacetime' geometry*—the one *defined* by the (dynamical) algebraic interrelations between the 'geometrical objects' themselves (*ie*, the physical fields and their particles[100])—with *our* mathematical model of spacetime as a differential manifold and, concomitantly, of the fields that inhabit it as being, in

[99]That is to say, our basic contention that *Nature has no singularities, and in no way physical laws are limited by them.*

[100]See concluding section.

the classical sense, $\mathcal{C}^\infty$-smooth. As noted earlier, the shortcomings, inadequacies and limitations—ultimately, the breakdown—of the CDG-theoretic analysis of the latter should not be attributed to Nature, that is, to the physical fields that actually comprise it. For surely, as long as one understands by "*the breakdown of physics*" simply the fact that *the physical theory is out of its depth when dealing with singularities* (R2.?), and if one tends to 'identify' the physical theory with the mathematical model for spacetime (and the fields dynamically propagating and interacting on it),[101] then the aforesaid breakdown of physics is just an indication that *the smooth manifold model of spacetime—ultimately, CDG—has reached the limits of its applicability and validity* (Q?.?, R2.3, R2.4). Accordingly, the said 'misattribution' (of the differential geometric breakdown to a breakdown of the physics itself) may be justified on the grounds that *so far we have been unable to do differential geometry without a base differential (spacetime) manifold*—what we earlier coined '*the $\mathcal{C}^\infty$-smooth manifold conservative attitude and its associated CDG-monopoly*' (R2.6, Q2.2).

At the same time, if the physically unacceptable 'behavior' of $\mathcal{C}^\infty$-singularities is supposed to be not just due to the manifest breakdown of differentiability or smoothness *per se* (as for example with DGSs), but that in the vicinity of their *loci* certain physically important smooth fields 'grow without bound' (diverge) (Q?.?) and, ultimately, become infinite (Q?.?) (as for example with SFSs which do not admit even 'blown up' or 'smeared out' solutions), it behoves one, who still insists though at confronting singularities with the CDG-theoretic panoply in a $\mathcal{C}^\infty$-conservative manner, to search for the physical (field) culprit that 'misbehaves' at singularities. Thus, following [87], we mention *en passant* that, in the usual $\mathcal{C}^\infty$-spacetime manifold context, one could attribute the physically anomalous behavior at singularities to the following three fields:

1. The smooth spacetime metric g whose ten components, as noted earlier, physically represent the gravitational potentials.[102]

2. The smooth affine (Levi-Civita) connection ∇, which is taken to be compatible with the metric in the usual Riemannian geometry (Q3.1) [199], and which, gauge-theoretically speaking, may be thought of as representing the gravitational gauge potential field [272].[103]

[101] This 'identification' may be justified as follows: in a general Wheelerian sense, a physical theory '*is*' the dynamical laws expressed within its mathematical framework (R2.?). For the particular physical theory, GR, the (mathematical) expression of the law of gravity is the Einstein equations, which are (non-linear, hyperbolic, partial) *differential equations* and which in turn vitally depend on our *ab initio* modelling spacetime after a differential manifold.

[102] For example, explicit allusion to the 'breakdown' of the metric, whose components $g_{\mu\nu}$ are of finite differentiability class C^k, was involved in the definition of DGSs above. Without any constraints on the order of differentiability (*ie*, assuming a $\mathcal{C}^\infty$-inextensibility of M), and as it was briefly alluded to in footnote ??, it is plain that the components of $g_{\mu\nu}$ belong to $\mathbb{A} = \mathcal{C}^\infty(M)$, or locally, and in a sheaf-theoretic sense, they are local sections of the 'classical' structure sheaf of smooth coordinates $\mathbf{A} = \mathcal{C}^\infty_M$ of M (write $g_{\mu\nu} \in \Gamma(U, \mathbf{A})$, U open in M) [259, 260, 271, 272].

[103] Similarly to the case of the metric, and after assuming that the smooth affine connection is *metric*, or equiva-

3. The smooth Riemann curvature tensor R (or its partial Ricci tensor $\mathcal{R}$, and full Ricci scalar $\mathfrak{R}$, contractions) of the connection ∇ (or equivalently, of the meric and its first partial derivatives), which, again gauge-theoretically speaking, may be thought of as representing the gravitational gauge field strength [272].[104]

With [87] as compass, a general and brief comparison of DGSs, VESs and SFSs, concentrating more on the relation between the two 'extreme' kinds (*ie*, DGSs and SFSs) and always in the context of CDG, follows below:

- First, in view of the fact that the expression for R involves the metric as well as its first and second derivatives, it would *prima facie* appear that an SFS could be automatically coined a C^2-DGS.[105]

- Physically speaking, R is usually held to be more important than either g or ∇; consider for example:

(Q3.5) "*...From some points of view, the Riemann tensor is of more direct physical importance than either the connection or the metric, in the sense that an unbounded Riemann tensor would suggest unbounded tidal forces, while an unbounded connection or metric component could be a purely coordinate effect...*" [87]

although at the same time we find Hawking in [186], in view of the unnaturalness of the method of surgically removing singular regions out of the spacetime manifold, tending more towards adopting the DGS-characterization of singularities rather than as *loci* where the curvature tensor diverges, as follows:

lently *torsionless* (*ie*, that it obeys the metric compatibility condition $\nabla g = 0$), one could also define in the context of CDG 'connection based DGSs'—that is, a C^k-inextensible M admitting a connection whose Christoffel components $\Gamma^\lambda_{\mu\nu}$ in a local frame are C^k-differentiable functions. Again, with no differentiability constraints imposed on the extensibility of M, $\Gamma^\lambda_{\mu\nu} \in \Gamma(U, \mathbf{A} \equiv \mathcal{C}^\infty_M)$, U open subset of M.

[104] Similarly to the metric and connection cases above, if one does not require up-front that M be inextensible at a particular finite order of differentiation, the (local) components of R (and as a result, of $\mathcal{R}$ and $\mathfrak{R}$ as well) are elements of $\Gamma(U, \mathbf{A})$, U open in M.

[105] Due to this, many people have maintained that in actual practice it is sufficient to view GR as a theory 'invariant' under C^2-diffeomorphisms, so that the assumption of full smoothness ($\mathcal{C}^\infty$) "*would be regarded as an idealisation convenient for geometrical purposes but of restricted validity*" [184, 87].

(Q3.6) "*...One normally thinks of a spacetime singularity as a region in which the curvature becomes unboundedly large. However, the trouble with that as a definition is that one could simply leave out the singular points and say that the remaining manifold was the whole of spacetime. It is therefore better to define spacetime as the maximal manifold on which the metric is suitably smooth...*[a]"

[a]Then he goes on and defines singularities as in (Q?.?), in terms of (smooth or analytic) inextensibility and incompleteness (of causal geodesics).

The same dissatisfaction with the physically 'forced' and *ad hoc* character of surgically excising singularities out of the smooth and otherwise regular spacetime manifold was expressed earlier in [155], who writes:

(Q3.7) "...We originally introduced geodesic incompleteness because the concept appeared to give a precise statement of [the following property]:[a] *In a nonsingular spacetime, one should like to be sure that 'no regions have been deleted from the spacetime manifold'*[b]..."

[a]Our addition. Geroch actually calls it 'Property 1'.
[b]Our emphasis.

Arguably, such both Geroch's and Hawking's dissatisfaction with removing from the spacetime manifold, as it were by 'theoretical fiat', singular *loci* is tautosemous with Einstein's 'disbelief' in (Q?.?).

- On the other hand, in line with Hawking's reservations about the relevance of the unboundedness of the curvature tensor in defining singularities, it must also be stressed that "*an unbounded Riemann tensor does not in itself indicate a singularity*" [87], while at the same time, the boundedness of R does not imply that the metric is C^2-differentiable either.

Indeed, it appears that one could 'go in circles and argue till one is blue in the face' whether the breakdown of the differentiability of the metric (*ie*, DGS) or whether the unboundedness of the (even if of a distributional character) curvature field solution of Einstein's equations (*ie*, SFS) is the physically more meaningful definition of singularities within the confines of the CDG-framework. On the other hand, since at the bottom of both DGSs and SFSs lies the notion of analytic spacetime manifold inextensibility, and in accord with Hawking's lose 'definition' of singularities in (Q?.?), we will content ourselves with the following concise, almost 'apofthegmatic' and 'aphoristic', conception of singularities.

3.1.6 Calculus is dead!; long live a background manifoldless differential geometry: ADG-theoretic forebodings

Singularities are *loci* where 'analyticity', and in particular, 'differentiability' or 'smoothness', in the classical CDG- or Calculus-theoretic sense of these words, 'breaks down'—*ie*, it just does not work. In turn, this means that at singularities, CDG simply ceases to be an effective mathematical framework within which to analyze and do gravitational physics, and concomitantly, a base $\mathcal{C}^\infty$-smooth manifold altogether ceases to be a sound model of 'physical spacetime', since the law of gravity appears to 'halt', stop being in force, or even 'break down', at them.

At the same time, by smooth (or analytic) inextensibility of the spacetime manifold—the notion that as we saw above underlies all the aforementioned definitions of $\mathcal{C}^\infty$-singularities—and the concomitant relegation of the singular *loci* at the boundary or edge of an otherwise regular 'physical' spacetime manifold M, we simply understand that the algebra $\mathcal{C}^\infty(M)$ (or its associated structure sheaf $\mathcal{C}^\infty_M$) cannot be further employed so as to coordinatize those 'differentially pathological' points (DGSs). Indeed, supposedly, if that could be done, from the $\mathcal{C}^\infty$-smooth perspective (CDG) the purported singularities would be characterized as being merely 'coordinate' or 'virtual' ones, of no physical significance. *In summa*, and in connection with (1), it is precisely $\mathcal{C}^\infty(M)$ (*ie*, M itself!) that 'inherently' carries the seeds for those 'differential geometric pathologies', thus, insofar as $\mathcal{C}^\infty_M$ (or the differential manifold M itself) and CDG is employed to deal with singularities, the problem is insuperable simply because, in a strong sense, it is 'self-referential'.[106] Another way to say this, if one tries to change coordinate algebra (sheaf)

$$\mathbf{A}_M \longrightarrow \mathbf{A}'_{M'} \tag{7}$$

so as to smoothly extend M (to M') past the singular *loci*, albeit, always staying within the classical one $\mathcal{C}^\infty_M$ of $\mathcal{C}^\infty$-smooth functions, singularities will persist (*ie*, M will remain smoothly inextensible), since they are 'innate' to the very coordinate 'structure algebra' $\mathcal{C}^\infty(M)$ (or the classical structure sheaf $\mathbf{A} \equiv \mathcal{C}^\infty_M$).

On the other hand, it must have become now fairly transparent that if one did not stay within $\mathcal{C}^\infty(M)$ (*ie*, within M) and the classical differential geometric framework, but could change altogether the structure algebra sheaf of coordinates (arithmetics)—with a concomitant, of course, change of the classical notion of 'differentiability' or 'smoothness'—while at the same time one was able to retain the full conceptual and technical panoply of CDG and still have in one's hands the latter's full calculational power,[107] the differential geometric pathologies (singularities) of the differential spacetime manifold could be totally evaded. We may coin this change '*differential*

[106] That is to say, the 'germs' of singularities are already built into $\mathcal{C}^\infty(M)$!

[107] What we cumulatively referred to before as the 'inherent' differential geometric mechanism or machinery of Calculus (CDG).

geometric extensibility', which means of course that we enlarge or generalize (in fact, *abstract*!) the CDG-mechanism in a way that is not vitally (in fact, not at all) dependent on classical smoothness, which is secured by $\mathcal{C}^\infty_M$ (*ie*, by the very nature of the base locally Euclidean manifold M). Of course, in line with what was said before, this 'differential geometric extensibility' is an abstract analogue of 'coordinate (structure sheaf) change' and, *in extenso*, the purported total singularity-evasion will reveal that, in a deep sense, *all singularities are effectively 'coordinate singularities'*. This last point lies at the heart of ADG and foreshadows our ADG-theoretic musings in the sequel.

All in all, in the smooth manifold context (*ie*, working always within the coordinate structure sheaf $\mathcal{C}^\infty_M$), by '*analytic extension*', and for all intents and purposes, one essentially means (7), so that

- when one is able to include by such a procedure a singular *locus* with the other regular points, the said singularity is said to be a 'virtual' or 'coordinate' one, and classical differentiability (CDG) is essentially retained, but

- when no such extension past the offensive point is possible, the *locus* in focus is regarded as a 'real' or 'true' singularity, and classical smoothness apparently breaks down—*ie*, CDG simply becomes inadequate for coping with the singularity.

In contradistinction, as briefly noted above and as we shall see subsequently in the light of ADG, (7) above acquires a new, completely different meaning (than the notion of smooth extension in CDG) on two accounts:

- On the one hand, and more from a mathematical perspective, not just remaining within the confines of differential manifolds and the associated (both technical and conceptual) $\mathcal{C}^\infty$-conservatism, (7) can be interpreted not only as a (smooth or analytic) coordinate change aimed at integrating apparently problematic *loci* with the other regular points of a smooth manifold M,[108] but, changing (in fact, enlarging!) categories (of differential spaces) themselves—as it were, to change altogether *differential geometric framework* (theory).[109] Indeed, (7) may be taken to mean not just a smooth coordinate change, but as a change

[108] Analytic or smooth extension perceived here effectively as an enlargement, an embedding of M ('charted' by $\mathcal{C}^\infty_M$) into a 'larger' manifold $M^{'}$ ('charted' by $\mathbf{A}^{'}_{M'}$) [184], always remaining though within the category of smooth manifolds ($\mathcal{C}^\infty$-conservatism).

[109] What we coined above '*differential geometric extensibility*. In fact, as we shall see in the next section (3.1.7), the category of smooth manifolds is a subcategory of the category of differential triads—the abstract differential (sheaf) spaces of ADG.

from, say, the usual ('classical') smooth coordinate functions (in $\mathcal{C}^\infty_X$),[110] to another functional algebra structure sheaf[111] which might be significantly different from the 'classical' one $\mathcal{C}^\infty_M$ of algebras of infinitely differentiable functions, provided of course the new structure sheaf of generalized arithmetics furnishes us with a 'viable' differential geometric mechanism as versatile and potent as the classical one of Calculus (CDG). On such changes of differential geometric framework rests the so-coined *Principle of Relativity of Differentiability*(PRD) that we shall encounter in the sequel.[112]

- On the other hand, and more from a physical viewpoint, such changes ('enlargements') of algebras of differentiable functions (generalized coordinate or coefficient arithmetics), while still retaining in full force and effect the (inherently algebraic) differential geometric mechanism of Calculus (albeit, in the manifest absence of base $\mathcal{C}^\infty$-manifolds!) [271], may prove to be of great physical import since what might have appeared to be insuperable, (*ie*, 'analytically inextendible'—something that cannot be included with the rest of the regular interior points of M) 'truly' singular *loci* in a manifold ($\mathbf{A} \equiv \mathcal{C}^\infty_M$), now from the novel differential geometric perspective afforded by the new $\mathbf{A}^{'}$ may appear to be completely 'regular' and in no way inhibiting (let alone collapsing) the said (inherently algebraic) 'differential geometric mechanism' that $\mathbf{A}^{'}$ also possesses in full effect. Moreover, what might have appeared as 'genuinely' singular points (or functions) from the classical vantage (of $\mathbf{A} \equiv \mathcal{C}^\infty_M$) may now be integrated in (absorbed or engulfed by) $\mathbf{A}^{'}$, while at the same time the physical law of gravity, which is still modelled after a differential equation nourished and sustained by the mechanism afforded by $\mathbf{A}^{'}$, may hold in full force in the presence of or at the *locus* of those very singularities. Thus the latter sites in no way represent breakdown points of the law of gravity (Einstein equations), and hence they are completely evaded or bypassed (algebraically).

3.2 CDG Cannot Cope with $\mathcal{C}^\infty$-Singularities: $\mathbf{A} \equiv \mathcal{C}^\infty_M(\equiv M)$ is the Culprit

The crux of the foregoing presentation and discussion is on the one hand that $\mathcal{C}^\infty$-singularities cannot be thought of independently of $\mathcal{C}^\infty_M$ (in point of fact, of M itself!) as they are built into that classical structure sheaf of differentiable functions, and on the other, that the manifold based

[110] Structure functions that define (the base topological space) X as a differential manifold M, placing it thus within the category of smooth manifolds.

[111] Of function-like entities which the theorist declares (assumes) up-front as being 'differentiable'.

[112] This principle has been already anticipated in [272].

CDG breaks down at (*ie*, it proves to be grossly inadequate for coping with) them. Hence, in this light we vindicate what we have been arguing for in the past two sections, namely, that

(R3.11) it is not gravity *per se* (*ie*, the physical law that the gravitational field obeys) that carries within it the seeds for its own destruction (Q?.?, Q?.?, Q?.?), but rather that CDG itself—which is geometrically effectuated (or expressed) via the background spacetime manifold—hosts its 'autocatastrophe'. At the same time, GR appears to set its own limit of validity by predicting singularities exactly because it relies on Calculus (*ie*, on base manifolds) for its (differential) geometrical expression (modelling). *In toto*, in a philological, self-referential sense, *it is not that 'gravity collapses under its own weight', but rather, that it is the manifold that breaks down due to its own $\mathcal{C}^\infty$-smooth foundation; singularities are not pathologies of the gravitational law (field), but of the (differential) geometrical medium (background) employed by CDG—the differential manifold M—in order to express that law differential geometrically, as a differential equation.* Thus, it is the mathematics to 'blame', not the physics.[a]

[a]Alas, unfortunately the maths gets intimately entwined and 'confused' with the physics that it is supposed to model, since one assumes up-front that *physical spacetime* is (modelled after) a smooth manifold supporting a smooth metric satisfying the differential equations of Einstein.

On the other hand, it is well appreciated by now that the smooth coordinates labelling the points of M do not have a direct metrical meaning; hence, since the smooth spacetime metric is the sole dynamical variable in GR as originally formulated by Einstein, the coordinates play no role in the gravitational dynamics, thus they have no physical significance either (PGC).[113] We turn now to discuss in more detail precisely this issue.

3.2.1 The metric insignificance of coordinates in GR

Here we would like to recall that Einstein himself 'struggled' for a long time to understand the metric meaning of spacetime coordinates—that is, to clarify what is the relation from the per-

[113]This is in line with our earlier Wheelerian remark that *no theory is a physical theory unless it is a dynamical theory.* Dynamics is everything!

spective of gravitational dynamics between the smooth coordinates (in $\mathcal{C}^\infty(M)$) of the manifold's points, which *prima facie* appear to have only a kinematical significance, and the sole dynamical variable in GR, the spacetime metric $g_{\mu\nu}$ representing the ten gravitational potentials in the original relativistic theory of gravity. This quest was essentially motivated by his desire to give a precise (mathematical) formulation of the PGC, given the satisfaction of the Equivalence Principle (EP),[114] and it was eventually answered in the negative (*ie*, that in GR, the spacetime coordinates have no direct dynamical meaning), as the following quotation from [128] shows:

[114]Which, coupled to the assumption of (differential) *locality*, requires that the spacetime manifold of GR is locally flat—*ie*, that locally $g_{\mu\nu}$ reduces to the $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ of the Minkowski space of SR.

(Q3.8) "...We start with an empty, field-free space, as it occurs—related to an inertial system—within the meaning of the special theory of relativity, as the simplest of all imaginable situations. If we now think of a noninertial system introduced by assuming that the new system is uniformly accelerated against the inertial system in one direction, then there exists [according to the EP][a] with reference to this system a static parallel gravitational field. The reference system may be chosen to be rigid, Euclidean in its three dimensional metric properties. But the time in which the field appears as static is not measured by equally constituted stationary clocks. *From this special example one can already recognize that the immediate metric significance of coordinates is lost once one admits nonlinear transformations of the coordinates.*[b] *To do the latter is, however, obligatory if one wants to do justice to the equality of gravitational and inertial mass through the foundations of the theory*[c] [*ie*, the EP]...[d]
If, then, one must give up the notion of assigning to the coordinates an immediate metric [and operational!][e] *meaning (differences of coordinates = measurable lengths, or times), one cannot but treat as equivalent all coordinate systems that can be created by continuous transformations of the coordinates. The general theory of relativity, accordingly, proceeds from the following principle: Natural laws are to be expressed by equations that are covariant under the group of continuous coordinate transformations*[f]*...*"

[a]Our addition.
[b]Our emphasis.
[c]Einstein's own emphasis.
[d]Again, our addition.
[e]Our addition.
[f]Our emphasis.

while, in an even more direct, straight-out manner, Einstein justified the long time it took him to construct GR starting from SR exactly on the grounds that he had to shed first his 'prejudice' that coordinates have direct metrical significance [120]:

(Q3.9) "...Why were another seven years required for the construction of the general theory of relativity? *The main reason lies in the fact that it is not so easy to free oneself from the idea that coordinates must have an immediate metrical meaning...*"

Thus, in GR, the spacetime coordinates have no (dynamical) metrical significance, and one would intuit that their role in the theory is relegated merely to a 'kinematical' one. Albeit, in the sequel we will argue that their role is not just kinematical either; it is a deeper one, having to do with the very classical differential geometric (in fact, analytic) foundations of the theory. We discuss this point now.

3.2.2 $\mathcal{C}^\infty$-coordinates, 'geometrical points' and intervening space(time) are sacrosanct in CDG and GR for differentiability's sake

We shal start with some remarks of Chern from [81] and [82], respectively:

(Q3.10) *"...From the proliferation of coordinate systems it is natural to have a theory of coordinates. General coordinates need only the property that they can be identified with points; i.e., there is a one-to-one correspondence between points and their coordinates—their origin and meaning are inessential..."* [81]

culminating in a 'definition' of '*geometric objects or properties*':

(Q3.11) *"...A property is geometric, if it does not deal directly with numbers or if it happens on a manifold, where the coordinates themselves have no meaning..."* [82]

After having established in 2.2.1 that the smooth coordinates of the differential spacetime manifold have no direct metrical significance in GR, hence also since $g_{\mu\nu}$ is the sole dynamical variable in that theory, that they play no dynamical role either, one is tempted to ask what exactly is the 'operative' role played by the smooth coordinates of the spacetime manifold's points in the relativistic field theory of gravity.

The answer we suggest is that, although the coordinates of the $\mathcal{C}^\infty$-smooth spacetime manifold M, as encoded in $\mathcal{C}^\infty(M)$, have no dynamical meaning in the theory, they play a vital role in setting up the theory within a CDG-framework. That is, since one wishes, as Einstein did for the sake of locality,[115] to model physical laws—in particular, the law of gravity—by *differential equations*, and since as it has been mentioned emphatically earlier, the differential spacetime manifold M, as

[115] See below.

a pointed space, is nothing else but $\mathcal{C}^\infty(M)$ (or the structure sheaf $\mathcal{C}^\infty_M$ thereof) (1),[116] the latter algebra provides the very differentiability property for the metric in GR[117] so that it can partake into the dynamical *differential* equations of Einstein in the first place.

Another way to say this is that the smooth coordinates (in essence, M itself) provide the *the vital 'precondition' of differentiability or 'smoothness' for GR—ie*, so that one can apply the methods and techniques of Calculus on Manifolds (CDG) in the theory of gravity, so that in turn the dynamical equations of motion of Einstein that *define* the latter (as a physical theory) are *differential equations* proper. In a metaphorical sense, while smooth coordinates (*ie*, the manifold) are not active parts in the dynamics of GR, they are crucial 'initial conditions' for it, as the assumption of a smooth spacetime manifold is posited (fixed or 'postulated') up-front by the theorist in order to place the theory in the (familiar) CDG-framework, thus represent physical laws (here, the Einstein equations) by differential equations within the latter (and secure locality as we will see soon).

These arguments then apply *mutatis mutandis* for the points of the manifold, which can also be recovered by Gel'fand duality from $\mathcal{C}^\infty(M)$ or $\mathcal{C}^\infty_M$.[118] Once one fixes the algebra of coordinates to $\mathcal{C}^\infty(M)$ (or correspondingly, the structure sheaf to $\mathcal{C}^\infty_M$), one automatically obtains M as a pointed manifold with its 'built in' topological and differential structures.

All in all, from this viewpoint, points and the smooth coordinates labelling them appear to be sacrosanct in GR, and, *in extenso*, in any theory that wishes to represent the dynamical laws that define it by differential equations. A corollary is, of course, that CDG (Calculus on Manifolds) is vital for the (mathematical) formulation of GR.

Smoothness at the service of locality. To recapitulate, coordinates (and points) are neither dynamical nor kinematical proper entities in GR. Of course, one wishes to secure *ab initio* 'smoothness' in one's theory in order to implement the 'classical' notion of locality (Q?.?), which in the CDG (classical spacetime continuum) context can be expressed as follows:

[116] Some clarifying technical remarks are due here: underlying the 'identification' (better, equivalence) in (1) is the general notion of *Gel'fand duality*—whose usual application is the so-called Gel'fand representation theory—according to which, M, as a pointed set endowed with its usual topology and differential structure, is the (real) Gel'fand spectrum $\mathfrak{M}$ of the topological algebra $\mathcal{C}^\infty(M)$, or its sheaf $\mathcal{C}^\infty_M$ (write: $\mathfrak{M}(\mathcal{C}^\infty(M))$ or $\mathfrak{M}(\mathcal{C}^\infty_M)$) [254, 258, 259, 272]. This duality is supposed to be captured by the amphidromous arrow in (1).

[117] Again, whose components in a local frame are elements of $\mathcal{C}^\infty(M)$ (*ie*, smooth functions) since it is by definition a $\otimes_{\mathbf{A}\equiv\mathcal{C}^\infty_M}$-tensor.

[118] One may wish to recall that the points $p \in M$ are represented by maximal, or Zariski prime (since $\mathcal{C}^\infty(M)$ is a regular topological algebra), ideals in $\mathfrak{M}(\mathcal{C}^\infty_M)$ [254, 258, 259, 272]; see also 4.2.3 for an extended discussion of the application of Gel'fand duality to $\mathcal{C}^\infty(M)$ in order to recover the $\mathcal{C}^\infty$-smooth manifold M as a pointed differential continuum.

(R3.12) physical processes, corresponding to dynamical (field) actions, connect infinitesimally separated point-events—a statement equivalent to the assumption that the physical laws of dynamics should be (modelled after) differential equations [270, 271, 272].

As noted in the introductory section, Einstein, for example, was well aware of the fact that the *a priori* conception and assumption of spacetime as a continuum (manifold), with the differentiability (smoothness) properties that the latter offers in order to implement the 'classical' notion of (infinitesimal) locality, is the last relic of an ether-like, absolute structural element in GR—a structure which, in a philological sense, 'acts, but is not acted upon'[119] [116, 118, 270, 272].

How come not a 'variable' differentiability? In this paragraph we would like to address the issue of still not possessing (or even having intuited yet) in theoretical physics a 'variable' notion of differentiability or 'smoothness'. The following discussion will anticipate the ADG-theoretic '*algebraic relativization of differentiability*', culminating in the *Principle of Algebraic Relativity of Differentiability* (PARD), to be discussed in detail in 3.2 subsequently.

Our initial considerations in this respect are motivated the following tower of structures 'postulated' in GR:[120]

[119]'Acts, but is not acted upon' here meaning, as explained above, that differentiability (*ie*, the differential structure of the spacetime continuum) is a vital precondition for setting up the dynamics (as differential equations) in GR, but this dynamics does not in turn affect it in any way, as 'differentiability' is fixed up-front by the theorist to the classical notion of $\mathcal{C}^\infty$-smoothness of CDG—a notion which is part and parcel of the algebra $\mathcal{C}^\infty(M)$ of coordinates labelling the manifold's points.

[120]The following diagram is borrowed almost unaltered from [147]. We wish to thank David Finkelstein for allowing us to do so.

$$\begin{array}{c}
\boxed{\textbf{Level 4. Quantization} \overset{?}{\Rightarrow} \textbf{Quantum General Relativity}}\\
\uparrow\\
\boxed{\textbf{Level 3. } \mathcal{C}^{\infty}-\textbf{Smooth Lorentzian Metric Structure}}\\
\uparrow\\
\boxed{\textbf{Level 2. Classical } \mathcal{C}^{\infty}-\textbf{Differential Structure}}\\
\uparrow\\
\boxed{\textbf{Level 1. Classical } \mathcal{C}^{\mathbf{0}}-\textbf{Manifold Topology}}\\
\uparrow\\
\boxed{\textbf{Level 0. Classical Set Theory and its Logic (Continuum Cardinals)}}
\end{array} \tag{8}$$

Finkelstein used this diagram in [147] to make the following point: the attempts to arrive at a genuinely quantum theoresis of spacetime and gravity by a direct quantization of the smooth metric field on a pointed spacetime manifold, as for example the various attempts in the context of the so-called Quantum General Relativity (QGR)[121] aspire to arrive at QG, appear to be 'superficial', 'non-economical' and 'contrived' since only the 'surface structure' of GR is involved (*ie*, the smooth metric at level 4 in (8)), but the deeper ones, such as the differential and topological ones, let alone the set-theoretic (logical) one at the bottom, are *a priori* fixed to the ones of the classical picture—*ie*, the classical spacetime continuum (manifold) regarded as a classical (albeit uncountable) point-set. Accordingly, he favors a 'bottom-up' approach to quantum spacetime structure and dynamics whereby, on *ab initio* manifestly finitistic and quantal grounds, one should first attain a quantum relativistic picture of set theory and its logic (with a 'discrete' version of relativistic causality

[121] That is, generally speaking, the application of the usual continuum based QFT formalism, canonically (via Hamiltonian methods) or covariantly (via Lagrangian, action methods), to GR. Such an approach to QG, especially via the Lagrangian, action based way, would in general be favored by particle physicists who use predominantly field-theoretic methods, rather than by general relativists who by and large use classical differential geometric methods [140, 410], although after the advent of Ashtekar's new (self-dual connection) variables approach to GR [7], as well as the loop variables approach to QG based on smooth holonomies of Ashtekar's connections [330, 30, 382, 383, 351], differential geometric methods (on the moduli spaces of those connections) have been revived in the context of QGR [14, 15, 16, 8, 9, 10].

still in force), and then build on it the higher level spacetime structures such as the topological for instance [145, 146, 150], an ambitious endeavor which eventually developed into a full-fledged Quantum Causal Net and, ultimately, Quantum Relativity theory [148]. Finkelstein's basic thesis is essentially that the deeper structures of GR, not only the top level metric one, should be subjected to some kind of 'quantization' and 'relativization', and altogether starting from scratch—*ie*, from the logical one at level 0 in (8).

En passant, and in view of the essentially pointless algebraico-categorical ADG-machinery supporting a background spacetimeless ADG-field theory in general, and ADG-gravity in particular, as it will be exposed in great detail in the coming sections, it should be noted here that Finkelstein already in his pioneering paper [142] (which laid the foundations for and marked the commencement of his Quantum Set and Net Theory, as well as his inspired Quantum Relativity Theory subsequently) had emphasized the 'unreasonableness' of the spacetime manifold supported GR, as well as any attempt at directly quantizing its 'surface' (metric) structure in order to arrive at QGR (regarded as a QFT on the classical spacetime continuum) and QG. As we read below, what he set out to do in [142] was 'antipodal' to the tower of structures in (8) above:

"...This approach [*ie*, the 'Space-Time Code'],[a] which attributes to space-time points an intricate internal structure, seems upside down from the point of view of general relativity, and general relativity seems upside down from here, seems too complicated a theory of too simple a thing to be fundamental rather than phenomenological. *What is too complicated about general relativity is the delicate vertical structure of laws that would have to be legislated on the first day of creation: set theory, holding up topology, holding up differential manifolds, holding up pseudometric geometry, with a precarious topping of quantization.*[b] *What is too simple about general relativity is the spacetime point. It looks as if a point*
(Q3.12) *might be an enormously complicated thing. Each point, as Feynman once put it,*[c] *has to remember with precision the value of indefinitely many fields describing indefinitely many particles; has to have data inputs and outputs connected to neighboring points; has to have a little arithmetic element to satisfy the field equations; and all in all might just as well be a complete computer*[d]... "

[a]Our addition for textual continuity.

[b]See again (8) above.

[c]See quotation (Q7.54) taken from [137] in section 7 and comments therein in the light of our ADG-theoretic perspective on (gauge) field theory and especially (quantum) gravity.

[d]Our emphasis.

Leaving Finkelstein's reticular 'quantum set theory' at the deepest, bottom logical level in (8) aside for a moment, it should be mentioned that, starting from Wheeler's original foam conception of a 'quantal', dynamically fluctuating spacetime 'microtopology' at sub-Planckian scales [416], theoretical physicists have worked on various finitistic and algebraic 'quantization' scenarios for spacetime topology [202, 203, 150, 318, 319], with Isham's recent 'quantizing on a general category' scenario [209, 210, 211] being the category-theoretic evolution of the 'group quantization of spacetime topology' (*à la* Mackey's imprimitivity) ideas put forth initially in [201] and further elaborated subsequently in [203]. At a more abstract level, mathematicians too have worked on 'quantal', noncommutative conceptions of topology, with Mulvey's quantales in the context of Gel'fand representation theory for non-abelian C^*-algebras [281, 283, 284], as well as Van Oystaeyen's noncommutative topologies in the context of noncommutative algebraic geometry [404, 401, 402, 403], being the two most prominent paradigms in our view.

At the next level of *differential* structure (level 2 in (8))—in which we are predominantly interested in the present paper, while, as noted above, physicists have intuited that both in the classical context of GR [198], but more importantly, in a genuinely quantum theoresis of spacetime structure and dynamics [416], one's theory should be able to account for dynamical processes of spacetime topology change, no one has intuited yet similar changes in the differential structure of spacetime.[122] Presumably, this is because the smooth manifold provides the sole theoretical framework (CDG) we possess so far within which we know how to differentiate, and the automatic, almost 'reflex'-attitude of the theorist is to assume and fix it up-front in her theory. This 'smooth manifold monopoly' and associated 'CDG-solipsism'—especially in theoretical physics where CDG methods and applications reign supreme, with impressive experimental success and support(!), from the microcosm of the Standard Model (SM) to the macrocosm of GR—is the epitome of the manifold conservative attitude criticized throughout this paper, which attitude, in turn, as mentioned earlier, appears to be mandatory if one wishes to represent the laws of physics by differential equations proper.

Prima facie, one could readily counter the aforesaid almost religious commitment to the (commutative)[123] spacetime manifold by bringing up the by now well developed and quite popular Noncommutative (Differential) Geometry (NDG) of Connes [91, 222] already enjoying numerous applications in high energy (SM) theoretical physics [79, 80] and potentially in (quantum) spacetime structure and gravity research [92].[124] However,[125] NDG, *à la* Connes, may be perceived as an attempt to '*quantize Calculus*' by functional analytic means, *while the notion of manifold still survives in it, fixed as it were at the background*, of course, with all the classical differential geometric anomalies and diseases in the guise of 'singularities' that such a 'manifold conservatism' entails. On the other hand, the whole algebraic (categorical), being sheaf-theoretic, machinery of ADG has been developed independently of any notion of base differential manifold [259]. Being free of any smooth manifold dependent concept and construction, ADG is able to cope with problems pertaining to 'singularities' still by applying methods of NLPDEs ('generalized functions').[126] It is also worth noting here the conceptual simplicity of the machinery of ADG, as well as that, in

[122]And this ellipsis is kind of 'short-sighted' in our opinion, since the differential is normally regarded as a higher level structure than the topological (8). In other words, if 'continuity' should be somehow 'quantized' and 'relativized'—*ie*, be regarded as a non-fixed, *variable structure* that partakes into dynamics [148], why not also 'differentiability' or 'smoothness'?

[123]Commutativity here pertains to the fact that $\mathcal{C}^\infty(M)^{\mathbb{K}}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) is an abelian algebra.

[124]It must be noted at this point that NDG appears to be particularly 'popular' nowadays in the (super)string-theoretic approach to QG, for which however we provide no direct references herein. The reader can go via [328] to find references about how NDG appears to be of import in current string theory research.

[125]And this is meant as a mild critical remark about NDG *vis-à-vis* ADG and what we are trying to accomplish in the present paper.

[126]Indeed, for a similar *en passant* critique of NDG on the face of ADG's successful application to the so-called spacetime foam dense singularities associated with Rosinger's algebraic theory of non-linear generalized functions (distributions), the reader is referred to [274]. We will revisit in detail this application of ADG in subsection 4.2.

general terms, resorting to noncommutativity,

- "*does not mean a loss of interest in, let alone, the abandonment of commutative theories. Indeed, in many important problems, the latter turns out to be both more simple and far more effective*" [274].[127]

- Related to the point above, (differential geometric) theories are noncommutative "*only on their so-called 'global' level, while in the last instance of their detailed calculations, that is, 'locally',*[128] *they reduce to commutative ones*" [274].

- The latter point, and from a quantum-theoretic perspective, becomes even more important *visà-vis* concrete calculations from direct application of differential geometric ideas in physics if one recalls Bohr's correspondence principle according to which, the values of our measurements of the so-called 'observables' of quantum systems—the noncommuting in general q-numbers of quantum mechanics—are 'classical', commutative c-numbers [274, 270, 271, 272].

- Finally, even more incisive is Finkelstein's remark that "*noncommutativity is by no means the quintessential characteristic of 'quantumness'; rather, the essence of the latter is 'relativization' and dynamical variability of physical attributes*".[129] Equivalently, *noncommutativity does not necessarily imply 'quantumness'.*[130]

After this short critical interlude about NDG, returning to (8) with its 'differential manifold monopoly' and the associated 'CDG solipsism' at level 2, we would like to prepare here the ground for (a) the ADG-theoretic 'dynamicalization' of coordinates and spacetime (points),[131] and (b) the associated Principle of Algebraic Relativity of Differentiability (PARD), both to be expounded in more detail in subsection 3.2.

[127]Recently for example, Jet Nestruev (*ie*, Vinogradov *et al.*) [285] have been rather critical of noncommutative theories as regards their conceptual and effective calculational import in the theory of PDEs.

[128]And it should be emphasized that 'differentiability' is, by definition, a local notion.

[129]David Finkelstein quoted roughly from an old private communication with the second author. In other words, 'quantization' (if one would still want to use this term in a theory such as Quantum Relativity, and also our approach [272], in which it is fundamentally assumed that "*all is quantum*") is '(quantum) relativization' entailing dynamical variability [148].

[130]For instance, think of the non-commuting boosts in SR, which is a classical theory.

[131]By the term 'dynamicalization' of coordinates and spacetime (points) we mean the dynamical 'consequence' of 'spacetime'—in other words, that spacetime is the result of the gravitational dynamics, not fixed up-front to a manifold as in the classical theory, and that this dynamical dependence is in turn due to a process of 'relativization of differential structure' as the latter is effectively encoded in the structure sheaf $\mathbf{A}$. The latter is essentially the PARD alluded to before.

3.2.3 Anticipating the central field-axiomatic of ADG: 'field-solipsism' and the associated Principle of Field Realism (PFR)

At this point, one must highlight and admire the intuition, imagination and resourcefulness of the physicist in attempts to transcend the technical (mathematical) rigidity and monopoly of the differential spacetime manifold. For example, ever since the first 'toddling' steps of GR, Einstein had intuited that the point-events (and, *in extenso*, their coordinate labels) of the spacetime continuum are the 'results' of gravitational dynamics—in a strong sense, *spacetime is 'inherent' and foreshadowed in the dynamical gravitational field itself*, and that the (empty) spacetime continuum, devoid of (the dynamics of) the gravitational field, is a physically meaningless concept and structure. Characteristically, we quote:[132]

(Q3.13) "...According to general relativity, the concept of space detached from any physical content does not exist. *The physical reality of space is represented by a field whose components are continuous functions of four independent variables—the coordinates of space and time. It is just this particular kind of dependence that expresses the spatial character of physical reality.*[a]" [235][b]

[a]Our emphasis.
[b]Page 175 and reference therein.

while

(Q3.14) "...If the laws of this field are in general covariant, that is, are not dependent on a particular choice of coordinate system, then the introduction of an independent (absolute) space is no longer necessary. That which constitutes the spatial character of reality is simply the four-dimensionality of the field. *There is no 'empty' space, that is, there is no space without a field.*[a]" [235][b]

[a]Again, our emphasis.
[b]Again, page 175 and reference therein.

as well as

[132]The following quotations can be also found in [272], but due to their importance we recast them here.

(Q3.15) "*...No space and no portion of space can be conceived of without gravitational potentials; for these give it its metrical properties without which it is not thinkable at all. The existence of the gravitational field is directly bound up with the existence of space*[a]*...*" [116]

[a]Emphasis is ours.

and

(Q3.16) "*...according to the general theory of relativity even empty space has physical qualities, which are characterized mathematically by the components of the gravitational potential.*[a]" [235][b]

[a]That is, even in the vacuum Einstein equations, it is the dynamical gravitational field that gives spacetime its physical identity. We will return to elaborate more on this in the final section when we discuss Einstein's hole argument.

[b]Again, page 111 and reference therein.

moreover,

(Q3.17) "Thus, once again 'empty' space appears as endowed with physical properties, *i.e.*, no longer as physically empty, as seemed to be the case according to special relativity. One can thus say that the ether is resurrected in the general theory of relativity, though in a more sublimated form." [235][a]

[a]Page 111 and reference therein.

so that we come to the 'apotheosis' of this '*no gravitational field, no spacetime*' apofthegma of Einstein:

(Q3.18) "*...There is no such thing as empty space, i.e., a space without field. Space-time does not claim existence on its own, but only as a structural quality of the field*[a]*...*" [124]

[a]Our emphasis.

which can be augmented by his later remark that

(Q3.19) "*...space has lost its independent physical existence, becoming only a property of the field*[a]*...*" [125]

[a]Again, our emphasis.

Subsequently, Bergmann and Komar, working on (a possible extension of) the diffeomorphism symmetries of GR, which implement the PGC in that differential spacetime manifold based theory [43], were led to the conclusion that points (and their coordinates) are not independent structural entities in GR, that they have no actual physical meaning, but rather that in a sense they derive their 'identity' from the field equations themselves:[133]

(Q3.20) "...Such transformations might point in a direction in which *fields get unified under an invariance group in which the space-time manifold no longer plays a pre-eminent role. A world point would derive its identity from its dynamic environment, or it might possess no identity at all.*[a]" [42]

[a]Emphasis is ours.

While, the climax of this 'dynamicalization' of spacetime points and their coordinates appears to be the principal aim in Stachel's latest work, as we read from [374]:

[133]Peter Bergmann's life-long work on the role of Diff(M) in GR has repeatedly emphasized that points (and coordinates) are not absolute, 'sacrosanct', self-sustaining entities in the theory, but rather that they are 'consequences' of the dynamical symmetries of the theory (David Finkelstein in private e-correspondence).

"...I shall then emphasize the ways in which the space-time structure of the general theory of relativity, by virtue of its dynamization[a] of the chrono-geometry, differs radically from that of all previous physical theories. I shall stress two fundamental differences...
2. the role of diffeomorphism invariance in precluding the existence of any pre-assigned (kinematical) spatio-temporal properties of the points of the manifold (even locally) that are independent of the choice of a solution to the field equations (no kinematics before dynamics).[b] *The physical points of space-time thus play a secondary, derivative role in the theory, and cannot be used in the formulation of physical questions within the theory (they are part of the answer, not part of the question)*[c]*...*[d]"

(Q3.21)

[a]Here we use the slightly different term 'dynamicalization' instead of Stachel's 'dynamization' in order to prepare the reader for a significantly different ADG-theoretic stance about these issues, to be expounded in the sequel (*cf.* last section when we discuss Einstein's hole argument motivated Stachel's extensive work on it).

[b]The reader should wait for our arguments about the ADG-theoretic reversal, from '*kinematics-before-dynamics*', to '*dynamics-before-kinematics*', to be given in 3.2.3. Note also the comments about 'differentiables'—properties (better, 'initial conditions') pertaining to the 'differentiability attributes' of the structures involved in the theory.

[c]Our emphasis.

[d]From our point of view, this is not quite; rather: *they are part of the 'initial (pre)conditions for differentiability'*, which will be relativized ADG-theoretically below (again, see subsection 3.2). 'Differentiables' lie with the 'observer'/'measurer', who, like initial conditions, chooses what **A** to employ, thus differential geometrize her theory.

All this points at the following *'axiom' of field-solipsism* in ADG:

(R3.13) In ADG, the concept of (gravitational) field—represented by the pair $(\mathcal{E}, \mathcal{D})$[a]—is a fundamental, 'ur', primitive notion. *There is no background spacetime structure;*[b] *all there is and is of physical significance are the 'geometric objects'—ie, the fields themselves.* In an Aristotelian [5, 6, 306], but more explicitly, Leibnizian sense, the fields, like Leibniz's monads [246, 247, 384], are autonomous, self-sustaining entities, without being in need of an external, background spacetime for their subsistence, but unlike monads, our fields are not 'windowless', as they interact with each other. Thus, *the notion of field in ADG comes to replace that of the spacetime manifold and its point-events in the classical theory of gravity (GR). In a strong sense, the 'events' in our theory are the fields (in) themselves, and there is nothing else except fields and the dynamics that they define as differential equations.*

[a] This is a 'superfluous' representation. Strictly speaking, for us the field (of gravity) is simply the connection $\mathcal{D}$, but since the latter is a map—*ie*, the $\mathbf{C}$-linear sheaf morphism $\mathcal{E} \longrightarrow \mathcal{E}(\Omega) \equiv \mathcal{E} \otimes_{\mathbf{A}} \Omega$—we also include for completeness its domain $\mathcal{E}$ (or even its range $\Omega := \mathcal{E}^*$ [272]).

[b] Or rather, this background surrogate 'space(time)'—whether a continuum or a discretum—plays no role whatsoever in the algebraic differential geometric mechanism offered by ADG.

All in all, and in contradistinction to Einstein's GR, in which too "[In the theory of relativity,] *the field is an independent, not further reducible fundamental concept...and the theory presupposes the independence of the field concept*" [125], our 'solipsistic' field

(R3.14) *does not depend at all on a background spacetime continuum (manifold) for its support and (differential geometric) subsistence.*[a]

[a] See concluding section.

In [272] we argued that the aforesaid 'field-solipsism' sets the stage for the ADG-theoretic formulation of a *genuinely 'unitary' field theory—one dealing solely with the fields themselves and which therefore is in no need of an external (background) geometrical spacetime, whether the latter is modelled after a continuum (manifold) or a discretum.*[134]

[134] Again, see the concluding section 7 for further discussion of the possibility that ADG, with its fundamental field-solipsism, is an appropriate mathematical framework in which to materialize Einstein's unitary field vision.

3.3 Section's Résumé

Although there is a plethora of issues raised in the past section that we could summarize now, here is a short list of points made that we would like the reader to keep in mind as the paper unfolds:

1. $\mathcal{C}^\infty$-smooth singularities in GR are inherent in $\mathcal{C}^\infty_M$; in effect, they are built into M itself. One cannot think of gravitational singularities apart from the assumption (mathematical model) that (physical) spacetime is a differential manifold.

2. Singularities signify a breakdown and end of the manifold based CDG (Calculus or Analysis), not of the dynamical gravitational field laws of GR, but since the latter is inextricably tied to the base spacetime manifold, we are misled into thinking that singularities are pathologies of the physical theory itself (*ie*, of the gravitational dynamics—Einstein differential equations) and not just 'anomalies' of the mathematical model for spacetime (M) and of the general differential geometric framework fundamentally based on it (CDG).

3. From the ADG-theoretic perspective, all singularities are virtual, 'coordinate' ones, as they are part and parcel of the coordinate structure sheaf $\mathbf{A}$ one uses in the theory (in the classical case, $\mathbf{A} \equiv \mathcal{C}^\infty_M$).

4. Spacetime (manifold) points and their (smooth) coordinates are not fundamental, that is to say, physical (*ie*, dynamical) entities in GR. Traditionally, they (*ie*, the pointed manifold and the smooth coordinates labelling its points) are thought of as being part of the kinematics of GR. Here we regard them as vital structural preconditions for setting up the physical theory differential geometrically in the first place. At the same time, if there is any 'space(time) geometry' at all, then it is inherent in $\mathbf{A}$ that *we* assume in order to (differential) geometrically represent the gravitational field $\mathcal{D}$.

5. What is normally regarded as being 'irreducible' in the smooth manifold and *in extenso* CDG-based GR, namely, the manifold's points and the gravitational field g defined on them—totally known as '*spacetime events*', in ADG-gravity is replaced by the 'it ur'-notion of field, namely, a connection $\mathcal{D}$ on a vector sheaf $\mathcal{E}$.[135] Moreover, the causal nexus between events—which in the standard GR is supposed to be a dynamical variable[136]—is replaced in ADG-gravity by the dynamical law (vacuum Einstein equations) that the (vacuum) ADG-gravitational field $\mathcal{D}$ defines on $\mathcal{E}$.[137] As a pun-type of metaphor, the dynamical gravitational

[135] Equivalently, the ADG-gravitational field is the pair $(\mathcal{E}, \mathcal{D})$ [272].

[136] Since $g_{\mu\nu}$—the sole dynamical variable in the original formulation of GR due to Einstein—defines (locally) the Minkowski lightcone, which in turn delimits the local causal relations between spacetime points (events).

[137] See (29) below.

connection field $\mathcal{D}$ represents in ADG-gravity the dynamical causal connection [270], and the primitive notion of causality is replaced by *the* fundamental ADG-notion of (algebraic) connection. *Connection* $\mathcal{D}$ *is causality.*

4 GR from the ADG-Theoretic Point of View

We start this section with the simple but fundamental observation that in order to do differential geometry one needs to possess some sort of *differential operator* d. There is no DG proper without d. In turn, to be able to differentiate one needs somehow to *form differences* (*ie*, possess some kind of *linear structure*), as well as to *take limits* (*ie*, possess some sort of *topology*). Thus, *prima facie*, all one needs to have in hand in order to differentiate (*ie*, to define a differential d) is a type of *topological vector space* structure. On top, if one further needs one's derivative d to obey a Leibnizian condition (*ie*, behave in a particular way regarding products of 'functions'—the sort of 'animals' on which d acts), one should possess a kind of *topological algebra* structure, with $\mathcal{C}^\infty(M)$ (and the coordinate structure sheaf $\mathcal{C}^\infty_M$) underlying differential manifolds M and on which the entire CDG edifice is based (1) being the archetypal example of a 'non-normable' topological algebra (structure sheaf). All in all,

(R4.1) *'differentiability' is a topologico-algebraic notion.*

Comment: It is important to note here that ADG's original aim was to formulate the central differential geometric notion of *connection* (*viz.* 'generalized differential operator') *entirely algebraico-categorically*, in a way that explicitly does not depend at all on a background geometrical $\mathcal{C}^\infty$-smooth manifold (thus show that the classical case of a smooth connection arises only when one chooses $\mathcal{C}^\infty_M$—*ie*, a base differential manifold (1)—for one's structure sheaf of generalized 'coordinates' or 'coefficients').

(R4.2) *Alas, one can get a differential* d *(or a generalized differential* $\mathcal{D}$*), thus a generalized notion of differentiability or 'smoothness', from structure sheaves glaringly different from* $\mathcal{C}^\infty_M$ *—and more importantly, from ones that may be teeming with singularities or ones that are more suitable for addressing and tackling one's (physical) problem at issue.*

What is important to stress here is that, in effect, ADG's main aim is to '*algebraicize Calculus*'. That is to say, in view of the topologico-algebraic character of d mentioned above, in ADG there is the tendency to underplay and 'atrophize' the 'spatial', topological (:geometric) aspect (and dependence) of Analysis, and to pronounce its purely algebraic aspect. Indeed, as we shall see below, the arbitrary base topological space X on which the algebra and differential module sheaves of interest are soldered plays no role in the *'inherently' algebraic* differential geometric mechanism of ADG, which, as it were, derives from the stalk of the said sheaves—*ie*, from the algebraic structures inhabiting the relevant sheaf spaces. But it is high time now to expound the basic tenets of ADG.

4.1 Rudiments of Abstract Differential Geometry: The ABC of ADG

In this section we present the basic concepts and mathematical structures involved in ADG. We begin by giving several *physico-mathematical* 'shortcomings' and 'philosophical blemishes' of differential (*ie*, $\mathcal{C}^{\infty}$-smooth) manifolds and, *in extenso*, of the CDG based on them, 'inadequacies' which could be regarded *a posteriori*[138] as 'reasons' for developing ADG, as a mathematical theory with potent physical applications (and being supported by sound conceptual/philosophical foundations), in the first place, although the principal *a priori* (mathematical) motivation of the first author for constructing the theory will be clearly highlighted and distinguished from those 'lateral', 'by-the-way' and secondary reasons. The following exposition is taken mainly from [315, 317].

4.1.1 Manifold reasons against the (spacetime) manifold: 'lateral' mathematical and physical motivations for developing ADG

Mathematical 'deficiencies' of manifolds. As in [315, 317], we follow mainly Papatriantafillou [289, 290, 291, 292] and first list some mathematical shortcomings of (the category $\mathcal{M}an$ of) finite-dimensional smooth manifolds:

1. An arbitrary subset of a (smooth) manifold is not a (smooth) manifold. In categorical terms, *$\mathcal{M}an$ has no canonical subobjects*. For example, a curve with a corner or a cusp in a differential manifold M is not itself a smooth submanifold of M.

2. The quotient space of a manifold by an (arbitrary) equivalence relation is not a manifold. Consider for instance the 'discrete' T_0-poset obtained by a suitable equivalence relation from a continuous manifold M relative to a locally finite open cover of the latter in [355].[139]

[138] That is, now after ADG has been developed, applied in both mathematics and physics, and contrasted against CDG.

[139] This example will be of special interest to us subsequently (section 5), when we 'resolve' the inner Schwarzschild singularity by finitistic-algebraic, ADG-based means.

3. In general, $\mathcal{M}an$ is not closed under projective (inverse) or inductive (direct) limits.[140] In categorical terms, *$\mathcal{M}an$ is not bicomplete*. To be sure, in general, the projective limit of an inverse system of finite-dimensional manifolds is not a manifold; while, in certain cases, as in the theory of jets, where the inverse limit exists, it is infinite dimensional.

4. $\mathcal{M}an$ has no initial or final structures. For instance, one cannot pull-back or push-out a smooth atlas by a continuous map. As a consequence, there is no what one might coin 'differential geometry of topological spaces'.

5. In general, *there are no categorical products and co-products in $\mathcal{M}an$*.

Physical shortcomings of manifolds. We have already alluded to the physical 'anomalies' in the guise of singularities and associated unphysical infinities of smooth fields on a spacetime modelled after a $\mathcal{C}^\infty$-continuum. We briefly recapitulate them below and add a few more physically problematic features of differential manifolds:

1. In GR, singularities are due to our *a priori* assumption of modelling spacetime after a pointed 4-dimensional differential manifold M.

2. But then, anyway, there is no well defined notion of singularities in GR, if anything, because one needs Diff(M) to implement the PGC within the confines (and limitations!) of a smooth spacetime framework (CDG). We may coin this '*the Geroch singularities-versus-differential spacetime manifold vicious circle effect*' [155, 317].

3. Similarly, in Quantum Field Theory (QFT), the weaker, but still troublesome unphysical infinities of the quantum fields of matter may be ultimately attributed to the fact that Minkowski space is, after all, a pointed $\mathcal{C}^\infty$-smooth manifold in which one can pack infinite degrees of freedom (field modes) in a finite spacetime volume.

4. In QG, string theory aside, the occurrence of a 'natural' length scale—the so-called Planck length/time ($\ell_P = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-33} cm$), and dually, momentum/energy—could be interpreted as indicating that the smooth continuum model for 'macroscopic' spacetime becomes inadequate for studying very energetic processes such as those expected to underlie quantum gravitational effects. In section 6 we will comment more on the potential evasion of ℓ_P by the background manifold-free approach to DG that ADG offers, as regards potential ADG-applications to QG.

[140] In category-theoretic jargon, these are known as categorical limits and colimits, respectively.

5. And finally, due to some robust problems that various smooth manifold and CDG-based (canonical or covariant) approaches to QG have, such as the *inner product problem* and *the problem of time*, both of which essentially involve $\mathrm{Diff}(M)$)—the 'structure group' of the $\mathcal{C}^\infty$-smooth spacetime manifold M [272]—the current 'trend' or 'tendencies' in QG research (especially in the loop QG approach to QGR) is to *develop the theory in a manifestly and genuinely background spacetime manifold independent way* [8, 9, 11, 12, 20, 351, 382, 383]. Again, we will return to comment more on these issues in section 6.

In summa, *vis-à-vis* the envisioned physics of the 'quantum deep', where quantum theory and GR are united into a conceptually sound and consistent (and moreover, a calculationally finite!) QG, theoretical physicists have time and again questioned and criticized explicitly the smooth spacetime continuum and, inevitably, the whole edifice and technical panoply of CDG that goes hand in hand with it.[141]

4.1.2 A brief prehistory of ADG culminating in ADG

To account briefly about different ideas that preceded the inception and development of ADG, we follow mainly [259]. Thus, *circa* 1970, some of the aforesaid differential manifold 'deficiencies' led to the theory of *differential spaces* [345, 346, 280] in which new classes of smooth functions, generalizing those in $\mathcal{C}^\infty(M)$ defining a smooth manifold M, were considered. In theoretical physics, this theory was applied, by also using sheaf-theoretic means, to GR, singularities, and to some extent in QG research, by the Polish school of Heller *et al.* [188, 189, 190, 191, 192, 193, 194]. It is fair to say that the original impetus along this algebraic line of approach to GR was given in Geroch's celebrated paper [157].

In the late 80s–early 90s, and quite independently of the aforementioned antecedents, the first author conceived of an entirely algebraic (:sheaf-theoretic), axiomatic scheme for doing differential geometry without at all the use of any Calculus, or what amounts to the same, without using at all any base differential (smooth) manifold. This scheme was subsequently developed into a full-fledged theory, which was coined ADG or '*Geometry of Vector Sheaves*'—the title of a recent monograph [259, 260].

In retrospect, one can argue that the central aim of ADG was originally *to arrive at an entirely algebraic notion of derivative (viz. 'connection') without any commitment to a background locally Euclidean space—or what amounts to the same, without using at all any notion of Calculus* [255, 256]. The basic observation in this direction was that *the geometry of a $\mathcal{C}^\infty$-smooth manifold M is actually deduced from the algebraic structure of its structure sheaf $\mathcal{C}^\infty_M$, thus relegating the local structure of M and the functional character of $\mathcal{C}^\infty_M$ to a secondary, auxiliary role.* Thus, the usual

[141]See for example the three quotations (Q?.?)–(Q?.?) by Einstein, Feynman and Isham in section 7.

CDG assumptions about the local structure of M, such as charts, atlases *etc*, are replaced in ADG by assumptions on the existence of an algebraic derivation on an arbitrary sheaf of algebras[142] on some arbitrary base topological space X, with the latter sheaf playing a role analogous to the structure sheaf $\mathcal{C}^\infty_M$ of (germs of) smooth functions on M in the usual CDG.

The essence of that original intuition of the first author is that *in principle any sheaf of algebras*, provided that they furnish us with a suitable differential operator d in order to be able to do differential geometry in the first place, *may be employed as the structure sheaf in ADG*—as it were, to replace the 'classical' one $\mathcal{C}^\infty_M$ defining a differential manifold M on which CDG is based. This brings us to the two key notions in ADG.

4.1.3 K-algebraized spaces and their associated differential triads: the heart and soul of ADG

The two primitive, fundamental, *ur* as it were notions in ADG are those of **K**-*algebraized space* and *differential triad*, which we now define.

Let X be an arbitrary topological space, and $\mathbf{A}_X$ a sheaf of abelian, associative, unital $\mathbb{K}$-algebras $\mathbb{A}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$)[143] on it, called the *structure sheaf*. The pair $(X, \mathbf{A}_X)$ is called a **K**-*algebraized space*, and it may be regarded as the (abstract) differential geometric analogue of the notion of a *ringed space* in algebraic geometry [178, 344].[144]

Now, a *differential triad* $\mathfrak{T}$ relative to $(X, \mathbf{A}_X)$ is the triplet

$$\mathfrak{T} := (\mathbf{A}_X, \partial, \mathbf{\Omega}^1(X)) \tag{9}$$

consisting of the structure sheaf $\mathbf{A}_X$ involved in the **K**-algebraized space $(X, \mathbf{A}_X)$ above, a sheaf $\mathbf{\Omega}^1$ of (first-order differential) $\mathbb{A}$-modules Ω^1 over X,[145] and a derivation map ∂ defined as the following $\mathbf{C} \equiv \mathbf{K}$-*sheaf morphism*

[142] Algebras which in turn may be 'non-functional'—*ie*, non-function-like (*eg*, [270, 271, 272]), or even if they are functional, possibly be far from smooth in the standard sense of $\mathcal{C}^\infty(M)$ (*eg*, [273, 274, 275]).

[143] In fact, one could in principle use any number field as linear coefficients for the elements of $\mathbb{A}$.

[144] The reader should note that the constant sheaf $\mathbf{C} \equiv \mathbf{K}$ of the real or complex numbers is embedded into $\mathbf{A}_X$ (*ie*, $\mathbf{C} \hookrightarrow \mathbf{A}_X$). It is tacitly assumed that for every open set $U \subset X$, the algebra $\mathbf{A}(U) \equiv \Gamma(U, \mathbf{A})$ of continuous local sections of $\mathbf{A}_X$ is a unital, commutative and associative algebra $\mathbb{A}$ over $\mathbb{K}$ (in fact, even globally it is assumed that: $\mathbb{A} \simeq \mathbf{A}(X) \equiv \Gamma(X, \mathbf{A})$!) [259]. Furthermore, for sheaf-cohomological purposes, $\mathbf{A}_X$ is usually assumed to be *fine* and *flabby* [259]. Fineness and flabbiness are two properties of $\mathbf{A}$ that will prove to be of great import later, in the next section, when we discuss the application of ADG to spacetime foam dense singularities [273, 274, 262, 275]. In the sequel, when it is rather clear what the base topological space X is, we will omit it from $\mathbf{A}_X$ and simply write $\mathbf{A}$.

[145] Ω^1 is the ADG-theoretic analogue of the usual sheaves of germs of smooth 1-forms over a differential manifold M—*ie*, when $\mathbf{A} \equiv \mathcal{C}^\infty_M$.

$$\partial : \mathbf{A} \longrightarrow \mathbf{\Omega}^1 \tag{10}$$

which is $\mathbf{K}$-linear and satisfies the Leibniz (product) rule

$$\partial(s \cdot t) = s \cdot \partial(t) + t \cdot \partial(s) \tag{11}$$

for any local sections s and t of $\mathbf{A}$ (*ie*, $s, t \in \mathbf{A}(U)$, with $U \subset X$ open).[146]

With the definition of the operator ∂ in $\mathfrak{T}$, we come to the central definition in ADG: that of an $\mathbf{A}$-connection $\mathcal{D}$. As noted before, the purely algebraic and background manifold-free notion of connection in ADG is the key concept in the theory.

4.1.4 A-connections and their curvatures

In ADG, the operator ∂ in (10) readily generalizes to an $\mathbf{A}$-connection $\mathcal{D}$, as follows: given a differential triad $\mathfrak{T} = (\mathbf{A}, \partial, \mathbf{\Omega}^1)$ as above, let $\mathcal{E}$ be an $\mathbf{A}$-module sheaf on X.[147] Then, one first defines $\mathcal{D}$ as a map

$$\mathcal{D} : \mathcal{E} \longrightarrow \mathbf{\Omega}(\mathcal{E}) \equiv \mathcal{E} \otimes_{\mathbf{A}} \mathbf{\Omega} \cong \mathbf{\Omega} \otimes_{\mathbf{A}} \mathcal{E} \tag{12}$$

which is a $\mathbf{C}$-linear morphism of the $\mathbb{K}$-vector sheaves involved, and then one requires that this map satisfies the following condition

$$\mathcal{D}(\alpha \cdot s) = \alpha \cdot \mathcal{D}(s) + s \otimes \partial(\alpha) \tag{13}$$

for $\alpha \in \mathbf{A}(U)$, $s \in \mathcal{E}(U)$, and $U \subset X$ open. The connection $\mathcal{D}$, as defined above, is commonly known as a *Koszul linear connection*.[148]

[146]One must emphasize here the intimate link between $\mathbf{K}$-algebraized spaces and differential triads as expressed by the following result: "*every* $\mathbf{K}$*-algebraized space defines a triad* $\mathfrak{T}$" [259, 290, 291]. This is but a preliminary indication of the purely algebraic (:sheaf-theoretic) conception of 'differentiability' ('smoothness') that ADG supports, one that is significantly wider, more general and versatile than the traditional $\mathcal{C}^\infty$-one associated with differential manifolds (CDG), which is just a particular case when one assumes $\mathbf{A} \equiv \mathcal{C}^\infty_X$ (*ie*, $X \equiv M$). Also, about a conventional-notational 'ambiguity': so far in the ADG-literature one encounters differential triads defined as in (9) above (*eg*, see [260]), or equivalently, as triplets $(X, \mathbf{A}, \partial)$ (*eg*, see [274])—*ie*, in the first case one includes in the definition both the 'domain' ($\mathbf{A}$) and the 'range' ($\mathbf{\Omega}^1$) of ∂ (10); while in the second, while regards as important to include the base topological space X of the corresponding $\mathbf{K}$-algebraized space diad, and exclude the target sheaf $\mathbf{\Omega}^1$ of ∂. In what follows we will use both definitions interchangeably.

[147]For $\mathcal{E}$, regarded as a vector sheaf of rank n, one has by definition the following $\mathbf{A}|_U$-isomorphisms: $\mathcal{E}|_U = \mathbf{A}^n|_U = (\mathbf{A}|_U)^n$ and, concomitantly, the following equalities section-wise: $\mathcal{E}(U) = \mathbf{A}^n(U) = \mathbf{A}(U)^n$ (with $\mathbf{A}^n$ the n-fold Whitney sum of $\mathbf{A}$ with itself). One says that $\mathcal{E}$ *is a locally free* $\mathbf{A}$*-module of finite rank* n—an appellation synonymous to *vector sheaf* in ADG. We further assume that for the vector sheaf $\mathcal{E}$ endowed with the $\mathbf{A}$-connection $\mathcal{D}$ as above, the $\mathbf{A}$-module sheaf $\mathbf{\Omega}^1$ in the given differential triad $\mathfrak{T}$ is its dual (*ie*, $\mathbf{\Omega}^1 = \mathcal{E}^* \equiv \mathcal{H}om_{\mathbf{A}}(\mathcal{E}, \mathbf{A})$).

[148]For more about the algebraic $\mathbf{A}$-connection $\mathcal{D}$ above, as for example its local gauge-split form $\mathcal{D} = d + \mathcal{A}$ as well

The curvature of $\mathcal{D}$. In order to define the curvature $R(\mathcal{D})$ of the $\mathbf{A}$-connection $\mathcal{D}$, we first need $\mathbf{A}$-modules $\mathbf{\Omega}^2$ of 2-form-like entities. To this end, one defines inductively $\mathbf{A}$-modules of higher-order 'differential forms' by iteration of the completely antisymmetric homological tensor product functor $\wedge_{\mathbf{A}}$, as follows: $\mathbf{\Omega}^0 := \mathbf{A},\ \mathbf{\Omega}^1 := \mathbf{A} \wedge_{\mathbf{A}} \mathbf{\Omega},\ \mathbf{\Omega}^2 = \mathbf{A} \wedge_{\mathbf{A}} \mathbf{\Omega}^1 \wedge_{\mathbf{A}} \mathbf{\Omega}^1, \cdots \mathbf{\Omega}^i \equiv (\mathbf{\Omega}^1)^i := \wedge^i_{\mathbf{A}} \mathbf{\Omega}^1$. Then, one introduces the first-order extension of $\partial \equiv d^0$ as a $\mathbf{K}$-linear exterior differential operator $\mathtt{d} \equiv d^1$ effecting sheaf morphisms

$$\mathtt{d} :\ \mathbf{\Omega}^1 \longrightarrow \mathbf{\Omega}^2 \tag{14}$$

and obeying relative to ∂ the following 'cohomological nilpotency' condition[149]

$$\mathtt{d} \circ \partial \equiv d^1 \circ d^0 \equiv d^2 = 0 \tag{15}$$

Then, in complete analogy to the 'extension' of the connection $\partial \equiv d^0$ in 10) to $\mathtt{d} \equiv d^1$ in (14) above, given a $\mathbf{A}$-module $\mathcal{E}$ endowed with an $\mathbf{A}$-connection $\mathcal{D}$, one can define the first prolongation of $\mathcal{D}$ to be the following $\mathbf{K}$-linear vector sheaf morphism

$$\mathcal{D}^1 :\ \mathbf{\Omega}^1(\mathcal{E}) \longrightarrow \mathbf{\Omega}^2(\mathcal{E}) \tag{16}$$

satisfying section-wise relative to $\mathcal{D}(\equiv \mathcal{D}^0)$

$$\mathcal{D}^1(s \otimes t) := s \otimes \mathtt{d}t - t \wedge \mathcal{D}s, \tag{17}$$

with $s \in \mathcal{E}(U), t \in \mathbf{\Omega}^1(U)$.

We thus come to define the curvature R of an $\mathbf{A}$-connection $\mathcal{D}$ by the following commutative diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\ \mathcal{D}^0\ } & \mathbf{\Omega}^1(\mathcal{E}) \equiv \mathcal{E} \otimes_{\mathbf{A}} \mathbf{\Omega}^1 \\ & R \equiv \mathcal{D}^1 \circ \mathcal{D}^0 \searrow \quad \swarrow \mathcal{D}^1 & \\ & \mathbf{\Omega}^2(\mathcal{E}) \equiv \mathcal{E} \otimes_{\mathbf{A}} \mathbf{\Omega}^2 & \end{array} \tag{18}$$

as its affine local gauge transformation properties, the reader is referred to [272], and of course to the exhaustive treatment [259, 260]. Additionally, in case the reader would like to learn more about the standard theory of Koszul connections from the traditional geometrical viewpoint of smooth fiber bundles, (s)he can refer to [94].

[149]This nilpotency of d^i $(i \geq 0)$ marks the starting point in ADG of abstract (de Rham) sheaf-cohomology, which in turn relates to the exactness of the following complex: $\mathbf{0}(\equiv \mathbf{\Omega^{-2}}) \stackrel{\imath \equiv d^{-2}}{\longrightarrow} \mathbf{C}(\equiv \mathbf{\Omega^{-1}}) \stackrel{\epsilon \equiv d^{-1}}{\longrightarrow} \mathbf{A}(\equiv \mathbf{\Omega^0}) \stackrel{d^0 \equiv \partial}{\longrightarrow} \mathbf{\Omega^1} \stackrel{d^1 \equiv \mathtt{d}}{\longrightarrow} \mathbf{\Omega^2} \stackrel{d^2}{\longrightarrow} \cdots \mathbf{\Omega^n} \stackrel{d^n}{\longrightarrow} \cdots$, with each $d^i :\ \mathbf{\Omega}^i \longrightarrow \mathbf{\Omega}^{i+1}$ $(i \geq 2)$ being defined like $\partial \equiv d^0$ and $\mathtt{d} \equiv d^1$ before, *ie*, as a sheaf morphism between the relevant $\mathbf{A}$-modules of higher-order $(i \geq 2)$ differential forms involved [259, 260, 271].

from which one reads directly that

$$R \equiv R(\mathcal{D}) := \mathcal{D}^1 \circ \mathcal{D} \tag{19}$$

and from which it follows that $R(\partial) = d^2 = 0$—*ie*, that ∂ *is a flat* **A**-*connection*.[150]

For the sake of our exposition here, the property of the curvature R of an **A**-connection $\mathcal{D}$ that we would like to highlight, in contradistinction to the connection $\mathcal{D}$ itself, is its *geometrical character*, which can be expressed in a concise manner ADG-theoretically as follows:

(R4.3) *while R is an* **A**-*morphism, $\mathcal{D}$ is only a constant sheaf* **K**-*morphism* [272].

In turn, (R2.?) above has the following 'algebraico-geometrical implications' first noted in [272]:[151]

(R4.4) *Since R is an* **A**-*morphism, our 'local measurements', our gauge acts of coordinatization or localization (of the fields involved), which are organized into the 'geometry-encoding algebraic apparatus'* **A** *of ADG—the sheaf of generalized coefficients or 'arithmetics' in the theory, respect it. Thus, R is a geometrical object (ie, an* **A**-*tensor) in our theory, while $\mathcal{D}$, which is respected only by the constant sheaf* **K** *but not by our (local) measurements in* **A**, *is not a geometrical object and it eludes our acts of trying to measure (localize) it. Rather, $\mathcal{D}$ is an algebraic entity not captured by our 'geometrical/measuring apparatus' ('space') which is effectively encoded in* **A** *(Gel'fand duality). This is reflected in the affine or inhomogeneous (local gauge non-covariant) and the homogeneous (local gauge covariant) transformation laws of $\mathcal{D}$ and $R(\mathcal{D})$ under a local change of gauge (coordinates), respectively* [259, 260, 272].

[150]Again, for more about $R(\mathcal{D})$ on a vector sheaf $\mathcal{E}$ of rank n, as for example its local gauge character as a 0-cocycle of local $n \times n$ matrices having for entries local sections of $\mathbf{\Omega}^2$, as well its covariant local gauge transformation properties, the reader is referred to [272], and of course to the exhaustive treatment [259, 260].

[151]These implications will be of great import in 3.2.2 and 3.2.3 where we discuss the novel ADG-perspective on the PGC of GR as well as on (gravitational) kinematics and dynamics. Moreover, they will play a significant role when we interpret quantally (quantum theoretically) the inherently algebraic formalism of ADG in section 6.

4.1.5 Manifold-free algebraic (pseudo)-Riemannian structures in ADG

In complete analogy to the smooth (pseudo-)Riemannian structures on the differential spacetime manifold of GR, one can define ADG-theoretically, in an entirely algebraic (:sheaf-theoretic) and manifestly base manifold-free way, the following structures:

The A-metric ρ. By an **A**-valued (pseudo)-Riemannian inner product ρ on a vector sheaf $\mathcal{E}$ we mean a *sheaf* morphism

$$\rho : \; \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathbf{A} \tag{20}$$

which is **A**-*bilinear* between the **A**-modules concerned, *symmetric indefinite* (*ie*, $\rho(s,t) = \rho(t,s), \; s,t \in \mathcal{E}(U)$ and of indefinite signature), as well as iii) *strongly non-degenerate*.[152] For any two local sections s and t of $\mathcal{E}$ in $\mathcal{E}(U)$, ρ is implemented via the canonical isomorphism

$$\mathcal{E} \overset{\tilde{\rho}}{\cong} \mathcal{E}^* \tag{21}$$

between $\mathcal{E}$ and its dual $\mathcal{E}^*$,[153] in the following way

$$\tilde{\rho}(s)(t) := \rho(s,t) \tag{22}$$

with (21) being true up to an **A**-isomorphism [272].

Christoffel-Riemannian metric A-connections. In complete analogy with the usual Christoffel theory, we can define a *linear connection* ∇ as follows

$$\nabla : \; \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{E} \tag{23}$$

acting on $\mathcal{E}(U)$ as

$$\nabla(s,t) \equiv \nabla_s(t) := \mathcal{D}(t)(s) \tag{24}$$

Now, one says that $\mathcal{D}$ *is a pseudo-Riemannian* **A**-*connection* or that it is compatible with the indefinite metric g of the inner product ρ in (20), whenever it fulfills the following two conditions:

[152] For more description of these three defining properties of ρ, see [272].

[153] As noted earlier, we assume that for the vector sheaf $\mathcal{E}$ endowed with the **A**-connection $\mathcal{D}$, the sheaf $\mathbf{\Omega}$ of differential 1-form-like entities in the given differential triad $\mathfrak{T}$ is the dual of $\mathcal{E}$ appearing in (21)—*ie*, $\mathbf{\Omega} = \mathcal{E}^* \equiv \mathcal{H}om_{\mathbf{A}}(\mathcal{E}, \mathbf{A})$.

- *Riemannian symmetry*: $\nabla(s,t) - \nabla(t,s) = [s,t]$; for $s,t \in \mathcal{E}(U)$ and $[\,.\,,\,.\,]$ the usual Lie bracket (product).

- *Ricci identity*: $\partial(\rho(s,t))(u) = \rho(\nabla(u,s),t) + \rho(s,\nabla(u,t))$; for $s,t,u \in \mathcal{E}(U)$, as usual.

In particular, for a Lorentzian ρ and its associated g,[154] an **A**-connection $\mathcal{D}$ is said to be compatible with the Lorentz **A**-inner product ρ on $\mathcal{E}$[155] when its associated Christoffel ∇ in (23) satisfies

$$\nabla\rho = 0 \tag{25}$$

which, in turn, is equivalent to the following '*horizontality*' condition for the canonical isomorphism $\tilde{\rho}$ in (21) relative to the *connection* $\mathcal{D}_{\mathcal{E}\otimes_{\mathbf{A}}\mathcal{E}^*}$ in the tensor product vector sheaf $\mathcal{H}om_{\mathbf{A}}(\mathcal{E},\mathcal{E}^*) = (\mathcal{E}\otimes_{\mathbf{A}}\mathcal{E})^* = \mathcal{E}^*\otimes_{\mathbf{A}}\mathcal{E}^*$ induced by the **A**-connection $\mathcal{D}$ on $\mathcal{E}$

$$\mathcal{D}_{\mathcal{H}om_{\mathbf{A}}(\mathcal{E},\mathcal{E}^*)}(\tilde{\rho}) = 0 \tag{26}$$

It is worth reminding the reader who is familiar with the usual theory that (26) above implies that the Levi-Civita **A**-connection $\mathcal{D}$ induced by the Lorentz **A**-metric ρ is *torsion-free* [262].

The manifold-free algebraic (pseudo)-Riemannian structures above have enabled us to formulate the vacuum Einstein equations for gravity as *differential equations proper*, but in the prominent absence of a background differential spacetime continuum. We briefly recapitulate this ADG-based formulation next.

4.1.6 ADG-vacuum Einstein gravity without the base spacetime manifold

The Calculus-free ADG-machinery has been applied to writing the Einstein equations for the 'pure' (*ie*, vacuum) gravitational field over space(time)s that differ dramatically from the classical, featureless locally Euclidean spacetime manifold, and ranging from the 'ultra-smooth' (and 'ultra-singular'!) [273, 274, 262, 275],[156] to the 'discrete' ('discontinuous') [270, 271, 272].[157]

However, in order to actually write down the ADG-theoretic version of the vacuum Einstein equations, together with the pseudo-Riemannian notions above we also need the *Ricci curvature*

[154] With respect to a *local (coordinate) gauge* $e^U \equiv \{U;\ (e_i)_{0\leq i\leq n-1}\}$ of the vector sheaf $\mathcal{E}$ of rank n, $\rho(e_i,e_j) = g_{ij} = \mathrm{diag}(-1,+1,\cdots)$ [259, 260].

[155] Such a metric connection is commonly known as *Levi-Civita connection.*

[156] With [262] the key reference. We will encounter this 'ultra-singular' case in the next section and, regarding the evasion of the inner Schwarzschild singularity, also in 5.2.3.

[157] With [272] the key reference. Again, concerning the evasion of the interior Schwarzschild singularity, we will encounter this 'discrete', finitary case in 5.2.2.

tensor and its *scalar* contraction—structures that are readily available in ADG again in a manifestly manifoldless guise [259, 260, 272].

Thus, given a (real) Lorentzian vector sheaf $\mathcal{E}^\uparrow := (\mathcal{E}, \rho)$ of rank n equipped with a non-flat $\mathbf{A}$-connection $\mathcal{D}$, one can define the following *Ricci curvature operator* $\mathcal{R}$ relative to a 'local gauge' U of $\mathcal{E}$

$$\mathcal{R}(\,.\,, s)t \in (\mathcal{E}nd\mathcal{E})(U) = M_n(\mathbf{A}(U)) \tag{27}$$

for local sections s and t of $\mathcal{E}$ in $\mathcal{E}(U) = \mathbf{A}^n(U) = \mathbf{A}(U)^n$. $\mathcal{R}$ is an $\mathcal{E}nd\mathcal{E}$-valued operator, thus called a '*curvature endomorphism*'.

Since $\mathcal{R}$ is matrix-valued with (local) matrix-entries in $\mathbf{A}(U)$, we can take its trace, thus define the following *Ricci scalar curvature operator* $\mathfrak{R}$

$$\mathfrak{R}(s, t) := tr(\mathcal{R}(\,.\,, s)t) \tag{28}$$

which is $\mathbf{A}(U)$-valued.

With the ADG-version of the Ricci scalar in hand, we read from [262, 272] that the ADG-based *vacuum Einstein equations* for Lorentzian gravity are:

$$\mathfrak{R}(\mathcal{E}) = 0 \tag{29}$$

for a given choice of $\mathbf{K}$-algebraized space and its associated differential triad—or essentially, for a given choice of structure sheaf $\mathbf{A}$ of generalized coefficients or coordinates.[158]

It is also important to mention here from [262, 272] that *the equations* (29) *can be obtained from the variation of an* $\mathbf{A}$*-valued Einstein-Hilbert action functional* $\mathfrak{E}\mathfrak{H}$ *on the affine space* $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$ *of Lorentzian metric* $\mathbf{A}$*-connections on* $\mathcal{E}$*, which are the sole dynamical variables in the ADG-theoretic perspective on gravity.*[159]

[158] And it is precisely this *freedom to choose* $\mathbf{A}$—generalized coefficients or 'coordinates' quite different from the classical $\mathcal{C}^\infty$-smooth ones in $\mathbf{A} \equiv \mathcal{C}^\infty_M$—that ADG allows us, which qualifies the title of the present sub-subsection '*base spacetime manifoldless vacuum Einstein gravity*'.

[159] This last remark is made in order to prepare the reader for the fact, to be further corroborated in the sequel, that *in ADG*—in contradistinction to the original CDG based GR where the basic dynamical variable is supposed to be the smooth spacetime metric—*the only gravitational dynamical variable is the connection*, not the metric (see 3.3 next). That is to say, *according to ADG, gravity is another gauge theory*, but unlike the other spacetime manifold (and CDG) based gauge theories of matter (electromagnetism, Yang-Mills theories), *a background spacetime manifoldless one*.

4.1.7 The categorical imperative: the category $\mathfrak{DT}$ of differential triads, its properties and versatility compared to the category $\mathcal{M}an$ of smooth manifolds

In the foregoing sections we have mentioned numerous times that ADG is an *algebraico-categorical* scheme for doing differential geometry. Its *algebraic* (:sheaf-theoretic) character has already been amply manifested, thus here we reveal in some detail its purely *category-theoretic* features.[160] We itemize this presentation into four paragraphs, **i–iv**, briefly borrowing the basic concepts, constructions and results from [292] (**i**), [291] (**ii**), [289] (**iii**) and [290] (**iv**), respectively.[161] The last paragraph (**iv**) in particular will prove to be of great import later when we 'resolve' ADG-theoretically the interior Schwarzschild singularity by finitistic-algebraic means—a culmination of our finitary application of ADG to classical and quantum gravity in the past trilogy [270, 271, 272].

i. Morphisms of differential triads: generalized (abstract) differentiable maps. The first observation in the endeavor to explore and study categorical features of ADG is that one could regard differential triads as objects in some category—*the category of differential triads*, which we will denote by $\mathfrak{DT}$. This aim could at least be motivated on the following grounds: since a differential triad purports to abstract and generalize (the differential structure of) a differential manifold, and since the latter is an object in the category $\mathcal{M}an$ of (finite-dimensional) smooth manifolds, having for arrows the usual arbitrarily differentiable (smooth) maps between them, in the same way differential triads could be viewed as objects in a category. Thus, the issue arises of what are the arrows in $\mathfrak{DT}$, a question that brings us to the definition of *morphisms of differential triads* which we borrow from [292].

Let X and Y be topological spaces, which are base spaces of the $\mathbf{K}$-algebraized spaces $(X, \mathbf{A}_X)$ and $(Y, \mathbf{A}_Y)$, respectively. In addition, let $\mathfrak{T}_X = (\mathbf{A}_X, \partial_X, \mathbf{\Omega}_X)$ and $\mathfrak{T}_Y = (\mathbf{A}_Y, \partial_Y, \mathbf{\Omega}_Y)$ be differential triads over them. Then, a *morphism* $\mathcal{F}$ between $\mathfrak{T}_X$ and $\mathfrak{T}_Y$ is a triplet of maps $\mathcal{F} = (f, f_A, f_{\mathbf{\Omega}})$, such that:

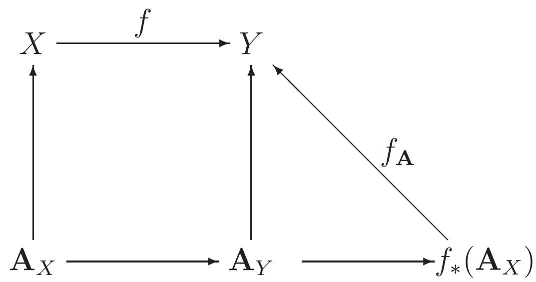
1. the map $f: \ X \longrightarrow Y$ is *continuous*;

2. the map $f_{\mathbf{A}}: \ \mathbf{A}_Y \longrightarrow f_*(\mathbf{A}_X)$ is a *morphism of sheaves of* $\mathbb{K}$-*algebras over* Y preserving the respective algebras' unit elements (*ie*, $f_{\mathbf{A}}(1) = 1$);[162] and the following categorical diagram is obeyed:

[160] Of course, the categorically minded reader has perhaps already noticed terms like *sheaf morphism* (*eg*, $\mathcal{D}$ and $R(\mathcal{D})$) and *homological functor* (*eg*, sheafification and tensor product functor), which already indicate that we have implicitly been working all along in a *category of sheaves* of some kind.

[161] The reader should refer to the reference list at the back and note that the order in which the categorical features of ADG are presented below does not accord with the chronological order in which these were originally studied by Papatriantafillou.

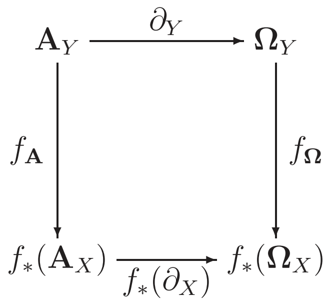
[162] In the expression for $f_{\mathbf{A}}$ above, f_* is the *push-out* along the continuous f, a map which carries each element



(30)

3. the map $f_{\mathbf{\Omega}}: \ \mathbf{\Omega}_Y \longrightarrow f_*(\mathbf{\Omega}_X)$, as noted in the last footnote, is a *morphism of sheaves of* $\mathbb{K}$*-vector spaces over* Y, with $f_{\mathbf{\Omega}}(\alpha\omega) = f_{\mathbf{A}}(\alpha) f_{\mathbf{\Omega}}(\omega), \ \forall \alpha \in \mathbf{A}_Y, \ \omega \in \mathbf{\Omega}_Y$; and finally,

4. with respect to the $\mathbf{C} \equiv \mathbf{K}$-sheaf morphism (*viz.* flat connection) ∂ in the respective triads, and as it has also been alluded to in the last footnote, the following diagram is commutative:



(31)

of a differential triad into a like element in the sense that, for any triad $\mathfrak{T}$, $f_*(\mathfrak{T}) := (f_*(\mathbf{A}), f_*(\partial), f_*(\mathbf{\Omega}))$ is also a differential triad—the one 'induced' by f [292, 291]; whence, term-wise for our triads $\mathfrak{T}_X$ and $\mathfrak{T}_Y$ above (and omitting the base topological space subscripts): $f_*(\mathbf{A}) := (f_*(\mathbf{A})(U) := \mathbf{A}_{f^{-1}(U)}), \ (U \subseteq Y$ open) is a sheaf of unital, abelian, associative $\mathbb{K}$-algebras over Y, $f_*(\mathbf{\Omega}) := (f_*(\mathbf{\Omega})(U) := \mathbf{\Omega}_{f^{-1}(U)}), \ (U \subseteq Y$ open) a sheaf of $f_*(\mathbf{A})$-modules (of 1st-order differential form-like entities), and $f_*(\partial) := (f_*(\partial)(U) := \partial_{f^{-1}(U)}), \ (U \subseteq Y$ open) an induced $\mathbf{K}$-linear, Leibnizian sheaf morphism [292].

which is readily read as $f_{\mathbf{\Omega}} \circ \partial_Y = f_*(\partial_X) \circ f_{\mathbf{A}}$.

Thus, the category $\mathfrak{DT}$ has $\mathfrak{T}$s as objects and $\mathcal{F}$s between them as arrows. Furthermore, we saw above in the definition of a morphism $\mathcal{F}$ of differential triads what an important role a continuous map f between their respective base spaces plays via its push-out f_*. In fact, it has been shown that the (essentially algebraic) differential (geometric) mechanism encoded into a differential triad can be transferred forward and backwards by any continuous map; moreover, the final and initial structures obtained by such push-forward and pull-back actions induced by f enjoy certain *universal properties* that qualify f to a *differentiable map* [291]. This result opens up new possibilities of developing what one might coin '*the differential geometry of topological spaces*'. Below, we discuss briefly this 'universality' of differential triad morphisms and its potent consequences, especially in the application of ADG to our finitary, causal and quantal approach to Lorentzian gravity, and more particularly, in the evasion of the inner Schwarzschild singularity by finitistic-algebraic means as it will be entertained in 5.2.2 later on.

ii. Towards a 'differential geometry of topological spaces'; Calculus 'reversed': continuity 'implies' differentiability. We begin with some general observations. We noted earlier that the notion of *sheaf* is essentially *topological*, while that of '*differentiability*' essentially *topologico-algebraic* (R3.1), and *local* (in the topological space involved). Thus, it is not surprising that ADG manages to capture so many (if not all!) the features of the usual Calculus on smooth manifolds by using *sheaves of algebras*—as it were, the harmonious combination and coherent intertwining of the local-topological character of sheaves with the algebraic structure of the (functional) objects localized sheaf-theoretically (in $\mathbf{A}$) results in the definition of a *differential* operator $d \equiv \partial$ (*viz.* connection), which is *the* basic object-tool with which one can actually do differential geometry. Furthermore, and perhaps more importantly, all this is achieved in the prominent absence of a base smooth manifold when, as a matter of fact (*ie*, of fundamental assumption in ADG), all that is needed is an in principle arbitrary topological space (to serve as a '*surrogate localization-scaffolding*' for the said algebras), a background stage which plays no role whatsoever in the said inherently algebraic differential geometric mechanism of ADG. In continuation of (R3.1), we may distill all this to the following general, but fundamental, *motto* of ADG:

(R4.5) *topological* + (sheaf-theoretically localized on it) *algebraic structure* = *differential* (geometric) *structure* (without any $\mathcal{C}^\infty$-smoothness $\equiv$ base differential manifold $M \equiv \mathcal{C}^\infty(M)$).

In what follows, we simply corroborate further (R2.?) above by following mainly [291].

The main result in [291], as summarized in its abstract, is that

(Q4.1) "the differential mechanism induced by a differential triad is transferred backwards and forward by any continuous map f. The initial and final structures thus obtained satisfy appropriate universal conditions that turn the continuous map f into a differentiable map."

In a nutshell, given a continuous $f: X \longrightarrow Y$, with X the base space of a differential triad $\mathfrak{T}_X$, it has been shown that f pushes forward the (essentially algebraic) differential mechanism of $\mathfrak{T}_X$, so that a new and unique differential triad—one that satisfies a *universal mapping* condition—is defined on Y, and thus f becomes differentiable. The relevant theorem,[163] which uses some ideas already expressed in the last footnote and paragraph (**i.**), can be stated as follows:[164]

(R4.6) Let $\mathfrak{T}_X = (\mathbf{A}_X, \partial_X, \mathbf{\Omega}_X) \in \mathfrak{D}\mathfrak{T}_X$,[a] and $f: X \longrightarrow Y$ continuous. If Y inherits $f_*(\mathfrak{T}_X) := (f_*(\mathbf{A}_X), f_*(\partial_X), f_*(\mathbf{\Omega}_X))$ from the push-out f_* of f, then there is a morphism of differential triads $\mathcal{F} = (f, f_{\mathbf{A}}, f_{\mathbf{\Omega}}): \mathfrak{T}_X \longrightarrow f_*(\mathfrak{T}_X)$ ($\in \mathfrak{D}\mathfrak{T}$)—*ie*, f becomes *differentiable*. Furthermore, the pushed-forward triad $f_*(\mathfrak{T}_X)$ satisfies the following universal (composition) property: given a triad $\mathfrak{T}_Y = (\mathbf{A}_Y, \partial_Y, \mathbf{\Omega}_Y) \in \mathfrak{D}\mathfrak{T}_Y$, as well as a morphism $\tilde{\mathcal{F}} := (f, \tilde{f}_{\mathbf{A}}, \tilde{f}_{\mathbf{\Omega}}): \mathfrak{T}_X \longrightarrow \mathfrak{T}_Y$, there is a unique morphism $(id_Y, g_{\mathbf{A}}, g_{\mathbf{\Omega}}): f_*(\mathfrak{T}_X) \longrightarrow \mathfrak{T}_Y$ such that:

$$\tilde{\mathcal{F}} = (id_Y, g_{\mathbf{A}}, g_{\mathbf{\Omega}}) \circ \mathcal{F}$$

[a] Plainly, $\mathfrak{D}\mathfrak{T}_X$ is the sub-category of $\mathfrak{T}$ consisting of all differential triads and triad morphisms with common base topological space X.

Accordingly, the 'dual' (converse) scenario involving f's *pull-back action* f^*, when now the range of f is a differential triad $\mathfrak{T}_Y$ on Y while X (f's domain) is merely a topological space not being endowed *a priori* with a differential (triad) structure, f^* too is seen to transfer (induce) backwards the differential mechanism encoded in $\mathfrak{T}_Y$ on X, thus rendering it a *differential* (not just a topological) space and in the process promoting f to a *differentiable* (not just a continuous) map [291].[165]

The 'aftermath' of the maths. The principal value of these push-forward and pull-back results lies with what one might call '*induced differential structures from topological structures*', or

[163] Theorem **3.1** in [291].

[164] For the corresponding proof, the reader is referred to [291].

[165] Although, as noted in [291], the proof of this 'dual', pull-back theorem is slightly more involved than the push-out one.

perhaps more strikingly, '*the differential geometry of topological spaces via a Calculus-reversal*'. Let us explain this issue a bit more. We saw earlier when we discussed the tower of structures imposed on (pseudo-)Riemannian manifolds (8) that, from the vantage of CDG, the topological ('continuous') structure on a manifold is usually regarded as being more basic (deeper) than its differential ('smooth') one, something that is already reflected in the so-called *Fundamental Theorem of Calculus* that we learn as undergraduates, namely, that

'differentiability implies continuity'

or, functionally speaking, that *a differentiable function is continuous.*[166] Here, however, we encounter exactly the reverse phenomenon, namely that,

(R4.7) as long as one has sheaf-theoretically localized an algebraic structure ($\mathbf{A}$) on a topological space (X) so that one is able to define a differential triad ($\mathfrak{T}_X$) on the latter *à la* ADG, then a continuous map (f) from the said topological space (X) to another topological space (Y) endows the latter with a differential structure (f_*-induced differential triad on Y), and in the process the map (f) becomes 'smooth' (differentiable). In a nutshell, and in a strong sense, *in ADG, continuity implies differentiability*—*ergo*, the aforementioned 'Calculus-reversal'.

These results indicate the 'superiority' and 'versatility' of $\mathfrak{DT}$ (and *in extenso* of ADG) relative to $\mathcal{M}an$ (and *in extenso* of CDG), since, to begin with,[167] the latter does not allow for such *initial* (pulled back) and *final* (pushed forward) *differential structures* in the sense that, and we quote *verbatim* Papatriantafillou from the introduction of [291],

(Q4.2) "if $(X, \mathcal{A})$ is a smooth manifold, Y a topological space (resp. X is a topological space and $(Y, \mathcal{B})$ a smooth manifold) and $f: X \longrightarrow Y$ is continuous, one cannot 'push-out' the atlas $\mathcal{A}$ (resp. pull-back the atlas $\mathcal{B}$), in order to make Y (resp. X) a smooth manifold and f a differentiable map."[a]

[a]This virtue of ADG compensates for item number 4 in the list of mathematical 'deficiencies' of $\mathcal{M}an$ and CDG given in 3.1.1.

[166]In footnote ?? we formally cast this 'power relationship' between C^i- $(i = 1 \ldots \infty)$ and $\mathcal{C}^0$-functions on M as $C^i > \mathcal{C}^0$.

[167]Shortly we will give more 'reasons' about the superiority of $\mathfrak{DT}$ compared to $\mathcal{M}an$—in effect, how ADG is able to resolve all the deficiencies and shortcomings of the smooth manifold based CDG mentioned earlier in 3.1.1.

Moreover, as also noted in [291] as a corollary of the theorem (R3.?) above,[168]

(Q4.3) "*Let* $(\mathbf{A}, \partial, \mathbf{\Omega})$ *be a differential triad over* X*, and let* X *be endowed with an* [in principle arbitrary][a] *equivalence relation* $\sim$*. Then the quotient space* $\tilde{X} := X/\sim$ *is provided with a differential triad, so that the canonical* [projection][b] *map* $\pi : X \longrightarrow \tilde{X}$ *is differentiable.*"

[a]Our addition.
[b]Our addition.

As the author of [291] observes in connection with this corollary, while the quotient space of a manifold X by an in principle arbitrary equivalence relation is not a manifold, thanks to the results above, in $\mathfrak{DT}$ the 'quotient-fold' $\tilde{X}$ itself acquires a differential structure (triad).[169] As a characteristic example, Papatriantafillou gives the differential triad that the orbifold (G-fold) or moduli space X/G inherits from X when a topological group G acts continuously on the latter.[170]

iii. The category of differential triads: further properties. Now that we have discussed above some of the advantages and virtues of (working with) $\mathfrak{DT}$ relative to $\mathcal{M}an$, in this paragraph and the next we want to give briefly without proofs some more properties of $\mathfrak{DT}$ that $\mathcal{M}an$ simply lacks—properties, such as categorical bicompleteness (**iv**), that not only will prove to be of great import in our finitary-algebraic resolution of the inner Schwarzschild singularity in the next section, but also ones that could be used in the future to promote $\mathfrak{DT}$ into an *elementary topos* (Lawvere-Tierney) [163, 253]—a structure which promises to enjoy great applications, both mathematical

[168]The corollary below is written in *emphasis*-typeface, as in the original paper [291].

[169]This virtue of ADG resolves item number 2 in the list of mathematical shortcomings of $\mathcal{M}an$ and CDG in 3.1.1 and it will be seen to figure prominently in the ADG-theoretic finitary-algebraic 'resolution' of the inner Schwarzschild singularity in 5.2.2 [317].

[170]This example is very important *vis-à-vis* our connection based ADG-theoretic approach to gravity since, as noted earlier in this paper and extensively in [272], the relevant gravitational configuration space in our theoresis is the affine space $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$ of $\mathbf{A}$-connections on $\mathcal{E}$; moreover, since the PGC of GR is implemented in ADG not via $\mathrm{Diff}(M)$ as in the usual M-based theory (GR), but via the group (sheaf) $\mathcal{A}ut(\mathcal{E})$ of automorphisms of the gravitational field $(\mathcal{E}, \mathcal{D})$, the 'effective' physical configuration space is the orbifold $\mathsf{A}_{\mathbf{A}}(\mathcal{E})/\mathcal{A}ut(\mathcal{E})$. Now, in the differential manifold based approaches to canonical QGR such as loop QG, it has become imperative to develop differential geometric ideas on the moduli space of Ashtekar's spin-Lorentzian connections modulo diffeomorphisms in $\mathrm{Diff}(M)$ [16]. Analogously here, ADG has been significantly developed on $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$ [259, 260, 272, 269]; therefore, Papatriantafillou's example above gives one a fleeting glimpse at the fact that one can straightforwardly transfer the (abstract) differential structure from $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$ onto the G-fold $\mathsf{A}_{\mathbf{A}}(\mathcal{E})/\mathcal{A}ut(\mathcal{E})$, thus be able to further develop differential geometric ideas on the latter [259, 260, 272, 269]—and what's more, unlike in the M-based loop QG approach, *all this is accomplished in the manifest absence of a background differential spacetime manifold*.

and physical alike, as anticipated in [311, 313, 314] and is currently under intense development [316], especially in exploring the (quantum) logical underpinnings of our 'finitary', ADG-theoretic approach to Lorentzian QG [270, 271, 272].

The first of those properties pertains to the existence of *canonical subobjects* in $\mathfrak{DT}$. Following [289], we note that $\mathcal{M}an$ simply lacks (canonical) subobjects, since an arbitrary subset of a differential manifold is not a manifold.[171] Again, just quoting Papatriantafillou from the abstract of [289],[172]

(Q4.4)

> "We prove that every subset of the base space of a differential triad determines a differential triad, which is a subobject of the former, a property missing from the category of manifolds."[a]
>
> [a]It must be noted here that in the aforementioned project to promote $\mathfrak{DT}$ to a topos, the result above (*ie*, that $\mathfrak{DT}$ has subobjects) might prompt one to look for the *subobject classifier* structure $\mathbf{\Omega}$ that would define $\mathfrak{DT}$ as a topos proper [163, 253]. Preliminary investigations [316] seem to indicate that in the same way that the 'quintessential' paradigm of a topos—the topos $\mathcal{S}\mathbf{h}_X$ of sheaves (of structureless, 'variable') sets (varying) over a topological space X (*eg*, a locale)—has a non-Boolean, Heyting algebra-type of 'generalized truth values object' $\mathbf{\Omega}$ (:a synonym for the subobject classifier) and hence an intuitionistic internal language (logic) [240, 253], so $\mathfrak{DT}$ too has a similar subobject classifier and, as a result, a non-classical (non-Boolean) kind of internal logic. This is glaring contrast to the classical, Boolean logic 'inherent' in the topos **Set** of 'constant' sets on which arguably the M-based GR rests (after all, M is a 'classical' point-set), and whose subobject classifier is the Boolean binary alternative $\mathbf{\Omega} = \mathbf{2} = \{\top, \bot\}$ of truth values (hence the denomination 'classical' given to M). In such a project the main aim is to do all the GR constructions *internally*, within the topos $\mathfrak{DT}$. But for more on this project, as well as for its physical implications, the reader should wait for [316].

Furthermore, Papatriantafillou shows that $\mathfrak{DT}$ has finite products (and dually, co-products).

iv. Bicompleteness of $\mathfrak{DT}$. This paragraph is about another property of $\mathfrak{DT}$ that $\mathcal{M}an$ does not possess, namely, that it is *bicomplete*—*ie*, closed under *projective* and *inductive limits*.[173]

[171]This pertains to item number 1 in the list of mathematical deficiencies of $\mathcal{M}an$ and CDG mentioned in 3.1.1.

[172]Again, for proofs of all statements made below, the reader is referred to the relevant papers cited.

[173]Projective limits are also known as *inverse* or *categorical limits*, while synonyms for inductive limits are *direct* or *categorical colimits*. The notion of projective limit is categorically dual to that of inductive limit, hence the prefix '*co-*' in front of the latter.

That is to say, in $\mathfrak{DT}$, projective (inverse) systems of differential triads have differential triads as projective (inverse) limits. The same holds dually for inductive (direct) systems.[174] As we did above, we quote directly the relevant result from the abstract of [290] (omitting the corresponding proofs), and also from its introduction we remark the associated deficiency of $\mathcal{M}an$:

(Q4.5)

> "We prove that in the category of differential triads projective and inductive systems have limits[a]... [That is,][b] we prove that $\mathcal{DT}$[c] is closed for the projective and inductive limits of differential triads. To this end, we prove that a projective (resp. inductive) system of differential triads over the same base space has a limit in $\mathcal{DT}$.[d] Then, for a projective (resp. inductive) system of differential triads over a projective system of base spaces,[e] we construct a new projective (resp. inductive) system of differential triads over the projective limit of the projective limit of the base spaces, and we prove that the limit satisfies the universal property of the projective (resp. inductive) limit for the initially given family[f]...[These results must be contrasted against the situation in $\mathcal{M}an$ where][g] the limit of a projective system of manifolds is not, in general, a manifold. In some cases, as in the theory of jets, it is a manifold, but it is infinite-dimensional. Thus, the category [$\mathcal{M}an$] of smooth, finite dimensional manifolds is not closed under projective limits. The same is true for inductive limits of manifolds...[h]"
>
> [a]From the abstract of [290].
> [b]Our addition for continuity of the text.
> [c]Papatriantafillou symbolizes the category of differential triads by $\mathcal{DT}$, we by $\mathfrak{DT}$.
> [d]Propositions 3.2 and 3.3 in [290].
> [e]This is precisely the result we will use in the sequel (5.2.2) to 'resolve' finitarily *à la* Sorkin [355] and by algebraic ADG-theoretic means the interior Schwarzschild singularity [317].
> [f]Theorems 4.4 and 4.5 in [290].
> [g]Our addition for continuity of the text.
> [h]From the introduction to [290].

Thus, in contradistinction to $\mathcal{M}an$, $\mathfrak{DT}$ is bicomplete (*ie*, both complete and co-complete). Also, in connection with our topos remarks just before, (finite) bicompleteness is another defining property for a category to qualify as an elementary topos [163, 253].[175] Finally, as noted in various footnotes

[174]This pertains to item 3 in the list of mathematical deficiencies of $\mathcal{M}an$ and CDG mentioned in 3.1.1.

[175]*In toto*, for a category to pass as an elementary topos *à la* Lawvere-Tierney, it must: (a) have canonical

above, concerning our finitary-algebraic, ADG-theoretic resolution of the inner Schwarzschild singularity later in 5.2.2, this bicompleteness of $\mathfrak{DT}$ (in its finitary analogue)[176] will prove to be an invaluable result.

v. A 'bonus' paragraph: the finitary analogue and the 'Newtonian spark' of ADG. The three main results-properties of $\mathfrak{DT}$ discussed above, namely, the universal mapping property of triad morphisms, their push-out and pull-back virtues *vis-à-vis* continuous maps (between their underlying topological spaces), as well as $\mathfrak{DT}$'s bicompleteness, have been particularly useful and fruitful in our ADG-based approach to a finitary, causal and quantal version of Lorentzian gravity [270, 271, 272]. In this last paragraph of the present subsection we make it precise and clear exactly when and in what way the results above were used, either explicitly or implicitly, in the said approach.[177]

To briefly recapitulate things, our starting point in this approach was Sorkin's finitary (:locally finite) replacements of continuous manifolds [355].[178] In a nutshell, Sorkin considered locally finite open coverings $\mathcal{U}_i$[179] of an open and bounded region X in a continuous (*ie*, topological or $\mathcal{C}^0$-) manifold M. With respect to one such covering, Sorkin then grouped X's points into equivalence classes according to the following equivalence relation:

$$X \ni x \overset{\mathcal{U}_i}{\sim} y \in X \Leftrightarrow \Lambda|_{\mathcal{U}_i}(x) = \Lambda|_{\mathcal{U}_i}(y) \tag{32}$$

where $\Lambda|_{\mathcal{U}_i}(x) := \bigcap\{U \in \mathcal{U}_i |\, x \in U\}$—the 'smallest' open set (in the subtopology τ_i of X generated by the open sets in $\mathcal{U}_i$[180]) containing x, which we here coin '*Sorkin's ur-cell of x relative to $\mathcal{U}_i$*'.

The *quotient space* $X/\overset{\mathcal{U}_i}{\sim} =: P_i$, consisting of $\overset{\mathcal{U}_i}{\sim}$-equivalence classes of points in X,[181] was then seen to be a T_0-topological space with the structure of a (locally finite) partially ordered set (poset), and it was pitched as "*the finitary substitute of the continuous topology of X*".

subobjects and hence a subobject classifier, (b) be (finitely) bicomplete, (c) have a so-called initial (and a terminal) object, and finally, (d) have an exponential structure [163, 253]. The last property means essentially that given any to objects A and B in the category, one can form their exponential A^B which represent the collection of all arrows (morphisms or maps) from B to A. For the case of $\mathfrak{DT}$ in particular, it can be shown that it possesses such a structure, since for any two differential triads $\mathfrak{T}$ and $\mathfrak{T}'$ (over, say, the same topological space), one can always group together, into the exponential $\mathfrak{T}^{\mathfrak{T}'}$, all the triad morphisms ('differentiable maps') from $\mathfrak{T}'$ to $\mathfrak{T}$.

[176] See next paragraph **v**.

[177] Such an explicit and direct presentation has not been given earlier in our trilogy [270, 271, 272], but have been partly exposed in the latest paper by the second author [317].

[178] The exposition below will be encountered again later, in 4.3.2, but for slightly different purposes.

[179] The index-set $I = \{i\}$ over which i varies being a poset or net—the 'topological refinement net' (see below).

[180] tau_i is generated by arbitrary unions and finite intersections of the covering open sets in $\mathcal{U}_i$.

[181] That is, the 'blown-up' points of P_i are Sorkin's *ur*-cells of the original X's points x.

The principal result in [355], and one that qualifies the P_is above as genuine 'discrete' approximations of the topological continuum, was that an inverse system (or net) $\overleftarrow{\mathcal{P}}$ of the P_i was seen to yield at the projective limit of infinite refinement of the said finitary posets[182] a space—call it P_∞—that is 'essentially' topologically equivalent (*ie*, homeomorphic) to the $\mathcal{C}^0$-manifold X that we started with.[183]

As highlighted in [355], the key result for setting up the projective system $\overleftarrow{\mathcal{P}}$ is that continuous surjections (corresponding to the canonical projection maps) from X to the $\overset{\mathcal{U}_i}{\sim}$-moduli T_0-spaces P_i enjoy a *universal mapping property* expressed by the diagram below:

$$\begin{array}{ccc} X & \xrightarrow{\ f_j\ } & P_j \\ & \searrow^{f_i} & \downarrow{\scriptstyle f_{ji}} \\ & & P_i \end{array} \tag{33}$$

That is, $f_i = f_{ji} \circ f_j$ ($i \leq j$ in I), and reading that *the map (canonical projection) of X onto the finitary substitutes is universal among maps into T_0-spaces*, with f_{ji} the *unique* map—itself a continuous surjection of P_j onto P_i[184]—mediating between the continuous projections f_i and f_j of X onto the T_0-posets P_i and P_j, respectively. With these canonical continuous projections of X onto the P_is, the said inverse system of finitary posets can be represented by $\overleftarrow{\mathcal{P}} := (P_i, f_{ij})$; while formally, the inverse limit result may be written as $\varprojlim \overleftarrow{\mathcal{P}} \equiv \lim_{\infty \leftarrow i} P_i \equiv P_\infty \overset{\text{homeo.}}{\simeq} X$ (modulo Hausdorff reflection) [271, 272].

At this point it must be noted however that Sorkin's considerations in [355] were purely topological, without any allusion at all to the differential (smooth) structure of the locally Euclidean X.

---

[182]Roughly, the act of topological refinement (or localization of X's points [310, 270, 271]) is represented by the relation $\mathcal{U}_i \preceq \mathcal{U}_j$ between the coverings involved, which reads: *$\mathcal{U}_i$ is a finer covering than $\mathcal{U}_j$* (or with respect to the index-net I: $i \leq j$)—which means in effect that $\mathcal{U}_j$ contains more and 'smaller' open sets than $\mathcal{U}_i$ (or equivalently, that the subtopology τ_j of X generated by the open sets in $\mathcal{U}_j$ is *finer* than the corresponding τ_i) [355].

[183]A minor detail here: the adverb 'essentially' above pertains to the fact that, actually, at the inverse limit of $\overleftarrow{\mathcal{P}}$ one does not get back X itself, but a (non-Hausdorff) space P_∞ having X as a dense subset. However, one can recover X from P_∞, by a procedure commonly known as *Hausdorff reflection*, as the set of the latter's closed points [234].

[184]Itself corresponding to the aforesaid act of topological refinement (or coarse-graining) $\mathcal{U}_i \preceq \mathcal{U}_j$ ($i \leq j$ in the index-net I). Above, the epithet 'continuous' for f_{ji} pertains to the fact that one can assign a 'natural' topology—the so-called Sorkin lower-set topology—to the P_is, whereby an open set is of the form $\mathcal{O}(x) := \{y \in P_i : \ y \longrightarrow x\}$, and where $\longrightarrow$ is the partial order relation in P_i (with basic open sets involving the links or covering—'immediate arrow'—relations in P_i). Plainly then, f_{ji} is a monotone (partial order-preserving) surjection from P_j to P_i, hence continuous with respect to the Sorkin topology.

The rich differential structure 'hybernating' in the finitary posets in Sorkin's scenario was unveiled subsequently when Zapatrin, in collaboration with the second author, passed to the Gel'fand-dual algebraic picture of the P_is above, involving the incidence (Rota) algebras Ω_i of those posets [318, 319]. Indeed, recognizing from the beginning that the P_is may be viewed homologically as (finitary) Čech-Alexandrov simplicial complexes (nerves) $\mathcal{K}_i$ [2, 78], their associated incidence algebras[185] were seen to be $\mathbb{Z}_+$-*graded discrete differential algebras*[186] $\Omega_i = \bigoplus^{j\in\mathbb{Z}_+} \Omega_i^j$,[187] having a nilpotent (exterior) Kähler-Cartan differential $\mathtt{d}$ as their basic differential operator,[188] which operator, in turn, can expressed via the homological border (boundary) and coborder (coboundary) operators of the $\mathcal{K}_i$s [432].

In fact, in [318, 319, 432] it was shown that the correspondence P_i or $\mathcal{K}_i : \longrightarrow \Omega_i$ is *functorial* in the sense that there is a (contravariant) functor from the category (finitary posets, monotone maps)[189], or equivalently, the category (finitary simplicial complexes, simplicial maps), to the category (incidence algebras, algebra homomorphisms). Subsequently in [310], and then in [270], it was further recognized that the aforesaid functorial correspondence is in fact an instance of a 'discrete' *sheafification functor*, in the sense that *the map* $P_i \longrightarrow \Omega(P_i)$ *is a local homeomorphism*—a *sheaf*—as the 'local topology' of the P_is is carried over to the 'local topology' of their corresponding Ω_is.[190] Thus, *finitary spacetime sheaves* (finsheaves) of incidence algebras over Sorkin's finitary substitutes of $\mathcal{C}^0$-manifolds were born.

Then, with ADG in mind, it did not take long to see that the said finsheaves define *finitary differential triads* (fintriads) $\mathfrak{T}_i$ [271].[191] So now we are in a position to expose how fruitful, in a finitary context, the results about $\mathfrak{DT}$ that we presented earlier are.

To begin with, the continuous surjections f_{ji} in Sorkin's system lift to *fintriad morphisms*—abstract *differentiable* maps, not merely topological (continuous) ones. Furthermore, the universal mapping property (33) that Sorkin's finitary poset morphisms enjoy translates directly to the universal mapping condition that triads satisfy in (R3.?). *In toto*, Sorkin's projective (inverse)

[185] Write $\Omega_i(\mathcal{K}_i)$, or simply Ω_i as above.

[186] Or equivalently *à-la* Dimakis *et al.*, *discrete differential manifolds* [97, 99, 98].

[187] With the $j = 0$-graded elements of Ω_i in $\Omega_i^0 \equiv A_i$ constituting an abelian subalgebra of the noncommutative in general Ω_i. In turn, $\mathcal{R}_i := \bigoplus^{j\geq 1} \Omega_i^j$ was seen to be an A_i-module—a module of discrete differential form-like entities [318, 319].

[188] $\mathtt{d}$ was seen to effect linear Leibnizian maps $\mathtt{d} : \Omega_i^j \longrightarrow \Omega_i^{j+1}$ between the linear subspaces Ω_i^j of Ω_i, raising their grade by 1 in the process.

[189] Monotone maps, or equivalently, poset morphisms (*ie*, partial order-preserving maps) are exactly the 'continuous' ones—maps preserving the aforementioned Sorkin T_0-topology of the P_is [271, 272].

[190] That is, the generating or covering relations (links) of the aforementioned Sorkin topology on the P_is was mapped by a procedure called *Gel'fand spatialization* (itself an instance of *Gel'fand duality*) to the generating relations of the so-called Rota topology on the Ω_is [431, 318, 319].

[191] $\mathfrak{T}_i$s are supposed to live in the category $\mathfrak{DT}_i$—the finitary analogue (in fact, a subcategory) of $\mathfrak{DT}$.

system $\overleftarrow{\mathcal{P}}$ of T_0-posets now translates to a projective system of fintriads. But the following question arises now:

How come the finitary differential triads $\mathfrak{T}_i$ in the first place?

The answer to this question follows straight from the aforementioned push-out result (Q?.?) and its quotient space corollary (Q?.?), namely that,

(R4.8) *the continuous canonical projection f_i from X to the moduli space $X/\overset{\mathcal{U}_i}{\sim}=: P_i$ in* (33) *induces (via its push-out f_{i*}) a differential triad on P_i—the fintriad $\mathfrak{T}_i$.*[a]

[a]We tacitly assume here that the locally Euclidean space (manifold) X carries the usual differential structure, which is encoded in the classical differential triad $\mathfrak{T}_\infty \equiv \mathfrak{T}_X = (\mathbf{A} \equiv \mathcal{C}^\infty_X, \partial, \mathbf{\Omega})$ (X a differential manifold).

This is a *direct* way to account for the differential structure that the finitary (originally taken to be purely) topological posets inherit from the continuum, instead of the *roundabout* way, via homological (simplicial) arguments, that we have given hitherto.[192] Moreover, this direct account is an instance of what we call the '*Newtonian spark*' conceptual paradigm of ADG, which we now briefly explain:[193]

[192]Of course, the (Cartan-Kähler) differential operator involved in the incidence algebras of the Euclidean 'triangulations' (nerve-simplices) of X is the same as the usual (de Rham) one of the smooth continuum X, since the Čech-covering sets are 'nice' [271]—for, after all, the Sorkin *ur*-cells (nerves) $\Lambda_{\mathcal{U}_i}(x)$ involved in (32) are nothing else but open subsets of the locally Euclidean X.

[193]We will return to discuss further this paradigm, and in more general terms, in 7.6.1.

(R4.9) In an 'ophelimistic' (or 'opportunistic') sense, while the differential involved in our application of ADG to the finitary context comes from the original continuum X, that manifold is not involved at all, as a background space,[a] in the development of a differential geometry *à la* ADG. Here in particular, the essentially algebraic differential geometric mechanism (in effect, the differential d) is abstracted (induced) from X (via f_{i*}) and used by the algebraic (:sheaf-theoretic) means of ADG in a finitary setting, while at the same time X is being disposed of (here, replaced by the P_is)—*ie*, it does not play any role whatsoever in the *aufbau* of that differential geometry in the reticular realm of the P_is and their associated Ω_is.

[a] As is the case in CDG.

As noted above, in general terms one may characterize this ADG 'attribute' as '*differential geometric forgetful opportunism*', in the following sense: in ADG we do not care where from (*ie*, from what kind of 'space') d comes, but once we have got hold of and secured it, we develop with it all our differential geometric *aufbau* purely algebraically (:sheaf-theoretically), independently of that original 'space'—as it were, regardless of 'the source of d'. Philologically put, from the Newtonian spark (d) we start, by ADG-means, the differential geometric fire, which then burns down that initial 'geometrical background' ('space'). Equivalently, to parallel in a metaphorical way the Tractarian Wittgenstein's words upon concluding [421],

(Q4.6) "...one must so to speak throw away the ladder, after he has climbed up on it..." [421],

similarly here, *for doing differential geometry à la ADG, one must throw away the underlying space(time) after she has gathered a differential from it*, for she does not actually need it (*ie*, here, the base manifold X), and thus, *a fortiori,* she avoids directly various (differential geometric) anomalies and pathologies that are inherent in it (*eg*, 'singularities'). Once we have obtained the differential—the 'Newtonian spark'—from X, we throw the latter away, and we build our differential geometric edifice purely algebraically, in a Leibnizian way, without the background manifold's presence or assistance via the mediation of its smooth coordinate structure (sheaf) $\mathcal{C}^\infty_X$, which anyway carries the differential geometric diseases of the classical theory (CDG).

Now, on with the translation of the inverse limit result in [355] to our finitary-algebraic ADG-theoretic setting involving the fintriads. First we note that, by Gel'fand duality,[194] since the P_is

[194] Or equivalently, by the categorical contravariant functor-duality between the categories of the P_is (and monotone maps) and their Ω_i (and incidence algebra homomorphisms) mentioned earlier.

comprise the *projective* system $\overleftarrow{\mathcal{P}}$, their corresponding incidence algebras Ω_i constitute an *inductive* system $\overrightarrow{\mathfrak{R}}$ [271, 272, 317].[195] Then, the resulting fintriads that the finsheaves of the Ω_is define, can be organized into an inverse/direct system $\overset{\rightleftarrows}{\mathcal{T}}$.[196] Then, thanks to $\mathfrak{DT}$'s bicompleteness,[197] in the same way that Sorkin's projective system $\overleftarrow{\mathcal{P}}$ 'converges', at the inverse limit of infinite refinement, to the original $\mathcal{C}^0$-manifold X,[198] the corresponding inverse system of fintriads yields (or perhaps better, *defines*) at the projective/inductive limit of maximal topological refinement of the underlying P_is and, as a result, of the Ω_i inhabiting the stalks of the finsheaves in the $\mathfrak{T}_i$s, an 'infinitary' differential triad, which we have fittingly coined in the past the `smooth` (or even, '`classical`') continuum differential triad $\mathtt{T}_{\infty} := (\mathcal{C}^{\infty}, \partial, \mathbf{\Omega})$ [271, 272, 317].[199] $\mathtt{T}_{\infty}$ comes as close as possible (via Sorkin's scheme) to the classical one $\mathfrak{T}_{\infty} = (\mathcal{C}^{\infty}_X, \partial, \mathbf{\Omega}_X)$ supported by the differential manifold X.[200] As noted earlier, this 'classical' continuum limit result will play a crucial role in

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[195] In other words, while the system $\overleftarrow{\mathcal{P}}$ of T_0-posets in Sorkin's scheme is an *inverse* one, by Gel'fand duality (or the contravariant functor in the respective categories noted above), the corresponding system of incidence algebras is a *direct* one. Moreover, since we are actually working with (fin)*sheaves* of incidence algebras (over Sorkin's T_0-posets), the inductive system $\overrightarrow{\mathfrak{R}}$ of the latter and its inductive 'continuum' limit may be interpreted as the act of infinite localization of the incidence algebras, defining in the process *stalks* of the '`classical`' continuum sheaves which are inhabited by *germs* of $\mathcal{C}^{\infty}$-`smooth` functions and differential forms over them (see below explanation for the adjective '`smooth`'). Indeed, as also briefly alluded to earlier, the aforesaid contravariant functor may be thought of as a *Gel'fand-type sheafification (sheaf-theoretic localization) functor* [258, 266].

[196] The joint epithet 'inverse/direct' to the system $\overset{\rightleftarrows}{\mathcal{T}}$ pertains precisely to the categorical (contravariant functor), Gel'fand-type of duality mentioned above: while the P_is—the base spaces of the $\mathfrak{T}_i$s—constitute an inverse system $\overleftarrow{\mathcal{P}}$ subject to a projective limit procedure, their categorically, Gel'fand-dual Ω_is—inhabiting the stalks of the finsheaf spaces in the $\mathfrak{T}_i$s—are organized into the direct system $\overrightarrow{\mathfrak{R}}$ which is subjected to an inductive procedure. Informally speaking, in a chiral sense, $\mathcal{U}_i$-refinement for the base P_is goes 'from-right-to-left', while for their associated Ω_is 'from-left-to-right'.

[197] And especially Papatriantafillou's result in [290] giving *the direct limit differential triad of a direct system of differential triads defined on a projective system of base topological spaces* (Q?.?).

[198] Again, modulo Hausdorff reflection.

[199] The denomination '*classical continuum*' for $\mathtt{T}_{\infty}$ comes from the physical interpretation of Sorkin's projective limit for the P_is (and correspondingly, the continuum direct limit of their Ω_is) as Bohr's Correspondence Principle (*ie*, as a *classical limit* process), because of the quantum interpretation that the spaces that the Ω_is represent enjoy [318, 319].

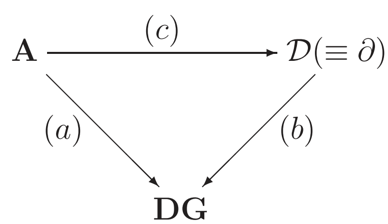
[200] As also pointed out in [272, 317], the expression 'comes as close as possible to $\mathfrak{T}_{\infty}$' pertains to the fact that, much in the same way that one does not actually recover X as the inverse limit space of $\overleftarrow{\mathcal{P}}$, one also does not exactly get $\mathcal{C}^{\infty}_X$ and the $\mathcal{C}^{\infty}(X)$-module sheaf $\mathbf{\Omega}$ of (germs of) smooth differential forms (over the differential manifold X's points) at the direct limit of (infinite sheaf-theoretic localization of) the Ω_is in $\overrightarrow{\mathfrak{R}}$. Rather, similarly to the fact that one gets a 'larger' inverse limit topological space P_{∞} having X as a dense subset in Sorkin's scheme (*ie*, roughly, P_{∞} has more points than X) [355], one anticipates $\overrightarrow{\mathfrak{R}}$ to yield at the inductive limit an abelian ('topological')

our ADG-based finitary-algebraic 'resolution' or 'evasion' of the inner Schwarzschild singularity (5.2.2).

In the next subsection we would like to digress a bit and discuss a basic 'aphorism' in ADG, namely, that all differential geometry is essentially based on the structure sheaf $\mathbf{A}$ of generalized arithmetics or coordinates—the space of 'differentiable' functions that one may freely choose *a priori* in order to base all her differential geometric constructions. From the point of view of CDG, as repeatedly noted earlier, the initial choice is mandatorily $\mathbf{A} \equiv \mathcal{C}^\infty_X$, which is tantamount to fixing a background differential manifold ($X \equiv M$) on which then the whole CDG-edifice is erected and vitally relies. On the other hand, the main didagma of ADG is that one can use structure sheaves glaringly different from the classical one $\mathcal{C}^\infty_M$ and still be able to develop all the traditionally continuum based differential geometric constructions without that locally Euclidean continuum being present in the background.

4.2 All Differential Geometry Boils Down to A: (A)DG Begins with (and Ends in) A

The fundamental motto of ADG can be diagrammatically expressed as follows:



(34)

Let us explain the schema above, starting with arrow (b):

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algebra $\mathcal{A}_\infty$ 'larger' than $\mathcal{C}^\infty(X)$ and consequently an $\mathcal{A}_\infty$-module $\mathcal{R}_\infty$ of differential form-like entities 'larger' than the standard $\mathcal{C}^\infty$-one. In Zapatrin's words, when he was working out continuum limits of incidence algebras of simplicial complexes [432, 433]: "*it is as if too many functions and forms want to be smooth in the continuum limit*" (Roman Zapatrin in private e-mail correspondence with the second author). One intuits that much in the same way that Hausdorff reflection does away with the 'extra points' of P_∞ to recover the $\mathcal{C}^0$-manifold X (as a dense subset of P_∞), so by getting rid of the extra functions and forms on P_∞ from Ω_∞ (*eg*, by factoring it by a suitable differential ideal [432, 433]), one should recover the usual smooth functions and forms over the differential manifold X. Anyhow, the important point here is that *one does indeed get a continuum differential triad*, which however, only in order to be formally distinguished from the classical $\mathcal{C}^\infty$-smooth one $\mathfrak{T}_\infty$ to avoid any minor technical misunderstanding, we might call '$\mathtt{C}^\infty$-smooth' and symbolize it by $\mathtt{T}_\infty$ [272, 317]. On the other hand, after having alerted the reader to this slight distinction between $\mathtt{T}_\infty$ and $\mathfrak{T}_\infty$, in the sequel, for all intents and purposes and in order to avoid proliferation of redundant symbols, we shall abuse language and notation and assume that $\mathtt{T}_\infty$ and $\mathfrak{T}_\infty$ are 'essentially isomorphic' (*ie*, effectively equivalent and indistinguishable) while both will be generically referred to as *the classical continuum differential triad* (CCDT), with the symbols $\mathfrak{T}_\infty$ and $\mathtt{T}_\infty$ used interchangeably.

- (b) All *Differential* Geometry (DG) is subsumed under the (existence of a) differential (operator) $\mathcal{D}$—the connection.[201] In other words, *there is no* **D**G *without a connection*: as a symbolic pun, '$\mathcal{D}$' stands for the letter 'D' in 'DG', and we write:

$$\underline{\mathrm{D}}\mathrm{G} \longleftrightarrow \underline{\mathcal{D}}\mathrm{G} \tag{35}$$

Arrow (b) in (34) above pictures this $\mathcal{D}$-input in DG.

- (a) Of course, $\mathcal{D}$, being an operator (a map!), acts 'somewhere', on some 'domain' or 'action-space'—the *representation* or *carrier space* of $\mathcal{D}$, so to speak. Insofar as the word '*geometry*' (G) enters into the term D**G**, it pertains to specifying the substrate—background as it were—on which $\mathcal{D}$ is soldered and operates. In ADG, that domain—ultimately, the source[202]—of $\mathcal{D}$ is provided by (specifying) $\mathbf{A}$.[203] However, it must be stressed at this point that in ADG if any 'space' is involved at all, it is not assumed *a priori*, but only evoked indirectly, via $\mathbf{A}$.[204] For in any case, in ADG $\mathcal{D}$ is termed an '*algebraic connection*' (or '**A**-*connection*' for short), emphasis being placed here on the epithet '*algebraic*', as the notion of connection is an entirely algebraic one, without any *a priori* geometrical commitment to (or dependence on) an underlying 'space(time)'—a background stage or medium which enables us to represent and interpret(!) $\mathcal{D}$ geometrically. The intention here is to clearly separate and distinguish up-front the purely algebraic $\mathcal{D}$ from its possible '*geometrical realization*' (or representation, and concomitant geometrical interpretation) accomplished by specifying (choosing) a specific $\mathbf{A}$ (and the space inherent in it).[205]

[201] Recall, the basic observation that initially motivated the development of ADG is the recognition that the usual differential $d \equiv \partial$ is a particular (*ie*, a flat) instance of a connection (and vice versa: $\mathcal{D}$ is a generalized, an abstracted and prolongated differential d).

[202] See explanation of arrow (c) below.

[203] And the reader should recall that in ADG the domain of ∂ is $\mathbf{A}$ (10), while of ∂'s generalization $\mathcal{D}$, it is $\mathcal{E}$ (12)—which vector sheaf anyway is (locally), by definition, a finite power of $\mathbf{A}$ ($\mathcal{E}|_U \equiv \mathcal{E}(U) \simeq \mathbf{A}^n$, U open in X).

[204] For we have time and again emphasized it here and in our previous trilogy [270, 271, 272], that if any 'geometrical space(time)' enters into our ADG-considerations at all, it enters through $\mathbf{A}$—in point of fact, *it is inherent in* $\mathbf{A}$ (Gel'fand duality/spectral theory).

[205] In much the same way for example that one should distinguish between an abstract algebraic structure (*eg*, a group) and its representations (*eg*, matrix-operator realizations on some linear action/carrier space); hence the jargon above. This geometrical representation of the algebraic $\mathcal{D}$, and the latter's geometrical interpretation (in that space chosen), recalls the Newton–Leibniz 'dispute' about the notion and meaning of derivative (*viz.* differential d, and in ADG, $\mathcal{D}$) in the context of the infant developmental steps of Calculus (CDG). Leibniz envisaged d as a *relational* (we would nowadays say, *algebraic*) entity independent of a surrounding space and its mediating coordinates *à la* Descartes. Newton on the other hand was more Cartesian, as d for him was heavily burdened by its geometrical representation and interpretation in an ambient space. Recall that for Newton the derivative of

To recapitulate the joint meaning of arrows (a) and (b): *all DG pertains to the activity (action) of some (generalized) differential* $\mathcal{D}$ *exercised (or carried out) on some 'space' ('geometry'), which in turn is inherent in some (freely chosen)* $\mathbf{A}$*—the 'source' (or 'representation') algebra of* $\mathcal{D}$. In turn, $\mathbf{A}$, being the 'geometrical source' of $\mathcal{D}$,[206] allows us to maintain the title of this subsection, namely that,

all DG boils down to $\mathbf{A}$*;*

under the important proviso that it is *us* that choose the representation algebra $\mathbf{A}$ (and, *in extenso*, the space inherent in it) for $\mathcal{D}$.[207]

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a function (at a point in its domain space) is tautosemous to the slope of the tangent line to the curve-graph of the function at the said point—a representation and interpretation that 'masks' in our view the purely algebraic, Leibnizian essence of d. Alas, the manifold based CDG followed Newton's steps, while ADG is markedly Leibnizian. For, as we also noted in [270, 271], if d was an 'inherently' geometrical object, would it not be redundant and meaningless—in fact, begging the question—to ask for its geometrical interpretation? One could imagine for example the following question being asked about a purely geometrical ('spatial') object like the triangle: '*What is the geometrical interpretation of the triangle?*'—plainly, the triangle, being 'inherently' a geometrical object, is in no need of a geometrical interpretation! In contradistinction, it is a meaningful question to ask for the geometrical realization and interpretation of the derivative, which goes to show that d is not an inherently geometrical object; it is an algebraic entity. In the last section (7.3), we will return to discuss further this all important distinction between the Newtonian (what we prefer to call, *Cartesian*) and the Leibnizian (what we coin, *Euclidean*) conceptions (and practices) of DG, by contrasting in the process CDG against ADG.

[206] Again, see explanation of arrow (c) in (34) below.

[207] Our maintaining that the term 'geometry' goes hand in hand with *our* choice of $\mathbf{A}$ becomes even more valid if one interprets (as we have done throughout the aforesaid trilogy) the elements of $\mathbf{A}$ as *generalized coordinates* or *measurements* (the words coefficients and arithmetics are synonyms) of the field $\mathcal{D}$. This is our 'geometrical capturing'—our representing, measuring and concomitant localizing in 'space(time)' (in the 'space' inherent in $\mathbf{A}$!)—of $\mathcal{D}$, which anyway exists independently of us (field realism; see below). For there is no geometry without measurement, and no measurement without us—the 'observers' and 'measurers' ('geometers') to carry it out. *The (algebraic) field ie,* $\mathcal{D}$*) is Nature's; the geometrical representation of it* (*ie*, $\mathbf{A}$) *ours*. Moreover, as it was also emphasized in [271, 272], in a quantum-theoretic sense, the algebraic $\mathcal{D}$ lies on the quantum side of the quantum divide (the so-called Heisenberg *schnitt*), while (the commutative) 'geometric' $\mathbf{A}$ on the classical side. In this sense, '*geometrization (or representation) is classicalization*' (Bohr's correspondence principle). At the same time, *the geometrical representation of* $\mathcal{D}$ *via* $\mathbf{A}$ *results in the geometrical interpretation of the derivative*, as noted two footnotes above. In CDG, at least in the usual Newtonian conception of Calculus, the essentially algebraic character of the differential is in a sense 'masked' by the intervention of (representation) space (manifold) in the guise of *our* choice of $\mathbf{A} \equiv \mathcal{C}^{\infty}_{M}$ for structure sheaf of coordinates, to the effect that one (misleadingly) tends to view $\mathcal{D}$ as a geometrical notion (see 'triangle oxymoron' two footnotes above). Of course, in the classical case (CDG), it is precisely our *a priori* assumption of a background locally Euclidean space (manifold) M that on the one hand mandates $\mathcal{C}^{\infty}_{M}$ for $\mathbf{A}$ and then furnishes us with the usual d (and *in extenso* with the standard smooth $\mathcal{D}$). This assumption masks the basic didactic point of ADG, namely that, *the differential (viz. connection) may come from (the assumption/choice of) structure sheaves* $\mathbf{A}$ *totally different from* $\mathcal{C}^{\infty}_{M}$ *and, as a result, without any a priori commitment to a geometrical base differential manifold* (recall, "*differentiability is independent of smoothness*"

In turn, as noted in the last footnote, all our (generalized) measurements of the (physical) fields (*ie*, the connections)—which measurement-records comprise our geometry, and concomitantly, our own ('mental') fiction of an ambient 'space(time)' (Q?.?, Q?.?)—take values in **A**.[208] In this sense, *(A)DG not only begins with* **A** ('representation-domain'), *but also it ends in* **A** ('value-range').[209]

- (c) This arrow simply reflects the basic ADG maxim mentioned above, namely, that *the differential* $d \equiv \partial$ $(\mathcal{D})$ *comes from algebra.*[210]

- One could compress all the (a)-(c) discussion above into the following symbolic 'pun-equivalence':

$$\underline{\mathrm{AD}}\mathrm{G} \longleftrightarrow \underline{\mathbf{A}\mathcal{D}}\mathrm{G}^{211} \tag{36}$$

4.2.1 The ADG-theoretic Principle of Algebraic Relativity of Differentiability

In the foregoing discussion we saw how 'differentiability' in ADG, freed from the topological shackles of Analysis (CDG), is a purely algebraic notion. Moreover, the geometrical or 'measurement' freedom—*our* freedom—to choose generalized coordinate structure sheaf **A** (other than the standard one $\mathcal{C}^\infty_M$ of CDG) mentioned in connection with arrow (a) in (34) above, reflects what we call the *Principle of Algebraic Relativity of Differentiability* (PARD) in ADG, namely, that

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[271]).

[208] As mentioned repeatedly in the foregoing trilogy [270, 271, 272], from an ADG-theoretic perspective, the 'geometrical objects', or physically speaking, the 'measurable'/'observable' entities in the theory, are basically **A**-valued $\otimes_{\mathbf{A}}$-tensors (or equivalently, **A**-morphisms) built out of the fundamental fields (*viz.* the connections $\mathcal{D}$), such as the curvature $R(\mathcal{D})$ of the connection.

[209] Characteristically, recall from [272] that in ADG the metric ρ—arguably, *the* entity in terms of which one can meaningfully speak about *geometry* proper—is a map (sheaf morphism) with domain $\mathcal{E}$ (in fact, $\mathcal{E}\oplus\mathcal{E}$), which locally reduces to a power of **A**, and range **A** again (*ie*, ρ is an **A**-*valued metric*). Thus ρ (geometry) begins (domain) and ends (range) with **A**.

[210] Also being implicit here that in the classical case of CDG where $\mathbf{A} \equiv \mathcal{C}^\infty_M$, the differential comes from our *a priori* assumption of a base manifold—a conflation of d or $\mathcal{D}$ with the mediation of a background 'space' which masks the differential's essentially algebraic character and misleads one into thinking on the one hand that the notion of connection is geometrical, and on the other that, in one way or another, all DG is supported by a locally Euclidean background space—what we called earlier, the manifold and *in extenso* CDG-conservatism and monopoly.

[211] With a Kleinian replacement of G (by the principal group sheaf $\mathcal{G} \equiv \mathcal{A}ut(\mathcal{E})$ of the field's $(\mathcal{E}, \mathcal{D})$-automorphisms) to be accomplished in sub-subsection 3.2.3 below, where we discuss an ADG-extended (abstract or generalized) version of the PGC of the manifold and CDG based GR.

(R4.10) DG—or how the algebraic field $\mathcal{D}$ is geometrically captured ('measured') and expressed (*ie*, represented as acting on some 'space(time)-geometry', which is anyway inherent in $\mathbf{A}$)—varies for different choices of our arithmetics $\mathbf{A}$: in other words, *choose an* $\mathbf{A}_1$, *and you get a* DG_1; *choose another* $\mathbf{A}_2$, *and you get a different* DG_2.

The import of PARD in applications of ADG to physical problems is significant. For example, concerning the problem of singularities in the spacetime manifold (and, *in extenso*, the CDG) based GR—the main problem addressed in the present paper, namely, changing arithmetics ('coordinates') from the classical ones $\mathbf{A}_1 \equiv \mathcal{C}^\infty_M$ defining M (and, in effect, the CDG) which in turn are responsible for (*ie*, they have built into them) the singularities of GR, to another $\mathbf{A}_2$ conveniently chosen so as to integrate, 'absorb' or 'engulf' those singularities but still retain the essentially algebraic differential geometric mechanism 'supplied' by the (gravitational) field $\mathcal{D}$ (which anyway exists independently of our geometry-defining measurements in the $\mathbf{A}$ we choose to geometrically represent it—field realism), *the problem of singularities simply disappears*. It appears appropriate to metaphorically 'paraphrase' again, in a suitably modified way, Wittgenstein from (an amalgamation of two quotations from) [421] and [422], respectively, *the solution of the problem of singularities is seen in the vanishing of this problem,*[a] *which is achieved simply by changing structure sheaf of generalized coordinate-arithmetics—one's observations-measurements (geometry); and more generally, by changing the way (the theoretical framework within which) one looks at DG as a whole and the so-called singularities troubling the 'conventional' way of looking at DG (ie, the CDG-way effectuated via the background* M*, or equivalently, via* $\mathbf{A} \equiv \mathcal{C}^\infty_M$*!).*[b]

(R4.11) Indeed, ADG is a totally new way of looking at DG, so that what appeared to be problematic from a CDG-standpoint (*viz.* singularities), now completely disappears. Of course, the cause of all these differential geometric anomalies and diseases—the base differential manifold—has been "*pulled out by the roots*", and the new, essentially algebraic, "form of expression" of ADG results in the effective discarding of the 'old' CDG-problems of singularities, which "*are discarded along with the old garment*" of CDG, namely, the 'alchemical' (*ie*, physically non-real or 'fiducial') background spacetime continuum (Q?.?, Q?.?, Q?.?).

[a] First quotation from [421]: "...The solution of the problem of life is seen in the *vanishing* of this problem..." (our emphasis).

[b] Second, more extensive, quotation from [422]: "...The way to solve the problem you see in life is to live in *a way that will make what is problematic disappear*. The fact that life is problematic shows that the shape of your life does not fit into life's mould. So you must change the way you live and, once your life does fit into the mould, *what is problematic will disappear*...Getting hold of the difficulty [or problem] *deep down* is what is hard. *It* [*ie*, the problem] *has to be pulled out by the roots; and that involves our beginning to think about these things in a new way*. The change is as decisive as, for example, that from the alchemical to the chemical way of thinking. The new way of thinking is what is so hard to establish, [but] *once the new way of thinking has been established, the old problems vanish...For they go with our way of expressing ourselves and, if we clothe ourselves in a new form of expression, the old problems are discarded along with the old garment...*" (our emphasis).

It is indeed as if what appeared to be problematic (and, quite paradoxically, of physical value) from the (old/traditional) viewpoint of the manifold based CDG (*ie*, the singularities), under the new prism of the (background spacetime) manifoldless ADG, it appears to be 'poor', of little (if not at all!) physical significance. We cannot refrain from quoting Wittgenstein [422] for the third time here:

(Q4.7) "*…The solution of philosophical problems can be compared with a gift in a fairy tale: in the magic castle it appears enchanted and if you look at it outside in daylight it is nothing but an ordinary bit of iron (or something of the sort)*[a]*…*"

[a] Our emphasis throughout.

and appropriately paraphrasing him to suit the title of the present treatise:

(R4.12) The solution of the problem of singularities can be compared with the various chimerical creatures in mythology: within the mythical realm of the (background spacetime) manifold (based CDG) it appears formidable (indeed, out of reach!), but if you look at it 'outside'—in the base manifoldless environment of ADG—it is nothing but part and parcel of the structure sheaf of generalized arithmetics one chooses to employ (freely and at will!) as generalized coordinates in the theory—*ie*, from the ADG-vantage all singularities are 'coordinate' ones (or something of the sort), while in view of the **A**-functoriality of the gravitational dynamics (Einstein equations) in ADG-gravity, the said singularities are of no physical significance.

In view of the three Wittgenstein quotations and concomitant remarks above, as well as in the two footnotes therein, we cannot resist quoting at this point Wallace Stevens from [376]:

(Q4.8) "*…Progress in any aspect is a movement through changes in terminology…*"[a]

[a] And in the case of ADG, *through changes in theoretical framework—the main change in ADG being to look at and actually do DG without a background manifold.*

In summa, the aforesaid generalized arithmetics' or coordinates' change is from the classical one involving the background differential manifold $\mathbf{A}_1 \equiv \mathcal{C}^\infty_M$ which supports CDG, to another $\mathbf{A}_2$ (thus also a different DG_2 altogether!), which is not only manifestly *not* supported by and effectuated via a base geometrical manifold like in CDG, but also it possibly incorporates the smooth singularities (of $\mathbf{A}_1$) in $\mathbf{A}_2$ and at the same time leaves the essentially algebraic differential geometric mechanism of the $\mathcal{D}$-fields intact and fully operative in their very presence!.[212]

It must be mentioned however that PARD is not a relativity principle 'proper'—*ie*, there is no *transformation theory* linking the different $\mathbf{A}$s [148].[213] This is to be expected since the choice of $\mathbf{A}$ (in effect, of DG!) lies with the 'external' (to the fields themselves) 'observer' or 'experimenter', and it is *she* that chooses how to 'geometrize'[214] the field $\mathcal{D}$, while at the same time, as we have already elaborated on in [272], *such 'generalized coordinate gauge choices' lie with the (macroscopic) exo- or epi-system and not with the (microscopic) endo-system*[215]—they are the experimenter's (measurer's or geometer's) free choices [148].

4.2.2 The field, the whole field, and nothing but the field: an ADG-generalized version of the PGC of the manifold based GR; the case for Synvariance

As we have already pointed out in the foregoing trilogy [270, 271, 272],[216] as well as earlier in the present paper, and as we will also elaborate on further in the sequel[217] since ADG is manifestly base differential manifold independent, and as a result Calculus-free, it manages to bypass without any difficulty various problems that GR, as well as some attempts to quantize it by still retaining though a background spacetime continuum for differentiability's sake,[218] encounter.

Concerning GR, the first 'problematic' issue that is completely evaded is the mathematical representation of the PGC by $\mathrm{Diff}(M)$—the (group of) automorphisms of the external (to the gravitational field) smooth M. On the face of the problems that this representation creates in

[212]See 5.2.2 and 5.2.3 for two explicit and concrete physical examples of this ADG-theoretic application.

[213]Except of course for the generalized 'general coordinate transformations' within the same chosen $\mathbf{A}$. For example, for a chosen, fixed as it were, $\mathbf{A}$ (and *in extenso* $\mathcal{E}$, which by definition is locally $\mathbf{A}^n$), the coordinate relativity/transformation group (supporting our generalized version of the PGC of GR as we will see numerous times in the sequel) is $\mathcal{A}ut\mathcal{E}$ (see 3.2.2 next). Accordingly, for the background manifold M and thus CDG-based GR, the said transformation theory implementing the PGC involves $\mathrm{Aut}M \equiv \mathrm{Diff}M$, it being tacitly assumed in this case that one is working within the category $\mathcal{M}an$ of differential manifolds having $\mathbf{A} \equiv \mathcal{C}^{\infty}_M$ as the chosen structure sheaf of coordinates. However, in the next sub-subsection we shall argue that, categorically speaking, PARD may be expressed via *natural transformations* [252, 253]. In other words, PARD is supported by some kind of '*categorical transformation theory*'—a 'super'-transformation theory since we are not just talking about generalized coordinate changes within the same (chosen) $\mathbf{A}$ (*eg*, $\mathbf{A}$ chosen to be $\mathcal{C}^{\infty}_M$ and the relevant category $\mathcal{M}an$), but about changes between different $\mathbf{A}$s altogether—thus effectively 'transcending' $\mathcal{M}an$ and working within the wider and more flexible (for doing differential geometry) category $\mathfrak{DT}$ of differential triads that we saw earlier.

[214]Here, synonyms to the word 'geometrize' are 'represent', 'coordinatize', 'arithmetize', or even 'localize' and 'particle represent' (as we will see from a geometric prequantization viewpoint subsequently).

[215]Which in ADG are the fields themselves (or perhaps better, '*in themselves*).

[216]Especially in [272].

[217]See for example our ADG-treatment of the Einstein hole argument in GR in 7.5.5.

[218]What we earlier coined 'manifold (and CDG) conservatism'. We shall discuss those problems later in this paper.

trying to cope with (actually, even to define!) singularities in GR as we discussed in the first two sections, the said evasion is more than welcome. However, we would still like to possess an ADG-theoretic analogue of the PGC of the M and CDG-based GR, even in the latter's prominent absence. The analogy is straightforward and goes hand in hand with the following *differential geometric 'correspondence principle'* with CDG that ADG allows us to draw: since CDG can be 'recovered' from (or even be regarded as a very particular—and to that, '*singular*'!—instance of) ADG when one chooses $\mathcal{C}_X^\infty$ for structure sheaf of coefficients ('coordinates') in the theory—a choice which automatically 'converts' the underlying, *a priori* arbitrary, topological space X to a smooth manifold M, and since as noted before in the classical theory $\mathrm{Diff}(M) \equiv \mathrm{Aut}M$, *in ADG the abstract version of the PGC is mathematically represented by the (principal) group sheaf* $\mathcal{G} \equiv \mathcal{A}ut\mathcal{E}$ *of automorphisms of the vector sheaf* $\mathcal{E}$ *involved*. Conversely, $\mathcal{E}$ *is the representation (associated) sheaf of the principal sheaf* $\mathcal{A}ut\mathcal{E}$ [405, 406, 407, 272, 408]—effectively, the action or 'carrier' (sheaf) space of the field $\mathcal{D}$. Thus, in the *Kleinian* sense of the word 'geometry',[219] the principal sheaf $\mathcal{A}ut\mathcal{E}$ of self-transmutations ('auto-symmetries') of $\mathcal{D}$—what we coin '*the esoteric geometry of the field*' in the sequel[220]—is represented (*ie*, acts) on (the local sections of) $\mathcal{E}$ which, from a geometric prequantization vantage, are the (local) particle states of the (prequantized) field [259, 260, 261, 263, 271, 272].[221]

(R4.13)

This is a completely *autonomous* conception and representation of GC—one that is self-sustained by the field and nothing else,[a] which we coin '*synvariance*'.[b]

[a]That is, by no structure other than the field itself—as it were, a structure 'external' to the field (and its 'innate' quanta/particles).

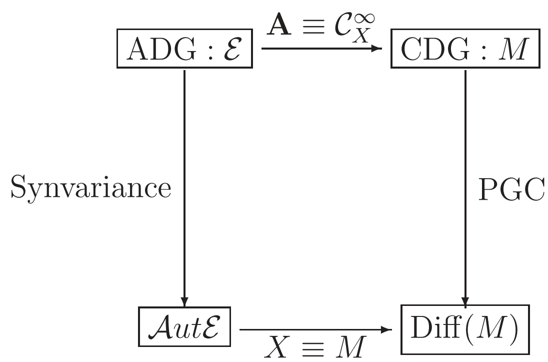
[b]The prefix '*syn-*' here is the Greek correspondent preposition of the Latin prefix '*co-*'. We use the new term 'synvariance', because 'covariance' is heavily 'loaded' with (and 'burdened' by) connotations from the usual, external spacetime continuum based GR, whereas ADG does not allude to, let alone employ, at all a(n external to the connection fields) base manifold in its constructions. Synvariance goes hand in hand with the '*genuinely unitary*', '*pure gauge*' conception and practice of field theory (especially of gravity) that ADG allows us to maintain—a field theory developed with the objects (fields) 'in-themselves', without recourse to a background spacetime structure [272], and *a fortiori*, regardless of whether the latter is a continuum or a discretum. See diagram (37) and relevant discussion next.

[219]Roughly, that the 'geometry' of an object is completely described by its (group of) symmetries.

[220]See (**??**) below. The epithet '*esoteric*' here essentially means '*the auto-transformations of the field in-itself, without recourse to a background spacetime structure (and especially, a manifold!) supporting it*'.

[221]For more about the ADG-based geometric prequantization of gravity, see 6.1 below.

We can formally picture the aforesaid *differential geometric 'correspondence principle'* between ADG and CDG by the following diagram:



(37)

However, in view of our remarks above about synvariance, this analogy/correspondence, and the associated difference in representation of the PGC, between ADG and CDG is far from trivial, as we explain now:

(R4.14) *In ADG, in striking contrast to CDG, no underlying space(time manifold) is employed (ie, mediates) to effectuate the essentially algebraic (ie, Leibnizian-'relational', between the connection fields in-themselves) differential geometric mechanism. All DG is carried out exclusively in the sheaf space $\mathcal{E}$—in point of fact, in terms of the (local) sections of $\mathcal{E}$[a]—with sole resource the algebraic relations between the 'geometrical objects' (fields and their particle-quanta) inhabiting it. As a result, in contradistinction to the manifold and CDG based GR, the PGC in our ADG-theoretic perspective on gravity—ie, the concept of synvariance—concerns solely and exclusively the self-transmutations of the fields in-themselves, without an allusion to or dependence on a background, external to those fields, spacetime manifold.*

[a]For after all, *a sheaf is its (local) sections* [259, 260].

Characteristically, to pronounce the contrast between the manifold and CDG-based GR thus also of the Diff(M)-implemented PGC, and the base space(time manifold) independent formulation of gravity *à la* ADG with the concomitant autonomous conception of field theory and the associated notion of synvariance, we first quote Stachel from [368]:

(Q4.9) "...The general theory of relativity provides a field-theoretical account of gravitation; moreover, *through its incorporation of the principle of general covariance, it seems to demand that any future physical theory that includes gravitation be built on the foundation of the space-time continuum...*"[a]

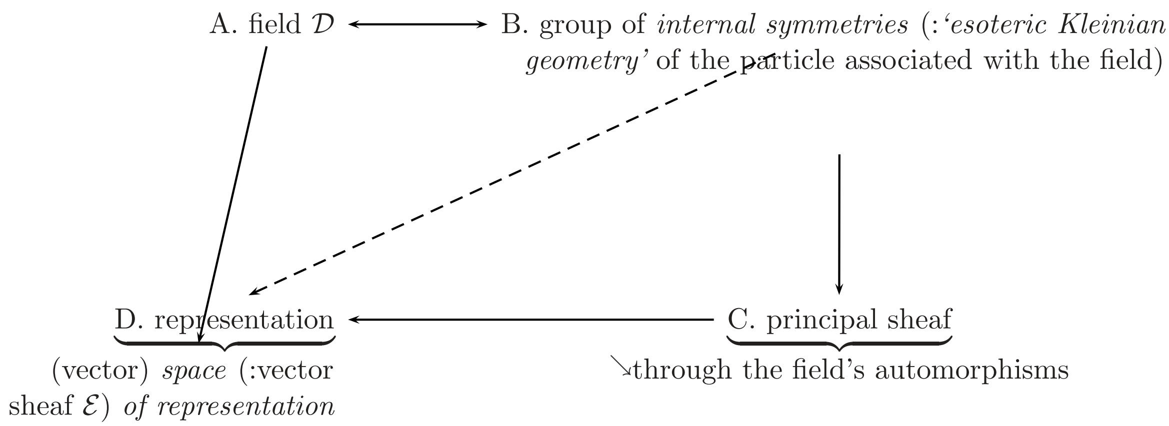
[a]Our emphasis. We will return to comment further on this point—that is to say, about doing field theory ADG-theoretically, independently of a background spacetime manifold, in subsection 7.5.

and juxtapose it with the following remarks of the first author in [264]:

(Q4.10) "...We see that [in ADG] fundamental notions (for example, connections) and relations of a similar nature, are indeed independent of the base space of the sheaves involved. The aforesaid *fundamental notions/relations are*, by definition, *referred to the sheaf-spaces themselves*, thus, *not to the corresponding base spaces...*"

We take the foregoing discussion as an opportunity to complete, in a Kleinian way, the third letter

(G) in the symbolic '<u>AD</u>G $\longleftrightarrow$ $\underline{\mathbf{A}\mathcal{D}}$G'-pun expressed in (36). We do this pictorially by virtue of the following diagram[222]



which we readily explain now clockwise:

- A. In ADG, a field is the pair $(\mathcal{E}, \mathcal{D})$, with the (local) sections of $\mathcal{E}$ representing, from a geometric prequantization viewpoint, the (local) quantum-particle states of the field.

- B. In a Kleinian sense, this is the group of 'internal' symmetries[223] of the particle (states) associated with the field—in effect, these are the self-transformations (technically speaking, the automorphisms) of the field, living in Aut$\mathcal{E}$.

- C. The said automorphisms are in fact organized into the principal (group) sheaf $\mathcal{A}ut\mathcal{E}$, which is thought of as being represented by the (associated) vector sheaf $\mathcal{E}$ of the field—the

[222] We wish to thank Mrs Popi Mpolioti—the secretary of the Algebra and Geometry section of the maths department of the University of Athens (Greece)—for constructing using LaTeX and supplying us with this diagram. This diagram appears in a slightly different guise in the first author's latest book [269].

[223] In keeping with the notion of synvariance, perhaps the epithet 'internal' or 'esoteric'(as opposed to 'external' or 'exoteric', which is usually reserved for the external symmetries of a background spacetime) is redundant here as *there is no external, base spacetime (manifold)—a realm independent of the fields themselves and the dynamics that they define* (as differential equations, being themselves connections). However, we will keep the epithet 'internal' if anything because it invokes ideas from gauge theory (*eg*, the gauge symmetries of a system are usually referred to as *internal symmetries* just to distinguish them from the external, base spacetime manifold ones), and our ADG-theoretic perspective on gravity has been coined '*pure gauge*' or '*genuinely unitary*' field theory [272]. We will comment further on ADG's genuinely unitary field theory, especially *vis-à-vis* the evasion of smooth gravitational singularities and its implications for QG, in the sequel (section 6).

carrier or representation (vector) space of $\mathcal{D}$, the space on which $\mathcal{D}$ acts.[224]

- D. This is the said representation sheaf $\mathcal{E}$ associated with the internal, 'Kleinian auto-symmetries of the particle-field' sheaf $\mathcal{A}ut\mathcal{E}$. Of course, since the field $\mathcal{D}$ *defines* the (vacuum Einstein gravitational) dynamics on $\mathcal{E}$ (29), one may think of the 'auto-transmutations' of the field in $\mathcal{A}ut\mathcal{E}$ as the 'invariances' (symmetries) of the said dynamics. In this sense 'synvariance' may be understood as Sorkin put it recently, in [358], in the context of the 'genealogical growth-dynamics' of causets: "*general covariance and becoming coexist*".[225]

All in all, in view of the above, and in the Kleinian sense of the word 'geometry', we may complete the symbolic pun-equivalence in (36) to:

$$ADG \longleftrightarrow \mathbf{A}\mathcal{D}\mathcal{G} \tag{38}$$

In view of synvariance and the points A-D above, some remarks are due now:

1. We first come to ask in an Einsteinian way, but rather rhetorically: *What 'space(time)' does the field* (or its quanta-particles) *'see' or 'feel' in the course of its dynamics?*[226] In view of synvariance, the answer is an emphatic '*no space(time)!*'. If there is any *physical* space(time) that $\mathcal{D}$ 'sees' or 'feels' in its dynamical self-propagation (autodynamics), it is the '*solution space(time)*' of the law that $\mathcal{D}$ obeys (in fact, that $\mathcal{D}$ *defines!* as a differential equation) in the first place. In a strong sense, *physical spacetime is inherent in the* $\mathcal{D}$*-dynamics, not an ambient 'kinematical' realm that* $\mathcal{D}$ *can see during its dynamical evolution—an ambient realm that can host and at the same time delimit* $\mathcal{D}$*'s possible moves (kinematics).*[227] In turn, that $\mathcal{D}$ sees no *a priori* existing 'space(time)' (like for instance a manifold)—a *background geometry* (which, as we noted above, is anyway inherent in $\mathbf{A}$)—is reflected on the fact

[224] A space which is, by definition, locally isomorphic to $\mathbf{A}^n$, and constituting the 'local geometry' of the field ($\mathcal{D}$)/particle ($\mathcal{E}$). (Parenthetically, we see here the traditionally quantum field-particle duality encoded in the fundamental ADG-conception of field as a pair $(\mathcal{E}, \mathcal{D})$. That is, *the ADG-notion of field is 'self-dual'*. Later on, in 6.2, we will make further remarks on its quantum-theoretic significance.) Anyway, as a result of the local isomorphism $\mathcal{E} \overset{\text{loc.}}{\simeq} \mathbf{A}^n$, locally $\mathcal{A}ut\mathcal{E}|_U = M_n(\mathbf{A}(U))$ (U open in X).

[225] The term 'becoming' here meaning, in analogy to the causal sequential growth of causets in Sorkin's scheme, the dynamical $\mathcal{D}$-connection of local ('graviton') states represented by the local sections of $\mathcal{E}$ on which the field $\mathcal{D}$ acts.

[226] And we call this kind of question Einsteinian, because it recalls one of the primitive, gedanken-questions that Einstein asked originally in formulating the theory of relativity. In a slightly modified way, he pondered on the question: '*If I was riding on a beam of light, what would I see—what would spacetime look like?*'.

[227] The priority of dynamics over kinematics will be discussed analytically in 3.2.5 next, and in connection with Einstein's hole argument in 7.5.5.

that *the ADG-theoretic expression of Einstein's equations involves the curvature* $R(\mathcal{D})$ *of the field (viz. field), which is an* $\otimes_{\mathbf{A}}$*-tensor, or better, an* $\mathbf{A}$*-sheaf morphism.* Thus, *the dynamics sees through our geometry (:'spacetime') built into* $\mathbf{A}$*, and is not affected at all by it* (synvariance)— *ie*, ultimately, $\mathbf{A}$ *(:our generalized measurements) plays absolutely no role in the gravitational field dynamics* (:field realism). Our geometry (in $\mathbf{A}$) does not affect the field-dynamics.[228]

2. We now come to emphasize again that $\mathcal{E}$, which by definition is locally $\mathbf{A}^n$, is the 'representation sheaf' associated with the principal group sheaf $\mathcal{A}ut\mathcal{E}$ of self-transmutations ('esoteric symmetries') of the field. In turn, since from a gemetric prequantization perspective $\mathcal{E}$ is the 'particle representation sheaf' of the field, in the sense that local sections of $\mathcal{E}$ correspond to local (*ie*, localized in 'spacetime', or what amounts to the same, measured in $\mathbf{A}(U)$) quantum-particle states of the field, this picture justifies our remark earlier that $\mathbf{A}$ is the 'geometric representation' of the field—the 'spati(otempor)al' (*ie*, localized in the 'space(time)' built into $\mathbf{A}$), particle (quantum) aspect of the field $\mathcal{D}$ which is being captured exactly by the $\mathbf{A}$-valued local sections of $\mathcal{E}|_U \simeq \mathbf{A}^n(U)$ (*ie*, vector n-tuples with entries in the arithmetics' structure sheaf $\mathbf{A}$ relative to a local gauge U in X [259, 260, 272]).

3. We come to stress again that since synvariance is modelled after the self-transformations of the field in $\mathcal{A}ut\mathcal{E}$, the ADG-theoretic conception of gravity is as a pure, genuinely external spacetime-free, gauge field theory. All there is 'out there' is the (field-$\mathcal{D}$ and particle-$\mathcal{E}$ aspect of the) field $(\mathcal{E}, \mathcal{D})$,[229] and no spacetime (especially a manifold) external to that field is involved at all. In turn, the action of the principal $\mathcal{A}ut\mathcal{E}$ on its associated $\mathcal{E}$ (*ie*, on the local particle states of the field) makes the field a 'quantum fuzzy', 'foamy' entity.

4. The last sentence above prompts to remark that *if any sort of 'noncommutative geometry' is involved at all in our scheme, that is a '*noncommutative Kleinian geometry*' being (locally) effectuated in our theory by* $\mathcal{A}ut\mathcal{E}(U) = (M_n(\mathbf{A}))^\bullet(U)$ (U open in X).[230]

[228]This 'extreme' field realism does not preclude a quantum interpretation for the 'self-dual' ADG particle-field pair $(\mathcal{E}, \mathcal{D})$, which we give in 6.2. It just puts into perspective the by now standard '*external (to the quantum system) observer or measurer dependence of physical reality*' that the standard quantum theory appears to have forced on us.

[229]Again, for a discussion of the self-quantum-dual ('self-complementary') particle-$\mathcal{E}$ and field-$\mathcal{D}$ aspects of the ADG-field pair $(\mathcal{E}, \mathcal{D})$, the reader should wait for 6.2.

[230]For, interestingly enough, a rather canonical example of a non-abelian C^*-algebra, which seems to crop-up frequently in Connes' noncommutative geometry [91], is $M_n(\mathcal{C}^0(X))$—the algebra of $n \times n$-matrices of continuous functions on a locally compact space X, vanishing at infinity [47]. Thus, if any 'noncommutative space(time) (geometry)' might creep into our theory, that may as well be through the spectrum of $M_n(\mathbf{A})$, in much the same way that, as noted earlier, a (commutative) space(time) geometry is spectrally built into the abelian structure sheaf

4.2.3 The issue of the A-functoriality of the gravitational dynamics and the PARD expressed categorically via natural transformations

Having in hand the remarks above about synvariance (the fields' '*external spacetimeless auto-covariance*'), one might go a bit further and claim that the 'bottom-line' of ADG concerning the expression of the (vacuum) Einstein equations in (29) is that,

(R4.15) The mathematical expression (:differential equation) for the gravitational dynamics (in vacuo) is *functorial* with respect to our generalized measurements (coordinates) in **A**. Equivalently, the said expression involves mathematical quantities (measurable dynamical variables commonly known as 'observables') that are $\otimes_{\mathbf{A}}$-tensors[a] (*alias*, 'geometrical objects' which 'by definition' our generalized arithmetics, our measurements defining as it were our 'spacetime geometry' encoded in **A**), respect. Or perhaps even better expressed in a sheaf-theoretic parlance, the said 'geometrical objects' are **A**-morphisms, the prime example being the curvature $R(\mathcal{D})$ of the connection (field) $\mathcal{D}$ involved in (29) [268]. Effectively, this is the content of the PGC of GR[b] when viewed ADG-theoretically.[c]

[a]Again, with $\otimes_{\mathbf{A}}$ the homological tensor product *functor*.

[b]As well as of the generalized Principle of Relativity in (R2.?) before, and of Einstein's 'definition' of (objective) physical reality associated with the latter (see footnote 5).

[c]The deeper physical meaning of this functorial expression of the PGC of GR will be unveiled in 7.5.5 where we discuss *à la* Stachel the deeper significance of Einstein's hole argument, albeit, from our ADG-theoretic perspective.

The remarks above about functoriality *vis-à-vis* the PGC of GR and the 'observables' involved in its dynamics[231] can be further supported by Baez's words taken from [27]:

A (Gel'fand duality). It is also interesting to note here that, in the general search for noncommutative spaces, what figures prominently is the formulation of a noncommutative version of the Gel'fand-Naimark representation theory for commutative C^*-algebras—a search culminating in noncommutative spaces known as C^*-quantales [281, 283, 284].

[231]Again, in our ADG-case, the Ricci curvature involved in (29), which is what we call a '*geometrical object*'—*ie, an* **A**-*sheaf morphism or* $\otimes_{\mathbf{A}}$-*tensor*—an object (better, map) which is respected by our generalized measurements (arithmetics) in **A**.

(Q4.11)

> "...We can also express the principle of general covariance and the principle of gauge-invariance[a] most precisely by saying that *observables are functorial.*[b] So physicists should regard functoriality as mathematical for 'able to be defined without reference to a particular choice of coordinate system'[c]..."
>
> [a]Two notions that are not distinct from each other (in fact, in a sense they are identical to each other!) in our ADG-theoretic perspective on GR, since gravity according to ADG is another gauge theory [272]—and what's more, a 'pure', 'genuine', external (background) spacetime-less (or spacetime-independent) gauge theory (*ie*, *a gauge theory of the 'third kind'*, as we shall see in 3.3 shortly).
>
> [b]Our emphasis. ADG has furthermore taught us that it is important to ask *relative to what* (*ie*, to what structure) *is something* (here the law of gravity) *functorial?* ADG's answer is: *relative to our generalized coordinates ('measurements') in* $\mathbf{A}$.
>
> [c]And what's more from an ADG-theoretic viewpoint, the law of gravity is possible to be defined, as a differential equation proper, without reference not only to a particular choice of a system of smooth coordinates, but also to *any* smooth coordinate frame with values in $\mathcal{C}^\infty_M$—*ie*, without reference to smooth background spacetime manifold M.

Functoriality is not merely 'freedom from coordinates'. There is a subtle point in the last sentence of the quotation above and in the footnote that follows it that we would like to discuss briefly here in order to avoid misunderstandings concerning the significance of functoriality in the application of ADG to gravity. The $\mathbf{A}$-functoriality of ADG is not just 'coordinate independence' or 'coordinate-free definition/description' as one might be tempted to read Baez's concluding remark above having in mind the so-called coordinate-free descriptions of modern CDG. To be sure, in the modern, coordinate-independent formulation of differential geometry one is able to write the Einstein equations in a coordinate-free manner—*ie*, without explicit commitment to a particular reference system of (still though, $\mathcal{C}^\infty$-smooth!) coordinates. However, in *actual calculations*, $\mathbf{A} \equiv \mathcal{C}^\infty_M$ (*ie*, the background manifold M) is invariably invoked, for after all, *the coordinate-free modern differential geometry still is 'CDG in disguise'*. For otherwise how else could one, for instance, write down Einstein's equations as *differential* equations proper?[232] The fact is that $\mathcal{C}^\infty_M$ is always implicitly present there, and when it is invoked to actually do CDG or Calculus (*ie*, in actual, concrete calculations!), it brings along all the differential geometric pathologies and anomalies (*eg*, singularities) that are inherent in the background manifold.

All this has already been anticipated and discussed in [265]. What we want to emphasize here

[232]Here is again the differential manifold conservatism and monopoly.

is that the **A**-functoriality of the gravitational dynamics formulated in ADG-theoretic terms goes much deeper than a superficial 'freedom from coordinates'. Essentially, it means that *the dynamical law of gravity is free from* any *coordinates—smooth* ($\mathbf{A} \equiv \mathcal{C}^{\infty}_{M}$) *or other*, which in turn means, via the interpretation that we have given to (the elements of) **A** as *our* generalized 'measurements' of the gravitational field and as *our* 'spacetime (differential) geometric' representation of it (which is inherent in **A**), on the one hand that *the law of gravity—ie*, the connection field $\mathcal{D}$ defining it—*is 'out there', unaffected by our 'measurement perturbations' and 'geometrical representations' (of it) in* **A** (PFR), and on the other, that (29) *is a genuinely background spacetime (manifold or not) independent description of gravitational dynamics*[233]—and what's more, *this still is a differential geometric description proper*.

We will return to discuss the issue of 'dynamical **A**-functoriality' (as opposed to merely a 'kinematical freedom from coordinates' one)—especially in the light of first, second, and what we call *third*, quantization—in 6.1.2, but here we would like to express the PARD via the categorical notion of *natural transformation*. The reader may already be aware of the fact that the *raison d'être*, or even *de faire*, of category theory is the notion of *natural transformation*. For one may recall Saunders MacLane, one of the founders of category theory, claiming in [252] that

(Q4.12)

> "...Category theory was not invented to talk about functors. It was invented to talk about *natural transformations*[a]..."
>
> [a]Our emphasis. The reader should note that in earlier drafts of the paper we inappropriately used the first-person, *ie*, '*I invented*', when, as a matter of well known fact, category theory was the making of both Samuel Eilenberg and Saunders MacLane. The second author apologizes for the misquotation.

Now then, having established that the categorical notion of *functor* plays a central role in the expression of (vacuum) gravitational dynamics *à la* ADG as it (*ie*, functoriality with respect to the structure sheaf **A**) effectively represents the PGC of GR in the more general and abstract, as well as external spacetimeless, terms of ADG, we further maintain that the even more important (at least according to MacLane) notion of *natural transformation* has a 'natural' (pun intended!) physical realization in the PARD discussed above. To explain this, recall first that, technically speaking, *a natural transformation is a map* $\mathcal{N}$ *between functors* (say for example, functors F and G between two categories C_1 and C_2) such that the following diagram commutes

[233]These two points will be further corroborated shortly when we express the PARD in terms of natural transformations.

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \Big\downarrow{\scriptstyle \mathcal{N}(A)} & & \Big\downarrow{\scriptstyle \mathcal{N}(B)} \\ G(A) & \xrightarrow[G(f)]{} & G(B) \end{array} \tag{39}$$

with $A, B \in \mathrm{Obj}(C_1)$, $f \in \mathrm{Hom}(A, B) \in \mathrm{Arr}(C_1)$, $[F(A), F(B), G(A), G(B)] \in \mathrm{Obj}(C_2)$, and $[F(f) \in \mathrm{Hom}(F(A), F(B)), G(f) \in \mathrm{Hom}(G(A), G(B))] \in \mathrm{Arr}(C_2)$. *In summa*,

$$G(f) \circ \mathcal{N}(A) = \mathcal{N}(B) \circ F(f) \tag{40}$$

Analogously, one may think of the aforementioned changes of structure sheaves $\mathbf{A}_1 \longrightarrow \mathbf{A}_2$ involved in the PARD as effecting natural transformation-type of changes between the functorial expression of (vacuum) gravitational dynamics (29) relative to $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively. Heuristically, one may cast this by the following natural transformation-type of commutative diagram:

$$\begin{array}{ccc}
\mathbf{A}_1(:\mathcal{E}_1 \overset{\text{loc.}}{\simeq} \mathbf{A}_1^n) & \xrightarrow{\otimes_{\mathbf{A}_1}} & \mathcal{R}(\mathcal{E}_1)=0 \\
\Big\downarrow \mathcal{N}_{\mathbf{A}} & & \Big\downarrow \mathcal{N}_{\mathcal{D}} \\
\mathbf{A}_2(:\mathcal{E}_2 \overset{\text{loc.}}{\simeq} \mathbf{A}_2^n) & \xrightarrow[\otimes_{\mathbf{A}_2}]{} & \mathcal{R}(\mathcal{E}_2)=0
\end{array} \tag{41}$$

which we explain below:

1. First (upper left corner of the commutative square above), one chooses $\mathbf{A}_1$ to 'catch' (solder/localize) and (differential) geometrically represent (on $\mathcal{E}_1 :\overset{\text{loc.}}{\simeq} \mathbf{A}^n$) the gravitational field (*viz.* connection $\mathcal{D}$).

2. Then (upper right corner), the gravitational dynamical law that the field *defines* is expressed *functorially*, with respect to the $\mathbf{A}_1$ chosen, via the homological tensor product functor $\otimes_{\mathbf{A}_1}$—or what amounts to the same, via the (Ricci) curvature tensor of the connection field, which is an $\mathbf{A}_1$-morphism: $\mathcal{R}(\mathcal{E}_1)=0$.

3. Then one considers a change $\mathcal{N}_{\mathbf{A}}:\ \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ of structure sheaf of generalized arithmetics—*ie*, one assumes a different $\mathbf{A}_2$ ($\mathbf{A}_2 \neq \mathbf{A}_1$) via which one can (differential) geometrically represent the gravitational connection field on another vector sheaf $\mathcal{E}_2 :\overset{\text{loc.}}{\simeq} \mathbf{A}_2^n$, and again express the gravitational law that the field obeys $\mathbf{A}_{\mathbf{A}_2}$-*functorially* relative to the new $\mathbf{A}_2$, as: $\mathcal{R}(\mathcal{E}_2)=0$ (again, with $\mathcal{R}(\mathcal{E}_2)$ an $\mathbf{A}_2$-morphism).

4. Finally, the 'natural transformation' $\mathcal{N}_{\mathbf{A}}$ of structure sheaves is supposed to 'induce' (or 'lift' to) a corresponding change $\mathcal{N}_{\mathcal{D}}$ between the $\mathbf{A}$-functorial expressions of the gravitational field law; formally: $\mathcal{N}_{\mathcal{D}}:\ \mathcal{R}(\mathcal{E}_1)=0 \longrightarrow \mathcal{R}(\mathcal{E}_2)=0$.

5. *In toto*, heuristically, and in loose formal analogy to (40) above:

$$\otimes_{\mathbf{A}_1} \circ \mathcal{N}_{\mathbf{A}} = \mathcal{N}_{\mathcal{D}} \circ \otimes_{\mathbf{A}_2} \tag{42}$$

All in all, the PARD can be represented in ADG as a '*natural transformation*' kind of change ('map') between the $\otimes_{\mathbf{A}}$-functorial expressions of the gravitational field ($\mathcal{D}$) dynamics (vacuum Einstein equations). The said transformation $\mathcal{N}$ is in fact a pair $(\mathcal{N}_{\mathbf{A}}, \mathcal{N}_{\mathcal{D}})$ of 'adjoint' maps, with $\mathcal{N}_{\mathbf{A}}$ the actual change of structure sheaves, and $\mathcal{N}_{\mathcal{D}}$ the induced passage from the expression of (vacuum) gravitational field ($\mathcal{D}$) dynamics in the generalized coordinates $\mathbf{A}_1$, to the corresponding expression of the *same* field[234] dynamics in $\mathbf{A}_2$.[235] As noted earlier in footnote ???, underlying PARD is this '*categorical transformation theory*' expressed via natural transformation-type of maps.

Of course, since, as we saw before, *the choice of generalized arithmetics* **A** *is ours and not Nature's own*, one should be cautious and 'critical' of the epithet 'natural' in the term 'natural transformations' (*ie*, 'natural' is not 'Natural'; pun intended). On the other hand, the PARD, viewed in the manner above as a natural transformation ($\mathcal{N}_{\mathcal{D}}$) between the **A**-functorial gravitational dynamics for different choices (changes) of **A** ($\mathcal{N}_{\mathbf{A}}$), reveals something quite natural indeed, namely, that *with respect to the gravitational dynamics*—essentially, with respect to the gravitational field $\mathcal{D}$ defining that dynamics (as a differential equation proper)—*all choices of structure/coordinate sheaf* **A** *are on an 'equal footing', to the effect that the dynamics itself is indifferent to and unaffected by* ('invariant' so to speak, *ie*, **A**-functorial with respect to) *any such choice*. To stress it once more, the gravitational field dynamics 'sees through' our coordinates, our generalized measurements (:perturbing acts of localization) of the gravitational field (*viz.* connection) which are organized in **A**. Which brings us to the deeper meaning of the PARD.

4.2.4 What is being 'relativized' and what remains 'invariant' by PARD?: the Principle of Field Realism (PFR) abolishing the differential manifold as the last relic of an ether-like substance in the manifold based GR

Plainly, what is being relativized by the PARD is **A**—the structure sheaf of our generalized coordinates ('field measurements'), while since all DG boils down to **A** as we argued earlier, *what*

[234] That is, same gravitational field $\mathcal{D}$. The PFR is implicit here (see next paragraph and sub-subsection): the (gravitational) field $\mathcal{D}$ (defining the law of gravity as a differential equation) exists 'out there' independently of (*ie*, it remains unperturbed, 'invariant' with respect to) our various 'geometrical representations' of it when we employ different kinds of **A** (and, in effect, different associated/representation sheaves $\mathcal{E} \overset{\text{loc.}}{\simeq} \mathbf{A}^n$).

[235] The expression (42) can be formally stated as follows: *the structure sheaf morphism (change)* $\mathbf{A}_1 \longrightarrow \mathbf{A}_2$ *'lifts' to a natural transformation-type of map between the corresponding homological tensor product functors* $\otimes_{\mathbf{A}_1}$ *and* $\otimes_{\mathbf{A}_2}$ *respectively, with respect to which the dynamics is expressed via the respective* $\otimes_{\mathbf{A}}$*-tensors* $\mathcal{R}$.

is in effect relativized is 'differentiability' itself. That is to say, the *a priori* (and in that sense, absolutely!) fixed 'classical' differentiability or smoothness that comes with assuming $\mathbf{A} \equiv \mathcal{C}^\infty_M$ for structure sheaf,[236] is now relativized by allowing other structure algebra sheaves, different from the classical one of $\mathcal{C}^\infty_M$, be used in (A)DG.[237] At the same time, as briefly noted above, what remains 'form-invariant' so to speak is the differential equation-expression of the field dynamics, with the gravitational connection field $\mathcal{D}$, 'underlying' the Ricci curvature **A**-tensor, 'remaining intact at the background'—being unaffected by these changes of **A**. It follows that the gravitational field $\mathcal{D}$—and, *in extenso*, the gravitational field law (vacuum Einstein equations) that it defines—remains unaffected, 'unperturbed' by our generalized 'measurements' (of it) in **A**. *In toto*, ADG allows us not only to generalize the PGC of GR to synvariance by abolishing the external, base spacetime continuum (manifold), but also, via the natural transformation-type of representation of the PARD, to generalize Einstein's PR[238] in a way that it suits the fundamental (gravitational) field solipsism underlying synvariance, in the following way:

(R4.16) **PFR:** *the gravitational field* ($\mathcal{D}$), at least in its differential geometric expression (representation) via its curvature $R(\mathcal{D})$ in the dynamical (differential) equation modelling the physical law of gravity in ADG, *remains unaffected ('unperturbed')*[a] *no matter what* **A** *one employs to 'geometrically represent', 'measure', 'localize', or 'arrest'/'freeze' it (in spacetime).* In this sense we say that at the base of the PARD lies the PFR.[b]

[a]And the corresponding dynamical law is **A**-functorial.

[b]Of course, as we shall argue extensively in 6.2 subsequently, the 'genuinely unitary' particle-field pair $(\mathcal{E}, \mathcal{D})$ is indeed 'internally perturbed'—in a quantum-theoretic sense—by our 'geometrization' or 'localization' actions in (each chosen) **A**. Accordingly, the field itself is '*self-quantum*', or even '*quantum foamy*', being (dynamically) self-transmuted by $\mathcal{A}ut\mathcal{E}$, but *our* (differential) geometric expression of the law that the field obeys (or defines as a differential equation) is **A**-invariant (functorial) indeed.

[236]Or equivalently, *a priori* fixing the base space(time) to a differential manifold (1) securing thus the CDG-conservatism and monopoly.

[237]Once again, "*differentiability is independent of smoothness*" [271].

[238]Again, see footnote 5.

4.2.5 Kinematics before dynamics, or vice versa?; 'dynamicalization' of coordinates and space(time): space(time) structure ('geometry') is 'inherent' in (is a consequence of) the dynamical field ('algebra'); '$\mathcal{D} \Longrightarrow R(\mathcal{D})$'

The fundamental background spacetimelessness and 'synvariance', the **A**-functoriality, the PARD, and its deeper consequence, the PFR (and field autonomy and solipsism), discussed above, especially in view of our remarks about the role that the smooth coordinates play in GR in 2.2.1 and 2.2.2 before, in a deep sense force us to posit that in the ADG perspective on gravity (GR), and contrary to the current theoretical physics 'consensus, *dynamics is prior to kinematics.* Let us explain this by first arraying certain pertinent quotations that apparently point to the opposite.

We read from [128] that Einstein, in formulating GR, in a clear and direct way posited what one would now call *the priority of kinematics over dynamics*—the basic idea there being that one should set up the kinematical variables *before* prescribing the dynamical equations of motion that these variables satisfy:

(Q4.13) "...The major question for anyone searching in this field is this: Of which mathematical type are the variables (functions of the coordinates) that permit the expression of the physical properties of space ('structure')? *Only after that:*[a] What equations are satisfied by those variables?"

[a]Our emphasis.

Einstein then goes on and describes the transition from kinematics to dynamics in GR as the theoretical procedure involving first the kinematical structure of SR—*ie*, the flat, (gravitational) field free Minkowski space, with its 'observable' (in his own words, "*measurable, with real physical meaning, quantity*") $ds^2 = g_{\mu\nu}dx_{\mu}dx_{\nu}$—the infinitesimal distance between spacetime events involving the quintessential chronogeometric structure of Minkowski space: $g_{\mu\nu} \equiv \eta_{\mu\nu}$. He then recognizes that the transition from SR to GR essentially involved the 'dynamicalization' of the metric structure of SR—that is, of $g_{\mu\nu}$. *In summa*, and in view of the quotation (Q?.?) above, (R4.17) Einstein's approach to GR (starting from SR) set the paradigm (or at least the trend) for almost all physical theory formation attempts in the sequel, namely, *by first delimiting what can possibly happen*—the structure of the nowadays called *kinematical space* of the theory—as well as the to-be-dynamically-variable quantities in it, *and then what actually happens*—the *dynamical equations* satisfied by those variables; or in other words, the *dynamics* of the theory. In this sense, it is fair to say that in every physical theory-construction procedures nowadays, *kinematics comes before dynamics.*[a]

[a]See (Q?.?) below where Sorkin in [357] 'quotes' Taketani from [379].

The description above is tautosemous to the by now well established fact that, while SR is in a strong sense all about (Minkowski) spacetime kinematics,[239] GR is a 'dynamicalization' of SR by making the latter's main kinematical entity—the spacetime manifold's metrical structure $g_{\mu\nu}$—a dynamical variable.[240]

More recently, this priority of kinematics over dynamics in setting up a physical theory has been emphasized by Sorkin in [357], where he describes 'historically' the conceptual development and construction of his 'discrete' causet theoretical scenario for QG. Sorkin, inspired by the writings of Taketani [379] on the various stages through which a physical theory (such as Newtonian mechanics) passes during its upbringing, notes characteristically:

(Q4.14) "...Equally important [in the construction of causet theory][a] however, was the influence of M. Taketani's writings, which convinced me that *there is nothing wrong with taking a long time to understand a structure 'kinematically' before you have a real handle on its dynamics...*"

[a]Our addition for completeness.

And even more recently, and importantly for what we want to highlight here in the context of the ADG-formulation of gravity, Sorkin clearly delineates the kinematical structure of GR [359]:

[239]That is, the chronogeometric structure of Minkowski space is fixed and purely kinematical, not dynamical (in pictorial geometrical terms, Minkowski space is flat).

[240]From which it follows that the kinematical space of GR, at least in its original formulation in terms of $g_{\mu\nu}$, is the space of Lorentzian metrics ('superspace').

(Q4.15) "...In the case of General Relativity, the Kinematics comprises a *differentiable manifold* M of dimension four, a *Lorentzian metric* g_{ab} on M, and a structure which, although it is closely intertwined with the metric, I want to regard as distinct, namely the *causal order-relation* $\prec$[a]..."

[a]All emphasis above is ours.

In what follows in the present paper, we will see that in the ADG-perspective on gravity, all the three 'traditional' kinematical structures of GR that Sorkin outlines above are replaced by the single notion of *connection* (field) $\mathcal{D}$, and more importantly, that this notion effectively reverses the roles of kinematics and dynamics in the sense that *in ADG-gravity dynamics comes before kinematics*. To dwell a bit more on these remarks without going into any detail for the time being, first, as we have been arguing numerous times above, *in ADG there is no base differential manifold*. Second, as we will see soon, the metric is not *the* fundamental dynamical variable in ADG-gravity, *the connection is*. And third, we shall see that 'causality', in a generalized (abstract) sense, is already represented by $\mathcal{D}$—*ie*, *the causal connection (nexus) is already encoded in the gravitational* $\mathbf{A}$*-connection* $\mathcal{D}$ (pun intended). In addition, we will see as a consequence of these replacements that even the fundamental in GR notion of *spacetime events* is substituted in ADG by the notion of the (autonomous) field $(\mathcal{E}, \mathcal{D})$. The bottom-line is that in ADG the basic notion of $\mathbf{A}$-connection (field) is a dynamical one, since $\mathcal{D}$ *defines* (via its curvature) the dynamical (differential) equations of Einstein; moreover, by virtue of the $\mathbf{A}$-functoriality of that dynamics, whatever 'background kinematical spacetime geometry' is inherent in $\mathbf{A}$, it is 'transparent' to $\mathcal{D}$—the field 'sees through it' and remains undisturbed by it.[241]

Of course, in ADG we still retain the traditional (in physics) Aristotelian categories of *possibility* (in the form of kinematics) and *actuality* (in the form of dynamics) in that we still possess a different, more abstract notion of gravitational kinematics in ADG-gravity, since now the possibility space is the affine space $\mathsf{A}_{\mathbf{A}}(\mathcal{E})$ of $\mathbf{A}$-connections;[242] furthermore, the said (vacuum) Einstein equations can be derived from varying (extremizing) the ADG-analogue of the Einstein-Hilbert action $\mathfrak{E}\mathfrak{H}$, which is an $\mathbf{A}$-valued functional on the said abstract kinematical space of 'gauge equivalent'

[241]The epitome here, especially *vis-à-vis* singularities, is that no matter what singularities are present in the structure sheaf $\mathbf{A}$ that one chooses to employ, these pose no obstacle or 'breakdown threat' to the dynamical law of gravity defined by the field $\mathcal{D}$. The latter 'sees through' singularities and remains virtually unperturbed by them (DGS beware!).

[242]Or to be more precise, in view of synvariance, the moduli space $\mathsf{A}_{\mathbf{A}}(\mathcal{E})/\mathcal{A}ut\mathcal{E}$ [272]. Of course, it must be noted here that in the current Ashtekar connection-based approaches to both classical and quantum gravity, the moduli space of smooth connections modulo $\mathrm{Diff}(M)$ is regarded as the relevant gravitational kinematical space, but still this has obviously not been entirely freed yet from the 'geometrical shackles' of a background differential manifold [272].

connections [272]. Yet, because there is no fixed background differential manifold in ADG-gravity, we are reluctant to give the terms gravitational kinematics and dynamics the same meaning that they have in the usual background M-dependent theory (whether classical or quantum). For one thing, in view of the PFR the dynamical gravitational field $\mathcal{D}$ exists autonomously 'out there' and it is *us*—the external (to the field) observers (experimenters or 'measurers'/'geometers')—that bring along our own $\mathbf{A}$, with the kinematical-geometrical space(time) built into it, in order to 'geometrize' (*ie*, geometrically represent) it. Arguably, kinematics lies with the (external) experimenter (the 'exosystem' [148]), who in the course of an *a priori* planned experiment 'constrains' or 'dictates' to the system its possible moves/paths (within the 'geometrical spacetime' framework inherent in the $\mathbf{A}$ that she chooses *ab initio*), not with the dynamical 'system' itself (the 'endosystem' [148])—the algebraic $\mathbf{A}$-connection $\mathcal{D}$ which *a priori* knows no (spacetime) geometry. Of course, in view of the $\mathbf{A}$-functoriality of dynamics, the field itself (and the dynamical law that it defines) is in no way actually constrained by the $\mathbf{A}$ that the external experimenter 'imposes' on it—*ie*, once again, the gravitational dynamics is in no way conditioned, let alone impeded, by any ambient, external (background to the field) kinematical/geometrical space(time) that *we*, the external 'observers', adopt to geometrically picture (realize) it. This is the essence of the PFR in ADG-gravity.

But without carrying any further this discussion, since anyway we are going to encounter again these issues in section 7 in connection with the ADG's perspective on Einstein's hole argument, let us summarize the traditional way of thinking about kinematics and dynamics.

The traditional or 'classical' analogy: 'kinematics before dynamics'. We may summarize the above by the following 'quotient':

$$\text{Dynamical histories} = \frac{\text{Kinematical histories}}{\text{Equations of motion}} \Leftrightarrow \frac{\text{Possibility space}}{\text{Dynamical law}} = \text{Actuality space}$$

which may be formally and pseudo-technically read as '*dynamics (what actually happens) is the 'moduli space' of the kinematical space (what can possibly happen) factored out by the dynamical equations of motion*', which in turn presupposes that the structure of the kinematical space has already been charted before the dynamical equations cut through (or select from) it the 'orbits' (dynamical histories or tracks) that are actually materialized—as it were, the geometrical footprints left by the system as it dynamically evolves through its 'pre-existent' (*a priori* determined and fixed) kinematical space.

Of course, regarding the notion of the geometrical background spacetime kinematics in GR, again we witness Einstein's remarkable insight into the fundamental concepts and workings of his relativistic field theory of gravity (GR) in the following words found in article 12 in [121], titled '*Time, Space, and Gravitation*':

(Q4.16) "*...In the generalized theory of relativity, the doctrine of space and time kinematics is no longer one of the absolute foundations of general physics. The geometrical states of bodies and the rates of clocks depend in the first place on their gravitational fields, which again are produced by the material system concerned*[a]*...*"

[a]Our emphasis throughout.

Arguably, the quotation above hints at the fact that, for Einstein too, the dynamical field (of gravity) is prior to an *a priori* fixed 'geometrical spacetime kinematics' (pun intended), which should thus *not* be regarded as being prior to dynamics and lying at the basis—as it were as an absolute, up-front fixed foundational structure—in the construction of any physical theory.[243]

Let us wrap up this traditional kinematics-before-dynamics 'principle' of theory-construction with the following words from [73] corroborating what has been said above:

[243]It is perhaps the aftermath of the gedanken '*hole argument*' that Einstein had originally come up with in order to 'test' the validity of the PGC of GR, that led him to review his theoresis of gravitational kinematics and dynamics as (Q?.?) and (Q?.?) show. It is on the basis of this argument, whose deeper meaning has been explored and explained in a long series of works by John Stachel [363, 364, 366, 367, 369, 371, 373, 374], that we shall later further support our claim that in ADG-gravity dynamics comes before kinematics (8.5.5, 8.5.6).

"...*Generally speaking, a dynamical theory, regardless of its being a description of fundamental interactions or not, must* <u>*presume*</u> *some geometry of space for the formulation of its laws and interpretation. In fact a choice of a geometry predetermines or summarizes its dynamical foundations, namely, its causal and metric structures.*[a] For example, in Newtonian (or special relativistic) dynamics, Euclidean (or Minkowskian) (chrono-) geometry with its affine structure, which is determined by the kinematic symmetry group (Galileo or Lorentz group) as the mathematical description of the kinematic structure of space(time), determines or reflects the inertial law as its basic dynamical law. *In these theories, the kinematic structures have nothing to do with the dynamics. Thus dynamical laws are invariant under the transformations of the kinematic symmetry groups. This means that the kinematic symmetries impose some restrictions on the form of the dynamical laws.* However, this is not the case for general relativistic theories.[b] *In these theories,* there is no a priori kinematic structure of spacetime,[c] *and thus there is no kinematic symmetry and no restriction*
(Q4.17) *on the form of the dynamical laws. That is, the dynamical laws are valid in every conceivable four-dimensional topological manifold, and thus have to be generally covariant. It should be noted, therefore, that the restriction of GC upon the form of dynamical laws in general relativistic theories is different in nature from the restrictions on the forms of dynamical laws imposed by kinematic symmetries in non-general relativistic theories*[d]..."

---

[a]Our emphasis.

[b]Our emphasis again. We will come back to comment on this important remark from the vantage of Einstein's hole argument (as exposed by Stachel) in the last section.

[c]Something that appears to contradict Sorkin's quotation (Q?.?) above. There is no real contradiction if one waters down the '*a priori*' above: a smooth manifold (and its diffeomorphism) group is indeed part of the kinematical structure of GR if one considers on the one hand the virtual (theoretical) constraint of requiring the Einstein equations to be *differential* equations proper, and on the other, the actual Diff(M)-constraints when (quantum) gravity is regarded as a constrained (quantum) gauge theory.

[d]Again, our emphasis. Again, please wait for our ADG-treatment of Einstein's hole argument in the last section.

Turning the tables around ADG-theoretically: 'dynamics prior to kinematics' from 'field solipsism'. [RETURN]

$$\text{Algebraic connection is the cause of geometric curvature} \Leftrightarrow \mathcal{D} \Longrightarrow R(\mathcal{D})$$
$$\text{The dynamical field entails geometry} \Leftrightarrow \text{Geometry is inherent in the dynamical field}$$
$$\text{Sheaf} - \text{cohomologically} : (\mathcal{E}, \mathcal{D}) \Leftrightarrow (\mathcal{D}, R(\mathcal{D}))$$

`Perhaps mention here Stachel's ideas on the 'dynamicalization of coordinates' from [374].`

4.3 GR à la ADG: The Connection $\mathcal{D}$ is Primary, the Metric g Secondary

We commence this subsection with the basic observation, already emphasized in [272], that *in the ADG-picture of GR the fundamental—in fact, the sole—dynamical variable is the connection* $\mathcal{D}$. This must be contrasted against the original, (pseudo)-Riemannian geometry based formulation of GR by Einstein, in which the gravitational potentials were identified with the ten components of the metric tensor $g_{\mu\nu}$. Of course, it is well known that soon after the original 'metric formulation' of GR, there were people, such as Eddington and Cartan for example, who tried alternative scenarios for gravity in which the fundamental dynamical variable was assumed to be the affine connection instead of the metric. Weyl, for instance, in early attempts to unite gravity with electromagnetism, thought of weakening the Riemannian idea of an invariant infinitesimal distance by replacing it with relative distance, in the sense that only ratios of distances, not ('absolute') distance itself, should be invariant, thus anticipating the ('local') gauge principle.

It must be emphasized here, however, that Einstein himself, on many occasions, alluded to the physical significance of the notion of (the Levi-Civita-Christoffel) connection in GR, and sometimes he went as far as to almost regard it as more important than the spacetime metric, as the following quotations show:

(Q4.18) "...The development just sketched of the mathematical theories essential for the setting up of general relativity[a] had the result that *at first the Riemannian metric was considered the fundamental concept on which the general theory of relativity and thus the avoidance of the inertial system were based. Later, however, Levi-Civita rightly pointed out that the element of the theory that makes it possible to avoid the inertial system is rather the infinitesimal displacement field* Γ^l_{ik}. *The metric or the symmetric tensor field which defines it is only indirectly connected with the avoidance of the inertial system in so far as it determines a displacement field*[b]..." [125]

[a]Here Einstein alludes to the work of Gauss and, more importantly, Riemann, on the (differential) geometry (*eg*, the study of the curvature) of (high-dimensional) metric spaces 'in themselves'—*ie*, independently of whether or not these spaces are embedded in higher-dimensional (Euclidean) spaces.

[b]Our emphasis.

At the same time, it is well known that in the CDG-based Riemannian geometry it is precisely the auxiliary requirement (or condition) that the connection be compatible with the metric[244] which establishes an 'equivalence' between the original metric formulation of GR and the latter's 'connection picture', as we read for example from [199]:

(Q4.19) "...In relativity theory it is convenient to distinguish three levels or layers of geometry, with increasing structure and complexity: (a) the underlying differentiable manifold, (b) the connection, (c) the metric...For Riemannian geometry (b) and (c) are made compatible, in a special sense, by the requirement $\nabla_a g_{bc} = 0$, with ∇_a torsion-free...*Remarkably, these conditions are sufficient to determine* ∇_a *uniquely in terms of* g_{ab}, *and it is this natural link between metric and connection that constitutes the foundation of Riemannian geometry*[a]..."

[a]Our emphasis.

Motivated by the quotation (Q?.?) above, we digress a bit and note that the bottom 'layer (a) of geometry', namely, the differential manifold ($\mathcal{C}^\infty$-) structure, is left intact and quite independent

[244]Which defines $\mathcal{D}$ as a '*metric connection*'. In turn, this metric compatibility condition on the connection implies that the gravitational field is *torsionless*.

by this 'identification' of the 'connection layer' (b) with the 'metric layer' (c), which are treated on a par (*ie*, they are viewed as being equivalent by the torsion-free connection condition). We may cast this CDG-based picture of Riemannian geometry by the following $\top$-shaped '*geometrical*' (*ie*, metric based) diagram:

$$\begin{array}{c} \boxed{\textbf{Connection } \nabla} \longleftrightarrow \boxed{\textbf{Metric } \mathbf{g}_{\mu\nu}} \\ \text{metric compatibility} \Big\uparrow \text{of the connection} \\ \boxed{\textbf{Differential manifold structure}} \end{array} \tag{43}$$

In contradistinction, in ADG, in a deep sense this $\top$-shaped diagram of the CDG and smooth manifold-dependent structure of Riemannian *geometry* is inverted (*ie*, it looks upside-down!) to the following $\perp$-shaped *algebraic* (*ie*, **A**-connection based) one

$$\begin{array}{c} \boxed{\mathbf{A}\textbf{–metric } \rho} \\ \text{connection compatibility} \Big\uparrow \text{of the metric} \\ \boxed{\textbf{Structure sheaf } \mathbf{A}} \dashleftarrow\dashrightarrow \boxed{\textbf{Connection } \mathcal{D}} \end{array} \tag{44}$$

which we briefly explain below:

1. First we note that the differential manifold M in (43) is replaced in (44) by the structure sheaf $\mathbf{A}_X$ of 'differential algebras' of generalized coordinates ('arithmetics') over an arbitrary topological space X, which itself carries no differential structure whatsoever *a priori*.[245] In (43), and in accordance with (8), the connection and the metric structures are imposed on the manifold's $\mathcal{C}^\infty$-smooth CDG-structure, inheriting thus the usual epithet '*smooth*' in front, as in '*smooth connection*' and '*smooth metric*'. By contrast, in ADG, the differential structure, primarily represented by the differential $\partial \equiv d^0$, 'derives' from **A** itself; moreover, more importantly, it was ADG that first appreciated that the differential operator ∂ is nothing else but a particular kind of *connection*—albeit, as mentioned in 3.1, a flat one.[246] The dashed bidirected arrow in (44) just signifies that in ADG the notion of connection (*viz.*, generalized differential $\partial \equiv d$) is an algebraic one. In other words, the connections defined in ADG are

[245]Of course, as noted earlier, and in the classical case, one recovers M (in effect, one identifies it with X) when $\mathbf{A}_X \equiv \mathcal{C}^\infty_M$. Yet, the same can still happen for suitably chosen 'geometric topological algebra sheaves' **A** [259, 266], $\mathcal{C}^\infty_X$ being of course a very special case.
[246]Then, as we saw in 3.1, one generalizes ∂ ADG-theoretically to a 'curved connection' $\mathcal{D}$.

'*algebraic connections*'[247]—*ie*, inextricably tied to (ultimately, coming from) the structure sheaf of differential algebras **A**. This is another corroboration of the title of 3.2, namely, that *all differential geometry boils down to* **A**, for it is plain that *one cannot do differential geometry without a differential operator d (or more generally, without a connection $\mathcal{D}$), and the latter originates with or has as its 'source' (action domain)* **A**.[248] These remarks are also in line with the brief historical account of the initial development of ADG and its 'precursors' we gave in 3.1 according to which the first author's original aim upon formulating ADG was *to arrive at an entirely algebraic notion of (generalized) differential (ie, connection) by sheaf-theoretic means, without the use of any underlying smooth manifold*, thus, in effect, *to do differential geometry without the use of any Calculus, or anyway, without any classical notion of smoothness in the traditional $\mathcal{C}^\infty$-sense—that supported and furnished by a locally Euclidean base space (manifold).*

2. Secondly, we come to the fundamental difference between ADG's perspective on GR and the CDG-based Riemannian geometry underlying GR: as noted above, while in the latter the main geometrical structure is the smooth spacetime metric $g_{\mu\nu}$ which is thus regarded as the basic dynamical variable in the corresponding physical theory (GR), in the former the basic, purely algebraic, structure is the connection $\mathcal{D}$, which is thus being regarded as the sole dynamical entity in the algebraic (*ie*, sheaf-theoretic) ADG-picture of GR [262, 272].

   This difference is partially reflected in the asymmetry between the $\perp$- and the $\top$-shaped diagrams above, as follows: *while in Riemannian geometry one makes the connection compatible with the metric, with the latter held as being more fundamental than the former, in ADG one goes the other way around and requires that the* **A**-*metric be compatible with the* **A**-*connection, which is thus assumed to be more basic than the former. Prima facie*, this reversal might appear to be only formal and rather superficial—as it were, it might remind one of the circular question

   which came first, the egg or the chick?,

   but *conceptually* we believe that there is much more to it. For recall that ADG-theoretically the **A**-metric ρ is an **A**-bilinear **A**-valued sheaf morphism. Being an **A**-morphism, like the curvature $R(\mathcal{D})$ of the connection $\mathcal{D}$, *ρ is a geometrical object* (*ie*, a $\otimes_{\mathbf{A}}$-tensor), as it respects our generalized arithmetics in **A** [272]. On the other hand, *the connection $\mathcal{D}$, being a*

[247]Indeed, the term used by ADG to refer to the $\mathcal{D}$s defined and studied in it is **A**-*connections* [255, 256, 259, 260, 405].

[248]For one thing, recall Kähler: essentially, *every commutative unital ring comes equipped with a differential* [260, 271, 272].

$\mathbf{C} \equiv \mathbf{K}$- *but not an* **A**-*morphism, is not a geometrical object*; rather, *it is an algebraic notion which eludes purely geometrical characterization* [272].[249] In the context of GR, the physical importance of the non-tensorial nature of the connection, as opposed to the tensorial character of either the (connection-compatible) metric or the (connection's Riemann) curvature (tensor) has been emphasized by Stachel [363, 364]:[250]

(Q4.20)

> "...Within a few years, Levi-Civita, Weyl and Cartan generalized the Christoffel symbols to the concept of affine connection. This concept served to make the relationship between the mathematical representations of various physical concepts much clearer. *Just because it is not a tensor field, the connection field provides an adequate representation of the gravitational-cum-inertial field required by Einstein's interpretation of the equivalence principle. There is no (unique) decomposition of the connection field into an inertial connection plus a gravitational tensor field*[a]..." [364]
>
> [a]Our emphasis.

who continues, in support of the quotation above, by quoting Einstein from a 1950 letter to Max Von Laue:

[249]As repeatedly emphasized in [272], from a gauge-theoretic viewpoint, the non-geometrical character of the connection, as opposed to its curvature, corresponds to the fact that, while the former transforms inhomogeneously (affinely) under a gauge transformation, the latter transforms homogeneously (tensorially). In other words, the connection is not a tensor under a symmetry (local gauge) transformation, hence it is not a geometrical object proper.

[250]The second author would like to thank cordially John Stachel for communicating to him, quite timely with respect to the writing of the present paper, the two papers [363, 364].

(Q4.21) "*...what characterizes the existence of a gravitational field from the empirical standpoint is the non-vanishing of the* [components of the affine connection][a]*, not the non-vanishing of* [the components of the Riemann tensor].[b] *If one does not think in such intuitive (Anschaulich) ways, one cannot grasp why something like curvature should have anything at all to do with gravitation. In any case, no rational person would have hit upon anything otherwise. The key to the understanding of the equality of the gravitational and inertial mass would have been missing*[c]*...*"

[a]Stachel's addition.
[b]Again, Stachel's addition.
[c]Our emphasis throughout.

In fact, Stachel goes a step further and argues, in retrospect of course, that had the notion of connection, in contradistinction to the metric of Riemannian geometry, been developed around when Einstein was laying the foundations of his relativistic field theory of gravitation, he would have chosen *it* instead of the metric as the 'natural' mathematical representation of the gravito-inertial field as mandated by the EP. In this respect, we read from [370]:

(Q4.22) "...With hindsight, one can see that Einstein's attempt to find the best way to implement mathematically the physical insights about gravitation incorporated in the equivalence principle was hampered significantly by the absence of the appropriate mathematical concepts. *His insight, as he put it a few years later, that gravitation and inertia are 'essentially the same'* ('*Wesensgleich*'), *cries out for implementation by their incorporation into a single gravito-inertial field, represented mathematically by a non-flat affine connection on a four-dimensional manifold.*[a] But the concept of such a connection was only developed *after*,[b] and largely in response to, the formulation of the general theory. *So Einstein had to make do with what was available: Riemannian geometry and the tensor calculus as developed by the turn of the century, i.e., based on the concept of the metric tensor, without a geometrical interpretation of the covariant derivative. As I have suggested elsewhere, this absence is largely responsible for the almost three-year lapse between the end of Act I and the close of the play*[c]*...*"

[a] Our emphasis.

[b] Stachel's emphasis.

[c] Our emphasis. Let it be noted here that the first act in the development of GR was, according to Stachel, the inception by Einstein of the EP, while the third and final act is the formulation of the correct (*ie*, generally covariant) field equations for gravity.

Furthermore, we would like to believe with the advent of ADG, the differential geometry (DG) that Einstein would have chosen for GR, had such a mathematical framework been around during the early, formative years of the latter theory, would not have been the CDG (*ie*, manifold) based (pseudo-)Riemannian geometry of $g_{\mu\nu}$, but a DG based primarily—if not *solely* and *exclusively*(!)—on the notion of connection, much like ADG is. Moreover, the additional virtue of ADG, in contradistinction even to some of Stachel's remarks above, is that its basic concept of (affine) connection does not depend at all on a smooth geometrical base manifold—it is a purely algebraic notion, without an *a priori* need of a geometrical interpretation as a covariant derivative (*eg*, parallel transporter of other structures/fields on the smooth spacetime manifold).[251] In turn, it is our contention that the dual role of the metric in Einstein's original formulation of GR was a 'misfortunate accident of labor' in the

[251] A background manifold independence that, as we argue throughout the present paper, not only proves to be invaluable *vis-à-vis* the singularities of GR, but one that is of potential import in QG research (see section 6).

aufbau of the theory, one that apparently necessitates the geometrical spacetime manifold based interpretation of the theory, and at the same time, always within the confines of the CDG-based Riemannian geometry, it unfortunately masked the purely algebraic character of the connection (which was put on a par with $g_{\mu\nu}$). The said dual role pertains of course to the well known fact that the components of $g_{\mu\nu}$ stand both for the gravitational potentials *and* for the entries in the infinitesimal line element ($ds^2 = g_{\mu\nu}dx^\mu dx^\nu$) of the smooth spacetime manifold marking the *chronogeometry* and delimiting the *local causal structure of spacetime* (lightcone soldered at each point of M). In this respect, we quote again Stachel from [364]:

(Q4.23) "...The main difficulty at this stage was to grasp the dual nature of the metric tensor: it is both the mathematical object which represents the space-time structure (chronogeometry) and the set of 'potentials', from which representations of the gravitational field (Christoffel symbols) and of the tidal 'forces' (Riemann tensor) may be derived..."

In a nutshell, our contention is that exactly because the original formulation of GR (accidentally) happened to be based on $g_{\mu\nu}$ rather than on $\mathcal{D}$, we have been misorientated, misdirected and ever since felt 'obliged' to look similarly for a geometrical (smooth) spacetime interpretation of the latter when, *a priori*, it is a purely algebraic entity. Moreover, we were misled to think that GR was a theory of the structure of spacetime—a theory of '*choro-chronogeometry*'—whereas, as we maintain herein and throughout our trilogy [270, 271, 272], it is a theory solely about the (purely algebraic) *gravitational field* (*viz.* connection), without any 'smooth geometrical background spacetime impurities' being *a priori* (and externally)[252] added to that picture.[253]

In toto, a combination of Einstein's quotations (Q?.?) and (Q?.?) point to the plausible conjecture that, for him too, at least intuitively (*Anschaulichkeit*), the notion of connection is more suitable for representing the gravito-inertial field than either the metric or the (Riemann) curvature; furthermore, assisted by Stachel's remarks in (Q?.?), we are led to infer that it is precisely due to the non-geometrical (purely algebraic in ADG [272]) character of the connection, in contradistinction to the tensorial character of either $g_{\mu\nu}$ or $R(\mathcal{D})$, that $\mathcal{D}$ may be viewed as *the* appropriate structure to implement the 'entire' or 'unitary'

[252] Here we simply mean that it is the experimenter (measurer or geometer), or at least the theoretician, that 'geometrizes' the gravitational connection field by bringing along $\mathbf{A} \equiv \mathcal{C}^\infty_M$.

[253] Impurities which then prove to be the main culprits for the singularities assailing the CDG and manifold based GR, for all smooth gravitational singularities are, in one way or another, inherent in $\mathcal{C}^\infty_M$.

gravito-inertial field that the EP mandates in GR.[254]

This is precisely how the algebraic connections $\mathcal{D}$ are viewed in ADG: as *integral wholes*—that is to say, as we have repeatedly emphasized in our previous trilogy [270, 271, 272], while by 'connection' theoretical physicists usually refer only to the gauge potential part $\mathcal{A}$ of $\mathcal{D}$,[255] we do rather mean $\mathcal{D}$ taken as an inseparable, integral whole. Of course, locally (*ie*, in a particular local gauge U), $\mathcal{D}$ splits into the flat '*inertial connection*' (Q?.?) ∂[256] and to the gravitational (gauge potential 1-form) $\mathcal{A}$, $\mathcal{D} = \partial + \mathcal{A}$, but, as Stachel noted in (Q?.?) above, in a manifestly non-geometrical way,[257] as $\mathcal{A}$ itself is not a 'geometrical object' *per se* (*ie*, an $\otimes_{\mathbf{A}}$-tensor or **A**-morphism in ADG),[258] because it transforms inhomogeneously or affinely under (local) gauge transformations. As briefly noted 3 footnotes ago, the potential, both technical and conceptual, import that the entirely algebraic (non-geometrical), non-smooth (*ie*, non-smooth base manifold dependent) and 'unitary'—what in [272] was coined '*fully covariant*'[259]—ADG-perspective on gravitational connections might have for current issues in QG research, especially in view of the $\mathcal{C}^\infty$-smooth background spacetime singularities that assail GR, will be discussed in more detail in section 6.

3. In this light, we are in a position to 'rationalize' about the *connection-compatibility of the metric* condition in ADG, in contrast to the reverse *metric-compatibility of the connection* in Riemannian geometry: in ADG, since the gravitational algebraic connection $\mathcal{D}$ is the basic dynamical structure regarded as being on a par (44) with the basic structure sheaf **A**

[254] The epithet '*unitary*' means here '*taken as an indivisible unit, an integral whole—one that is not (physically meaningfully) separable into its inertial and gravitational components*'. In the penultimate section, we will relate this notion of 'unitary gravitational field' with the 'unitary', 'purely gauge', or even 'fully covariant' (synvariant) conception of field theory propounded by ADG [272], as well as discuss the theoretical consequences that such a conception might have in QG research, especially *vis-à-vis* the smooth singularities of the classical theory (GR). (See also next paragraph.)

[255] Thus $\mathcal{D}$, as a whole, is usually referred to as the '*covariant derivative*', with all the 'extraneous geometrical baggage' that the latter term carries, such as being geometrically interpreted as a parallel transporter of various structures (*eg*, tangent vectors and other local higher-rank tensors) along smooth curves in the underlying manifold.

[256] And, it must be emphasized again, this is precisely how ∂ is viewed ADG-theoretically: as *a flat connection* [259]. In this respect, it must be recalled from [270] that the flat derivation ∂ is mathematically represented by the (smooth) vector fields in GR, which in turn are local sections of the tangent bundle TM to M. And one must keep in mind that the EP forces the curved M to be locally Minkowskian—*ie*, that the fibers of TM are isomorphic to (copies of) Minkowski space—so that its local sections are Minkowski vectors, which comprise *inertial frames* in the *basis* or *frame bundle* FM associated with TM.

[257] Here one could also use the terms 'non-tensorial', or perhaps even better, 'gauge non-covariant' way.

[258] This is unlike either the metric g or the curvature R, which are *the* 'geometrical objects' ($\otimes_{\mathbf{A}}$-tensors or **A**-morphisms) *par excellence* [272].

[259] Which has been coined here '*synvariant*' after the generalization in 3.2.2 of the PGC of GR in our background spacetime manifold free ADG-scenario called '*synvariance*'.

to which, after all, *all differential geometry boils down* (3.2), the **A**-metric that might be imposed on this differential geometric foundation as a secondary, non-fundamental ('derivative' or 'contingent') structure,[260] must presumably be such that it 'respects' the said basic structure—namely, the algebraic connection $\mathcal{D}$. By contrast, in the CDG-based Riemannian geometry on which GR rests, at the bottom one assumes up-front a fixed smooth base spacetime manifold (providing one with the usual CDG-machinery), and relative to that 'background absolute structure' one regards as the fundamental gravitational variable the smooth spacetime metric $g_{\mu\nu}$. Then one may adjoin to this structure the Levi-Civita-Christoffel connection ∇ in a way that respects the metric, thus defining a torsionless, metric connection. By this rationale one is able to view the metric-compatibility of the connection condition in the (pseudo)-Riemannian GR as *a physically non-fundamental, contingent condition that is somehow linked to the macroscopic spacetime continuum.*[261]

Talking about the fundamental, non-geometrical character of the gravitational connection, it is interesting to quote Ehlers from [111], where this virtue of $\mathcal{D}$ is emphasized, but shortly after it is remarkably 'down-played':[262]

[260] And it must be emphasized here that, since **A** (*ie*, 'geometry' or 'space(time)' in ADG) *is a free choice of the theorist, then so is the* **A***-metric relative to it*! [272]

[261] Indeed, "*Quantum spacetime will turn out to be not distortion-free. The torsionless condition on the covariant derivative is a classical* (spacetime continuum) *requirement*" (David Finkelstein in private e-communication with the second author 4 years ago).

[262] Something that is understandable however, since during the 'pre-connection', pre-Sen and Ashtekar [342, 7] period, theorists struggled to view gravity as a (pure) gauge theory (let alone QG as a quantum gauge theory) [215].

(Q4.24) "...the tensor field g, besides determining times, angles and lengths, also plays the part of a *gravitational-inertial*[a] potential...They [*ie*, the Christoffel symbols Γ^a_{bc}, formed from g_{ab} and its first derivatives,] are components of the Riemannian connection associated with g. One may call this connection the gravitational-inertial *field strength*,[b] and the Γ^a_{bc} its components relative to (x^a). *This is an example of a physically important object which is neither a tensor nor a spinor. It is non-local in the sense that it does not have a coordinate-independent 'value' a spacetime point*,[c] in accordance with Einstein's elevator argument showing that the gravitational field can be 'transformed away' at an event, like inertial forces in Newtonian theory..."

[a]Ehlers' own emphasis. Compare it with Stachel's denomination of $g_{\mu\nu}$ as the gravito-inertial field.
[b]Again, Ehlers' emphasis.
[c]Our emphasis this time.

However, as noted above, a few lines later we read:

(Q4.25) "...The formulation of the 'field kinematics' of GR in terms of principal bundles and their associated bundles allows one to consider GR as a *gauge theory*,[a] and thus to compare its structure—which differs considerably from Yang-Mills-type gauge theories—with other such theories. (As far as I can see such considerations have not led to a deeper understanding of GR as such, nor n particular of the role of the Poincaré group in GR. The physical role of translations in SR, as space-time displacements, is taken over in GR by *diffeomorphisms*[b] of spacetime. I at least fail to see that the use of affine bundles with affine (in Cartan's sense) connections changes this fact, nor does it help me to appreciate it more deeply)[c]..."

[a]Ehlers' emphasis.
[b]Ehlers' emphasis.
[c]This 'lack of appreciation' for $\mathcal{D}$ will be lifted by Feynman's words shortly.

After this digression, and following [272], we would like to make a few more remarks about our preference for the connection, rather than the metric, formulation of gravity. Our opting for $\mathcal{D}$

instead of $g_{\mu\nu}$[263], apart from the PFR which acknowledges the existence of the connection field 'out there' while it also recognizes that the 'geometrical' **A** and *in extenso* the **A**-metric that goes hand in hand with it is what *we* the external observers-*cum*-'geometers' impose on the gravitational field in order to 'measure' it, may be regarded as being tautosemous to the following fundamental statement:

(R4.18) *From the ADG-theoretic point of view, gravity is another gauge theory* [269].

To our knowledge, it was Feynman first, during his quantum field-theoretic attempts to quantize the gravitational field, to propose explicitly that gravity should be 'properly' viewed as a gauge theory, and, perhaps even more suggestively for our endeavors herein, that the fact that the gravitational field was originally identified with the spacetime metric was just a happy, albeit 'accidental', coincidence coming from the original formulation of GR within the theoretical confines of the "*fancy-schmanzy*" manifold-based Riemannian differential geometry [140]. For its worth and relevance to the discussion above, below we recall from [272] the excerpt from the "*Quantum Gravity*" forward to Feynman's "*Lectures on Gravitation*" written by Brian Hatfield:

(Q4.26) "...Thus it is no surprise that Feynman would recreate general relativity from a non-geometrical viewpoint. The practical side of this approach is that one does not have to learn some 'fancy-schmanzy' (as he liked to call it) differential geometry in order to study gravitational physics. (Instead, one would just have to learn some quantum field theory.) However, when the ultimate goal is to quantize gravity, Feynman felt that the geometrical interpretation just stood in the way. From the field theoretic viewpoint, one could avoid actually defining—up front—the physical meaning of quantum geometry, fluctuating topology, space-time foam, *etc.*, and instead look for the geometrical meaning after quantization...Feynman certainly felt that the geometrical interpretation is marvellous, '*but the fact that a massless spin-*2 *field can be interpreted as a metric was simply a coincidence that might be understood as representing some kind of gauge invariance*'[a]..."

[a]Our emphasis of Feynman's words as quoted by Hatfield.

Subsequently, and after the spin-connection formulation of GR had been established earlier, principally by the work of people like Cartan [74, 166], Bergmann [40] and Sen [342], the connection-based

[263]In ADG, we symbolize the metric by ρ, not g [272].

picture for GR (and potentially for QG!) found its 'apotheosis' with the work of Ashtekar on the new spin-Lorentzian (*ie*, $sl(2, \mathbb{C})$-valued) connection variables for classical and quantum gravity [7]. Currently, it is fair to say that the epicenter of main approaches to QG (modulo string theory), whether canonical (Hamiltonian) or covariant (Lagrangian), such as loop QG and cosmology for example [330, 13, 382, 383, 11, 20, 12, 351], concentrates mainly on arriving in one way or another at a quantum version of the connection formulation of GR—a scheme which is usually coined *Quantum General Relativity*. In other words, *QG is tackled as a quantum gauge theory*, and as the main quantum configuration arena—the relevant quantum configuration space for QG—one takes the affine space of smooth spin-Lorentzian connections, or more appropriately due to the diffeomorphism symmetries of the underlying differential spacetime manifold M implementing the PGC in the classical theory (GR), *the moduli space of* $\mathrm{Diff}(M)$*-equivalent smooth connections*.

Of course, this persistent abiding by the background smooth spacetime continuum of GR, even in the QG regime(!), creates all sorts of both technical and conceptual problems coming mainly from the infinite dimensional 'gauge' group $\mathrm{Diff}(M)$ of GR—the automorphism group of the $\mathcal{C}^\infty$-smooth base spacetime manifold. These problems seriously inhibit the actual carrying out of the (in some cases formal, in others, heuristic) quantization procedure, whether canonical (*eg*, problem of time) or covariant (*eg*, path integral measure problem), of the smooth gravitational connection field on the classical background spacetime continuum [272]. For this, many people have insisted upon *formulating the elusive QG in a manifestly background spacetime manifold independent way* [8, 9, 11, 20, 3, 12, 351].[264] However, the basic dynamical entities involved, namely, the gravitational connection variables, are still assumed to be *smooth*, and the necessity has arisen to develop a differential geometry—albeit, still manifestly along CDG-lines—on the moduli space of connections [16], an endeavor again motivated by loop QG [14, 15]. Thus, as critically pointed out in [272], the smooth spacetime manifold is still lurking (even if just dormant or in an apparently 'inert' disguise) at the background. Moreover, since, as we argued extensively earlier, the base spacetime M is the culprit for the singularities and their associated unphysical infinities in GR, while at the same time even the smooth connection-based approaches to both GR and QG appear essentially to retain M at the background for differential geometry's sake,[265]

[264]In 6.3 we will return to discuss in more detail the issue of 'background independence' in loop QG, how the latter has been recently used to resolve the interior Schwarzschild singularity [279] (in the wider context of quantum resolution of black hole singularities by Loop Quantum Gravity means [200]), and how it compares with what we regard as a '*genuinely background manifold independent*' formulation of gravity (whether classical or quantum) that ADG enables us to accomplish.

[265]Thus, QG too falling prey to the CDG and the associated differential manifold-conservatism and monopoly that we observed in the first two sections.

(R4.19) *what reasons are there for one to hope that a quantum theory of gravity will ever resolve*—or even more drastically, *evade—the problem of singularities in the classical theory (GR)?* In our view, as long as the approaches to QG abide by the $\mathcal{C}^\infty$-smooth base spacetime manifold of GR and the CDG-technology on it, the answer is an emphatic *None at all; the singularities of GR will persist and will inevitably have to be reckoned with in the quantum theory (QG)*!

We will return to address and elaborate further on this crucial issue in section 6. Now, we would like to conclude this discussion about the 'dominance' (more physical significance) of the connection over the spacetime metric by quoting from John Stachel's paper "*Eddington and Einstein*" in [371] a very pertinent passage indeed. As noted before, Eddington, primarily motivated by Weyl's so-called '*gauge theory of the first kind*' [414], in some sense neglected on purpose the fundamental theorem of Riemannian geometry which identifies the metric with the connection on a smooth manifold, and set out to formulate general relativity solely in terms of the affine connection.[266] The quotation below nicely describes Eddington's 'ahead of its time' effort:[267]

[266] See also [272] for a little discussion of Eddington's attempts.
[267] Below, all emphasis is ours.

"...The most important influence Eddington's work had on Einstein was in connection with the question of a unified theory. Einstein was searching for a physical theory that would not only provide an *organic*[a] unification of the gravitational and electromagnetic fields, but hopefully would also provide a theory of matter, in particular an explanation of the appearance of quantum effects. Eddington's motivation was rather more mathematical and epistemological. In particular, he interpreted his unified theory as providing an *explanation* of gravitation, as opposed to the mere *description* provided by Einstein's 1915 theory. Before going into this, I must say a few words about the mathematical side of the theory. After 1915, it had been realized by several mathematicians that metrical and affine structures on a manifold were logically independent, even though there exists a unique symmetric affine connection associated with each metric. Weyl was the first to take advantage of the possibilities arising from this distinction to set up a physical theory, in which an additional element—a vector field—is needed, besides the metric field, in order to determine the affine structure of space-time. He identified this vector field physically with the electromagnetic potentials.

(Q4.27) Noting the very special nature of Weyl's generalization, Eddington started by assuming that *there was no a priori connection between the metric and initially arbitrary symmetric connection.* He observed that the curvature or Riemann tensor and its contraction, usually referred to today as the Ricci tensor, may be formed from the affine connection without any metric. This Ricci tensor, however, will in general not be symmetric, even though the connection is. *What has this to do with physics?* Eddington noted that an affine connection enables us to compare tensors at neighboring points (in particular, to say when two neighboring vectors are parallel). *He regarded the possibility of such a comparison between quantities at neighboring points in space-time as the minimum element necessary to do any physics: "For if there were no comparability of relations, even the most closely adjacent, the continuum would be divested of even the rudiments of structure and nothing in nature would resemble anything else.*"[b]

The purely affine theory discussed so far involves no metric, and thus provides no basis for a comparison of lengths at different points, even neighboring ones. Such a comparison is provided by what Eddington, following Weyl's terminology, calls *gauge*. *Thus, the pure affine theory is a gauge-invariant theory...*"

[a]We emphasize the word 'organic' *vis-à-vis* Einstein's unitary field theory, because we will return and comment on it extensively in later parts of this paper (section 7, 7.5).

[b]Quotation from [107].

4.3.1 Gravitational singularities: 'idols' of the spacetime metric? 'Smooth misery' of the first and second kind.

Keeping in mind the question posed in (R3.2) above, and now that we have noted that there has recently been a sea-change in both classical and quantum general relativity research—especially after Ashtekar proposed the new spin-Lorentzian variables [7]—in *apparently* viewing the connection, and not the metric, as the fundamental gravitational dynamical variable,[268] thus tending to view GR, *à la* Feynman, more as a gauge theory and less as a physical application of the CDG-based Riemannian geometry (Q?.?), *what is the 'true' origin and problematic role of singularities in both the classical and the (potentially) quantum theory of gravity?*

Prima facie, this is a legitimate question that behooves the working classical general relativist, for it is fair to say that, at least originally, singularities in GR were conceived as *loci* in the spacetime continuum where (some of the components of) the solution-metric of Einstein's equations could not be continuously defined—*eg*, the Schwarzschild metric, the Friedmann metric, or even the de Sitter metric [87]—thus they were intimately connected with the original metric-formulation of the theory. Thus, by founding the theory on the connection rather than the metric, one would be tempted to enquire whether singularities could be evaded altogether in the connection-based picture, *as if gravitational singularities were due to our using the wrong dynamical variables in the first place*—as if Einstein's equations, in the original metric-formulation of GR, were inappropriately expressed in terms of $g_{\mu\nu}$, which in some sense 'inherently' carries these pathologies.[269]

Any such hopes, however, are deemed to be short-lived. They quickly get shattered by imposing the fundamental metric-connection condition of the CDG-based Riemannian geometry, which establishes the equivalence between $g_{\mu\nu}$ and ∇ (43)—that is, that the (smooth) connection is completely and uniquely determined by the (smooth) metric (and its partial derivatives) (Q?.?). For even if one does not wish to impose up-front, as it were 'by hand', the condition of metric-compatibility of the connection by committing oneself to the usual metric-picture of GR in which the (vacuum) Einstein equations are obtained from varying the Einstein-Hilbert action

$$\int \mathcal{R}\varpi \tag{45}$$

with respect to the metric,[270] but one rather wishes to resort to the connection-picture *à la*, say, Palatini or Ashtekar [272], in which the fundamental variables are the connection $\mathcal{A}$ and the

[268] The adverb 'apparently' will be justified below.

[269] In a metaphorical sense, this could be regarded as the dynamical analogue of the virtual, coordinate singularities we talked about earlier.

[270] The so-called *second-order formalism*.

vierbein e,[271] one on the one hand obtains the (vacuum) Einstein equations from varying the following Ashtekar action functional

$$S_{ash}[\mathcal{A}^{(+)}, e] = \frac{1}{2}\int_M \epsilon(e \wedge e \wedge R^{(+)}(\mathcal{A}^{(+)})) \tag{46}$$

with respect to the comoving 4-frame (*vierbein*) e (*ie*, one establishes that the connection is Ricci-flat), but on the other, by varying S_{ash} with respect to the (self-dual) connection $\mathcal{A}^{(+)}$,[272] one recovers the Riemannian metric-compatibility condition for the latter [272]. Thus, while in the second-order formalism (metric-picture of GR) the metric-compatibility condition of the connection is an additional, apparently *ad hoc*, requirement (added to the equations of motion for the metric), in the first-order formalism (the vierbein/connection picture of GR) it is derived, loosely speaking, as 'the equation of motion of the connection'.[273] Moreover, this observation, coupled to the fact that both the metric and the connection in either formalism are $\mathcal{C}^{\infty}$*-smooth structures* (*ie*, they are based on a background differential spacetime manifold M and their components in a local frame are elements of $\mathcal{C}^{\infty}(M)$),[274] shatters the hopes of anyone who would like to see singularities as being merely the artifacts of an inappropriately expressed dynamics (*ie*, one expressed in terms of the metric instead of the connection). For in any case, both the original second- and the more recent first-order formalism appear to have built-into them the Riemannian condition for the metric-compatibility of the connection, while at the same time in either formalism, in effect *the basic gravitational dynamical variable is the metric*—explicitly in the second-order formalism, and implicitly, in $g_{\mu\nu}$'s 'e-disguise', in the first-order one.

The manifold-free, 'half-order', purely algebraic connection-formalism of ADG. In contradistinction to either the first or the second-order formalism of GR, both of which are intimately bound to the manifold and hence CDG-based Riemannian geometry as we argued above, in ADG:

[271]The so-called *first-order formalism*.

[272]Again here the reader should note that we use the physicist's jargon according to which by the term 'connection' one means the gauge potential part $\mathcal{A}$ of the connection $\mathcal{D}$. $\mathcal{A}^{+}$ signifies *self-dual connection*.

[273]Thus, with respect to dynamics, the first-order (connection) formulation is like a second-order (metric) picture 'in disguise', since variation with respect to e yields the Einstein equations (like varying with respect to $g_{\mu\nu}$ in the second-order picture), and the connection is used only as an 'auxiliary variable' to secure its own Riemannian equivalence with the metric (43). In other words, in the first-order formalism the 'real' gravitational variable (*ie*, the one upon variation of which one obtains the Einstein equations for gravity) is the *vierbein*, not the connection. This is not surprising considering that e is the 'flat (locally Minkowskian/Lorentzian) square root of the metric': $e^a_\mu \eta_{ab} e^b_\nu = g_{\mu\nu}$!

[274]For, as we have emphasized throughout the present paper, it is the smooth base spacetime manifold that is the culprit for the gravitational singularities in GR.

- First of all, *the sole gravitational dynamical variable is the entire algebraic connection* $\mathcal{D}$ [272],[275] and not the metric as in the original second-order (metric) formulation of GR, or the *vierbein*-cum-gauge potential part $\mathcal{A}$ of the connection in the subsequent first-order (connection) one. *The ADG-picture of gravity is a genuinely and exclusively connection-based one.*

- Secondly, and more importantly regarding the $\mathcal{C}^\infty$-smooth singularities of GR, our algebraic gravitational connection field $\mathcal{D}$, unlike $g_{\mu\nu}$ (2nd-order) or $(e, \mathcal{A})$ (1st-order), is not dependent at all on a background differential manifold for its existence and sustenance. Since the background smooth manifold M is the culprit for the singularities in GR,[276] *the purely algebraic, $\mathcal{D}$-based description of gravity afforded by ADG can evade smooth gravitational singularities altogether*, as we will see in the sequel. For, in any case, in the context of the (pseudo-)Riemannian geometry on a differential spacetime manifold on which GR is founded, neither the smooth Levi-Civita connection, nor the smooth Lorentzian $g_{\mu\nu}$, appear to be the appropriate structures to study, in order to pin-point (*ie*, define precisely), gravitational singularities (Q?.?) [87].

- *In summa*, since by the connection alone, and without commitment to a background geometrical spacetime manifold, ADG represents the gravitational field, we may coin the ADG-theoretic formulation of GR *an entire, or unitary, algebraic field 'half-order' formalism—a 'pure' gauge theory* without at all a dependence on an external spacetime (in fact, whether the latter is a continuum or a discretum).[277]

[275]'Entire' above means '$\mathcal{D}$ as a whole, not just its local frame-dependent gauge potential part $\mathcal{A}$'. We will return to discuss the importance—especially for QG research—of this inseparable, 'holistic' conception of the gravitational field in section 6.

[276]That is, all singularities are inherent in $\mathcal{C}^\infty_M$.

[277]We explain it again: 'unitary algebraic field' pertains to the fact that in the ADG-theoretic formulation of GR, 'the gravitational field is represented solely by the algebraic connection $\mathcal{D}$ (on an appropriately defined vector sheaf—the 'representation' sheaf of $\mathcal{D}$ associated with the field's 'auto-symmetries' in $\mathcal{A}ut\mathcal{E}$), without this representation being dependent in any way on a smooth locally Euclidean base space(time)' [272], while the 'half-order' epithet pertains to the fact that half the variables of the first-order formalism are involved; namely, only the connection—but the *entire gravito-inertial connection* $\mathcal{D}$, not just its gauge potential part $\mathcal{A}$. This 'entirety' or 'non-separability' of the notion of connection in our ADG-perspective on gravity is another aspect of the 'unitary' or 'holistic' and (dynamically) autonomous field character of the ADG-approach to gravity, and as it will be argued in section 6, it might be of significant import to QG research.

4.3.2 ADG-gravity: gauge encounters of the third kind

We may subsume ADG-gravity under the heuristic term '*gauge theory of the third kind*'. Let us explain more this denomination. Historically, the terms *gauge (field) theory of the first and second kind* pertain to physical theories with global and local (over an external, background spacetime manifold) internal gauge symmetries, respectively. The epithets 'internal' and 'external' are traditionally used to distinguish between gauge and base spacetime symmetries, respectively. As briefly noted in (Q?.?) above, with the long abandoned and forgotten gauge theory of the first kind ('global scale theory') one normally associates the name Hermann Weyl [414]. As it is well known, Maxwellian electrodynamics and the (flat) Yang-Mills theories of matter interactions (weak and strong) are examples of (abelian and non-abelian, respectively) gauge theories of the second kind—so-called '*local gauge theories*', theories with (gauge) symmetries localized on a base spacetime manifold M (normally taken to be flat Minkowski space $\mathcal{M}$). They both emphasize the importance of the notion of connection (*viz.* gauge field).[278]

With the advent of the connection formulation of GR [7], people got to appreciate more the gauge-theoretic nature of the gravitational force, although it is perhaps 'wrong'[279] to think of the $\mathrm{Diff}(M)$-invariances of the (external spacetime manifold based) theory of gravity (GR) as gravitational gauge symmetries proper [410]. For plainly, technically speaking, the diffeomorphisms in $\mathrm{Diff}(M)$ are (by definition) the (external) 'symmetries' (automorphisms) of the (external) $\mathcal{C}^\infty$-smooth base spacetime manifold (*ie*, $\mathrm{Aut}(M) \equiv \mathrm{Diff}(M)$, with M smooth), *not elements of the automorphism group of a principal fiber bundle*–the structure commonly used to represent the other three gauge forces of nature. In other words, diffeomorphisms are external not internal 'invariances' of GR.[280]

In contradistinction, in the ADG-formulation of gravity (GR), no external, background spacetime manifold is involved whatsoever and the sole dynamical variable is the gravitational $\mathbf{A}$-connection part $\mathcal{D}$ of the ADG-gravitational field $(\mathcal{E}, \mathcal{D})$ [272].[281] Thus, for ADG too, gravity is a gauge theory, albeit, a '*pure gauge theory*', or equivalently, a '*gauge theory of the third kind*', one

[278] Usually, the said gauge fields are modelled after connections on principal fiber bundles (over a base manifold) having the local gauge (symmetry) groups as structure groups in their fibers. As noted earlier, in the usual manifold based gauge-theoretic jargon, by 'gauge field' one normally means the (local) gauge potential part $\mathcal{A}$ of the connection $\mathcal{D}$ [272].

[279] And certainly misleading, as we will argue in the context of QG in section 6.

[280] Indeed, especially *vis-à-vis* the problem of (canonical/Hamiltonian or covariant/Lagrangian) quantization of gravity, regarding $\mathrm{Diff}(M)$ as gravity's gauge symmetry group proper leads to a number of formidable problems, such as the so-called *problem of time* and the *inner product/quantum measure problem*. We will discuss them in more detail in section 6.

[281] This is reflected in the '*half-order*' formulation of GR *à la* ADG mentioned earlier, whereby, the dynamical (vacuum gravitational) equations of Einstein are obtained from varying the ADG-analogue of the Einstein-Hilbert functional (on the space of connections) *only with respect to* $\mathcal{D}$ [262, 272].

in which a base spacetime continuum arena plays no role at all and has no physical significance whatsoever;[282] while moreover, the local gauge 'auto-symmetries' of the ADG-gravitational field $(\mathcal{E}, \mathcal{D})$ 'in-itself' are indeed organized into the principal sheaf $\mathcal{A}ut\mathcal{E}$.[283]

In toto, ADG is concerned directly and solely with the gravitational field 'in-itself', without reference (or commitment) to (let alone the mediation of) a locally Euclidean background spacetime M (in the guise of smooth coordinates in $\mathcal{C}^{\infty}(M)$).[284] *ADG-gravity is fundamentally background (external) spacetimeless—ie*, it is only concerned with the gravitational field (*viz.* connection $\mathcal{D}$) itself. In turn, this is reflected in the fact that the PGC, which in the standard M-based theoresis of gravity (GR) is mathematically effectuated via $\mathrm{Diff}(M)$, in ADG is modelled after $\mathcal{A}ut\mathcal{E}$—the (principal) group sheaf of self-transformations ('auto-transmutations') of the gravitational field $(\mathcal{E}, \mathcal{D})$, glaringly without reference to the base space X.[285]

The last thing that should be noted before we conclude this section is that the said conception of ADG-gravity as a gauge theory of the third kind is closely related to our viewing our ADG perspective on gravity as being '*already quantum*' or '*self-quantum*' (*ie*, quantized by or in itself), a feature which we subsume under the term *third-* or *self-quantization*.[286] We will elaborate further on third quantization in 6.1 and 6.2.

[282]To be fair, the background topological space X in ADG plays only the role of a surrogate scaffolding for the sheaf-theoretic localization of the vector and algebra sheaves involved, on which then the gravitational connection fields are then defined and act, but as stressed earlier, X plays absolutely no role in the gravitational dynamics (Einstein equations), unlike M and its $\mathrm{Diff}(M)$ in the usual manifold based formulation of GR.

[283]Consequently, as noted earlier, 'covariance' is replaced by 'synvariance'.

[284]In fact, as we saw in this paper, regardless of any background spacetime whatsoever (whether continuous or discrete).

[285]It can be noted here that should one wish to study the manifold based GR as a gauge theory in bundle-theoretic terms, the 'natural' bundle that could be associated with it is the *frame bundle*, with 'natural' (local) structure (Lie) group (resp. algebra) of (local) gauge symmetries $GL(4, \mathbb{R})$ (resp. $gl(4, \mathbb{R})$)—the group of general frame (coordinate) transformations. In the purely gauge-theoretic formulation of GR *à la* ADG, the analogous statement is that the principal (group) sheaf $\mathcal{A}ut\mathcal{E}$ of 'local gauge auto-symmetries' of the (gravitational) field $(\mathcal{E}, \mathcal{D})$ is (by definition) locally isomorphic to $(M_n(\mathbf{A}))^{\bullet}$. On the other hand, in the external manifold based GR, $\mathrm{Diff}(M)$ and $GL(4, \mathbb{R})$ are two entirely different 'animals', often confused (and misused!) by physicists and philosophers of science alike, with unfortunately grave misconceptions and false directions hindering development in QG research as we will argue in section 6 [410].

[286]For instance, in [412] it is mentioned that exactly because $\mathrm{Diff}(M)$ is not a gauge group proper, one has trouble in defining observables in a possible quantization scenario for vacuum Einstein gravity (in a compact spacetime manifold) [394, 395]. In subsection 7.5, in connection with the ADG-theoretic resolution of the Einstein hole argument, we will argue that the 'third-gauged' and 'third-quantized' ADG-gravity enables us straightforwardly to view vacuum Einstein gravity as an external spacetimeless, pure gauge theory, with quantum traits already built into the fundamental formalism.

4.4 Section's Résumé

A brief summary of this section can be itemized as follows:

1. First we note that ADG is an algebraico-categorical way of doing DG, without commitment to a background smooth manifold. From a categorical viewpoint the category of differential triads $\mathfrak{DT}$ involved in ADG has significant advantages over the category $\mathcal{M}an$ of differential manifolds involved in CDG (*eg*, it is bicomplete, it has canonical subobjects, products and coproducts, and it behaves well under quotients—properties that $\mathcal{M}an$ simply lacks). These advantages may have significant physical implications and applications (*eg*, applying ADG to Sorkin's finitary topological spaces [270, 271, 272], or developing a topos-theoretic perspective on such a finitary gravity [316]).

2. From the ADG-theoretic viewpoint, all DG boils down to the structure sheaf $\mathbf{A}$ of generalized arithmetics (or 'differentiable coordinates') that one employs, as both d and its generalization, $\mathcal{D}$, are essentially 'derived' from it. In a (quantum) physical sense, our (generalized) measurements (in $\mathbf{A}$) induce dynamical changes ($\mathcal{D}$).

3. Further to the point above, as befits ADG's algebraico-categorical character, the (vacuum) gravitational dynamics (Einstein's equations) is functorial relative to $\mathbf{A}$. This entails that the dynamics remains 'invariant' under our general coordinate transformations, while since no external spacetime manifold M is involved (and hence no $\mathrm{Diff}(M)$ either), the PGC of the manifold and CDG-based GR is replaced in ADG-gravity by $\mathcal{A}ut\mathcal{E}$ and the principle of Synvariance (:dynamical auto-transmutations of the field). Moreover, $\mathbf{A}$-functoriality of the ADG-gravitational dynamics is pregnant to a natural transformation-type of 'phenomenon' for the Einstein equation, which can be (mathematically) interpreted on the one hand as the PARD, and on the other, (physically) as the PFR.

4. Finally, since the whole ADG-gravity formalism is manifestly *not* supported by a base locally Euclidean spacetime (:differential manifold), it should be contrasted against either Einstein's original smooth pseudo-Riemannian metric based second-order formalism, or the first-order, metric-affine Palatini formalism, or even more recently, Ashtekar's new variables scenario [7], in which both the *smooth* affine connection *and* the *smooth* metric in the guise of the *smooth vierbein* field are engaged. Fittingly, the 'ADG-gravity' formalism can been coined '*background smooth spacetime manifoldless, half-order, third gauge formalism*',[287] and it ap-

[287]And we shall see in the sequel, also '*third quantum field*' (or '*third quantized field*', or even '*self-quantum*') formalism.

pears to be a promising candidate scenario for solving the long standing problem of viewing (quantum) gravity as a (quantum) gauge theory proper [215, 412].[288]

5 ADG-Theoretic 'Algebraic Absorption' of Singularities

There have been recent attempts to deal with the singularities of GR in an algebraic and sheaf-theoretic way,[289] by also resorting to some sort of 'algebraically generalized differential spaces' [188, 189, 190, 191, 192, 193, 194]; albeit, all of them still rely essentially on a background $\mathcal{C}^\infty$-smooth spacetime manifold for 'drawing' their differential (and differentiable!) structures. One expected result of this manifold dependence, even in those generalized algebraic approaches, which also aspire to address structural issues in the QG regime, is that the singular *loci* are situated at the boundary of the spacetime manifold, them too stubbornly resisting (analytic) extensibility thus exposing the insuperable weaknesses of the classical theory on which GR is based (CDG) that we saw in section 2.

More recently, and quite explicitly, Stachel [374], in connection with the analytic inextensibility feature unambiguously characterizing 'real' spacetime singularities in the Calculus-based GR, notes rather 'pessimistically':[290]

(Q5.1) "...*sheaf theory*[a] might be the appropriate mathematical tool to handle the problem [of analytic manifold inextensibility *vis-à-vis* $\mathcal{C}^\infty$-spacetime singularities][b] in general relativity. *As far as I know, no one has followed up on this suggestion, and my own recent efforts have been stymied by the circumstance that all treatments of sheaf theory that I know assume an underlying manifold*[c]*...*"

[a]Our emphasis.
[b]Our addition for clarity.
[c]Again, our emphasis.

All in all, the base differential spacetime manifold, with its inherent $\mathcal{C}^\infty$-smooth singularities, still appears to persist in all the algebraic, sheaf-theoretic approaches to GR so far—a persistence that, taking into account the Einstein-Feynman-Isham 'no-go' of CDG in the quantum deep,[291] appears to halt the potential mathematical and physical applications of algebraic, sheaf-theoretic ideas to differential geometry and QG respectively already at the classical, $\mathcal{C}^\infty$-continuum level.

[288]See section 7 below.
[289]The algebraic approach to Einstein gravity was pioneered by Geroch in [157].
[290]We wish to thank Rafael Sorkin for communicating to us John Stachel's latest paper [374].
[291]See quotes (Q?.?–Q?.?) in the sequel.

This is not the case though in applications of ADG, since the latter employs sheaf theory alright, but in a manifestly base manifold free way. Parenthetically we digress a bit and note that in chapter 7, by way of contrast against the basic tenets of ADG, we will comment on and criticize the inappropriateness of the way we have so far applied differential geometric ideas, based essentially on a geometrical locally Euclidean spacetime, in QG research. This critique will lead us then to 'define' the notion of '*physical, Euclidean geometry*'—a notion that in our view appears to be fruitful in QG research—in contradistinction to the nowadays widely adopted and used conception of a '*mathematical, Cartesian geometry*'—a conception that in the context of *differential* geometry appears to be 'monopolized', with grave consequences for QG research, by the (use of the base) differential manifold.

In the next subsection we argue on general grounds that ADG, which uses sheaf theory in a way that Stachel would like (*ie*, without the presence of a base differential manifold), is able to evade the entire spectrum of $\mathcal{C}^\infty$-gravitational singularities: from the purely differential geometric DGSs, to the generalized, distributional SFSs.

5.1 Killing Two $\mathcal{C}^\infty$-Singular Birds, DGSs and SFSs, With One ADG Stone

On general grounds, and as we shall witness shortly in connection with both ways that we 'resolve' the interior Schwarzschild singularity by ADG-means, ADG, essentially due to its up-front abolishing the background differential spacetime manifold M carrying the $\mathcal{C}^\infty$-smooth singularities in its coordinate structure sheaf $\mathbf{A} \equiv \mathcal{C}^\infty_M$, evades all (and all sorts of) spacetime singularities that have been hitherto 'classified' by the usual Analytic (CDG) means [87]: from the DGSs, to the SFS, and of course the VESs 'in between' these two categories (6). That is to say, by the way we will evade ADG-theoretically the inner Schwarzschild singularity in the next section, it will become transparent that:

1. To begin with, since no base spacetime manifold is used at all in ADG, no issue of analytic extension of such a background past the singular *locus* arises whatsoever, thus the entire spectrum of $\mathcal{C}^\infty$-smooth singularities—from DGSs, to SFS—is evaded 'in one go'.

2. With respect to DGSs in particular, since to begin with the C^k-metric ($k = 0 \ldots \infty$) is *not* the basic dynamical gravitational field variable in ADG (but rather the background manifoldless, algebraic $\mathbf{A}$-connection $\mathcal{D}$ is),[292] no issue of its C^k-extension past the singular *locus* arises whatsoever, let alone, of course, that the law of gravity—essentially, the dynamical variable

[292] Again, see 3.3.

$g_{\mu\nu}$ entering the Einstein equations as formulated traditionally—breaks down at the C^k-order of differentiation.[293]

3. Also, concerning distributional SFSs in particular, we will witness in 5.2.3 that Einstein's equations hold over space(time)s that are densely packed with 'distributional singularities'—*ie*, the gravitational field law is formulated and holds entirely when Rosinger's differential algebras of generalized functions (non-linear distributions), hosting dense singularities in (locally) Euclidean space(time)s, are used as coefficient functions (structure sheaves of generalized coordinates) in ADG.[294]

4. Finally, it follows from (6) that VESs too, lying in between DGSs and SFSs, pose no problem to the application of ADG to vacuum Einstein gravity.[295]

In order to prepare the reader for the 'static', spatial point-resolution, as well as for the 'dynamic', temporal, extended line-resolution, of the interior Schwarzschild singularity by ADG-means in the next section, we recall below some basic results from the application of ADG to the so-called spacetime foam dense singularities in GR as originally treated in [273, 274, 275].

5.2 A Concrete Application of ADG: Uncountable 'Spacetime Foam Dense Singularities' in the Smooth Manifold's Bulk

In this subsection we will present elements from the application of ADG to Euclidean and finite-dimensional locally Euclidean (manifold) space(time)s X hosting the most numerous and unmanageable by CDG means singularities: the so-called 'spacetime foam' dense singularities [273, 274, 262, 275, 264].[296] In mathematics, these are singularities of *generalized functions* (distributions) which have been used as coefficients in and have been occurring as solutions of non-linear (both hyperbolic and elliptic) partial differential equations (NLPDEs), as originally treated entirely algebraically by Rosinger in [324, 327].[297] As briefly mentioned before, in physics, interest in such

[293]Again, see α) in 2.1.5.
[294]Again, see β) in 2.1.5.
[295]Again, see γ) in 2.1.5.
[296]Of these, the central references are [274, 275]. It must be noted here that a bit earlier, in [273], it was shown that ADG can be successfully applied so as to incorporate singularities situated on arbitrary closed nowhere dense subsets of Euclidean and locally Euclidean (manifold) space(time)s, sets that can have arbitrarily large Lebesgue measure [287]. As we will see below, the spacetime foam dense singularities in [274, 275] are much more numerous (and unmanageable by CDG-theoretic means) than the nowhere dense ones, albeit, the entire spectrum of ADG still applies to them without a problem.
[297]Hereafter we will refer to Rosinger's theory as *non-linear algebraic theory of generalized functions*, or better, as *non-linear algebraic distribution theory* (NLADT).

singularities has arisen in the study of 'spacetime foam' structures in GR and QG, as studied primarily by the Polish school of Heller *et al.* [190, 194].

Arguably, *these are the most robust and numerous singularities that have appeared so far in the theory of NLPDEs*, but two of their prominent features that we would like to highlight here are:

- First, *their cardinality.* These are singularities on arbitrary subsets of the underlying topological space(time) X. In particular, they can be concentrated on *dense* subsets of X,[298] under the proviso that their complements, consisting of non-singular (regular) points, are also dense in X. In case X is a Euclidean space or a finite-dimensional manifold, *the cardinality of the set of singular points may be larger than that of the regular ones.* For instance, when one takes $X = \mathbb{R}$, the dense singular subsets of it may have the cardinal of the continuum—*ie*, the singularities may be situated on the irrational numbers, while the regular ones are also dense but countable in $\mathbb{R}$ and situated on the rationals.[299]

- Second, *their situation in the manifold's bulk.* As it is evident from the above, the dense singularities, apart from their uncountable multiplicity, are not situated merely at the boundary of the underlying topological space(time) (manifold), but occupy 'central' points in its interior. This is in striking contrast to the usual theoresis of $\mathcal{C}^\infty$-smooth spacetime singularities that we revisited in section 2 [86, 87], which we may now coin 'separated and isolated', or 'solitary', or even '*spacetime marginal*' for effect.[300]

- And third, *the differential algebras of generalized functions in Rosinger's NLADT contain both the usual algebra $\mathcal{C}^\infty(X)$ of smooth functions and Schwartz's linear distributions* [274, 275]. Furthermore, the NLADs, either with nowhere dense, or even more prominently, with dense singularities, have proven to be more versatile (and potentially more useful in differential geometric applications) than the, lately quite popular in the theory of NLPDEs and its applications to various equations of mathematical physics, non-linear generalized functions of Colombeau [90].[301]

[298] That is why one generally refers to them as '*dense singularities*'.

[299] In 5.2.3 we will exploit this application of ADG to spacetime foam dense singularities and cast the interior singularity of the Schwarzschild solution of the Einstein equations as such an extended dense singularity extending along the 'wristwatch time-line' ($\mathbb{R}$) of a point particle of mass m at rest.

[300] This situation is also in contrast to the 'algebraically generalized differential spaces/spacetime foam' approach to GR and QG of Heller *et al.*, as they too assume that singularities—in fact, them too *nowhere dense singularities* like the ones in [273]—sit right at the edge of the spacetime manifold (see [188, 189, 192, 193, 194, 191], and especially [170]).

[301] See [274] for a discussion of the (differential geometric) virtues of the NLADs compared to the Colombeau distributions.

5.2.1 Dense singularities in Euclidean and locally Euclidean spaces

We first note that in [274] as the domain of definition of Rosinger's NLADs one takes any nonempty open subset X of a finite-dimensional Euclidean space ($\mathbb{R}^n$), or more generally, any non-void region of a space that is locally isomorphic to $\mathbb{R}^n$—an n-manifold M. Then, various collections of singularities in X are described by giving a family $\mathcal{S}$ of subsets $\mathfrak{S}$ of X, with each $\mathfrak{S}$ corresponding to a set of singularities of a certain given generalized function (NLAD).

Dense singularities are the most numerous ones. Arguably, cardinality-wise, the largest collection $\mathcal{S}$ of such Euclidean or manifold singularity-sets $\mathfrak{S} \subset X$ that one can encounter is

$$\mathcal{S}_{dns}(X) := \{\mathfrak{S} \subset X : \ X \setminus \mathfrak{S} \text{ is dense in } X\} \tag{47}$$

with the subscript '*dns*' denoting '*dense*'. It is plain that there are many such dense singularity-sets in X; moreover, for some of them, the cardinality of $\mathfrak{S}$ may be bigger than that of its complement $X \setminus \mathfrak{S}$, which in turn means that the set of singular *loci* is not only dense in X, but also larger than the set of X's regular points.

Due to the maximum cardinality property of $\mathcal{S}_{dns}$, the various families $\mathcal{S}$ of singularity-sets in X considered in [274, 275] are 'less numerous' than $\mathcal{S}_{dns}$—*ie*, they are taken to satisfy the condition $\mathcal{S} \subseteq \mathcal{S}_{dns}$.[302] Characteristically, two such collections that were considered in [273, 274, 275] are

$$\mathcal{S}_{ndns}(X) := \{\mathfrak{S} \subset X : \ \mathfrak{S} \text{ is closed and nowhere dense in } X\} \tag{48}$$

and

$$\mathcal{S}_{B_1}(X) := \{\mathfrak{S} \subset X : \ \mathfrak{S} \text{ is of first Baire category in } X\} \tag{49}$$

with $\mathcal{S}_{ndns}$ representing families of *nowhere dense singularity-sets* (hence the subscript '*ndns*') [273], while $\mathcal{S}_{B_1}$ singularity-sets of the first Baire category [274, 275]. In terms of their cardinality, they are ordered by inclusion as follows:

$$\boxed{\mathcal{S}_{ndns}} \subset \boxed{\mathcal{S}_{B_1}} \subset \boxed{\mathcal{S}_{dns}} \tag{50}$$

with $\mathcal{S}_{dns}$, as noted above, being the most numerous (and the most problematic from the usual CDG-viewpoint) ones, and the ones in which we are interested here.

[302] For a more detailed treatment of dense singularity-sets, the reader can refer to [325, 326, 327].

Spacetime foam algebras and nets of dense singularities with asymptotically vanishing differential ideals: a singularity analogue of Sorkin's topological finitary posets 'smeared point-quotient trick'. In this paragraph, and in view of the two-fold ADG-theoretic resolution of the interior Schwarzschild singularity that we are going to present in the sequel, by regarding it either as an 'extended' spacetime foam dense time-line singularity *à la* Mallios and Rosinger [274, 275] (5.2.3), or as a static spacetime 'localized' point-singularity and then treating it by Sorkin's 'blow-up' finitary topological posets methods [355], their corresponding differential incidence algebras [318, 319] and the finitary spacetime sheaves thereof [310] in the manner of ADG [270, 271, 272] (5.2.2), we present the basic ideas in the construction of spacetime foam algebras of generalized functions as originally developed in [325, 326, 327]. In the process, we will witness fundamental theoretical affinities, conceptual similarities and common technical 'tricks' between Rosinger's and Sorkin's ideas, especially when the latter are presented in an algebraic way as in [318, 319].[303]

The main thing that the reader should notice in the brief presentation below is that Rosinger, by the spacetime foam algebra construction, manages to encode the information on various singularity point-sets (*eg*, such as $\mathcal{S}_{ndns}$, or even more prominently, $\mathcal{S}_{dns}$) *entirely algebraically* in much the same way that the topological poset spacetime discretizations where cast again *purely algebraically* in [318, 319]. Then, the concomitant organization of the respective algebras[304] into sheaves, and after crucially realizing that both kinds of algebras are *differential algebras*,[305] the application of the manifold and Calculus-free sheaf-theoretic ADG-technology to either enables one to do differential geometry (and apply it to gravity!) over space(time)s that are far from being smooth (in the usual $\mathcal{C}^\infty$-manifold sense)—that is to say, space(time)s that are densely populated by singularities (of the most general type of functions) in Rosinger's case [273, 274, 275, 260, 262], and 'discrete' or 'reticular' space(time)s that are far from being smooth continua like the classical differential spacetime manifold M [270, 271, 272]. So far, these are the two main applications of ADG to mathematical physics—in particular, to classical and quantum gravity research.

There are three basic steps in the construction of Rosinger's spacetime foam algebras (*ie*, the SSTFDAs), steps that, as noted above, bear close similarities with the algebraic approach to

[303] We will see that, in a strong sense, the algebraic description of spacetime foam given in [318, 319] is closely related, both conceptually and technically, to Rosinger's spacetime foam algebras hosting dense singularities, while in the end, the differential geometric platform for unifying both is provided by the sheaf-theoretic means of ADG. Under the prism of ADG, this is more than a nominal, superficial analogy between the Raptis-Zapatrin algebraic conception of spacetime foam and the Mallios-Rosinger spacetime foam dense singularities, and this is what we wish to highlight by the presentation herein.

[304] Rosinger's spacetime foam algebras may be coined a mouthful '*singularity spacetime foam differential algebras*' (SSTFDAs), while Raptis and Zapatrin's, '*discrete spacetime foam differential algebras*' (DSTFDAs).

[305] Differential algebras of generalized functions (distributions) in Rosinger's case, and discrete differential algebras of generalized maps (poset arrows) in the Raptis-Zapatrin case.

spacetime foam (*ie*, the DSTFDAs) [319] which was originally based on Sorkin's finitary (locally finite) topological poset substitutes of continuous manifolds [355, 318]:

1. **'Coarse or fine graining of singularity-sets':** This step consists on imposing the following conditions on any collection $\mathcal{S}$ of singularity-sets $\mathfrak{S} \subset X$:

$$\textbf{Density of regular points} : \forall \mathfrak{S} \in \mathcal{S} :\ X \setminus \mathfrak{S} \text{ is dense in X} \tag{51}$$

   and

$$\text{`}\textbf{Coarse} - \textbf{graining}' \textbf{ of singularity}\textbf{-sets} : \forall \mathfrak{S}, \mathfrak{S}' \in \mathcal{S} :\ \exists \mathfrak{S}'' \in \mathcal{S} :\ \mathfrak{S} \cup \mathfrak{S}' \subseteq \mathfrak{S}'' \tag{52}$$

   We note that for such an arbitrary collection $\mathcal{S}$ of singularity-sets in X one has the inclusion $\mathcal{S} \subseteq \mathcal{S}_{dns}$; while plainly, both $\mathcal{S}_{ndns}$ and $\mathcal{S}_{B_1}$ satisfy (51) and (52) above.

   The condition (52) may be interpreted as signifying that '*the singularity-set* $\mathfrak{S}''$ *is the coarse-graining of* $\mathfrak{S}$ *and* $\mathfrak{S}'$';[306] moreover, it may be abstracted to the following *algebraic construction* involving not the singularity-subsets of X *per se*, but rather the $\mathcal{C}^\infty$-smooth functions that live on that space(time): so let $L = (\Lambda, \leq)$ be a right-directed partially ordered set—a net.[307] Then consider the algebra $(\mathcal{C}^\infty(X))^\Lambda$ of sequences of smooth functions on X indexed by Λ,[308] which brings us to the next basic step in the construction of spacetime foam algebras.

2. **'Differential ideals covering singular points':** For any singularity-set $\mathfrak{S}$ in an arbitrary singularity-family $\mathcal{S}$ in X, define the following *(differential) ideal* in $(\mathcal{C}^\infty(X))^\Lambda$:

$$\begin{gathered}\mathcal{I}_{L,\mathfrak{S}}(X) := \{(s_\lambda) \in (\mathcal{C}^\infty(X))^\Lambda : \\ \forall x \in X \setminus \mathfrak{S},\ \exists \lambda \in \Lambda \text{ such that, } \forall \mu \in \Lambda,\ \text{with } \mu \geq \lambda,\ \text{and } \forall p \in \mathbb{N}^n : \\ D^p s_\mu(x) = 0\}\end{gathered} \tag{53}$$

   with D^p in the last line standing generically for the smooth function ($p = 0$) entries in s_λ and their (term-wise) partial derivations of arbitrary order ($\mathbb{N} \ni p \geq 1$). The last line is an *asymptotic vanishing condition*, which, as noted in [275], secures that[309]

[306] In turn, $\mathfrak{S}''$ may be thought of as being refined (decomposed) into the 'smaller' (more localized!) singularity-sets $\mathfrak{S}$ and $\mathfrak{S}'$. Technically speaking, this condition is equivalent to positing that $\mathcal{S}$ is an 'upwards directed' $\cup$-semilattice, which may be coined '*the coarse-graining singularity semilattice in* X'.

[307] As noted in [274, 275], *prima facie* L may not be thought of as being dependent on $\mathfrak{S}$, but in certain special situations the two may be related [325].

[308] That is, $(\mathcal{C}^\infty(X))^\Lambda = \{(s_\lambda) :\ \lambda \in \Lambda\}$.

[309] In the quotation below emphasis is ours, as usual.

(Q5.2)

> "...the sequences of smooth functions in the ideal $\mathcal{I}_{L,\mathfrak{S}}(X)$ will in a way cover with their support the singularity set $\mathfrak{S}$, and at the same time vanish outside it, together with all their partial derivatives. In this way, the ideal $\mathcal{I}_{L,\mathfrak{S}}(X)$ carries in an *algebraic* manner the information on the singularity set $\mathfrak{S}$. Therefore, a *quotient* in which the factorization is made with such ideals may in certain ways *do away* with singularities, and do so through *purely algebraic means...*"[a]
>
> [a]In connection with the definition (53) above, we also read from [274] that the assumption that L is right-directed is actually used in proving the additive closure of $\mathcal{I}_{L,\mathfrak{S}}$, namely, that $\forall s, s' \in \mathcal{I}_{L,\mathfrak{S}},\ s + s' \in \mathcal{I}_{L,\mathfrak{S}}$, which is effectively a defining property for an ideal (as a linear subspace). This assumption will also be instrumental in exploring *singularity refinement* properties of spacetime foam algebras in the sequel. In connection with the last remark, we also note from [274, 275] that a judicious choice of L so that the resulting ideals are non-trivial ($\mathcal{I}_{L,\mathfrak{S}} \neq \{0\}$) may result in $\mathcal{I}_{L,\mathfrak{S}}$ being rather *large*—in fact, *uncountable*. Such can be, for instance, the lattice of all topologies on a $\mathcal{C}^0$-manifold [202], and we will encounter such an example shortly when we explore the close similarities between the Mallios-Rosinger [274, 275] and the Raptis-Zapatrin [318, 319] algebraic approaches to spacetime foam under the unifying sheaf-theoretic prism of ADG.

The last sentence in (Q?.?) above brings us to the third, *defining act* for spacetime foam algebras (with dense singularities in X).

3. **'Factoring out the singular loci'—the definition of spacetime foam algebras:** First we note that, in case the collection $\mathcal{S}$ of singularity-sets consists of only one such set $\mathfrak{S}$ (*ie*, $\mathcal{S} = \{\mathfrak{S}\}$), albeit, one that is dense in X (and moreover in the aforesaid sense that the set of regular points $X \setminus \mathfrak{S}$ is dense in X (47), which in turn identifies $\mathcal{S}$ with $\mathcal{S}_{dns}$),[310] the *spacetime foam algebra* is defined by the following quotient:

$$\mathfrak{B}_{L,\mathfrak{S}}(X) := (\mathcal{C}^\infty(X))^\Lambda / \mathcal{I}_{L,\mathfrak{S}}(X) \tag{54}$$

At the same time, by observing that when $\mathcal{S}$ consists of more than one dense singularity-set $\mathfrak{S}$ the set

[310]From now on we will omit the subscript '*dns*' from $\mathcal{S}$, since in what follows we will be interested exclusively in dense singularity-sets in X.

$$\mathcal{I}_{L,\mathcal{S}} := \bigcup_{\mathfrak{S}\in\mathcal{S}} \mathcal{I}_{L,\mathfrak{S}} \tag{55}$$

is also an ideal in $(\mathcal{C}^\infty(X))^\Lambda$, one defines the *spacetime multi-foam algebras* in complete analogy with (54) above, as follows:

$$\mathfrak{B}_{L,\mathcal{S}}(X) := (\mathcal{C}^\infty(X))^\Lambda / \mathcal{I}_{L,\mathcal{S}}(X) \tag{56}$$

noting along the way that, when $\mathcal{S} = \{\mathfrak{S}\}$, $\mathfrak{B}_{L,\mathcal{S}}(X) \equiv \mathfrak{B}_{L,\mathfrak{S}}(X)$—*ie*, spacetime multi-foam and foam algebras coincide.[311]

5.2.2 Spacetime foam algebras as differential algebras of generalized functions, their regularity, and the fine and flabby sheaves thereof

As it happens, the bottom-line of the preceding discussion is that *the spacetime foam algebras may be regarded as differential algebras of generalized functions*—'smeared', (non-linear) distribution-type of functions that enlarge the usual smooth ones in $\mathcal{C}^\infty(M)$, they contain the linear distributions of Schwartz, while for differential geometry's sake (and from the sheaf-theoretic perspective of ADG), they have certain structural (sheaf-theoretic) advantages over Schwartz's linear spaces, but more importantly, for example, over Colombeau's algebras of generalized functions. Below, we would like to discuss briefly these qualities of spacetime foam algebras.

To begin with, the main reason why special attention was paid in [274, 275] on the collection $\mathcal{S}_{dns}$ of dense singularity-sets in X is that if $X \setminus \mathfrak{S} \in \mathcal{S}$, then

$$\mathcal{I}_{L,\mathfrak{S}}(X) \cap \mathcal{X}^\infty_{\Lambda,\mathfrak{S}}(X) = \{0\} \tag{57}$$

where $\mathcal{X}^\infty_{\Lambda,\mathfrak{S}}(X)$ stands for the *diagonal* of the power set $(\mathcal{C}^\infty(X))^\Lambda$—that is to say,

$$\begin{aligned} \mathcal{X}^\infty_{\Lambda,\mathfrak{S}}(X) = \{\chi(\phi) := (\phi_\lambda :\ \lambda\in\Lambda)\} \text{ such that} \\ \forall\lambda\in\Lambda :\ \phi_\lambda = \phi,\ \text{as } \phi \text{ ranges over } \mathcal{C}^\infty(X) \end{aligned} \tag{58}$$

This in turn entails the following algebra isomorphism:

$$\mathcal{C}^\infty(X) \ni \phi \mapsto \chi(\phi) \in \mathcal{X}^\infty_{\Lambda,\mathfrak{S}}(X) \tag{59}$$

In summa, the following *algebra embedding* is secured by (59):

$$\mathcal{C}^\infty(X) \ni \phi \mapsto \chi(\phi) + \mathcal{I}_{L,\mathfrak{S}}(X) \in \mathfrak{B}_{L,\mathfrak{S}}(X) \tag{60}$$

[311] In line with [274, 275], we will call both spacetime foam and multi-foam algebras simply *spacetime foam algebras*.

And in words,

(R5.1) the algebra $\mathcal{C}^\infty(X)$ of smooth functions on X is *embedded* (included) into the spacetime foam algebras,[a] which in turn means that the latter are algebras of *generalized functions* (distributions).[b]

[a]And, of course, to multi-foam algebras as one obtains a similar embedding $\mathcal{C}^\infty(X) \ni \phi \mapsto \chi(\phi) + \mathcal{I}_{L,\mathcal{S}}(X) \in \mathfrak{B}_{L,\mathcal{S}}(X)$ in case the collection $\mathcal{S}$ consists of more than one dense singularity-set $\mathfrak{S}$ in X.

[b]In fact, the spacetime foam algebras are *unital*, with unit element $\chi(\mathbf{1}) + \mathcal{I}_{L,\mathfrak{S} \text{ or } \mathcal{S}}$ (where $\mathbf{1}$ is the constant unit-valued function on X).

In addition, the aforementioned asymptotic vanishing condition (53) entails the following '*closure*' of $\mathfrak{B}_{L,\mathfrak{S}}(X)$ and $\mathfrak{B}_{L,\mathcal{S}}(X)$ under (partial) differentiation of arbitrary order:

$$\begin{array}{c} D^p\mathfrak{B}_{L,\mathfrak{S}}(X) \subseteq \mathfrak{B}_{L,\mathfrak{S}}(X) \\ D^p\mathfrak{B}_{L,\mathcal{S}}(X) \subseteq \mathfrak{B}_{L,\mathcal{S}}(X) \end{array} \tag{61}$$

which means that $\mathfrak{B}_{L,\mathfrak{S}}(X)$, and *in extenso* $\mathfrak{B}_{L,\mathcal{S}}(X)$, are *differential* algebras of generalized functions.

We also read from [275] that

(Q5.3) "the [spacetime foam and] multi-foam algebras contain the Schwartz distributions, that is, [similarly to (60)], we have linear embeddings which respect the arbitrary partial derivation of smooth functions, namely $\mathcal{D}'(X) \subset \mathfrak{B}_{L,\mathfrak{S}}(X)$, for $\mathfrak{S} \in \mathcal{S}_{dns}(X)$ and $\mathcal{D}'(X) \subset \mathfrak{B}_{L,\mathcal{S}}(X)$.[a]"

[a]With $\mathcal{D}'(X)$ denoting the linear (vector) space of the Schwartz distributions.

For more details about the issues to be presented in the following two paragraphs, the reader is referred to [274, 275].

A note on regularity properties of the spacetime foam generalized functions: quotient regularity with a twist of Sorkin's finitary topological refinement; 'singularity-refinement'. In order to discuss briefly the regularity properties of the non-linear spacetime foam distributions, we first note the following set-theoretic inclusion of the aforesaid foam into multi-foam differential ideals:

$$\mathcal{I}_{L,\mathfrak{S}} \subseteq \mathcal{I}_{L,\mathcal{S}} \tag{62}$$

an inclusion which can be algebraically translated into the following *surjective algebra homomorphism* between the corresponding foam and multi-foam algebras:

$$\mathfrak{B}_{L,\mathcal{S}}(X) \ni v + \mathcal{I}_{L,\mathfrak{S}}(X) \mapsto v + \mathcal{I}_{L,\mathcal{S}}(X) \in \mathfrak{B}_{L,\mathcal{S}}(X) \tag{63}$$

Now, in the context of generalized functions, we read from [275] that (63) means that "*the typical generalized functions in* $\mathfrak{B}_{L,\mathcal{S}}(X)$ [(the multi-foam distributions)] *are more regular than those in* $\mathfrak{B}_{L,\mathfrak{S}}$ [(the foam distributions)]".[312] In a way, this is to be expected since, given two spaces $\mathcal{S}_1$ and $\mathcal{S}_2$ of generalized functions, with, say, $\mathcal{C}^\infty(X) \subset \mathcal{S}_1 \subset \mathcal{S}_2$, the elements of the *larger* space $\mathcal{S}_2$ normally appear to be *less regular* than those of the *smaller* one $\mathcal{S}_1$—a property that one may wish to coin *subset regularity*.[313]

A notion akin to subset regularity is, what is coined in [275], *quotient regularity*.[314] This notion can be briefly explained as follows. Again, let $\mathcal{S}_1$, $\mathcal{S}_2$ be as above, and also assume that they are obtained as quotient vector spaces (of some 'big' linear space $\mathcal{S}$) in the following way:

$$\mathcal{S}_1 = \mathcal{S}/\mathcal{W},\ \mathcal{S}_2 = \mathcal{S}/\mathcal{V};\ \text{with } \mathcal{V} \subseteq \mathcal{W} \tag{64}$$

Let also $\mathcal{F}:\ \mathcal{S}_2 \longrightarrow \mathcal{S}_1$ be the *canonical* surjective linear mapping between them given point-wise by

$$\mathcal{S}_2 = \mathcal{S}/\mathcal{V} \ni f + \mathcal{V} \mapsto f + \mathcal{W} \in \mathcal{S}/\mathcal{W} = \mathcal{S}_1 \tag{65}$$

Then, as noted in [275], one also has for the two quotient spaces $\mathcal{S}_1,\ \mathcal{S}_2$ of generalized functions the *quotient regularity power relation*

$$\mathcal{S}_1 \succ_{reg} \mathcal{S}_2 \tag{66}$$

reading that "*the typical elements of* $\mathcal{S}_1$ *are at least as regular as those of* $\mathcal{S}_2$". Thus, in the sense above, it is plain that

(R5.2) *the typical spacetime multi-foam generalized function is more regular than the typical foam one.*

[312] We may formally symbolize this '*regularity power-relation*' as $\mathfrak{B}_{L,\mathcal{S}}(X) \succ_{reg} \mathfrak{B}_{L,\mathfrak{S}}(X)$.

[313] Again, formally one may write $\mathcal{S}_1 \succ_{reg} \mathcal{S}_2$. In connection with subset regularity, a word of caution already noted in [275]: while subset regularity is satisfied, for example, by the inclusions $\mathcal{C}^\infty(\mathbb{R}) \subset \mathcal{C}^1(\mathbb{R}) \subset \mathcal{C}^0(\mathbb{R})$ (when $X \equiv \mathbb{R}$), there is also the linear surjective mapping $D:\ \mathcal{C}^1(\mathbb{R}) \longrightarrow \mathcal{C}^0(\mathbb{R})$, which by no means suggests that continuous ($\mathcal{C}^0$) functions (on $\mathbb{R}$) are more regular than the singly differentiable (C^1) ones.

[314] It too to be formally symbolized here by $\succ_{reg}$.

In other words,

$$\mathfrak{B}_{L,\mathcal{S}} \succ_{reg} \mathfrak{B}_{L,\mathfrak{S}} \tag{67}$$

in the sense of quotient regularity.[315]

By the foregoing discussion it has become clear that *the more dense singularity-sets* $\mathfrak{S}$ *a collection* $\mathcal{S}$ *includes, the more quotient regular the corresponding spacetime multi-foam algebra* $\mathfrak{B}_{L,\mathcal{S}}$ *is.* This prompts us to introduce the notion of *singularity refinement*, a notion which will prove to be very useful in comparing and exploring the close similarities between the two algebraic approaches to spacetime foam—namely, the Mallios-Rosinger approach in [274, 275] (SSTFDAs), and the Raptis-Zapatrin approach in [318, 319] (DSTFDAs), both of which, under the unifying perspective of ADG, will be used in the next section to 'resolve' the interior Schwarzschild singularity.

The basic idea behind singularity refinement is that as one employs 'smaller and more numerous'[316] (dense) singularity-sets in order to cover the (densely) singular point-*loci* of the underlying topological space(time) X—a procedure which may be formally symbolized by the *singularity refinement relation*

$$\mathcal{S} \prec_{sref} \mathcal{S}^{'} \tag{68}$$

between the corresponding (dense) singularity families $\mathcal{S}$—the respective differential ideals $\mathcal{I}_{L,\mathcal{S}}$ and $\mathcal{I}_{L,\mathcal{S}'}$ in $(\mathcal{C}^\infty(X))^\Lambda$ are (partially) ordered by inclusion according to (62) (*ie*, $\mathcal{I}_{L,\mathcal{S}}(X) \subseteq \mathcal{I}_{L,\mathcal{S}'}(X)$), which in turn entails *surjective spacetime multi-foam algebra homomorphisms* of the kind (63)

$$\mathfrak{B}_{L,\mathcal{S}}(X) \longrightarrow \mathfrak{B}_{L,\mathcal{S}'}(X) \Leftrightarrow \mathfrak{B}_{L,\mathcal{S}'}(X) \succ_{reg} \mathfrak{B}_{L,\mathcal{S}}(X) \tag{69}$$

In fact, (69) is equivalent to (68), but simply expressed in algebraic terms (*ie*, in terms of the corresponding multi-foam algebras instead of the dense singularity point-subsets of X themselves). *In toto*, and in view of the expression 68) and the right hand side of (69), we note that *singularity refinement and quotient regularity are 'order-covariant' notions.*[317]

The foregoing discussion gives us a hint that we are actually dealing with a kind of '*inverse system*' $\overleftarrow{\mathcal{N}} = (\mathcal{S}, \prec_{sref})$ of collections ($\mathcal{S}$) of dense singularity-sets ($\mathfrak{S}$) covering the densely singular

[315] For, plainly, $\mathcal{I}_{L,\mathfrak{S}} \subseteq \mathcal{I}_{L,\mathcal{S}}$ (62), and $\mathcal{S} \equiv (\mathcal{C}^\infty(X))^\Lambda$.

[316] The expression 'smaller and more numerous' may be cumulatively coined '*finer*', hence the term '*refinement*'. Shortly we will see the close link between this notion of singularity refinement and the one of *topological refinement* originally due to Sorkin [355].

[317] That is to say, as one refines one's dense singularity-sets, the corresponding spacetime multi-foam algebras become more regular, since, after all, in the process one does away with (*ie*, one factors out) more singular point-*loci* (the ones lying in the respective $\mathcal{I}_{L,\mathcal{S}}(X)$s).

point-*loci* of X. In turn, $\overleftarrow{\mathcal{N}}$ may be interpreted as a *singularity refinement net*. Now, having in hand the material above, we are in a position to explore the aforesaid close, both conceptually and technically, similarities between SSTFDAs and DSTFDAs, both of which were ultimately developed and presented by using the algebraic (*ie*, sheaf-theoretic) technology of ADG.

5.2.3 The conceptual 'leit motif' underlying both SSTFDAs and DSTFDAs: from an ADG-theoretic and physical perspective, more than a superficial, formal resemblance

First of all, we state it up-front that, geometrically speaking, the singularities and various other (differential) geometric pathologies of the (smooth) manifold are arguably due to its point-set character. For any of the points of M can be the *locus* of a singularity of some physically important (smooth) field. Thus, the common 'strategy' behind SSTFDAs and DSTFDAs is to somehow downplay (or perhaps even evade) the pointed nature of M, the former with differential geometrical aims[318] in mind, while the latter with topological issues in mind. Below, we briefly bring forth from 3.1.7 and recapitulate from [355] the basic steps in the construction of the so-called *finitary topological spaces*, which are structures originally intended to replace or 'approximate' the topology of continuous (spacetime) manifolds by finitistic means. In the process, we will see and highlight the close ties between the two approaches, especially when Sorkin's constructions are cast algebraically—the so-called combinatory-algebraic description of spacetime foam [318, 319].[319] For technical details about what is going to be mentioned below, the reader is referred to [355, 318, 319].

1. **Smearing points.** The first step in Sorkin's *finitary topological replacements of a continuous (spacetime) manifold*[320] is *the replacement of the latter's points by 'coarse', open sets about them*. The basic intuition underlying this substitution is that geometrical points represent ideal elements in the theory, when, in fact, what we actually determine by our 'coarse' space(time) measurements are never points (or 'instances'), but rather 'large' ('blown-up' or 'extended') regions (or time-intervals) about them. This 'smearing' of points by regions in the DSTFDAs theory is the analogue in the SSTFDA theoresis of concentrating on distributions rather than on the 'geometrical point-functions' $\mathcal{C}^\infty(M)$—*ie*, the main idea is to generalize (or enlarge) the (class of) smooth (differentiable) functions on the point-set M.

2. **Factoring out points set-theoretically (topologically).** As we saw in 3.1.7, after the said substitution, Sorkin effectively does away with points by grouping them into '*indistin-*

[318] In fact, with the theory of non-linear PDEs.

[319] The reader should note that in 5.2.1, before we give a finitary-algebraic and ADG-based resolution of the inner Schwarzschild singularity, we are going to discuss further the virtues of DSTFDAs.

[320] Say, spacetime regarded as a topological (*ie*, $\mathcal{C}^0$-) manifold.

guishability equivalence classes'. Namely, he first covers a bounded region $X \subset M$ with a *locally finite (finitary) open covering*[321] $\mathcal{U}_i$, and then he defines the following 'indistinguishability[322] equivalence relation' between X's points:[323]

$$X \ni x \overset{\mathcal{U}_i}{\sim} y \in X \Leftrightarrow \Lambda|_{\mathcal{U}_i}(x) = \Lambda|_{\mathcal{U}_i}(y) \tag{70}$$

where $\Lambda|_{\mathcal{U}_i}(x) := \bigcap\{U \in \mathcal{U}_i | \; x \in U\}$ is the 'smallest' open set in the subtopology τ_i of X generated by the open sets in $\mathcal{U}_i$[324] that contains x.

Then, it is shown that *the quotient space $P_i := X/ \overset{\mathcal{U}_i}{\sim}$, consisting of $\overset{\mathcal{U}_i}{\sim}$-equivalence classes of points in X, is a partially ordered set (poset) with the structure of a T_0-topological space*, and it is coined *the finitary substitute of X*.

As noted in section 3, the P_is can also be viewed as *simplicial complexes* $\mathcal{K}_i$ when one regards each poset as a family of abstract k-simplices ($k = 1, 2, 3, \ldots$), with each k-simplex being defined as a collection of $k+1$ open sets $U_\alpha \in \mathcal{U}_i$ ($\alpha = 0 \ldots k$) of non-trivial common intersection; that is to say,

$$\{U_0, \ldots, U_k\} \in \mathcal{K} \;\Leftrightarrow\; U_0 \cap U_1 \cap \ldots \cap U_k \neq \emptyset \tag{71}$$

The k-simplices in $\mathcal{K}_i$ are then partially ordered by the homological incidence relation '*is the face of*' to the effect that $\mathcal{K}_i$ itself can be viewed as a poset.[325]

3. **Factoring out points algebraically.** One can get a clearer idea of the close affinities between Sorkin's work on topological locally finite poset discretizations of $\mathcal{C}^0$-manifolds and the spacetime foam dense singularity constructions, by passing to the *Gel'fand-dual algebraic picture* of the finitary topological posets/simplicial complexes above involving the so-called *incidence algebras* associated with the said P_is, or equivalently, with the $\mathcal{K}_i$s [431, 318, 319, 432].

   Without going into any technical detail, as also mentioned earlier when we were discussing the finitary differential triads, one associates with Sorkin's P_is associative, but in general

[321] Shortly we will see how this assumption of 'finitarity' (local finiteness) is also of great import in the SSTFDAs theory.

[322] Relative to our coarse observations or 'measurements' $U \in \mathcal{U}_i$.

[323] This expression is identical to (32) given earlier. In fact, the short exposition of Sorkin's scheme given below is the same as the one given in the bonus paragraph in 3.1.7, but with different interpretational emphasis.

[324] That is, $\tau_i(X)$ is generated by finite intersections of countable unions of open sets U in $\mathcal{U}_i$.

[325] Indeed, one may recall that $\mathcal{K}_i$ is sometimes referred to as the Alexandrov-Čech '*nerve complex*' associated with the locally finite covering $\mathcal{U}_i$ of X [2, 78]. For short, and for obvious reasons, we call it a '*finitary complex*'.

noncommutative, $\mathbb{K}$-algebras ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) $\Omega_i(P_i)$s, which are called *the incidence algebras of the finitary topological posets*. The main characteristic of the Ω_is is that they are *discrete differential algebras of generalized maps (arrows)* [97, 99, 26, 98], which can be decomposed into a direct sum of an abelian subalgebra $\mathbb{A}_i \equiv \Omega_i^{p=0}$ of 'coordinate', position-like maps, and a $\mathbb{Z}_+$-graded $\mathbb{A}_i$-bimodule $\bigoplus_p \Omega_i^{p\geq 1}$ of discrete differential, momentum-like maps comprising p-graded linear subspaces in Ω_i. At the same time, there is a *linear, $\mathbb{A}_i$-Leibnizian and nilpotent Cartan-Kähler differential operator* $\mathtt{d}$ effecting maps $\mathtt{d}: \Omega_i^p \longrightarrow \Omega_i^{p+1}$.[326]

One then, by virtue of a procedure called *Gel'fand spatialization* [318, 319], recovers the finitary poset's points (vertices) and the Sorkin graph-topology that the arrows in its Hasse diagram define by going to the *primitive spectrum* of Ω_i[327] and by assigning the so-called *Rota topology* on it.[328] Moreover, one can show that the said ideals are *differential ideals* relative to the Cartan-Kähler differential $\mathtt{d}$ [432].

4. **X's points are 'blown-up' or smeared, and then quotiented-out: Gel'fand duality on $\mathcal{C}^\infty$-manifolds; forced 'surgery' on singular points.** The conceptual distillation of the steps 1–3 above is that first the geometrical points of the original (topological) space(time) manifold X were 'blown-up', 'smeared', or 'enlarged' by being replaced by the Us in $\mathcal{U}_i$ about them,[329] to the effect that the 'points' of the resulting P_i are 'coarse' equivalence classes of points. Moreover, by going to the algebraic realm of the corresponding Ω_is, on the one hand the points (:$\overset{\mathcal{U}_i}{\sim}$-equivalence classes) of P_i are replaced by (differential) ideals in its Ω_i, and on the other, one uncovers differential geometric, not just topological, information encoded in the graded discrete differential incidence algebras.[330]

   The aforedescribed enlargement and the concomitant 'modding-out' (substitution) of the geometrical points of X by ideals *à la* Gel'fand may be viewed as a discrete analogue of the basic result from the application of Gel'fand duality to $\mathcal{C}^\infty$-smooth manifolds, namely, the reconstruction of a $\mathcal{C}^\infty$-smooth manifold M as the spectrum $\mathfrak{M}$ of its topological algebra

[326] Indeed, $\mathtt{d}$ defines the Ω_is as discrete *differential* algebras. As noted in section 3, $\mathtt{d}$ can be expressed in terms of the homological boundary (border) and coboundary (coborder) operators defined on the respective $\mathcal{K}_i$s [432].

[327] Whose points, by definition, are *primitive ideals* in Ω_i, which in turn may be identified with the *(kernels of the) Ω_i's irreducible representations.*

[328] In fact, one first defines algebraically the *generating relation* of the Rota topology (corresponding to the *immediate arrows*, or *links*, or even *covering relations* in the Hasse diagram of the P_i), and then recovers the finitary poset topology as the *transitive closure* of this generating relation.

[329] In Sorkin's scheme in particular, by the minimal open cells $\Lambda(X)|_{\mathcal{U}_i}$—what we coined Sorkin's *ur*-cells (relative to $\mathcal{U}_i$) in 3.1.7.

[330] Indeed, here we witness, in a finitary setting, one of the basic tenets of ADG: that *differentiability* (or differential structure) *comes from algebraicity* (or algebraic structure). This is in the spirit of Leibniz's relational (:algebraic) conception of differentiation (derivative).

${}^{(\mathbb{R})}\mathcal{C}^\infty(M)$ of smooth ($\mathbb{R}$-valued) functions [254, 258].[331] To recall briefly this construction, let M be a differential manifold and p one of its points. One considers then the following collection of smooth $\mathbb{R}$-valued functions on M

$$\mathcal{I}_p = \{\phi : \ M \longrightarrow \mathbb{R} | \ \phi(p) = 0\} \subset {}^{\mathbb{R}}\mathcal{C}^\infty(M) \tag{72}$$

It is easy to verify that $\mathcal{I}_p$ is a *maximal ideal* in ${}^{(\mathbb{R})}\mathcal{C}^\infty(M)$ and that the quotient of the latter by the former is isomorphic to the reals:

$${}^{\mathbb{R}}\mathcal{C}^\infty(M)/\mathcal{I}_p \simeq \mathbb{R} \tag{73}$$

Accordingly, the set $Spec[{}^{(\mathbb{R})}\mathcal{C}^\infty(M)]$[332] of all maximal ideals $\mathcal{I}_p$ ($\forall p \in M$) of ${}^{(\mathbb{R})}\mathcal{C}^\infty(M)$ such that

$$\mathbb{R} \hookrightarrow {}^{\mathbb{R}}\mathcal{C}^\infty(M) \longrightarrow {}^{\mathbb{R}}\mathcal{C}^\infty(M)/\mathcal{I}_p \tag{74}$$

(within an isomorphism of the first term), is called *the (real) (Gel'fand) spectrum of* ${}^{(\mathbb{R})}\mathcal{C}^\infty(M)$. Furthermore, if ${}^{(\mathbb{R})}\mathcal{C}^\infty(M)$—regarded algebraic geometrically as a commutative ring—is endowed with the so-called Zariski topology, or equivalently, with the usual Gel'fand topology [178],[333] then the 'point-wise' map

$$M \ni p \mapsto \mathcal{I}_p \in \mathfrak{M}[{}^{(\mathbb{R})}\mathcal{C}^\infty(M)] \tag{75}$$

proves to be a *homeomorphism* between the usual $\mathcal{C}^0$-topology of M (*ie*, M being regarded simply as a topological manifold) and the Gel'fand (Zariski) topology on $\mathfrak{M}[{}^{\mathbb{R}}\mathcal{C}^\infty(M)]$.[334] All in all, the essential idea of Gel'fand duality here is to substitute the underlying space(time)

[331] Gel'fand duality is the basis for the equivalence between M and $\mathcal{C}^\infty(M)$ (or $\mathcal{C}^\infty_M$) noted in (1).

[332] As noted above, often symbolized by $\mathfrak{M}[{}^{(\mathbb{R})}\mathcal{C}^\infty(M)]$.

[333] That the Gel'fand and the Zariski topology on $\mathfrak{M}[{}^{\mathbb{R}}\mathcal{C}^\infty(M)]$ are identical is due to the fact that ${}^{(\mathbb{R})}\mathcal{C}^\infty(M)$ is an abelian *regular* topological algebra—*ie*, maximal ideals in ${}^{(\mathbb{R})}\mathcal{C}^\infty(M)$ are also its prime ideals, so that the maximal spectrum of ${}^{(\mathbb{R})}\mathcal{C}^\infty(M)$ coincides with its prime spectrum [254].

[334] Of course, $\mathfrak{M}[{}^{\mathbb{R}}\mathcal{C}^\infty(M)]$, initially regarded as a structureless point-set (having for points the ideals $\mathcal{I}_p$), inherits from the one-to-one and onto map in (75) not only the usual locally Euclidean topological structure of M, but also the differential (*ie*, the $\mathcal{C}^\infty$-smooth) one. It is precisely this identification (both in topological as well as in differential structure) of M with $\mathfrak{M}[{}^{\mathbb{R}}\mathcal{C}^\infty(M)]$ that underlies the left-pointing arrow in the tautesis of M with $\mathcal{C}^\infty(M)$ in (1), which in turn supports our claim in section 1 that *a differential manifold M is nothing else but the algebra $\mathbb{A} = \mathcal{C}^\infty(M)$ of $\mathcal{C}^\infty$-smooth functions on (coordinates of) its points* (or equivalently, M *is* the structure sheaf $\mathbf{A} = \mathcal{C}^\infty_M$).

continuum by the (algebras of) objects (functions/fields) that live on it, and then recover it by a suitable technique, which we have coined *Gel'fand spatialization* in the past [318, 319, 272].

Furthermore, *vis-à-vis* singularities, which, as noted earlier, are arguably due to the pointed nature of M, the factoring-out of $\mathcal{I}_p$ from ${}^{(\mathbb{R})}\mathcal{C}^\infty(M)$ in (73) (to yield $\mathbb{R}$—the reals, which, in turn, induce the usual Euclidean topological and differential structure to the quotient space), in a way factors out singularities completely analogously to the differential ideal-smearings and 'modding-outs' involved in the definition of spacetime foam algebras in (56). This is straightforward to see if for example one considers briefly the interior Schwarzschild singularity associated with the gravitational field of a point particle p of mass m.[335] There, as it is well known, the Schwarzschild solution to the vacuum Einstein equations has a singularity right at the point mass m—that is to say, when the differential spacetime manifold M is charted by a smooth coordinate system[336] with origin at the point particle p, the Schwarzschild metric solution g_S has a singularity at distance $r = 0$ from the particle, since the smooth function $\phi(p) = r(p)(= \sqrt{x^2 + y^2 + z^2})$, $\forall p \in M$ [337] appears as a coefficient in the denominator in the expression for g_S.[338] But the smooth functions like $r(p)$ are precisely the ones that *define* the maximal ideals $\mathcal{I}_p$ in ${}^{(\mathbb{R})}\mathcal{C}^\infty(M)$ (72)[339] and, moreover, precisely the ones that are factored out of $\mathcal{C}^\infty(M)$ in (72).

Of course, one cannot claim that the procedure of Gel'fand duality exercised on the smooth manifold actually does away with point-singularities, since, simply set-theoretically speaking, the factoring out of the ideals from $\mathcal{C}^\infty(M)$ is tantamount to removing by fiat—as it were, surgically dissecting in an *ad hoc* and forced fashion—from M the offensive points. In the Schwarzschild case, this corresponds to removing by fiat the point mass m placed at p, leaving thus $M - \{p\}$—a '*punctured spacetime*'—as an effective (regular) manifold in the theory. Equivalently, as we will see in more detail in 5.1, as an effective, regular spacetime manifold in the Schwarzschild case, one may regard the space $M - L_t$ corresponding to the original (total) spacetime manifold M less the singular continuous 'wristwatch time-line' $L_t := \{p \in M : \; x_i(p) = 0\}$ of the point-mass placed at the origin of an (analytic)

[335] We will come back to tackle this example from a DSTFDA and a SSTFDA ADG-perspective in the next section. This is just a sketchy discussion of the inner Schwarzschild singularity here just to illustrate this potential doing away with singularities by algebraic 'ideal-smearing' and factoring-out points of the $\mathcal{C}^\infty$-smooth spacetime continuum.

[336] In fact, by *any* $\mathcal{C}^\infty$-smooth coordinate system!

[337] In parenthesis above, is the distance function expressed in the usual smooth cartesian coordinates. Also, the expression '$\forall p \in M$' above may be read '*on any point p of the smooth spacetime manifold M one decides to place the point-mass m*'.

[338] See (83) in 5.1.

[339] Algebraic geometrically speaking, the points of $\mathfrak{M}[{}^{\mathbb{R}}\mathcal{C}^\infty(M)]$.

Cartesian coordinate system (x_μ) charting M [141].[340] In either case, this is a forced exclusion of singularities that *prima facie* appears to accord with Einstein's dictum for excluding singularities in (Q2.1), but which, from our ADG-theoretic viewpoint, simply indicates both the inadequacy of the $\mathcal{C}^\infty$-smooth (spacetime) manifold based CDG in describing the (here gravitational) field and the law (differential equation) that it obeys right at its singular source-*locus*. In contradistinction,

(R5.3) the factoring out of spacetime foam dense singularities expressed in (56), and the concomitant application of the differential manifold-free ADG to sheaves of spacetime foam algebras over the (in principle) arbitrary base space(time) X, in no way represents such a theoretically *ad hoc* surgical exclusion of the singular *loci* from X; rather, *when sheaves of the differential spacetime foam algebras are used as structure sheaves* $\mathbf{A}$ *of generalized coefficient functions in the theory, one simply witnesses on the one hand the absorption or integration of* $\mathcal{C}^\infty$*-smooth singularities in those generalized 'coordinate-arithmetics',*[a] *while on the other, that the entire, essentially algebraic, differential geometric mechanism of the theory still holds, and in no way breaks down, in their presence* [260, 273, 262, 274, 275].[b]

[a] For after all, the algebra of $\mathcal{C}^\infty$-smooth functions is manifestly embodied in the foam algebras (60).

[b] We will briefly recall the results of the application of ADG to spacetime foam dense singularities shortly, in 4.2.4 below.

Topology refinement versus singularity refinement: some common grounds and close ties resting on (sheaf-theoretic) localization. In the foregoing we argued that the points of the smooth spacetime continuum M are the 'bearers' or '*carriers*' of its singularities.[341]

In a topological context—and in particular in the aforementioned T_0-poset discretizations of $\mathcal{C}^0$-spacetime manifolds—Sorkin too basically assumes that "*the points of the continuum are the carriers of its topology*" [355]. Moreover, and in order for the T_0-poset substitutes to qualify as genuine replacements of the continuum that are coarsely approximating its continuous ($\mathcal{C}^0$)

[340] That is, in this case, the dissected effective regular spacetime is $M - L_{x_0=t}$ $(t \in \mathbb{R})$.

[341] Or equivalently, by Gel'fand duality, the smooth coordinates of M's points in $\mathcal{C}^\infty(M)$, which themselves are the bearers of M's points (75), are the carriers of its singularities. In other words, any of M's points can serve as the host-*locus* of a singularity of some physically important $\mathcal{C}^\infty$-smooth field, which is just a $\otimes_{\mathbb{A}\equiv\mathcal{C}^\infty(M)}$-tensor.

topology, as we saw in section 3 Sorkin provides an *inverse limit* (re)construction of X, regarded as a point-set, from an inverse system (or net) $\overleftarrow{\mathcal{P}} = \{P_i\}$ of finitary T_0-posets. The basic notion here is that of *topological refinement*,[342] which we may formally symbolize as

$$\mathcal{U}_i \prec_{tref} \mathcal{U}_j \Leftrightarrow \tau_i(X) \prec_{tref} \tau_j(X) \tag{76}$$

and read directly as follows:[343] *the subtopology τ_i of X generated by the open sets of the locally finite open covering $\mathcal{U}_i$ of X is coarser than the subtopology $\tau_j(X)$ generated by $\mathcal{U}_j$.*[344] As we noted in section 3, the physical semantics of this inverse limit procedure, as also Sorkin points out in [355], is that as one employs more numerous and 'smaller' or 'finer' open sets in order to cover (the points of) X—as it were, as one employs higher (microscopic) power of resolution to locate (or 'localize'[345]) and effectively separate (or distinguish between) X's points by using open sets about them,[346] at the (ideal[347]) of infinite refinement (or localization) of the finitary posets, one obtains a space that is topologically indistinguishable (*ie*, effectively homeomorphic, modulo Hausdorff reflection [234]) to the original topological space(time) manifold X that one started with. Formally, one writes:

$$\varprojlim \overleftarrow{\mathcal{P}} = \lim_{\infty \leftarrow i} P_i = P_\infty \overset{\text{homeo.}}{\simeq} X \tag{77}$$

[342] Which may be also coined *covering refinement*.

[343] The reader should note the use of the same 'precedence' or 'order' relation '$\prec$' for both (t)opological ($\prec_{tref}$) and (s)ingularity ($\prec_{sref}$) (ref)inement. Below, we will show the close links between the two notions, which in turn reflect the intimate technical and conceptual affinities between the algebraic DSTFDA and SSTFDA approaches to spacetime foam respectively, especially when viewed from the unifying algebraic (:sheaf-theoretic) perspective of ADG.

[344] And *vice versa*, τ_j is *finer* than τ_i. One may represent the said inverse system $\overleftarrow{\mathcal{P}}$ of finitary posets by the pair $(P_i, \prec_{tref})$. In fact, we read from [355], it is this (partial order) relation of topological refinement, which can be also read simply as '$\tau_i(X)$ *is a subtopology of* $\tau_j(X)$', that makes one think of the set $\{\tau_i(X)\}$ as a *net*—essentially, *a right-directed set of elements* (like the $L = (\Lambda, \leq)$ we encountered earlier in defining spacetime foam algebras; also, '*right-directed*' here meaning that $\forall \tau_i, \tau_j$, or simply index-wise i, j, in the net, $\exists k$ in the net, such that $i, j \prec_{tref} k$)—which in turn indexes the elements P_i of the projective system $\overleftarrow{\mathcal{P}}$. However, by abuse of terminology, and without causing any confusion, in what follows we may use the terms 'inverse system' and 'net' interchangeably for $\overleftarrow{\mathcal{P}}$.

[345] That is, determine the *locus* of a point in X. We will come back to the issue of (sheaf-theoretic) localization shortly.

[346] This for instance may be regarded as the essence of the, in this case, T_0-axiom of separation of point-set topology, namely, that the points of a set are the carriers of the open sets (about them), which sets, in turn, *separate* them and at the same time *define* X's topology—*ie*, qualify the *a priori* structureless point-set X as a *topological space* proper.

[347] And we use the epithet '*ideal*', because the points of the continuum are operationally non-pragmatic, non-realistic theoretical artifacts; while, the operationally realistic entities—what we actually determine by our acts of localization of 'events'—are 'large', 'extended', coarse regions about them [355, 357, 318, 319].

Now, a functional representation of topological refinement and of the inverse system $\overleftarrow{\mathcal{P}}$ bears a close resemblance to how we represented singularity refinement (and its 'order-covariant' notion of quotient regularity $\succ_{reg}$) in (69), namely, we read from [355] that

$$\mathcal{U}_i \prec_{tref} \mathcal{U}_j (\equiv \tau_i(X) \prec_{tref} \tau_j(X)) \Leftrightarrow P_j \xrightarrow{f_{ij}} P_i \tag{78}$$

which says that '$\mathcal{U}_j$ is topologically finer than $\mathcal{U}_i$' is equivalent to 'the existence of a *continuous surjection* f_{ij} between the corresponding T_0-posets (33).[348] In fact, in terms of such continuous surjections, the inverse system can now be equivalently written as the pair $\overleftarrow{\mathcal{P}} = (P_i, f_{ij})$. This inverse system of finitary T_0-posets and continuous epi-maps between them enjoys a *universal mapping property* (33), which in turn guarantees an inverse limit topological space P_∞ appearing in (76)[349] as a continuous surjection $f_{i\infty}: \ P_\infty \longrightarrow P_i$.

In a similar functional way, and by Gel'fand duality, topological refinement corresponds to a *continuous surjective algebra homomorphism* $\hat{f}_{ij}$ *between the finitary posets' respective incidence algebras* Ω_j *and* Ω_i: $\hat{f}_{ij} : \ \Omega_j \longrightarrow \Omega_i$ [318, 319].[350] Since *the* Ω_i*s are categorically-dual to the* P_i*s* [271, 272, 432], it follows that the collection $\overrightarrow{\mathfrak{R}} = \{\Omega_i\} = (\Omega_i, \hat{f}_{ij})$ is an *inductive system* under topological refinement. As such, and in complete analogy to the inverse limit procedure exercised on $\overleftarrow{\mathcal{P}}$ in (77), it possesses a *direct* or *inductive limit* incidence algebra Ω_∞, which we may formally write as follows

$$\varinjlim \overrightarrow{\mathfrak{R}} = \lim_{i\to\infty} \Omega_i = \Omega_\infty \tag{79}$$

The relevance and utility of this inductive limit process becomes even more transparent when one views, in the sheaf-theoretic context of ADG, *finitary spacetime sheaves (finsheaves) of incidence algebras* [310, 270, 271, 272] and one emphasizes the fact that they define (finitary versions of) *differential triads* (the fintriads we saw in section 3), which are the basic building blocks of ADG [259, 260, 271, 272].

Finsheaves. To recall the relevant concepts and constructions briefly, initially, the basic intuition behind [310] was to do the same to the sheaf ${}^{\mathbb{R}}\mathcal{C}^0_X$ of (germs of) continuous ($\mathbb{R}$-valued) functions

[348] Or equivalently, order-theoretically speaking, an onto monotone map (*ie*, a partial order preserving epimorphism).

[349] It too a T_0 topological space—*ie*, non-Hausdorff (not T_2).

[350] Here *continuous* means *respecting the aforementioned Rota topology on the corresponding incidence algebras' primitive spectra*—a topology which derives from the algebraic structure of the incidence algebras, which in turn is preserved by the said homomorphisms. In fact, as also noted in section 3, this correspondence $((P_i, f_{ij}) \longrightarrow (\Omega_i, \hat{f}_{ij}))$ is *functorial* between the respective categories (of finitary posets/simplicial complexes–order preserving maps/simplicial maps and their finitary incidence algebras/algebra homomorphisms) [318, 319, 432].

on the topological manifold X as Sorkin did for the base space X itself in [355]. Thus, *emphasis was placed on the functions that 'live' on spacetime rather than on the background space(time) per se*, which is traditionally the domain of definition of those functions.[351] The sheaf-theoretic analogues of the finitary substitutes P_i were then coined *finitary spacetime sheaves*[352] and were seen to be sheaves of continuous functions over Sorkin's locally finite topological posets[353]—as it were, the finitary substitutes of the aforementioned 'continuum sheaf' $\mathcal{C}_X^0$. With our discussion above in mind, in a sheaf-theoretic context the most important notion is that of *localization*, a notion that is essentially tautosemous to the technical one of the process of *sheafification* (of a presheaf) [266].[354] Indeed, the very definition of the *stalks* of a sheaf—the elementary, 'point-like' building blocks of the sheaf space, each erected over each point p of the base topological space X[355]—involves in an essential way a *direct limit* process exercised on an inductive system of open subsets of X relative to which a presheaf has been first defined [259, 310].[356] Moreover, from a physical point of view,

[351]See the last section for a discussion of the traditional '*geometric*' ('domain dependent') *versus* the modern '*algebraic/categorical*' ('domain independent') conception of functions.

[352]And symbolized by $\mathcal{S}_i$ in [310].

[353]Write formally, $\mathcal{S}_i(P_i)$.

[354]In fact, we read from [266] that "*sheafification is localization*".

[355]For recall that for a general sheaf $\mathcal{S}_X$, the sheaf space is given by $\mathcal{S} = \bigcup_{p\in X}^{\text{dis}} \mathcal{S}_p = \bigoplus_{p\in X}$, where $\bigcup^{\text{dis}}$ denotes *disjoint union* [259].

[356]This direct limit procedure may be (topologically) interpreted as *fine graining*—the resolution or analysis of the sheaf space into its '*ultra local*' elements, its stalks (fibers). Thus, we can roughly say here, in complete analogy to what we said earlier for the topological space X and its points, that *the topology of the sheaf space is carried by its stalks*—the stalks which, in turn, like the *a priori* thought of as being completely disconnected points of the point-set X, inherit from $\mathcal{S}$ the *discrete relative topology*. Of course, this is more than a formal analogy (between the points of X and the stalks of $\mathcal{S}$) if one considers the very definition of a sheaf $\mathcal{S}_X$ as a *local homeomorphism* (between X and $\mathcal{S}$) [259, 310].

(R5.4) *localization*[a] *may be thought of as the process of 'relativization', 'gauging' and concomitant 'dynamicalization'*[b]; hence, so is sheafification.[c] In other words, *a sheaf (of objects of any kind) by definition entails the dynamical variability of those objects.*

[a]Roughly speaking, making the objects/structures one is dealing with U-dependent ($U \subset X$), with U 'varying' over the lattice of open subsets of X which *defines* the latter as a topological space in the first place.

[b]That is to say, making the objects/structures one is dealing with *dynamical variables* that are formally varying with respect to the background topological space, which thus acts as a 'virtual', 'surrogate' external 'parameter space' [270, 271, 272]. (The epithets 'virtual' and 'surrogate' for the external 'carrier' topological space X will be qualified shortly, after the ADG-theoretic notion of a connection on a sheaf is introduced.)

[c]Let us furthermore note at this point Finkelstein's motto in the context of his Quantum Relativity theory that, in essence, "*(quantum) relativization is dynamicalization*". We will return to address quantum (gravity) issues in the sheaf-theoretic context of ADG in section 6.

With the remarks in (R4.4) in mind, and as noted in section 3, we stress that the second author's original *mathematical* aim in defining finsheaves as noted at the end of [310], was to organize the discrete differential incidence algebras that we saw earlier into sheaf-like structures so as to apply the first author's ADG-theoretic concepts and differential geometric machinery to a finitistic or combinatory-algebraic setting quite remote from the realm of the $\mathcal{C}^\infty$-smooth spacetime continuum, something that *prima facie* befits ADG's differential calculus (CDG) and differential manifold-free character [259]. At the same time, the *physical* idea underlying this sheaf-aggregation of incidence algebras was that, when the locally finite posets were interpreted not as finitary topological spaces proper, but as *causal sets* (causets) [54, 353, 354, 357, 359, 360], while their associated incidence algebras as *quantum causal sets* (qausets) [309], the said agglomeration of qausets into (fin)sheaves would provide one with a natural setting in which to study the 'localization', 'gauging', 'curving' and the concomitant dynamical variation of 'discrete quantum causality' [270]. All in all, finsheaves of qausets were intuited as being 'natural' models for developing a *'discrete' version of Lorentzian QG*; moreover, one which is *manifestly background spacetime manifold independent*,[357] quite unlike the usual (*eg*, canonical or covariant) $\mathcal{C}^\infty$-smooth spacetime manifold based approaches to QG which still effectively employ a background differential manifold [270, 271, 272].

As also briefly alluded to in section 3, the five essential observations (results) for actually setting

[357]Yet still employing the full differential geometric panoply of CDG; albeit, in a reticular setting and entirely by algebraico-categorical means [271].

up finsheaves of incidence algebras (ultimately, with an eye towards applying ADG to them) were that:

1. The correspondence finitary posets/simplicial complexes $\longrightarrow$ incidence algebras is *functorial* (Gel'fand duality functor) [318, 319, 432].

2. In fact, first in [270] it was noted that the map $P_i \longrightarrow \Omega_i$ is *by construction* (of the Rota topology on Ω_i via Gel'fand spatialization/duality [431, 318, 319]) a local homeomorphism—*ie*, a sheaf [259, 310].[358]

3. Finsheaves of the *differential* incidence algebras define *finitary differential triads* $\mathfrak{T}_i$ [271].

4. Categorically speaking, and by Gel'fand duality, *inverse limit* 'classical continuum localizations' of the P_is[359] correspond to *direct limit* 'classical continuum localizations' of the corresponding incidence algebras and, *in extenso*, of the fintriads $\mathfrak{T}_i$ that their finsheaves define [271].[360]

By now, and after this short digression on (fin)sheaf-theoretic localizations (of the DSTF-DAs), we hope that the close conceptual and structural similarities between the SSTFDA and the DSTFDA approach to spacetime foam have become more transparent. In the next couple of paragraphs we bring the two approaches even closer together by first exploiting, in a sheaf-theoretic context, the issue of finitarity (local finiteness) and its import in the SSTFDA scheme, as well as by giving heuristic arguments linking topological and singularity refinement.

'Finitarity' (local finiteness): the condition underlying the fineness and flabbiness of (structure) sheaves of multi-foam algebras. Before we proceed with the next paragraph, where we unite SSTFDAs and DDSTFAs under the sheaf-theoretic roof of ADG, and before the sub-subsection 4.2.4 below, where we outline the results of the successful application of ADG to spacetime foam dense singularities and the sheaves of spacetime foam algebras thereof, we would

[358] From now on we write $\mathbf{\Omega}_i(P_i)$, or simply $\mathbf{\Omega}_i$, for finsheaves of incidence algebras. When we wish to emphasize the qauset interpretation of the incidence algebras dwelling in the stalks of the said finsheaves, we will use the symbol $\vec{\mathbf{\Omega}}_i$ for the latter, as we did throughout the trilogy [270, 271, 272].

[359] The epithet '*classical*' above comes from [318] when the continuum inverse limit of finitary posets and, dually, of their associated incidence algebras, was physically interpreted as *Bohr's correspondence principle or limit*.

[360] As explained in section 3, fintriads comprise in fact an inverse-*cum*-direct system $\overset{\rightleftarrows}{\mathfrak{T}}$. In 5.2.2, based precisely on Papatriantafillou's inverse/inductive limit results of of differential triads as applied to our fintriads, we will show how to totally evade the inner Schwarzschild singularity—regarded as a localized, 'static' point-singularity—solely by finitistic-algebraic/categorical means.

like to dwell for a while on the issue of *finitarity* or *local finiteness*. In the SSTFDAs context, the assumption of local finiteness plays an important role in actually showing that the sheaves of spacetime foam algebras—employed as structure sheaves of generalized arithmetics or 'coordinates' to replace the 'classical' one $\mathcal{C}^\infty_X$ of smooth functions on the topological space X—are *fine* and *flabby*. In turn, the flabbiness and fineness properties of those sheaves secure the application of very basic sheaf-cohomological constructions and associated applications of ADG—applications which are of great importance in mathematical physics, as for instance the construction of a *short exact exponential sheaf sequence* and the consequences that this construction has for the process of *geometric (pre)quantization* (*eg*, Weil's integrality theorem) [259, 260, 271]. Putting it in a negative way, in the case of Colombeau algebras for example, the sheaves of which manifestly lack the flabbiness property,[361] a short exact exponential sheaf sequence cannot be constructed; hence they would be of little (abstract) differential geometric utility if one wished to use them ADG-theoretically as structure sheaves of generalized coefficients. So, let us see briefly how finitarity or local finiteness appears as an important assumption in the theory of spacetime foam algebras.

Once again we follow [274, 275]. As before, one considers a collection $\mathcal{S}$ of (dense) singularity-sets in a Euclidean or locally Euclidean (manifold) space(time) X, the elements $\mathfrak{S} \subset X$ of which are also assumed to satisfy the two conditions (51) ('regularity density') and (52) ('singularity coarse-graining') that we saw earlier. Then, in order to show that the structure sheaf $\mathbf{A}_X \equiv \mathfrak{B}_{L,\mathcal{S},X}$ of spacetime foam algebras on X is *fine* and *flabby*, one assumes that $\mathcal{S}$ is *locally finitely additive*. By the latter, and in complete analogy to the locally finite open coverings of X assumed in [355], one roughly means that *every point x of X has an open neighborhood about it that non-trivially intersects a finite number of the singularity-sets*.[362] More precisely, one considers any sequence $\mathfrak{S}_k$ ($k \in \mathbb{N}$) of singularity-sets in $\mathcal{S}$ and takes their union $\mathfrak{S} = \bigcup_k \mathfrak{S}_k$, which belongs to $\mathcal{S}$ by (52).[363] Then, for $U \subseteq X$ open, $\mathfrak{S} \cap U \in \mathcal{S}|_U$, whenever

$$\begin{gathered}\forall x \in U,\ \exists V \subseteq U\ (V \text{a neighborhood of } x):\\ \text{the set } \{k \in \mathbb{N}:\ \mathfrak{S}_k \cap V \neq \emptyset\} \text{ is finite}\end{gathered} \tag{80}$$

where $\mathcal{S}|_U$ denotes *the restriction of $\mathcal{S}$ to U*. With respect to these restriction mappings, one then

[361] In a paragraph in 4.2.4 below, we will mention that, actually, the failure of sheaves of Colombeau's non-linear distributions [90] to be flabby is due to the imposition of several *growth conditions* that these generalized functions must satisfy at the vicinity of their singularities. Moreover, as noted in [275], "*this lack of flabbiness of the Colombeau algebras is quite closely related to a number of deficiencies*" [221]. By contrast, the non-linear spacetime foam distributions considered here do not have to obey any such growth conditions around open neighborhoods of their dense singularities in X, and one of the advantages of that 'growthlessness', *vis-à-vis* ADG and their potential import in mathematical physics applications, is that the sheaves thereof are indeed flabby.

[362] Compare with Sorkin's definition of a locally finite open covering $\mathcal{U}$ of X that we saw earlier: "*for every point x of X there is an open set about it that meets a finite number of the covering sets in $\mathcal{U}$*".

[363] Write formally, $\bigcup_k \mathfrak{S}_k = \mathfrak{S} \subseteq \mathcal{S}$.

proceeds and constructs a *complete presheaf*[364] $\mathcal{B}$ of spacetime foam algebras $\mathfrak{B}$ over the topological space X, which, moreover, with the help of (79), proves to be fine and flabby.[365]

Heuristic remarks about 'continuum' inverse/inductive limits under topological/singularity refinement. We begin with Sorkin's assumption that X is a relatively compact (open and bounded) region of the spacetime manifold M, that it admits locally finite open covers $\mathcal{U}_i$, and that it is inhabited by dense singularities (whose set-theoretic complements in X are also dense). In turn, the open sets comprising the covers are densely packed with singularities of 'all sorts'. The basic physical idea behind Sorkin's open coverings' refinement is that as one employs higher and higher power of resolution (of X into is points—"*the carriers of its topology*" [355]) so as to localize them with higher accuracy by using smaller and more numerous open sets about them, at the inductive or projective limit of infinite refinement (:infinite power of resolution and localization) one effectively recovers (modulo Hausdorff reflection) the continuous point-set topology of X. For a singular *locus* (:point) in particular,...

Growthlessness suitable for ADG's 'background spacetimelessness'. One could argue that the main reason why global topological (and differential topological) tools and methods were extensively developed and used from the 60s until the 80s in the analysis of spacetime singularities—in particular, in endowing $\overline{M} = M \cup \partial M$ with a certain topology while at the same time relegating the singular *loci* of the spacetime manifold M 'asymptotically' or 'marginally', to its boundary ∂M (relative to the chosen topology)—was that one wished to study and make precise (as it were, rigorously 'quantify' in an analytic way) for instance how physically observable geometrical properties of the gravitational field, such as the Riemann curvature tensor[366] as well as its Ricci-tensor ($\mathcal{R}$) and Ricci-scalar ($\mathfrak{R}$) contractions, *grow* (*ie*, 'converge' or 'diverge') as one (say, a gravitated test particle of mass m) approaches by a continuous (or more fittingly, smooth) path the singular *locus* at the manifold's boundary.[367] This is clearly stated in the following passage from Clarke's book [87]:[368]

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[364]Or equivalently, a *sheaf* [259].

[365]For more details about this construction, the reader is referred to [274, 275].

[366]Which represents gravitational tidal forces.

[367]In a mathematical sense, one would like to model the growth of the gravitational field strength R on the particle, as the latter approaches the singular point, after a *continuous limit-convergence* relative to the chosen topology.

[368]By the way, the following quotation further corroborates our arguments in 2.1.2 (based on (Q?.?)) about singularities, situated at the boundary ∂M of the otherwise smooth and regular spacetime manifold 'bulk' M, as being sites where CDG (Calculus or Analysis) ends, or more graphically, 'breaks down'.

(Q5.4)

> "...We must now give the definition of the noun 'singularity'. The fundamental idea is that space-time itself (the structure (M, g)) consists entirely of regular points at which g is well behaved [*ie*, regular],[a] while singularities belong to a set ∂M of additional points—'ideal points'—added onto M. *We denote the combined set $M \cup \partial M$ by ClM, the closure of M, and define the topology of this set to be such that phrases like 'a continuous curve in M ending at a singularity p in ∂M', or 'the limit of R^b as x tends to a singularity p is...' all have meanings corresponding to one's intuitive picture of what they ought to mean*[c]..."
>
> [a]Our addition.
> [b]Here, by 'R' Clarke symbolizes the Riemann curvature tensor.
> [c]Our emphasis.

Indeed, and from a more general perspective, in the theory of (non-linear) PDEs where, apart from the linear distribution theory of Schwartz and the related theory of generalized functions *à la* Sobolev, the non-linear Colombeau distributions have recently enjoyed much popularity, (differential) *growth conditions* are invariably imposed in the neighborhood of all those functions' singularities. For instance, we witness in [87], where an entire section is devoted to Sobolev spaces and their manifold applications to the analysis of spacetime singularities, a plethora of conditions that the *à la* Sobolev functionally generalized g and R must satisfy locally, *ie*, in the vicinity of a singularity.[369] On the other hand, as we read from [274], in the context of the application of ADG to the spacetime foam dense singularities of Rosinger's non-linear distributions, the imposition of *such growth conditions manifestly lifts the flabbiness property of the respective structure sheaves.* That is to say, if for example one wished to employ sheaves of Colombeau algebras (instead of Rosinger's) as structure sheaves of generalized arithmetics in the manner of ADG, precisely due to the growth conditions near their singularities that the Colombeau distributions are demanded to satisfy, the said sheaves would simply fail to be flabby. In turn, as noted before, exactly because of this shortcoming, the non-flabby Colombeau structure sheaves do not allow for basic differential geometric constructions such as that of a short exact exponential sheaf sequence—a vital construction in the application of ADG to geometric (pre)quantization (via Weil's integrality) for example [260, 261, 271].[370] Thus, in glaring contradistinction to the CDG-based analysis of

[369]Such conditions are usually imposed in order to control and classify the 'differential blow-up' (*eg*, give upper bounds to the order of differentiability) of the metric and its Riemann curvature tensor in the proximity of a singularity (*Sobolev 'differentiability' classes* [87]).

[370]We will return to discuss in more detail the application of ADG to geometric (pre)quantization of gravity in 6.1.

spacetime singularities in [87], we read from [274]:

(Q5.5) "*...Furthermore, as in the case of the nowhere-dense algebras and also in space-time foam algebras, no kind of condition is asked on the generalized functions in the neighborhood of their singularities*[a]*...*"

[a] Emphasis is ours.

Indeed, this *growthlessness* of Rosinger's non-linear generalized functions appears to be tailor-cut for ADG's fundamental *spacetimelessness*. That is to say, the purely algebraic, background geometrical spacetime manifold-free ADG-theoretic perspective on gravity (GR) and its singularities (*eg*, via Rosinger's entirely algebraic theoresis of spacetime foam dense singularities) is not concerned at all with the standard (differential) geometric-analytic question above whether and in what manner the Riemann curvature tensor grows without bound ('diverges') as a singularity (situated on ∂M) is approached in a continuous manner by a (smooth or regular) path that a particle follows in M. ADG totally bypasses, by purely algebraic means, the CDG-based analysis of gravitational singularities, while the geometrical manifold-based reasoning (and picturization!) that goes hand in hand with that analysis is simply rendered obsolete and is of no import in our theory. Of course, all this is not surprising, because in a '*cutting the Gordian knot*' fashion ADG evades all the problems that $\mathcal{C}^\infty$-smooth singularities present to CDG (Calculus or Analysis):

(R5.5) *Since there is no background differential spacetime manifold in ADG's perspective on gravity (GR), there are no singularities either, and it is begging the question to try to classify, let alone attempt to define, singularities by analytic, CDG-theoretic means when they do not 'exist' in the first place.*[a]

[a] Again, by '*do not exist*' we mean that singularities, which are built into the structure sheaf $\mathbf{A} \equiv \mathcal{C}^\infty_M$ of the underlying differential manifold M, do not perturb the slightest bit the inherently algebraic, base manifold independent differential geometric mechanism of ADG and therefore, unlike CDG, they are not perceived as *loci* where differentiability breaks down or is limited in one way or another as DGSs appear to indicate.

5.2.4 Differential geometry à la ADG in the presence of the most numerous and 'wildest' from the CDG perspective singularities: the versatility of ADG

As repeatedly mentioned in the previous sections, and from a sheaf-theoretic perspective, the CDG of $\mathcal{C}^\infty$-smooth manifolds M is entirely captured by employing $\mathcal{C}^\infty_M$ as the structure sheaf of coefficients (or coordinates labelling M's points). At the same time, the gist of ADG consists in showing that one can actually do differential geometry by using structure sheaves **A** other than the classical one $\mathcal{C}^\infty_M$, of course, as long as these generalized coefficient algebra sheaves provide one with the essential(ly algebraic) 'differential mechanism'[371] by virtue of which one actually does differential geometry, and which in the classical theory (CDG) is furnished by $\mathcal{C}^\infty_M$ or, what amounts to the same, by the locally Euclidean character of the underlying space(time) M.[372] Thus, as noted earlier, the first successful application of ADG was to do differential geometry over space(time)s that are dense with singularities of the most general, and 'problematic' or 'anomalous' from the viewpoint of smooth Euclidean or locally Euclidean spaces on which CDG vitally relies, kind by using, instead of the $\mathbf{A} \equiv \mathcal{C}^\infty_M$ of the classical theory (CSDG), sheaves of Rosinger's spacetime foam algebras of generalized functions as structure coefficients.

Indeed, one witnesses in a series of papers [273, 274, 275] fundamental differential geometric constructions, normally being associated exclusively with the presence of $\mathcal{C}^\infty$-smooth base space(time manifold)s, to apply, completely unaltered and uninhibited in any way, on space(time)s teeming with singularities of the classically most unmanageable sort—the aforedescribed dense singularities encoded in the (structure sheaves of) spacetime foam algebras of generalized functions. A long list of results of this application of ADG includes among other things:

1. Poincaré's lemma and, *in extenso*, de Rham's theorem [259, 260, 273, 274, 275, 271],

2. Maxwell's equations: [259, 260, 269, 264, 269],

3. Yang-Mills equations [259, 260, 262, 264, 272, 269],

4. Einstein's equations [262, 272, 264, 265, 267, 268, 269],

5. Geometric Prequantization and field (second) quantization [260, 261, 263, 271, 272, 269],

6. General remarks and philosophical implications on (gravitational) singularities [262, 265, 267, 268, 269].

[371] In point of fact, with a linear, Leibnizian differential operator ∂, which is the 'canonical' example of a flat **A**-connection in ADG [259].

[372] In other words, as repeatedly noted in the previous sections, ADG has taught us that in order for one to be able to do differential geometry, one does not have to exclusively commit oneself to the structure sheaf $\mathcal{C}^\infty_M$ (or equivalently, to classical differential manifolds M), but one can explore other structure sheaves of coefficients that may possibly be *far from smooth*. To stress it once again, the principal didactic of ADG is that "*differentiability is independent of $\mathcal{C}^\infty$-smoothness*" [271].

5.3 Section's Résumé

The epitome, 'bottom line' as it were, of this section is that in ADG, by absorbing into the structure sheaf **A** of generalized arithmetics ('differentiable coordinates') singularities of the most unmanageable (at least from the CDG-vantage) kind, like Rosinger's spacetime foam dense singularities of non-linear generalized functions (distributions), one is able to develop the entire differential geometric conceptual panoply and technical constructions in their very presence, as if singularities were not there. In this sense ADG bypasses or evades singularities, and moreover, unlike the DGSs or the SFSs that we saw in section 3, which are regarded as pathologies and differential geometric anomalies (or 'breakdown points' for the differential equations corresponding to the CDG-implemented gravitational field law) uncircumventable by the manifold-grounded CDG (Analytic or Differential Calculus-based) means. *In summa*, ADG 'sees through' **A**;[373] hence also through the singularities that are built into it.

6 A Concrete Toy Model and Playground Application of ADG: a Spatial Point-Localized Finitary-Algebraic and a Temporal Line-Distributional Spacetime Foam Resolution of the Interior Schwarzschild Singularity

First of all, we state it up-front that in the context of the usual CDG-analysis of $\mathcal{C}^\infty$-manifolds, the three most 'canonical' and familiar examples of 'true' DGSs and, perhaps more appropriately, SFSs (even though distributional solutions are not normally associated *ab initio* with them), are those of the Schwarzschild, the de Sitter, and the Friedmann solutions to the (vacuum) Einstein equations [184, 87]. Thus, in both cases the Ricci curvature scalar appears to grow without bound as smooth causal curves, that material particles are supposed to follow in the differential spacetime manifold, approach the corresponding singular *loci*. Even more strikingly, (certain components of) the respective 'solution metrics' (*ie*, the Schwarzschild, the de Sitter, and the Friedmann metric) appear not to be able to be continuously defined on the singular points of the locally $\mathbb{R}^4$ differential spacetime manifold M carrying them [184, 87].

In this section we choose the interior singularity of the Schwarzschild solution as a playground physical model to illustrate its ADG-theoretic 'absorption' or 'dissolution'. At the same time, having also in hand the exterior, so-called 'virtual' or 'coordinate', Schwarzschild singularity and its successful evasion by Finkelstein in [141], we will make it clear how *both* the superable exterior *and*

[373]That is, the gravitational field law (29) in ADG-gravity is **A**-functorial (Synvariance).

the supposedly insuperable interior singularities[374] are, in a subtle ADG-theoretic sense, '*virtual*' or *coordinate* ones, and that *in no way they inhibit the 'inherent', essentially algebraic (ie, sheaf-theoretic), differential geometric mechanism of ADG, which still applies galore over them.* Thus, we will maintain that *the $\mathcal{C}^\infty$-smooth singularities of GR in no way indicate a halting or breakdown of differentiability, contrary to what CDG has 'forced' us to believe so far; while, we will also show how to 'absorb' the inner Schwarzschild singularity into suitable*[375] *structure sheaves* **A** *of generalized arithmetics (coordinates) and still possess at our disposal the vacuum Einstein equations in full force over the classically offensive* $(r = 0)$ *locus.*

In more detail, below we will revisit, with an ADG-theoretic eye, the anomalies of the Schwarzschild solution to the vacuum Einstein equations. To begin with, we will briefly recall from [141] how Finkelstein showed that the exterior Schwarzschild singularity is not a 'true', 'genuine' singularity, but only a 'virtual', coordinate one. Essentially, he showed that when the Schwarzschild spacetime manifold associated with a stationary gravitating point-particle of mass m is suitably recoordinatized (*ie*, in effect, *analytically extended*)—from (analytic) Schwarzschild coordinates, to (analytic) Finkelstein-Eddington coordinates charting its point events—the exterior singularity disappears and the 3-spherical surface of Schwarzschild radius $(r = 2m)$—what is usually referred to as the *Schwarzschild horizon*, appears to behave like a unidirectional membrane—a temporally semi-permeable shell allowing only future or past-directed causal signals to cross its surface. Finkelstein then inferred that this '*gravitational osmosis*' may be regarded as a classical model for distinguishing (future propagating) particles from their time-reversed (past moving) antiparticles, a phenomenon which, as he observes, is essentially due to the non-linear nature of the gravitational force field. We will then use Finkelstein's rationale to outline key technical and conceptual features of our ADG-theoretic approach to 'resolving' $\mathcal{C}^\infty$-smooth spacetime singularities.

Of course, Finkelstein's work also pronounced, even if just indirectly—as it were, by elimination—the unavoidability and 'unevadability' of the interior Schwarzschild singularity right at the point-mass m $(r = 0)$, namely, it made it clear that the gravitational field becomes unmanageably infinite (in fact, that it becomes undefinable, that the law that it obeys breaks down, and that there is no further analytic extension of the smooth spacetime manifold past it!) right at its point-source. From a CDG-theoretic or Analytic point of view, the latter *locus* is deemed to be characterized as a 'real', 'genuine' singularity [141, 87], and one is tempted to infer (like most physicists do infer!) that GR, as a physical theory, is out of its depth when trying to describe the gravitational field and the law that it obeys right at the point where its mass source is situated.

Motivated by this apparently insuperable shortcoming of GR, *we will apply the ADG-technology and show how to evade completely the interior Schwarzschild singularity*, in two different, but

[374] In point of fact, the interior singularity is manifestly insuperable by CDG-theoretic (analytic) means!

[375] The epithet 'suitable' here means 'appropriate to the physical problem at issue' (see below).

closely related both in their technical aspects and their underlying philosophy, ways:

- (a) The internal Schwarzschild singularity will be resolved by finitistic-algebraic means stemming from our previous work on extending algebraically and by using the sheaf-theoretic means of ADG Sorkin's *topological* discretizations of continuous ($\mathcal{C}^0$) spacetime manifolds [355] to the *differential geometric* regime, thus arriving at a finitary (locally finite), causal and quantal description of Lorentzian gravity [270, 271, 272]. This is a natural continuation and extension of the said trilogy to a tetralogy so as to show, in a straightforward, and by virtue of a concrete physical example, way, the sense in which ADG totally evades, in a finitary-algebraic, sheaf-theoretic fashion the commonly regarded as being 'real', 'true' or 'genuine' spacetime singularities of GR [273, 274, 262, 264, 275, 265]. In a sense, to be contrasted against (b) below, this is a *'static-point resolution'* of the inner Schwarzschild singularity, regarded as a '*stationary or static point-singularity*' (of the DGS kind). Physically speaking, its essence is that, since only a (locally) finite number of 'degrees of freedom' (spacetime point-events) are involved (or 'excited' by the gravitational field) , and apart from the fact that the vacuum gravitational field equations will be manifestly shown to hold over the interior singularity,[376] it is not actually the case, in contradistinction to the smooth spacetime continuum based GR, that the gravitational field becomes infinite (let alone undefined!) in the vicinity of (or right at) the singularity. In fact, by suitable inverse and direct limit techniques and results of ADG, we will show that in a suitable 'continuum limit' the vacuum Einstein equations still hold over the interior singular *locus*; while, furthermore, we will also argue that there is nothing to suggest that the continuum gravitational field strength, represented by the Ricci scalar, becomes unbounded (infinite) at the singular point. This last argument will put into perspective the nowadays supposedly mandatory transition from a continuous (manifold) to 'discrete', 'cut-off' spacetime below Planck length in various QG approaches [355, 54]—a transition which is apparently necessitated by a need to render finite, even if 'by force of hand', the gravitational path integral (regularization) or the entropy of the horizon of a black hole which is supposed to conceal in its core the singularity under focus [357]. In fact, these arguments will pave the way towards our questioning of the 'physicality' of the Planck length and, ultimately, of the endeavor to quantize spacetime itself (section 6) in a theory, such as ADG, which deals solely with the fields themselves and the laws that they obey (differential equations), independently of a background (base) spacetime structure, whether the latter is assumed up-front to be a 'classical continuum' (manifold) or a 'quantal discretum' (section 7).

- (b) The second way in which the interior Schwarzschild singularity will be evaded is by pre-

[376]That is, they do not actually breakdown in the (differential) geometric sense of DGSs [87].

senting the latter, not as a localized, 'isolated or solitary' as it were, point-singularity as in (a) above, but rather as a *distributional* one of SFS-type—a singularity continuously extending along the point-particle's time-line and viewed from the perspective of the spacetime foam dense singularities of Mallios and Rosinger [274, 275]. Stemming from Finkelstein's work [141], one realizes that the 'effective' (analytic) Schwarzschild manifold X associated with a gravitating point-particle is the total spacetime manifold M minus the singular continuous 'wristwatch time-line' $L_t := \{p \in M : \ x_i(p) = 0\}$ of the point-mass placed at the origin of an (analytic) Cartesian coordinate system (x_μ) charting M: one writes $X = M - L_t$.[377] Without a loss of generality, we then regard L_t as a (locally) Euclidean space (*ie*, locally homeomorphic to $\mathbb{R}$), which is everywhere dense with $\mathcal{C}^\infty$-singularities in the 'spacetime foam' sense of generalized functions (distributions) of [274, 275],[378] thus apply the ADG technology as in the latter papers[379] so as to show that the vacuum Einstein equations still hold over the entire L_t; hence in effect, over all M, without the need of "*reducing the manifold of solutions*" (Q?.?),[380] by 'forced surgery' as it were, to the effective X. This is a concrete example of the application of ADG to GR by using a sheaf of Rosinger's algebras of generalized functions as structure sheaf of generalized arithmetics (coordinates), an idea originally entertained in the context of GR and gravitational singularities in [262, 264, 265, 267].

The physical moral of both of these resolutions can be appreciated in the light of two of Einstein's negative remarks about singularities encountered in [125]: "*It does not seem reasonable to me to introduce into a continuum theory points (or lines etc.) for which the field equations do not hold...*" (Q2.1) and "*...we cannot judge in what manner and how strongly the existence of singularities reduces the manifold of solutions.*" (Q?.?). Namely, with the two ADG-theoretic evasions of the inner Schwarzschild singularity anticipated above, we will vindicate Einstein by showing that actually the (vacuum) gravitational field equations actually hold over singular points[381] and lines[382] in the spacetime continuum M,[383], and in no way, the occurrence of either (*ie*, of singular points or lines), reduces the manifold of solutions.[384] Quite on the contrary, we will maintain in the

[377]With $x_0 = t$, $x_1 = x$, $x_2 = y$ and $x_3 = z$, as usual.

[378]For instance, we may think of Schwarzschild-type of singularities situated on the irrational points of the 'time-line' L_t of m.

[379]See subsection 4.2.

[380]See also next paragraph.

[381]The localized static Schwarzschild point-singularity in the resolution foreshadowed in (a) above.

[382]The extended 'time-line' Schwarzschild singularity in the resolution anticipated in (b) above.

[383]Let alone that the law that the (vacuum) gravitational field obeys breaks down in any (differential geometric) sense at the singular *loci*.

[384]For example, in case (a), the reduced spacetime, consisting of regular points on which the vacuum Einstein equations hold, is a smooth manifold M punctured at the locus of the point mass; while in case (b), the reduced

sequel, it is not that GR, as a physical theory (defined by local physical-dynamical laws—here, the vacuum Einstein equations), breaks down, but rather, that the CDG mathematical framework (and its associated base differential spacetime manifold) within which we model GR differential geometrically[385] is of limited applicability and validity *vis-à-vis smooth*-smooth singularities, since, as noted above, the vacuum Einstein equations still hold over the classically (*ie*, from the manifold based CDG-viewpoint) offensive *loci*.

6.1 Finkelstein's Resolution of the Exterior Schwarzschild Singularity, its Physical Interpretation and its 'Aftermath' Viewed Under the Prism of ADG

As it is well known, the Schwarzschild solution represents the spherically symmetric gravitational field outside a massive, spherically symmetric body of mass m. On grounds of physical utility alone, our choice of this particular solution on which to exercise our ADG-machinery and results may be justified on the fact that experimentally all the differences between non-relativistic (Newtonian) gravity and GR have been based on predictions by this solution. Also, since comparison with Newtonian gravity allows us to interpret the Schwarzschild solution as the gravitational field (in empty spacetime) produced by a point-mass source m viewed from far away (*ie*, from infinity) [184], Finkelstein's original treatment of the Schwarzschild gravitational field as being produced by a point-mass in an otherwise vacuous spacetime manifold [141] appears to be a good starting choice.[386]

So first, following Finkelstein in [141], one assumes that spacetime is a smooth ($\mathcal{C}^\infty$) or even analytic ($\mathcal{C}^\omega$) manifold X,[387] and then one places at its 'center' (interior) a point-mass m. The 'effective' spacetime manifold of this point-particle becomes X minus the particle's 'wristwatch'

manifold is the smooth regular space $X = M - L_t$, with the singular wristwatch time-line of the particle excised. In fact, any forced 'surgical' procedure removing by hand and in an *ad hoc* fashion the singular points, lines or surfaces where the gravitational field equations do not hold, will simply not (and actually does not!) do from an ADG-theoretic viewpoint.

385 That is, we model the physical law, that *defines* GR as a dynamical theory of the gravitational field, after a *differential* equation proper.

386 The following recollection of results about the Schwarzschild singularities can be also found in the recent paper [317].

387 In this paper we do not distinguish between a $\mathcal{C}^\infty$- and a $\mathcal{C}^\omega$-manifold (or for the same reason, between CDG and Calculus or Analysis). From an ADG-theoretic viewpoint, as noted earlier, a smooth manifold X corresponds to choosing $\mathcal{C}^\infty_X$ for structure sheaf, while an analytic one has $\mathbf{A} \equiv \mathcal{C}^\omega_X$—the structure sheaf of coordinate functions (of X's points) each admitting analysis (expansion) into power series. Admittedly, $\mathcal{C}^\omega$- is a slightly stronger assumption for a manifold than $\mathcal{C}^\infty$-, but this does not change, let alone inhibit, the points we wish to make here about the S-sing and its bypass in the light of ADG.

time-line $L_t := \{p \in X : x_i(p) = 0, (i = 1, 2, 3, t \equiv x_0)$ (expressed in a Cartesian coordinate system with m situated at its origin); that is,

$$X_S = X - L_t{}^{388} \tag{81}$$

Then, one observes that m is the source of a gravitational field, represented by a smooth (or analytic) spacetime metric $g_{\mu\nu}$, satisfying the vacuum Einstein equations (29) which are cast here as follows

$$^\infty\mathcal{R} = 0 \tag{82}$$

with the pre-superscript '∞' indicating the $\mathcal{C}^\infty$-smoothness of the base spacetime manifold X (and therefore also of the smooth Ricci tensor, which is a function of the smooth $g_{\mu\nu}$ and its partial derivatives)[389] on which the partial differential equations above hold.

The Schwarzschild solution of the said equations is the Schwarzschild metric $g^S_{\mu\nu}$ (expressed in Cartesian-Schwarzschild coordinates), which in turn defines an infinitesimal proper time interval, as follows:

$$ds^2_S = (1 - r^{-1}_S)(dx^0_S)^2 - (1 - r^{-1}_S)^{-1}dr^2_S - (dx^i_S dx^i_S - dr^2_S) \tag{83}$$

expressed in 'natural units' in which the so-called Schwarzschild radius ($r = 2m$) and the speed of light ($c = 10^8 m/s$) are equal to 1.[390]

Evidently, $g^S_{\mu\nu}$ has two singularities: one right at the *locus* of the point-mass—the Cartesian origin ($r = 0$), and one at the Schwarzschild radius ($r = 1$) delimiting a spacelike 3-dimensional unit-spherical shell in X, commonly known as the Schwarzschild horizon. The two singularities are usually pitched as the *interior* (inner) and *exterior* (outer) Schwarzschild singularities, respectively.[391]

In [141], Finkelstein initially considered an analytic metric $g^{EF}_{\mu\nu}$ on X, expressed in what is

[388]The subscript 'S' stands for '(S)chwarzschild'.

[389]One, like Finkelstein, could also use the pre-superscript 'ω' to indicate an analytic X, metric and Ricci tensor. Of course, from an ADG-perspective, all this essentially boils down to choosing $\mathbf{A} \equiv \mathcal{C}^\infty_X$ for structure sheaf.

[390]Also, in (83) above, $r_S = \sqrt{x^i_S x^i_S}$ and $dr_S = r_s^{-1} x^i_S dx^i_S$. The more familiar (*ie*, not in 'natural units') expression for the Schwarzschild line element in cartesian coordinates is $(1 - \frac{2m}{r})dt^2 + dx^2 + dy^2 + dz^2 + \frac{2m}{r(r-2m)}(xdx + ydy + zdz)^2$, while in spherical-Schwarzschild coordinates (again not in natural units), it reads $-(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$.

[391]The said hypersurface is *the horizon of the Schwarzschild black hole*, which is supposed to have the inner singularity in its core, as it were, 'beyond the horizon', which screens it from the view of an external observer.

nowadays usually called Eddington-Finkelstein coordinates (frame),[392] defining the following infinitesimal spacetime line element

$$\begin{aligned} ds_F^2 = (1 - r_F^{-1})(dx_F^0)^2 + 2r_F^{-1} dx_F^0 dr_F - (1 + r_F^{-1}) dr_F^2 - (dx_F^i dx_F^i - dr_F^2) = \\ -(1 - \tfrac{2m}{r})(dn^{\pm})^2 \pm 2dn^{\pm} dr + r^2(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned} \tag{84}$$

and he then showed that, for the region of X outside the Schwarzschild horizon ($r_F > 1$), the following simple 'logarithmic time coordinate change' from the analytic Finkelstein ${}^{\omega}\mathbf{A}_F = {}^{\omega}(x_F^{\mu})$ coordinates to the also analytic Schwarzschild ones ${}^{\omega}\mathbf{A}_S = {}^{\omega}(x_S^{\mu})$

$$\begin{gathered} \underline{{}^{\omega}\mathbf{A}_F \longrightarrow {}^{\omega}\mathbf{A}_S :} \\ x_F^0 \longrightarrow x_S^0 = x_F^0 + \ln(r_F - 1) \\ x_F^i \longrightarrow x_S^i = x_F^i \end{gathered} \tag{85}$$

transforms the line element ds_F^2 (and the associated $g_{\mu\nu}^F$) in (84) to the Schwarzschild one ds_S^2 (and its associated $g_{\mu\nu}^S$) in (83).

Conversely, he argued that since ${}^{\infty}\mathcal{R}$ in (82) is a tensor with respect to the ${}^{\omega}\mathbf{A}_F$ coordinates, the vacuum Einstein equations hold in all X (now coordinatized by ${}^{\omega}\mathbf{A}_F$[393])—in particular, they hold on the horizon unit-shell.

In toto, Finkelstein showed that the analytic coordinate change

$$X_S \equiv (X, {}^{\omega}\mathbf{A}_S) \longrightarrow (X, {}^{\omega}\mathbf{A}_F) \equiv X_F \tag{86}$$

amounts to an *analytic extension* of X_S (coordinatized by the Cartesian ${}^{\omega}\mathbf{A}_S$ and carrying the analytic $g_{\mu\nu}^S$ defining ds_S^2 above—which is singular at $r = 1$), to X_F (coordinatized by the analytic ${}^{\omega}\mathbf{A}_F$ and carrying the analytic $g_{\mu\nu}^F$ defining ds_F^2, which is *not* singular at the Schwarzschild radius!).

In fact, Finkelstein showed that the said analytic extension of X_S to X_F can be carried out in two distinct ways,[394] each one being the time-reversed picture of the other, which in turn means that the $r = 1$ horizon, far from being a site of singular *loci*, acts as "*a true unidirectional membrane*" in the sense that "*causal influences can pass through it only in one direction*" and, moreover, he gave a particle-antiparticle interpretation of this gravitational time-asymmetry [141].[395]

[392] The Eddington-Finkelstein frame consists of so-called *logarithmic-null spherical coordinates* $(n^{\pm}, r, \theta, \phi)$, with the null coordinate $n^{\pm}$ being either *advanced* $n^{+} := t + r^{'}$ or *retarded* $n^{-} := t - r^{'}$, and $r^{'}$ defining a logarithmic radial coordinate $r^{'} := \int \frac{dr}{1-2mr^{-1}} = r + 2m\log(r - 2m)$.

[393] Which we may just as well symbolize by X_F.

[394] Depending of course on whether one chooses advanced or retarded logarithmic-null coordinates.

[395] The null (in the Finkelstein frame) hypersurface at $r = 1$ is also known as a *closed trapped surface* [184], which 'traps' past- (resp. future-) directed causal (*ie*, timelike or null) signals depending on whether one chooses advanced

On the other hand, about the inner Schwarzschild singularity Finkelstein concluded that *the theory* (*ie*, the CDG-based GR) *is out of its depth* as there is no (analytic) coordinate change that can remove it like the outer one.[396] In other words, the interior singularity, right at the point-particle m, is regarded as being a 'genuine', 'true' singularity of the gravitational field, not removable (or 'resolvable') by analytic (*ie*, CDG-theoretic) means [141, 184, 88, 87].[397]

Characteristically, Rindler [323] draws a sharp distinction between 'true', 'genuine', 'non-coordinate' singularities, and 'virtual', coordinate ones—while more importantly for our ADG-based exposition here, he (implicitly) regards the former as being *physically real*—by using both the exterior and the interior Schwarzschild singularities as 'canonical' examples of the said distinction:

(resp. retarded) Finkelstein coordinates to chart the original manifold. Let it be also noted here that it can be easily seen that inside Schwarzschild horizon the original time and radial coordinates exchange roles.

[396]Indeed, the maximal analytic extension of X, which is unable to include the $r=0$ *locus*, is the well-known one constructed by Martin Kruskal [237, 184].

[397]Indeed, in the n^+-picture, any future-directed causal curve crossing the horizon can reach $r=0$ in finite affine parameter distance. Moreover, it can be shown that as $r \longrightarrow 0$ the Ricci scalar curvature $\mathcal{R}$ in (82) blows up as $\frac{m^2}{r^6}$, while there is no further analytic extension (in a C^2-, or even in a C^0-, fashion!) of the Schwarzschild spacetime manifold across the $r=0$ *locus*.

> "...For many years it was believed that there was a real singularity at $r = 2m$, in the sense that the local physics would become unusual there...However, in 1933 Lemaître found that the [exterior][a] Schwarzschild singularity is not a *physical*[b] singularity at all, but merely a *coordinate*[c] singularity, i.e., one entirely due to the choice of the coordinate system...Of course, in general it will not be so obvious how to remove a coordinate singularity, or even whether a given singularity is due to coordinates or not. One way of deciding this last question is to calculate the invariants of the curvature tensor and testing whether these remain finite as the singularity is approached: *if they do, the singularity is probably not a physical one*[d]..."

(Q6.1)

[a]Our addition.

[b]Rindler's emphasis.

[c]Rindler's emphasis.

[d]Our emphasis. Thus, implicitly, Rindler regards the inner Schwarzschild singularity as being a *physical* one—precisely what we want to challenge in the present paper (at least in the way that Rindler puts it above, namely that: "*local physics becomes unusual at a real singularity*"—*ie*, that the differential geometrically expressed, as a differential equation, law of gravity 'breaks down' or 'ceases to hold' at a true singularity).

Which, having ADG in mind, brings us to the aftermath of Finkelstein's resolution of the exterior Schwarzschild singularity.

The moral of the story from an ADG-theoretic vantage. Finkelstein's resolution or evasion of the outer Schwarzschild singularity, always within the confines of CDG, on the one hand showed that the said singularity is not 'genuine', 'true' or 'physical', but merely a coordinate one—an indication that the manifold (and the smooth gravitational metric field on it) was expressed (charted) by the physicist in a 'wrong' system of coordinates; while on the other, it showed that the inner singularity is a 'true' or 'physical' one, as it resists all smooth coordinate changes (extensions past it) intended to include it with the other regular points of the smooth X. This then set the paradigm of how to go about, within the realm of Analysis, and distinguish between 'virtual' (coordinate, unphysical) and 'genuine' (physical) singularities that we discussed in section 2, namely, the method (and subsequent trend) of analytic extension and (causal) completion accompanied with the construction of various singular topological and causal boundaries on which 'genuine' singularities are 'asymptotically' situated.[398] *In toto*, Finkelstein's analysis of the Schwarzschild singularities, when suitably abstracted (generalized), points to a general Analytical (CDG-based) method of dealing with (smooth) spacetime singularities [87]—at least it provides one with a general method for deciding when a singularity is a coordinate or a genuine one, something that was 'frustratedly' demanded by Eddington [108] as the following quotation from [88] shows:

(Q6.2) "...Eddington deplored the lack of a method to distinguish between a real and a coordinate singularity: '*It is impossible to know whether to blame the world-structure or the coordinate system*'[a]..."

[a]Our emphasis of Eddington's quotation.

On the face of all this, the serious conflict between the PGC of the manifold based GR and the 'definition' of (and distinction between) coordinate and true gravitational singularities with CDG-methods may be expressed thus: on the one hand the dynamics—Einstein's equations—treats all coordinate systems on an equal footing (being in fact indifferent with respect to what frame the said equations are written down), while on the other, a metric-solution to those equations possesses singularities whose character (*ie*, whether they are coordinate or 'real' ones) actually depends on what coordinate system is used.[399] In other words,

[398]With the concomitant general classification of 'true' singularities into DGSs, VESs and SFSs, as we reviewed in section 2.

[399]More precisely, coordinate singularities appear to be frame-dependent (*ie*, singular in one frame, but non-singular in another), while genuine ones are in a negative sense coordinate-independent (always within the CDG-theoretic framework): no matter what $\mathcal{C}^\infty$-smooth system of coordinates is used to chart the spacetime manifold, true singularities remain singularities and cannot be removed or 'gauged away' by any coordinate ('gauge') change!

> *how can the gravitational field equations, which purport to be generally covariant, admit solutions that appear to be 'coordinate-sensitive'?*[a]
>
> [a]In a metaphorical sense, the singularities of GR may be perceived as analogues of the anomalies of QFT. Anomalies are usually thought of as indicating the 'breakdown' (or lifting) of a classical symmetry of (the Lagrangian defining the dynamics of) a classical theory *upon quantization* [216]; whereas here, singularities may be thought of as the 'breakdown' of the Diff(M)-'symmetry' of the (classical) theory (*ie*, of the M-based Einstein equations defining GR) *upon solution* (of those equations, while
> (R6.1) still remaining in the classical domain of CDG where no quantization is invoked). In this sense, singularities (especially DGSs) may be coined '*differential geometric solution-anomalies*'. A bit later on, in 5.3, we are going to challenge, having the ADG-theoretic evasion of gravitational singularities in hand, the traditional viewpoint that the concrete *solutions* (solution-metrics) of the gravitational field equations are the physically significant entities in GR. Further later on, in section 6, we will challenge, again based on the ADG-didactics learned from the evasion of the inner Schwarzschild singularity, the nowadays well-established intuition (or expectation!) that a quantization of GR (or of spacetime structure) will remove singularities.

From an ADG-vantage however, the Schwarzschild singularities' story acquires a whole new meaning (interpretation) and teaches us a very different moral. To begin with, since no background manifold is involved at all in the theory, the CDG-based techniques of analytic extensibility (of a manifold), smooth (causal) geodesic completion (of a manifold) and the associated construction of various (causal/topological) boundaries (to the manifold) in order to accommodate true singularities altogether lose their meaning and import.[400] In particular, they are not used to distinguish between coordinate and genuine singularities, hence the latter distinction loses its significance in ADG-gravity. Indeed, from an ADG-theoretic perspective all singularities are virtual, coordinate ones as they all are inherent in the particular structure sheaf $\mathcal{C}^\infty_M$ of generalized arithmetics (coordinates) that uniquely characterizes (defines) a differential manifold M and *in extenso* the CDG that is based on it. Moreover, Finkelstein's 'method' of changing coordinates in order to show that the exterior Schwarzschild singularity is a virtual one, followed by the novel *physical interpretation* (time-asymmetry) that he gives to the situation now 'pictured' in the new coordinate system (the Schwarzschild horizon acts as a unidirectional membrane effecting a 'causal osmosis'),

[400]For in ADG we do not do Calculus 'geometrically', in a Cartesian-Newtonian fashion—*ie*, by 'picturing' (or representing) differential geometrically physical situations with the assistance (mediation in our calculations) of a smooth background space(time) in the guise of coordinates in $\mathcal{C}^\infty M$—but rather *purely algebraically* (relationally), in a Leibnizian way. (See concluding section.)

is abstracted by ADG to the following 'method' of dealing with (in fact, evading completely!) arbitrary singularities of any kind:

(R6.2) *Upon encountering a singularity (or in general, 'non-smooth' situation), one should look for an 'appropriate' structure sheaf* **A** *of generalized arithmetics (coordinates)*[a] *that incorporates the (function that misbehaves at) the said singularity (or more generally, a structure sheaf that somehow represents faithfully the 'non-smooth' situation in hand), yet it is still able to furnish one with the essentially algebraic differential geometric mechanism one needs in order to apply differential geometric ideas to the physical situation or problem that one wishes to tackle. In summa, the basic general didagma here is that upon encountering a differential geometric problem (eg, a singularity or a non-smooth situation) one could evade it (or even include it in one's now 'generalized Calculus'!) by judiciously changing (or enlarging) one's structure algebra sheaf* **A** *of 'differentiable functions';*[b] *moreover, in the new differential geometric situation afforded by the new* **A**, *one, like Finkelstein, should look for a novel* physical interpretation *for how the physical situation 'looks' now under the prism of the new "coordinatizations' (generalized 'observations' or 'measurements').*[c]

[a]As noted earlier, the epithet 'appropriate' to **A** in the context of ADG means that the said (functional) structure sheaf provides one with a differential d to the extent that the functions involved in **A** can be branded 'differentiable', like in the classical case of CDG where $\mathbf{A} \equiv \mathcal{C}^\infty_M$.

[b]Within the confines of the differential manifold based CDG, the procedure of analytic extension does precisely this: *one tries to enlarge one's algebra of differentiable functions so as to include the 'pathological' one having the singularity.*

[c]Metaphorically speaking, **A** is like one's reading-glasses: if something appears to be problematic (*eg*, blurry and unreadable), one should change one's glasses, but by no means attribute his failure to read to the text he is trying to read—to emphasize it again: *singularities lie with* **A**, *which is ours; not with Nature.*

Of course, the CDG-conservatism and monopoly (*ie*, the fact that so far the only way we knew how to do differential geometry is, in one way or another, by involving a base manifold) virtually 'mandated' the distinction between true and coordinate singularities as it identified the latter with *loci* resisting Analysis—sites where the manifold based CDG ceases to apply (and hence the law of gravity, modelled after a differential equations, appears to 'break down'—DGSs). In contradistinction, in view of the discussion and (R5.?) above, *from the ADG-viewpoint all singularities are in a deep sense coordinate ones, being inherent in (or built into)* **A** (which in the classical case is $\mathcal{C}^\infty_M$—*ie*, singularities are innate anomalies of the background differential spacetime manifold).

Gravitational 'differential geometric solution-anomalies': no sufficient reason there. The question we posed in (R5.?) above and the implicit reply we gave in its footnote, namely, that singularities (or at least DGSs) may be interpreted as 'differential geometric solution-anomalies', with our subsequent regarding all $\mathcal{C}^\infty$-smooth spacetime singularities as virtual, coordinate ones from the perspective of ADG-gravity, recalls Finkelstein's fundamental question upon resolving the exterior, 'virtual' (coordinate) Schwarzschild singularity in [141]:[401]

(Q6.3)

> "...Thus it seems that T [time] invariance and general relativity are incompatible for a spherical point particle: although the requirements (*abcde*) [that Finkelstein imposes in the beginning of [141] on the gravitational field and its background spacetime (manifold) symmetries] do not distinguish between past and future, the only universe which obeys them does. *How is it possible that causes which are symmetric can have effects that are not? Such a violation of the principle of sufficient reason must be attributed to the nonlinear nature of gravitational theory...*[a]
>
> [a]And then Finkelstein goes on to anticipate a theoretical situation in which the nonlinear character of gravity will 'choose' one of the two alternative time-asymmetric Schwarzschild universes, namely, the phenomenon of spherical gravitational collapse of a spherical body (*eg*, a star) past the critical point when its radius equals the Schwarzschild radius.

In complete analogy, as we argued above, from the CDG-viewpoint singularities too appear to violate the principle of sufficient reason in the following sense: while the Einstein equations (the '*causes*' in our case) are generally covariant (no preferred coordinate system) and thus 'coordinate-insensitive', their solutions (the '*effects*' in our case)[402] appear to be 'coordinate-sensitive'. We then

[401]In the excerpt below, emphasis is ours; while in square brackets are our own additions for clarity and completeness.

[402]And it must be stressed here that from an ADG-viewpoint this appellation of the gravitational field (*viz.* algebraic **A**-connection), defining the field equations as differential equations proper, as '*the cause*', while on the

coined this 'Diff(M)-breaking' phenomenon associated with the appearance of singularities of exact solutions of the Einstein equations '*gravitational differential geometric solution-anomalies*'.[403] On the other hand, we argued above, from the vantage of ADG-gravity there is no violation of sufficient reason simply because in the first place our *a priori* assumption of spacetime as a differential manifold is not sufficiently reasonable (pun intended).[404]

6.1.1 A Finitary-Algebraic and ADG-based 'Resolution', 'Evasion', 'Dissolution', or 'Engulfment', of the Interior Schwarzschild Singularity

In this central part of the present paper we provide arguments that lead us straightforwardly to a finitary (:locally finite), algebraic and ADG-based 'resolution' (better, 'total evasion' and 'dissolution') of the inner singularity of the Schwarzschild solution of the vacuum Einstein equations for the smooth gravitational field of a point-particle of mass m situated at the 'center' of the differential spacetime manifold . We will show, by finitistic-algebraic and ADG-based (*ie*, sheaf-theoretic and categorical) arguments,

- that this singularity is not actually a breakdown *locus* of the differentiability (smoothness) of the gravitational field [86, 87] (Q2.?), which anyway in our ADG picture is not identified with a smooth metric, but with an algebraico-categorically defined and manifestly background manifold independent $\mathbf{A}$-connection variable $\mathcal{D}$ [270, 271, 272],

- that there is no divergence of, no infinity associated with, the Ricci tensor as one approaches m [184, 87] (Q2.?), since it is plain that, physically speaking, only a finite number of events

other hand, of a solution to these equations as '*the effect*', is not just a formal matter. For us, as we shall argue in detail later on, the *algebraic* connection $\mathcal{D}$ defining the gravitational field law is, abstractly speaking, 'the cause' of the '*solution-geometry*' (*ie*, '*solution space*') of that law. In turn, the latter is the effect (result) of that law—it is *the realm where the field (law) holds*. This is another way to say what we mentioned earlier and we will further discuss in the last section, namely, that 'geometry' (in the generalized sense of 'solution space') is the effect of the 'primary cause'—the field ($\mathcal{D}$) itself defining the gravitational law (differential geometrically) as a differential equation in the first place. *In toto*, as it has been argued in 3.2.5 and will be further corroborated in section 7, *dynamics (algebra) comes before kinematics (geometry), and the former is the 'cause' of the latter.*

[403] Arguably, this lies at the heart of the glaring conflict between the PGC of GR and the existence od $\mathcal{C}^\infty$-smooth spacetime singularities mentioned in section 1, an asymphony that in turn makes a precise definition (by classical Analytic means) of singularities a difficult, slippery issue indeed [155, 87].

[404] Thus, in contradistinction to Finkelstein in [141], we did not attribute the apparent violation of the principle of sufficient reason to the gravitational field itself (*ie*, for example, to its non-linear nature), but to the smooth background spacetime manifold M which carries in its coordinate structure sheaf $\mathcal{C}^\infty_M$ singularities, now perceived as gravitational 'differential geometric solution-anomalies'. That is, in our case, we 'blame it' on the external base spacetime (manifold), and by no means on the field itself (which anyway in ADG-gravity is not the smooth metric, but the base manifold independent algebraic $\mathbf{A}$-connection $\mathcal{D}$; see below).

(or 'degrees of freedom' of the gravitational potential field—the aforesaid 'finitary spin-Lorentzian connection' [270, 271, 272]) are involved or 'excited' (at least locally—*ie*, in the 'immediate neighborhood' of the point mass source m),

- that this singularity is 'isolated'[405] and 'weak' compared to the classically (*ie*, by CDG means) most numerous, unmanageable, robust, non-smooth and non-pointed (*ie*, 'smeared out' or distributional) spacetime foam dense singularities we encountered in subsection 4.2, with which ADG copes without a problem in the sense that they are not at all impediments to its 'inherent' algebraico-categorical differential geometric mechanism [274, 275], while the (vacuum) Einstein equations, which are expressed via the ADG-machinery, hold in full force over them [262, 264, 265], and

- that we give a 'classical continuum limit' construction, with a concomitant new and generalized, because abstract, definition of 'classical smoothness', which shows that the (new) differential spacetime continuum vacuum Einstein equation still holds over its point mass source m—which, in turn, in the usual $\mathcal{C}^\infty$-smooth manifold case, is supposedly a 'real' or 'true' spacetime singularity. Thus for example, there is neither a need to subject M to a physically *ad hoc* 'surgery' [155, 186] in order to remove the offensive *locus* m, nor to push the latter to the boundary of M, both of which in turn point to the fact that the inner singularity, as far as the 'innate differential geometric mechanism' (of ADG) is concerned, is neither "*a point where the (gravitational) field equations do not hold*" (Q2.1) (let alone 'break down' (Q2.?) in any differential geometric sense of the DGSs), nor a *locus* at the edge of spacetime beyond which M cannot be smoothly (or analytically) continued. Moreover, this bypass of the inner Schwarzschild singularity will show that ADG effectively offers us a direct, Gordian knot-cutting answer to Einstein's quest to "*judge in what manner and how strongly the existence of singularities reduces the manifold of solutions*" (Q2.?), namely, *not at all!*; while at the same time, it goes a long way towards offering "*a method to derive systematically solutions that are free of singularities*" (Q2.?), as singularities are not impediments to, let alone breakdown points of, the physical law of gravity, which holds galore in their very presence.[406]

The basic theoretical background material for this 'finitary' and sheaf-theoretic resolution can be found in our trilogy [270, 271, 272] and it has been briefly reviewed in the previous sections of

[405] Or better, 'solitary', as the term 'isolated singularity' has a different, by now rather standard, meaning in the usual GR and singularities' terminology.

[406] To be precise, ADG does not offer a method of deriving (new) solutions of the Einstein equations that are singularity-free, but rather, a (differential geometric) method of showing that the Einstein equations can be written, as differential equations proper, in the presence of *any* singularity, *as if singularities were not there*. Not only gravity, but differential geometry in general, has been freed from singularities!

this paper. But first, as a suitable preamble to the promised evasion, we wish to make some preliminary points about the virtues of '*pointlessness*' and '*algebraicity*' (as well as the related notions of '*categoricity*' and '*functoriality*'), which figure prominently in the said trilogy, in what we have said so far herein, and in what follows.

6.1.2 The point of pointlessness, algebraicity and of sheaf-theoretic localization ('gauging') of structures and their concomitant 'dynamicalization'

We commence by first noting that the point-like character of the events of the spacetime continuum is an ideal mathematical assumption. In other words, the geometric manifold, as a point-set, is an operationally unrealistic or non-pragmatic model for 'spacetime' [318, 319]. As a theoretical alternative to the 'pointed nature' of the manifold, one could suggest to substitute its points by some kind of 'open sets' about them. Such a move is supposed to be of an operationally more pragmatic flavor, since operationally realistic determinations or 'measurements' (localizations) of events—"*what we actually do to produce spacetime by our measurements*" [357]—involve 'coarse' (because instrumentally limited, perturbing and indeterminate, as they either carry 'experimental error' or inflict uncontrollable changes onto the observable fields) acts of 'observation'. This operationally sound assumption lies at the heart of the so-called 'finitary' approach to substituting the spacetime continuum by 'discrete' spaces, as originally pioneered by Sorkin well over a decade ago [355].

'Operational pragmatism' aside for a moment, since also the pointed character of the background spacetime manifold is arguably *the* 'geometrical reason' for (or origin of) its singularities and their associated infinities,[407] the idea to substitute the points of the manifold by coarse regions about them appears to be in line with the general intuition that these potentially singular points must somehow be 'smeared out'—here for instance, be substituted by 'fatter', extended 'open sets'.[408]

On the other hand, from a physical point of view, there is another glaring theoretical problem associated with Sorkin's so-called finitary topological substitutes of continuous ($\mathcal{C}^0$) point-set manifolds in [355]. To begin with, we mention that the T_0-topological posets obtained from covering (a relatively compact region X of a) $\mathcal{C}^0$-manifold M by locally finite open covers are based on the assumption that "*the points of X are the carriers of its topology*" [355]. If at the same time

[407] For any point in the spacetime manifold may be the host of a singularity for a physically important (smooth) field on it—a *locus* where the smooth field grows unbounded and its differentiability ('smoothness') appears to break down.

[408] In spirit at least, this is akin to the by now bread-and-butter 'blow-up' idea and associated technique for resolving singularities in algebraic geometry whereby one smears and, ultimately, resolves the offensive geometrical point by erecting and spreading a bundle (or better, sheaf!) of 'directions' over it [178]. But more about such ADG-based sheaf-theoretic resolutions of such differential geometric 'point-anomalies' shortly.

one wishes to carry theoretically this assumption to its extreme, by suggesting for example that the spacetime topology itself is an 'observable' (*ie*, a dynamical and in principle experimentally 'measurable') entity [168, 63, 318, 319], one could infer that a theory for 'observable topology' is essentially a theory of spacetime (point) measurements. However, again from an operational viewpoint, *we never actually measure spacetime itself*—let alone its ideal point events, but only the physical fields on (*ie*, triggering those events in) it [49, 50, 356].[409] In effect, and this is one of the central didagmas of ADG,

(R6.3) the fields themselves are the 'carriers' of (the geometry of) 'spacetime'—in particular, of its 'topology'—and the dynamical (and 'observable'!) character of the former entails the dynamical variability of the latter [96, 272]. In a Leibnizian-Machian sense, physical 'space(time)' (geometry) derives from the dynamical relations (which in turn can be mathematically represented combinatory-algebraically) between the 'geometrical objects' (*ie*, the physical fields) themselves.

In other words, there is no *a priori* 'spacetime' as such, a fixed absolute ether-like entity, an 'out there' pre-existent container or carrier of the dynamical fields. Turning the tables around, *the geometric properties of spacetime (eg, its topology) are inherent in the (dynamical) fields.* Of course, the operationally ideal character of spacetime points, and the more realistic idea of regions about them, can be retained in this 'field picture'—whether one assumes those fields to be classical or quantum, as we read for instance from [50]:

(Q6.4) "*...Classical electrodynamics operates with the idealization of field components defined at every point of space-time. Although in the quantum theory of fields these concepts are formally upheld, it is essential to realize that only averages of such field components over finite space-time regions have a well defined meaning*[a]*...*"

[a]Our emphasis throughout.

with the notion of 'average' highlighting the aforesaid non-sharpness or 'coarseness' and the

[409]To be clear about this point, what we usually measure is not the spacetime location of fields and their particles (quanta)—*ie*, our measurements are almost never 'spacetime localizations' proper of fields, particles, but only physical processes of energy-momentum transfer between the observable and interacting fields (particles) [49, 50]. One is tempted to say here that the (quantum) dual ('complementary') picture of the space-time continuum, which (locally) involves differential (co)tangent vector energy-momentum quantities, is the experimentally significant one, not the space-time ('position') picture itself. Loosely put, (dynamical) changes not (static) locutions are physically important, because measurable.

smeared out character of our field determinations.[410]

Taking things one at a time, we note that all our ADG-based work so far [270, 271, 272] has concentrated on de-emphasizing the importance of a background spacetime—whether it is 'discrete' or a 'continuum'—while placing emphasis on the dynamical physical fields themselves, the 'geometrical objects' living on that background 'surrogate'[411] space(time). The conceptual and methodological development of this 'de-emphasization' of 'static' (pointed) background geometrical space(time) structures with the concomitant concentration solely on the (algebraically represented) dynamical fields *per se*, may be cast progressively in the following steps:

$$x \xrightarrow{\text{(a)}} U \xrightarrow{\text{(b)}} \mathbb{A}(U) \xrightarrow{\text{(c)}} \mathbf{A}_U \tag{87}$$

which we briefly explain below:

- (a) This is essentially Sorkin's step going *from points to regions*, with a concomitant substitution of the continuum by a 'discrete', locally finite, directed graph—a T_0-topological poset. However, as noted above, while this scheme appears to be manifestly pointless and results in substituting the continuum by a reticular, finitistic structure with an operational flavor, points, in an ontological sense, are still tacitly assumed to be the carriers of space(time) topology, and the aforesaid 'finitary substitutes' of continuous topology are regarded as 'coarse approximations' of (the $\mathcal{C}^0$-topology of the) pointed space(time) manifold, not as 'autonomous' structures anyway.

- (b) This second step pertains essentially to the 'algebraic spatialization', based on a discrete version of Gel'fand duality, procedure followed by Raptis and Zapatrin in [318, 319]. In a nutshell, as also noted in some detail in section 3, the topological information encoded in the finitary T_0-posets mentioned above was transcribed to an algebraic setting—the so-called incidence (Rota) algebras associated with those posets. The basic idea was to substitute the 'spatial' posets (carrying the 'discretized' $\mathcal{C}^0$-manifold topology) above by relational, combinatory-algebraic structures effectively encoding the same (topological) information.

[410] At this, independently of whether it is classical or quantum, level of field description, this indeterminateness may be thought of as being either instrumental (classical 'fuzziness') or fundamentally epistemic/systemic (quantum indeterminacy).

[411] The epithet 'surrogate' essentially means in our work that the sole role played by the base (topological) space(time) in our ADG-based scheme, is as a background (topological) space(time) enabling the (sheaf-theoretic) soldering or localization (see step (c) below) of the said 'geometrical objects'—the physical dynamical fields—without actually participating at all in their dynamics—the differential equations that they obey (or better, that they *define*). In turn, this is a consequence of the fact that, in ADG, unlike the manifold based CDG, 'differentiability' does not derive from a smooth geometrical base space(time), but from the algebra inhabited stalks of the sheaves engaged in the theory.

This was the preliminary, albeit important, step of the general idea of *substituting* ('discrete') *space(time) by* (finite dimensional and noncommutative) *algebras*, something that enabled us to attain a finitistic and to a certain extent 'quantal' picture of spacetime (topology) [318, 319]. The welcome bonus we got in going from pure 'geometry' ('spatial topology') to algebra in a finitistic setting was that the corresponding incidence algebras were seen to be discrete *differential* algebras (manifolds)—that is, they proved to encode information not only about the topological structure proper of discretized space(time), as their corresponding finitary posets do, but also about its *differential* structure [318, 319, 432].

- (c) This last step is about making the combinatory-algebraic structures of step (b) *variable*, by *localizing* them in a (finitary spacetime) sheaf-theoretic way ([310]) over their corresponding finitary topological posets of step (a) [270]. This sheaf-theoretic localization procedure, when the relevant locally finite posets (and their incidence algebras) are not interpreted as finitary topological spaces proper, but as *causal sets* or 'causets' for short (and their incidence algebras as *quantum causal sets* or 'qausets' for short) [357, 309], has been physically interpreted as the process of '*gauging qausets*', hence the aforesaid conception of the relevant algebraic structures as being variable [270]. In the latter paper, it was shown that this localization or gauging procedure amounted to endowing the relevant finitary spacetime sheaves (finsheaves) of qausets with a non-trivial ('discrete', spin-Lorentzian) connection, which generalizes (in effect, 'curves') the flat Cartan-de Rham-Kähler differential (it too a connection, albeit flat! [259, 260]) carried by the incidence algebras (qausets) [318, 319, 432] 'stalk-wise' in the respective finsheaves. This connection variable sets the stage for the finitary gravitational dynamics of qausets, as a finitary, causal and quantal version of the vacuum Einstein equations of Lorentzian gravity is expressed solely in terms of it (in fact, of its curvature) [272].[412] Moreover, since the said connection is categorically (in fact, functorially) defined *à la* ADG as a (fin)*sheaf morphism* [270, 271, 272], the base topological space plays absolutely no role in the said qauset dynamics. This is in line with the general ADG-didactics that the base topological space(time)—no matter what its character (*ie*, whether it is assumed to be 'discrete' or 'continuous')—does not influence at all the 'inherently algebraico-categorical' differential geometric mechanism in terms of which the gravitational law (*ie*, the differential equation obeyed by the gravitational connection field and its curvature) is expressed (in ADG, as an equation between the relevant sheaf morphisms).[413] Physically speaking,

[412]Indeed, on very general grounds, "*sheafification is localization*" [266], which in turn amounts to 'gauging' and 'dynamicalizing' the relevant (algebraic) structures thus endow them with a non-trivial connection $\mathcal{D}$ with respect to which the said dynamics is represented (as a differential equation; for gravity in particular, the Einstein equations are expressed via the curvature of the connection). All this is the epitome of ADG.

[413]See [272] for an analytical discussion of this very important point, as well as section 3 where we discuss the

in ADG, the base topological space(time), whether it is assumed to be a continuum or a discretum, serves solely as a 'surrogate' background stage for the localization or 'gauging' of the 'geometrical objects' (the dynamical fields) themselves, without contributing at all in ADG's inherently algebraico-categorical differential geometric mechanism in terms of which the said dynamical laws are expressed as equations between the relevant sheaf morphisms.[414]

6.1.3 The interior Schwarzschild 'dissolved' finitistically and algebraically in an ADG-theoretic manner: a 'static' ('spatial') point-resolution

This is arguably the *neuralgic* part of the present paper and it has been also presented (more laconically and in a slightly different guise) in a recent paper by the second author [317]. In the ADG-theoretic 'resolution' (or perhaps better, evasion) to be presented below the interior Schwarzschild singularity is regarded as a 'static', 'spatial', localized point-singularity, and it is evaded in two different ways. The first way, for reasons to be given below, may be coined *direct* and *immediate*, while the second *indirect* and *step-wise constructive*. So here is a brief outline of the stages of a 'syllogism' leading in a straightforward manner to the finitistic-algebraic 'resolution' of the inner Schwarzschild singularity. In the course of this 'syllogism' or 'algorithm', all the virtues of our ADG-approach to gravity highlighted earlier throughout the paper, such as pointlessness, background spacetime independence ('spacetimelessness'), **A**-functoriality and synvariance, algebraicity, categorical bicompleteness of differential triads *etc*, come to play their role in one way or another.

- First we let X be an open and bounded region of a spacetime manifold M, from which initially, in the manner of Sorkin [355], we consider only its topological (*ie*, $\mathcal{C}^0$-continuous) structure.[415]

- We then let a point-particle of mass m be situated at the 'center' of X,[416] as in [141].

- Next, we cover X[417] by a locally finite open covering $\mathcal{U}_i$.[418]

- Subsequently, as we recalled in the last paragraph (**v**) of 3.1.7, we first discretize X relative to $\mathcal{U}_i$ in the manner of Sorkin, and then pass to the Gel'fand-dual representation of the resulting finitary posets P_i in terms of discrete differential incidence algebras Ω_i.

A-functoriality of the gravitational dynamics in ADG. We will return to it in more detail shortly, in 5.2.2 next.

[414] See footnote 82 earlier.

[415] That is, without committing ourselves *ab initio* to its differential (*ie*, $\mathcal{C}^\infty$-smooth) structure.

[416] In any case, we assume that m is a point in X's interior (bulk) without evoking any boundary ∂X construction.

[417] Or in the jargon of ADG, '*locally gauge*' X [259, 260, 270, 271, 272].

[418] Indeed, the Us in $\mathcal{U}_i$ are called '*open local gauges*' in ADG [259, 260].

- Then we consider finsheaves [310] of incidence algebras $\mathbf{\Omega}_i$ in the manner first studied in [270].[419]

- We then recall from 3.1.7 the fintriads $\mathfrak{T}_i$ (of qausets) that the said finsheaves define.

- We would like to digress a bit here and remind the reader of the two different ways in which we can obtain $\mathfrak{T}_i$ from X: the 'roundabout', '*constructive*' one starting from P_i and proceeding step-wise via the Ω_is and the finsheaves $\mathbf{\Omega}_i$ thereof as described above, and the '*direct*' or '*immediate*' one, via Papatriantafillou's categorical push-out and pull-back (of differential triads) results,[420] going directly, by what we called the 'Newtonian spark' in 3.1.7, from X (now regarded not just as a topological, but also as a differential manifold) and the classical differential triad $\mathfrak{T}_\infty$ that it supports, to the fintriad $\mathfrak{T}_i$ on the base P_i.[421]

- Having $\mathfrak{T}_i$ in hand either by direct or indirect means, we then bring forth from [272] the result that, on the said triads the vacuum Einstein equations of a (f)initary, (c)ausal and (q)uantal version of Lorentzian (v)acuum Einstein gravity hold;[422] write:

$$\mathcal{R}_i(\mathcal{E}_i) = 0 \tag{88}$$

- Then, from section 4 we recall that the said finitary differential triads comprise an inverse/direct system $\overset{\rightleftarrows}{\mathfrak{T}}$ possessing, following Sorkin via Papatriantafillou's categorical perspective on ADG, the CCDT $\mathfrak{T}_\infty \equiv \mathsf{T}_\infty$ as a projective/inductive limit (??).[423]

- Moreover, a plethora of finitary ADG-theoretic constructions, vital for the formulation of a finitary version of Lorentzian gravity regarded as a gauge theory, are based on those $\mathfrak{T}_i$s. These include for example the aforementioned $fcqv$-Einstein equations, the *$fcqv$-Einstein-Hilbert action functional* $\mathfrak{EH}_i$ from which these equations derive from variation with respect to the Lorentzian gravitational fcq-connections $\mathcal{D}_i$, and the *fcq-moduli spaces* $\mathsf{A}_i(\mathcal{E}_i)/\mathcal{A}ut\mathcal{E}_i$

[419] Parenthetically, one may wish to bring forth from [309] the causet and qauset interpretation that the P_is and their associated Ω_is may be given, as well as the finsheaves thereof [270].

[420] Especially 'via her push-out result whereby the classical differential triad supported by a manifold may be pushed forward to the 'moduli space' obtained from that manifold by an (arbitrary) equivalence relation—in Sorkin's case in particular, the quotient of X by the $\overset{\mathcal{U}_i}{\sim}$-relation we saw in (32).

[421] These two different routes that one can follow in order to arrive at the fintriad $\mathfrak{T}_i$ were coined at the beginning of this section *indirect* ('constructive') and *direct* ('immediate'), respectively.

[422] In [272] we abbreviated this model by the acronym $fcqv$-Lorentzian gravity.

[423] Recall, inverse limit for the base P_is and direct limit for their (Gel'fand) dual Ω_is inhabiting the stalks of the $\mathbf{\Omega}_i$s.

of those gauge-equivalent (self-dual) fcq-spin-Lorentzian connections—the gauge-theoretic 'configuration spaces' of our $fcqv$-version of Lorentzian (vacuum) Einstein gravity. Thus, it is fitting at this point to recall from [272][424] the "*11-storeys' tower of $fcqv$-inverse and direct systems*" based on the $\mathfrak{T}_i$s in $\overset{\rightleftarrows}{\mathfrak{T}}$:[425]

[424] Expression (150) there.

[425] In the table below, the important notion of '*curvature space*' is mentioned. The reader can refer to [272] (or of course to the 'originals' [259, 260, 262]) for a more detailed discussion of this notion, which however we will not be needing in the sequel. Finally, the expression '*genuinely covariant*' at the top level (7) is what we called earlier '*synvariant*' and we will encounter it again later on.

‘Standing on the shoulders of triads’

$$\boxed{\textbf{Level 7}: \text{Inverse system } \overleftarrow{\mathcal{Z}} \text{ of ‘genuinely covariant}' \text{ fcqv} - \text{path integrals}}$$
$$\uparrow$$
$$\boxed{\textbf{Level 6}: \text{Inverse system } \overleftarrow{\mathfrak{E}} \text{ of fcqv} - \text{connection fields and their curvature spaces}}$$
$$\uparrow$$
$$\boxed{\textbf{Level 5}: \text{Inverse system } \overleftarrow{\mathcal{E}} \text{ of fcqv} - \text{Einstein equations}}$$
$$\uparrow$$
$$\boxed{\textbf{Level 4}: \text{Inverse system } \overleftarrow{\mathcal{M}} \text{ of (self} - \text{dual) fcqv} - \text{moduli spaces}}$$
$$\uparrow$$
$$\boxed{\textbf{Level 3}: \text{Inverse system } \overleftarrow{\mathcal{EH}} \text{ of (self} - \text{dual) fcqv} - \text{Einstein} - \text{Hilbert action functionals}}$$
$$\uparrow$$
$$\boxed{\textbf{Level 2}: \text{Inverse system } \overleftarrow{\mathfrak{A}} \text{ of affine spaces of (self} - \text{dual) fcqv} - \text{connections}} \qquad (89)$$
$$\uparrow$$
$$\boxed{\textbf{Level 1}: \text{Inverse system } \overleftarrow{\mathcal{G}} \text{ of principal finsheaves and their (self} - \text{dual) fcqv} - \text{dynamos}}$$
$$\uparrow$$
$$\boxed{\textbf{Level 0}: \text{Inverse} - \text{direct system } \overset{\rightleftarrows}{\mathcal{T}} \text{ of fcq} - \text{differential triads}}$$
$$\uparrow$$
$$\boxed{\textbf{Level } -\textbf{1}: \text{Inverse system } \overleftarrow{\mathcal{S}} \text{ of finsheaves of continuous functions}}$$
$$\uparrow$$
$$\boxed{\textbf{Level } -\textbf{2}: \text{Direct system } \overrightarrow{\mathfrak{R}} \text{ of incidence Rota algebras or qausets}}$$
$$\uparrow$$
$$\boxed{\textbf{Level } -\textbf{3}: \text{Inverse system } \overleftarrow{\mathcal{P}} \text{ of finitary substitutes or causets}}$$

Papatriantafillou’s results secure that all these inverse-direct systems yield, like Sorkin’s original projective system $\overleftarrow{\mathcal{P}}$, their classical 'continuum' counterparts at the limit of infinite resolution of the (base) P_is.[426]

- Of special interest to the proposed finitistic-algebraic 'resolution' of the inner Schwarzschild singularity here, is the inverse system $\overleftarrow{\mathcal{E}}$ at level 5 in (89) above. *The projective limit of this system recovers the classical continuum vacuum Einstein equations over the whole (ie, over all the points of) X* (29). In particular, we wish to emphasize that

  > *the (vacuum) Einstein equations hold over the, offensive (ie, genuinely singular) from the CDG-theoretic vantage, point-mass m in the interior of X, and in no sense*[427] *do they appear to break down there.*

  In this sense we say that the interior Schwarzschild singularity has been 'resolved' by finitary-algebraic ADG-theoretic means.

Below, we wish to make some further points in order to qualify more this evasion:

- First, as noted in section 3.3, since in the ADG-theoretic perspective on GR it is the algebraic **A**-connection $\mathcal{D}$ and not the smooth metric g (as in the original formulation of the theory) that is the sole, fundamental (dynamical) variable, and since moreover ADG is genuinely smooth background manifold independent, the usual conception of the inner Schwarzschild singularity as a DGS (with all the classical Analytic baggage that the latter notion carries, such as analytic inextensibility, incompleteness, topological boundary constructions to place that singularity *etc*) is not valid in our theoresis, since *neither the metric nor the C^k-extensibility*[428] *of the manifold supporting it are relevant, let alone important, issues in the theory.*

- Related to the point above is the fact that in ADG we replace the usual CDG-based GR conception of a genuinely non-singular spacetime '*the solution metric holds (ie, it is non-singular) in the entire manifold X*' by the expression that '*the field law (ie, the differential equation of Einstein that $\mathcal{D}$ defines via its curvature $\mathcal{R}$) is valid throughout all the field's carrier (sheaf) space $\mathcal{E}$ (over the base topological space(time) X), which in turn can possibly*

[426] Or what amounts to the same, at the limit of infinite (topological) $\mathcal{U}_i$-refinement [355] (or even, at the limit of infinite sheaf-theoretic localization of qausets—inhabiting the stalks of the respective finsheaves at the finitary level—over X's points).

[427] At least in the *differential geometric* sense of the DGSs in which we are especially interested in the present paper.

[428] For any order of differentiability $k = 0, 1, \ldots \infty, \omega$.

host sings'. *Alias*, there is no breakdown whatsoever of 'differentiability', that is, of the differential equation that $\mathcal{D}$ defines, in our scheme. The field $(\mathcal{D}, \mathcal{E})$, and the dynamical differential equations that it defines via its curvature, $\mathcal{R}(\mathcal{D})(\mathcal{E}) = 0$[429] is not impeded at all by any sings that the background topological space X might possess.

- One should note that the particular finitary-algebraic inner Schwarzschild singularity 'resolution' presented above is closely akin to (or one might even say that it 'follows suit' from) the topological resolution of X (into its points) *à la* Sorkin [355], in the following sense: as the ur-cell $\Lambda(m)|_{\mathcal{U}_i}$ 'smearing' the classically offensive point $m \in X$ becomes 'smaller' and 'smaller'[430] with topological $\mathcal{U}_i$-refinement, the law of gravity holds as close to the point-singularity m as one wishes to get (*ie*, at every level 'i' of resolution or refinement of X by the open coverings $\mathcal{U}_i$); furthermore, at the (projective) limit of infinite topological resolution (refinement) of X into its points,[431] one gets that (29) actually holds on (over) m itself.

- In connection with the last remarks, it is also worth pointing out that the law of vacuum Einstein gravity holds *both* at the 'discrete', fcq-level of the P_is $(\forall i)$ *and* at the classical level limit corresponding to X,[432] which further supports our *motto* that *the ADG-picture of (vacuum) GR, and the fcq-version of it, is genuinely background independent*—*ie*, whether that background is a continuum or a discretum. *In toto*, this emphasizes that our ADG-perspective on classical or 'quantal' gravity is manifestly (base) spacetime free [270, 271, 272, ?].[433] Furthermore, concerning the CDG-problem of the inner Schwarzschild singularity and the usual divergence of the gravitational field strength ($\mathcal{R}$) in its immediate vicinity, this freedom may be interpreted as follows: the vacuum Einstein equations hold *both* when a (locally) finite and an uncountable continuous infinity of 'degrees of freedom' of the gravitational field are excited;[434] moreover, unlike the CDG-based picture of inner singularity, no infinity at all (in

[429] Read: '*the curvature (gravitational field strength) $\mathcal{R}$ of the gravitational connection field $\mathcal{D}$ on the carrier, representation (associated) sheaf space $\mathcal{E}$ (over X) vanishes identically (over the whole base topological space X)*'.

[430] Equivalently, the topology τ_i generated by the open sets in the $\mathcal{U}_i$s becomes finer and finer.

[431] Which, as noted earlier, in the spirit of point-set topology are supposed to be the carriers of the $\mathcal{C}^0$-topology of the continuum X [355].

[432] Indeed, as noted earlier, this inverse limit procedure may be interpreted as the classical correspondence limit *à la* Bohr effecting the 'transition' from the fcq-level of the $\Omega_i(P_i)$s (and their finsheaves $\mathbf{\Omega}_i$), to the classical one of the continuum X [318, 319, 270, 271, 272].

[433] As alluded to towards the end of section 3, the ADG-theoretic formulation of GR views gravity as a *pure gauge field theory* (or equivalently, as a '*synvariant gauge theory of the third kind*')—one with no allusion to (dependence on) an external continuous spacetime manifold (continuum) or even a 'discrete $\overset{gauge_i}{\sim}$-orbifold' ('discontinuum'). The reader should wait for the next section where numerous important implications that this 'gauge theory of the third kind' might have for QG research are presented.

[434] As it were, when the gravitational field 'occupies' and effectuates a finite and an infinite number of point-events

the analytical sense of CDG)[435] for $\mathcal{R}$ appears as m is 'approached' (in the categorical limit sense of $\infty \leftarrow i$) by $\mathcal{R}_i(\mathcal{D}_i)$ upon (topological) refinement (of $\Lambda(m)|_{\mathcal{U}_i}$). Plainly, there is no unphysical infinity associated with this ADG-picture of the interior Schwarzschild singularity, and in this sense the latter is genuinely 'resolved' (as it were, into locally finite 'effects').

- Of course, all this can be attributed to the fact that the base topological space(time) X (whether a continuum or a 'discretum') plays no role whatsoever in the inherently algebraic differential geometric mechanism of ADG, which, as noted earlier, derives from the algebra inhabited stalks of the (fin)sheaves involved and not from the base space. Technically speaking, this is reflected by the fact that the categorical in nature ADG-formulation of the relevant differential equations (here, the Einstein equations) involves (equations between) *sheaf morphisms*[436] which by definition 'see through' the generic base topological space X, which in turn serves only as a surrogate scaffolding, without any physical significance,[437] used only for the localization of the algebraic objects in the relevant (fin)sheaves.

- Even more important than the remarks about the physical insignificance of the base space X, but closely related to them, is the issue of the **A**-functoriality of dynamics already alluded to in section 3. Namely, the fact that the vacuum Einstein equations (29) are (local) expressions of the curvature $\mathcal{R}$ of the gravitational connection $\mathcal{D}$, which curvature is an **A**-morphism (or **A**-tensor)—a 'geometrical object' (an $\otimes_{\mathbf{A}}$-tensor) in ADG jargon [272], means that our generalized coordinates (or 'measurements') in the structure sheaf **A** (that *we* assume to coordinatize and geometrically represent the gravitational field $\mathcal{D}$, and solder it on $\mathcal{E}$, which is anyway locally $\mathbf{A}^n$) respect the gravitational field (strength); or equivalently, it indicates that the field dynamics 'sees through' our (local) measurements in $\mathbf{A}(U)$. As all the singularities are inherent in **A**—the structure sheaf of generalized algebras of 'differentiable' coordinate functions,[438] it follows that the **A**-functorial field dynamics 'sees through' the singularities built into **A**,[439] or equivalently, but in a more philological sense, **A** (and the singularities that

in the background space(time) X.

[435] For example, when m is relegated to X's boundary ∂X and a suitable topology is given to $\overline{X} = X \cup \partial X$, as $\mathcal{R} \longrightarrow m$, $\mathcal{R}$ diverges (as $1/r^6$).

[436] And in particular, **A**-morphisms such as $\mathcal{R}$.

[437] As it plays no role whatsoever in the gravitational dynamics—the (vacuum) Einstein differential equations (29).

[438] As noted earlier epithet 'generalized' pertains to the fact that in ADG one is free to use for structure sheaf algebras different from the classical one $\mathcal{C}^\infty(M)$ of smooth functions on a differential manifold. However, as it is the case in the classical case too ($\mathbf{A} \equiv \mathcal{C}^\infty_M$), it is the structure sheaf that carries the singularities—*ie*, the singularities are singularities of certain 'differentiable' functions in **A**, and they are 'geometrically' localized (situated) on some *locus* in the base space X [265, 267]. In turn, ideally, X itself derives from **A** (Gel'fand duality and spectral theory) [254, 258, 259, 266].

[439] A **A** that, we emphasize it again, *we* assume anyway to coordinatize the gravitational field $\mathcal{D}$ and localize it on

it carries) is 'transparent' to the $\mathcal{R}(\mathcal{D})$ engaging into the gravitational field dynamics—the differential equations of Einstein (29). *In summa*, the field $(\mathcal{E}, \mathcal{D})$ (and the differential equation that it defines via its curvature) does not stumble on or break down at any singularity inherent in $\mathbf{A}$ since it 'passes through' them. In this sense, the term 'singularity-resolution' is not a very accurate name to describe how ADG evades singularities; perhaps a better term is 'dissolution' or 'absorption' in $\mathbf{A}$.

A good example of the aforesaid singularity 'dissolution' in or 'absorption' by $\mathbf{A}$ is the ADG-theoretic evasion of the inner Schwarzschild singularity regarded as a *time-extended* or '*time-smeared*' ('temporally distributional') *spacetime foam dense singularity* in the sense of Mallios and Rosinger [273, 274, 262, 275]. In the next sub-subsection we present this distributional 'dissolution'.

6.1.4 The Schwarzschild singularity regarded as an extended spacetime foam dense singularity: a 'dynamic' ('temporal') time-line resolution

Alternatively to the 'static', 'spatial', point-resolution of the inner Schwarzschild singularity above, which was situated right at the *locus* of the point-mass ('particle') source of the (otherwise externally vacuum) spherically symmetric gravitational field in X, as noted before we can also regard the Schwarzschild region $X_S = X - L_t$ in (81) as an 'effective' regular spacetime, where L_t is the 'wristwatch' time-line ($\mathbb{R}$) of the particle at which the gravitational field is normally considered to be singular [141]. Of course, by excluding *by hand* L_t from X, one unfortunately appears to violate by fiat Einstein's 'disbelief' in (Q2.1) earlier that the gravitational field (*ie*, the equations that it obeys) do not hold on the line L_t, and also one appears to go against the grain of Hawking's negative remarks about singularity-exclusion by surgical excision or simply by elimination (omission) in (Q?.?). Thus, in this subsection we will argue, based on ADG-grounds, that the (vacuum) Einstein equations actually hold in full force over L_t (*ie*, over all X!) when that Euclidean time-axis is thought of as being occupied by space*time* foam dense singularities in the sense of Mallios-Rosinger [274, 262, 275] and 4.2 above. In fact, as promised in the previous section, we go a bit further and combine the methods of Mallios-Rosinger in [274, 275] and Sorkin in [355] effectively by bringing together the notions of *singularity-refinement* and *topological-refinement*. All in all, we shall regard the inner Schwarzschild singularity as a distributional one in the sense of SFS, and 'resolve' it in two different ways—one 'direct', the other 'indirect'—similarly to what we just did for the 'static', point-resolution in 5.1.3:

1. First, we can directly view L_t as being inhabited by spacetime foam dense singularities in the manner of [274, 275] and then straightforwardly borrow from [262] its main result, namely,

$\mathcal{E}$, which by definition is locally of the form $\mathbf{A}^n$.

that the vacuum Einstein equations hold over all L_t (hence, over all X!) when sheaves of Rosinger's differential algebras of generalized functions are used as structure sheaves of generalized arithmetics (thus defining *spacetime foam differential triads*) in the manner of ADG; and,

2. Second, we let the densely singular $L_t \simeq \mathbb{R}$ like in 1 above be covered by suitably defined locally finite '*dense singularity open coverings*',[440] thus combining the Sorkin and the Rosinger schemes. We then naturally and constructively arrive at *spacetime foam fintriads* by Papatriantafillou's push-out results, and have for each such dense singularity-cover[441] the vacuum Einstein equations holding *à la* [262, 272] on the respective spacetime foam fintriad. Finally, again via Papatriantafillou's inverse limit results, we pass to the 'continuum' projective limit of maximum (infinite) *topological-cum-singularity refinement* to show that the vacuum Einstein equations hold over the whole (space)time—and in particular, over the densely singular $X \subset L_t$), like in 1 above.[442]

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Direct distributional 'resolution' without topological discretization. The direct evasion of the inner Schwarzschild singularity in a distributional way begins by realizing that the singular (locally) Euclidean 'wristwatch time-axis' $L_t \simeq |_{\text{loc.}}\mathbb{R}$ above may as well be thought of as being occupied by spacetime dense singularities in the sense of [274, 275] as also exposed in section 4 earlier. This effectively means that instead of assuming $\mathbf{A} \equiv \mathcal{C}^\infty_{X \equiv L_t}$ for structure sheaf in the theory (as would befit the classical theoresis of the wristwatch time-line of the point-particle as being a smooth or analytic manifold [141]), we let the fine and flabby sheaf $\mathfrak{B}_{L,\mathcal{S},X}$ of Rosinger's differential algebras of generalized functions (non-linear distributions) that we talked about in section 4 to take its place. We then recall straight from [274, 275] that the $\mathbf{K}$-algebraized space $(X, \mathbf{A} \equiv \mathfrak{B}_{L,\mathcal{S},X})$ supports *à la* ADG a differential triad—the so-called *(s)pace-(t)ime (f)oam differential triad*

[440] Let us note here that it is perhaps more sensible to consider a bounded region X of the 'wristwatch' (proper) time-axis L_t of the particle as befits a physically realistic assumption about an actual *physical* particle having a finite lifetime. This assumption is also suitable for applying Sorkin's ideas from [355] as he too considered a bounded region of a $\mathcal{C}^0$-spacetime manifold, as we saw earlier in section 3.

[441] With those coverings assumed of course to comprise an inverse or projective system as in [355].

[442] To be precise, as we will see below in accordance with the remarks in footnote 150 about Hausdorff reflection, the vacuum Einstein equations hold over the 'large' projective limit non-Hausdorff space (*ie*, formally, the poset P_∞ corresponding to the maximally refined open cover $\mathcal{U}_\infty$ in Sorkin's scheme). Then, plainly, the vacuum Einstein equations also hold over the densely singular $X \subset P_\infty$, which as noted in footnote 150 can be recovered by Hausdorff reflection [234, 355] as a *dense subset* of (closed points of) P_∞, with each point in turn hosting a singularity for some generalized function in Rosinger's spacetime foam algebra.

$$\mathfrak{T}_{stf} := (X, \mathfrak{B}_{L,\mathcal{S},X}, \partial) \tag{90}$$

with ∂ the usual flat connection we defined abstractly in (10) having now a concrete realization as the following $\mathbf{K}$-linear sheaf morphism

$$\mathfrak{B}_{L,\mathcal{S}}(U) \xrightarrow{\partial} \mathbf{\Omega}^1(U) \tag{91}$$

for a suitably defined $\mathfrak{B}_{L,\mathcal{S}}(U)$-module sheaf $\mathbf{\Omega}^1(U)$, regarded along the basic ADG-lines as a free $\mathfrak{B}_{L,\mathcal{S}}(U)$-module of finite rank n freely generated by the usual coordinate differentials dx_1, $dx_2 \dots dx_n$. Point-wise in $\mathfrak{B}_{L,\mathcal{S}}(U)$, that is, for one of its local sections $\mathcal{F}$ (:a generalized function)

$$\mathcal{F} \mapsto \sum_{i=1}^{n} (\partial_i \mathcal{F}) dx_i \tag{92}$$

with $\partial_i \equiv \frac{\partial}{\partial x_i}$, as usual.

Then, again borrowing directly the result from [262],[443] the vacuum Einstein equations hold on $\mathfrak{T}_{stf}$; write

$$\mathcal{R}(\mathcal{E}_{stf}) = 0^{444} \tag{93}$$

and, in particular, over the densely singular L_t. Thus, once again the inner Schwarzschild singularity cannot be thought of either as a DGS, or even in this example, as a SFS [86]—both kinds of singularities meant here in an abstract, generalized sense, since no base manifold is involved in ADG,[445] while at the same time $\mathcal{C}^\infty(M)$ is already included in Rosinger's spacetime foam algebras of generalized functions (60).

Distributional 'resolution' via topological discretization and refinement. As noted above, an alternative distributional 'resolution' of the interior Schwarzschild singularity thought of as extending along L_t proceeds by combining Sorkin's topological with Rosinger's singularity-refinement ideas in a constructive sort of way (making explicit use also of Papatriantafillou's push-out and inverse limits of differential triads results).

First, like we did above, we assume that the bounded region X in L_t is densely packed with singularities, having sheaf-theoretically localized on it Rosinger's differential algebras of generalized

[443] Expression (5.7) there.

[444] The 'stf' subscript of $\mathcal{E}$ indicating that the relevant vector sheaf carrying (representing) the gravitational connection field $\mathcal{D}$ in this example is locally a finite power of Rosinger's structure sheaf of spacetime foam algebras: $\mathcal{E}_{stf}|_U = (\mathfrak{B}_{L,\mathcal{S}}(U))^n$ ($U \subset X$ open).

[445] That is, there is no issue here of smooth or analytic extension of a background spacetime manifold as in the CDG-based 'definition' of DGSs and SFSs that we saw in section 2 [86, 87].

functions ($\mathbf{A} \equiv \mathfrak{B}_{L,\mathcal{S},X}$). Then we evoke Sorkin's locally finite open covers of X, *a fortiori* assuming that the open sets that comprise them are also densely singular in Rosinger's sense.[446] This means that the open Us in Sorkin's $\mathcal{U}_i$s are viewed here as being '*dense singularity covering sets*'. Then we observe that topological refinement *à la* Sorkin 'induces' singularity refinement *à la* Rosinger, which in turn effectively means that we have identified the inverse systems (nets) $\overleftarrow{\mathcal{P}}$ (Sorkin)[447] and $\overleftarrow{\mathcal{N}}$ (Rosinger)[448] that we encountered in sections 3 and 4, respectively.[449]

Now, plainly, for every $\overset{\mathcal{U}_i}{\sim}$-partition (quotienting) of X, by Papatriantafillou's push-out (of differential triads on moduli spaces of arbitrary equivalence relations) results, we obtain a finitary version of the spacetime foam differential $\mathfrak{T}_{stf}$ in (90) above

$$\mathfrak{T}_i^{stf} := (P_i, \mathfrak{B}_{L,\mathcal{S}}(\Lambda(x)|_{\mathcal{U}_i}), \partial)^{450} \tag{94}$$

on which, of course, a finitary version of the vacuum Einstein equations in (93) holds

$$\mathcal{R}(\mathcal{E}_i^{stf}) = 0 \tag{95}$$

Finally, and quite interestingly, the inverse limit of the projective system of finitary spacetime foam differential triads (94) and, *in extenso*, of the vacuum Einstein equations that hold on them (95), is expected to yield (via Ppatriantafillou's inverse limit results) at maximum topological-*cum*-singularity refinement (of the covering dense singularity open sets) (90) and (93), respectively. Of course, like in Sorkin's case, this is true modulo Hausdorff reflection [234, 355], which here however enjoys a sound interpretation: much in the same way that in Sorkin's case the original topological manifold X is recovered as a *dense* subset of the inverse limit space obtained from $\overleftarrow{\mathcal{P}}$ upon infinite topological refinement, so here the *densely singular* $X \subset L_t$, together with the $\mathfrak{T}_{stf}$ (90) and the vacuum Einstein gravity (93) that it supports, are obtained at the said projective limit (of infinite singularity refinement). Evidently, in this distributional 'resolution' of the inner Schwarzschild singularity a plethora of results from Sorkin, Papatriantafillou and Rosinger were fruitfully combined under the roof of ADG.

[446] Refer again to section 4.2.
[447] This is Sorkin's '*topological refinement net*'.
[448] This is Rosinger's '*singularity refinement net*'.
[449] This rests on the fact about spacetime foam dense singularities mentioned in 4.2, namely, that singularity refinement $\prec_{sref}$ (and its associated notion of quotient regularity) of spacetime foam algebras of generalized functions is '$\prec_{tref}$-covariant'; where '$\prec_{tref}$' is the relation of topological refinement *à la* Sorkin in (78).
[450] For typographical clarity we have kept the finitarity index 'i' as a subscript as usual [270, 271, 272, 317], and have promoted the 'stf' acronym to a superscript.

6.1.5 The matter of the fact: the inclusion of matter fields and gauge interactions into the vacuum Einstein equations

Unitary fields with strong quantum undertones. So far in this paper, and in the past trilogy [270, 271, 272], we have been talking in the light of ADG about classical and potentially quantum aspects of *vacuum* Einstein gravity—the dynamics of the 'pure', 'free' gravitational field in empty space(time).[451] In the following couple of paragraphs we would like to entertain for a little while the possible inclusion of matter and, concomitantly, gauge fields, as well as their dynamical interactions, into our ADG-perspective on gravity. This inclusion, *prima facie* irrespective of classical and quantum distinctions, seems to be mandated by the fact that *gravity is a universal force* originating from and coupling (applying) to all mass-energy-momentum manifestations of matter in the world as well as interacting with the matter's various gauge radiation fields, let alone of course that because of its non-linearity, even in *vacuo* (*ie*, in the absence of matter), gravity is also self-coupling.

Traditionally, the inclusion of matter (source) fields into gravity is brought about by placing a non-vanishing stress-energy tensor $T_{\mu\nu}$ on the right hand side of Einstein's equations, while the incorporation of gauge (radiation) fields emitted from the said sources is usually represented by a so-called minimal coupling prescription—*ie*, basically by augmenting the gravitational Christoffel connection with the corresponding gauge potentials.[452]

Heuristically and tentatively speaking,[453] the ADG-theoretic 'recipe' for including matter and gauge (inter)actions into gravity is analogous to the standard one, but at the same time quite different in basic concepts, technique and theoretical scope, as well as in consequences and implications. To begin with, in keeping with the basic field-axiomatics of ADG, as noted earlier fields are (*by definition*) pairs $(\mathcal{E}, \mathcal{D})$, with $\mathcal{D}$ a sheaf morphism defining the dynamical (differential) equation of motion of the (free) field,[454] while $\mathcal{E}$ providing a local particle-states' representation of the field.[455] Then, as in the case of the 'free' (vacuum) gravitational Einstein equations, the dynamical

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[451]In fact, in our ADG-theoresis here, *in the manifest absence of any 'ambient', background spacetime altogether.*

[452]Accordingly, at the Lagrangian (action) level, by adding to the Einstein-Hilbert functional suitable expressions involving the curvatures (gauge field strengths) of those potentials (plus possible interaction cross-terms).

[453]For a more definitive and comprehensive ADG-theoretic treatment of gauge and matter fields (possibly adjoined to gravity), the reader is advised to wait for a forthcoming book [269].

[454]For example, for a (free or 'bare') fermion (*eg*, electron or quark) source field, $\mathcal{D}$ may be identified with (a suitable ADG-version of) the Dirac differential operator (*viz.* connection), while for a bosonic radiation field (*eg*, photon/Maxwell field or a general non-abelian Yang-Mills field), $\mathcal{D}$ may be identified with the usual gauge connections.

[455]From the ADG-theoretic perspective on second and geometric prequantization noted earlier [336, 259, 260, 261, 263, 271, 272, 269], (local) particle (representation) states of *bosonic fields* are identified with (local) sections of (associated) *line sheaves* (*ie*, vector sheaves of rank 1), while the (local) particle states of *fermion fields* are represented by (local) sections of *vector sheaves* of minimum rank 2.

(differential) equations for free fermionic matter (Dirac-like equations), or for their corresponding free bosonic gauge fields (Maxwell's equations, Yang-Mills equations), are again *equations between sheaf morphisms*. In addition, *vis-à-vis* the well known problem (of the Minkowski manifold based classical and quantum field theories of matter) of regarding those matter-sources (*ie*, the particles or quanta of matter) as singularities in their respective radiation gauge fields, the $\mathbf{A}$-functoriality of the said dynamical equations again enables one, like in the case of vacuum Einstein gravity that we saw before, to absorb whatever singularity is involved into the judiciously chosen structure sheaf $\mathbf{A}$ of generalized arithmetics without perturbing the law itself (*viz.* the differential equation defined by the corresponding connection field $\mathcal{D}$) the slightest bit [259, 260, 262, 272, 269].[456]

Thus, given the $(\mathcal{E}_G, \mathcal{D}_G)$ representation of the vacuum gravitational field satisfying (29),[457] the basic idea is to 'adjoin' or 'couple' to it another (g)auge or (m)atter field $(\mathcal{E}_{g,m}, \mathcal{D}_{g,m})$—as it were, to combine the two generally different kinds of fields into a joint, 'integral', '*unitary field*'[458] $\mathcal{D}' = \mathcal{D}_{\mathcal{E}_G \otimes_{\mathbf{A}} \mathcal{E}_{g,m}}$ acting on the *tensor product vector sheaf* $\mathcal{H}om_{\mathbf{A}}(\mathcal{E}_G, \mathcal{E}_{g,m}) = \mathcal{E}_G \otimes_{\mathbf{A}} \mathcal{E}_{g,m}$, which $\mathcal{D}'$ is in a sense 'induced' by the individual (free or just self-interacting)[459] $\mathbf{A}$-connections $\mathcal{D}_G$ and $\mathcal{D}_{g,m}$ on $\mathcal{E}_G$ and $\mathcal{E}_{g,m}$, respectively.

We would like to slightly digress here and make a couple of noteworthy observations about this gravity-*cum*-gauge/matter $\otimes$-'entanglement':

- First, regarding the base topological spaces X involved: it is tacitly assumed that the base topological localization-spaces (say, X_G and $X_{g,m}$) for the individual (free) fields *combine by identification*—*ie*, they are identified to one and the same base localization space $X_G \equiv X_{g,m} = X$ on which the combined tensor product vector sheaf $\mathcal{E}_G \otimes_{\mathbf{A}} \mathcal{E}_{g,m}$ is then taken to be soldered. That is, we assume that the two factors in the $\otimes$-combined system 'gravitational-*cum*-gauge/matter field' have a common external localization 'parameter space' (*ie*, their tensor-entanglement $\mathcal{D}'$ 'sees' a single, common base topological space X).

- Second, regarding the coordinate structure sheaves $\mathbf{A}$ engaged in the said combination: in principle, one can allow for the employment of different structure sheaves of generalized arithmetics to coordinatize the gravitational and the gauge/matter fields individually—*ie*, $\mathcal{E}_G :\overset{\text{loc.}}{\simeq} \mathbf{A}_G^n$, $\mathcal{E}_{g,m} :\overset{\text{loc.}}{\simeq} \mathbf{A}_{g,m}^l$, possibly with $\mathbf{A}_G \neq \mathbf{A}_{g,m}$.[460]

[456] Accordingly, like in the case of vacuum gravity, the 'self-invariances' ('synvariances') or 'auto-symmetries' of those bare field laws are organized into the principal sheaves $\mathcal{A}ut\mathcal{E}$. Thus, these field laws too are synvariant ('self-covariant'), without reference to a background spacetime (manifold) [272].

[457] The capital-'g' ('G'') subscript indicating *gravitational* field ($\mathcal{D}_G$) and its local particle-representation (associated) sheaf ($\mathcal{E}_G$).

[458] The word 'unitary' here is meant in the sense of Einstein (see below).

[459] Like gravity, non-abelian gauge (Yang-Mills) fields are non-linear and hence self-coupling.

[460] This is in keeping with the pragmatic or 'ophelimistic' aspect of ADG noted earlier in the context of the PARD,

- And third, regarding the connection fields $\mathcal{D}$ themselves: as noted above, the individual connections combine to one—the induced '$\otimes_{\mathbf{A}}$-*tensor product connection*' $\mathcal{D}^{'} = \mathcal{D}_{\mathcal{E}_G \otimes_{\mathbf{A}} \mathcal{E}_{g,m}}$ on $\mathcal{E}_G \otimes_{\mathbf{A}} \mathcal{E}_{g,m}$. We furthermore assume that, locally (*ie*, with respect to a local gauge $U \subset X$), $\mathcal{D}^{'}$ resolves into an analogue of the minimal coupling expression of the usual theory

$$\mathcal{D}^{'}|_{U \subset X} = \partial + \mathcal{A}_G(\equiv \Gamma^k_{ij}) + \mathcal{A}_g \tag{96}$$

with ∂—the flat, 'inertial' derivation (connection)—common to all three kinds of evolution: gravitational, gauge and matter.[461]

- As a 'bonus' remark, we note that the three observations above recall the way we actually combine individual (particle) systems' states in conventional mechanics (quantum or non-quantum), whereby, while we combine (normally in quantum theory, Hilbert space) states by tensor multiplication (*ie*, in quantum theory for example, $\psi_1 \in \mathcal{H}_1,\ \psi_2 \in \mathcal{H}_2 \longrightarrow \psi_1 \otimes \psi_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$), *we retain different spatial labels* (position coordinates), but at the same time *we identify their (external) temporal evolution parameters* (time coordinates).[462] *In summa*,

$$\psi_1(x_1, t_1) \in \mathcal{H}_1,\ \psi_2(x_2, t_2) \in \mathcal{H}_2 \longrightarrow \psi_1 \otimes \psi_2(x_1, x_2, t) \in \mathcal{H}_1 \otimes \mathcal{H}_2 \tag{97}$$

while, moreover, in relativistic quantum mechanics (QFT), when we combine quantum field systems (living in individual tensor product Hilbert=Fock spaces), we even merge the external *space*time coordinates (*ie*, we identify even their spatial coordinates), and suppose that they live in a joint tensor product Hilbert (=Fock)[463] space (localized and fibered, as a bundle,[464] over Minkowski space $\mathcal{M}$)

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namely, that one can choose freely how to 'measure' (and localize or 'geometrically represent') the gravitational and the matter/gauge fields, possibly by using different structure sheaves $\mathbf{A}$ of generalized arithmetics (coordinates) for each one. With respect to the ADG singularity-evasion in particular, one may choose for instance different structure sheaves of arithmetics—ones that accommodate different kinds of sings—to cast the Einstein equations for $\mathcal{D}_G$ and (the Yang-Mills equations for) $\mathcal{D}_g$ (Yang-Mills fields). Such a flexibility is prominently absent in the manifold based CDG treatment of both gravity and Yang-Mills theories, whereby a common structure sheaf $\mathbf{A} \equiv \mathcal{C}^{\infty}_X$ is inevitably employed (CDG-conservatism) so that all sings are branded as being $\mathcal{C}^{\infty}$-smooth in one way or another (and one is implicitly working within the category $\mathcal{M}an$ of differential manifolds).

[461] This identification of the ∂ is similar to the common 'external parameter localization/gauging topological space X' noted above.

[462] Pun intended.

[463] Strictly speaking, a Fock space is a direct sum of tensor-multiplied Hilbert spaces.

[464] Or perhaps even better, as a *sheaf*(!)—the associated, representation Fock sheaf of nets (or better, sheaves) of algebras of quantum field operators [176].

$$\phi_1(x_1,t_1),\ \phi_2(x_2,t_2) \longrightarrow [\phi_1 \otimes \phi_2](\mathbf{x}) \in \mathcal{F}|_{\mathcal{M}} = [\mathcal{F}_1 \otimes \mathcal{F}_2]|_{\mathcal{M}},\ (\mathbf{x} \text{ a point in } \mathcal{M}) \qquad (98)$$

Finkelstein has described this 'identification phenomenon' in detail in [148]*[465] when he talks about a similar identification of the imaginary unit ($i^2 = -1$ upon combination of systems in physical theory, quantum or not:[466]

[465] See footnote following the citation [148] for an explanation of the asterisk.

[466] By the way, the discussion in the following quotation arises when Finkelstein remarks that real ($\mathbb{R}$) quantum mechanics (*ie*, quantum mechanics with real coefficients/amplitudes) should be regarded as being more fundamental than the complex ($\mathbb{C}$) one we use today. In what follows, emphasis is ours. The first paragraph is really parenthetical and is included only for continuity of the text. The second and third paragraphs—the emphasized ones—are important for our discussion here.

(Q6.5)

> "...We expect that in nature the $\mathbb{R}$ theory underlies the $\mathbb{C}$. The operator i is a central or superselection operator from the $\mathbb{R}$ point of view, and we are familiar with the emergence of many such superselection operators when a microscopic quantum theory condenses into a non-quantum macroscopic one, as a result of random phases. Since i transforms into $-i$ under time reversal T, we expect that the superselection operator i arises when an underlying quantum spacetime structure condenses into the macroscopic non-quantum t variable of elementary quantum mechanics and field theory.
> *Imaginary units do not combine like symmetries under tensor multiplication, however. If we look at two input spaces V_1 and V_2, in which some group symmetry g is represented by two operators g_1 and g_2, then in the tensor product $V_1 \otimes V_2$ the same symmetry g acts as $g_1 \otimes g_2$. We say that under tensor multiplication, symmetries multiply. It follows that under tensor multiplication, infinitesimal symmetries add, unit factors* **1** *understood. However, if i is represented by i_n in V_n $(n = 1, 2)$, then to to multiply a product vector $\psi_1 \otimes \psi_2$ by i, it suffices to multiply either ψ_1 by i_1 or ψ_2 by i_2. Indeed, we have the identification $i = i_1 = i_2$. Under tensor multiplication, the imaginary units i of separate systems combine not by multiplication or addition, but by identification.*
> *They are not unique in this respect. When we combine systems in non-quantum or quantum mechanics, the times t also combine by identification, and in field theories the spacetime coordinates do. In general, when we combine systems we tensor multiply symmetry transformations but identify their parameters, which in a sense are dual. We understand this uniformly by regarding the entries being combined not as independent factors in a tensor product, but as subsystems of one embracing system, with unique operators for time t and imaginary i, among other group parameters...*"

And it must be further emphasized in view of the quotation above that:

1. In ADG too, the dynamical 'auto-symmetries' of the separate fields combine by tensor multiplication in the joint, 'unitary' gravity-*cum*-matter-*cum*-gauge field[467]

[467]Here, the epithet '*unitary*' pertains to what Einstein also referred to as the '*total*' field (see quotations below). To avoid any confusion with nomenclature used in previous work, in [272] the adjective 'unitary' for a (connection) field meant what we here call 'third gauged', namely, 'autonomous', 'external (base) spacetime (continuum) independent'. Below, the adjunction of the adverb 'genuinely' to 'unitary' is meant to capture precisely the said 'autonomous',

$$\mathcal{A}ut\mathcal{E}_G, \mathcal{A}ut\mathcal{E}_{g,m} \longrightarrow \mathcal{A}ut_{\mathcal{E}_G\otimes_{\mathbf{A}}\mathcal{E}_{g,m}} = \mathcal{A}ut\mathcal{E}_G \otimes_{\mathbf{A}} \mathcal{A}ut\mathcal{E}_{g,m} \tag{99}$$

[468]

while locally, in the same way that the fields' connections 'decouple', as a sum, into the the 'minimal coupling' expression (96), their 'self-transmutations' (auto-symmetries) too add (*eg*, they split into a direct sum of local 'auto-symmetries'). We write formally:

$$\begin{gathered}\mathcal{A}ut\mathcal{E}_G|_U \equiv \mathcal{GL}(n,\mathbf{A})|_U := gl(n,\mathbf{A});\ \mathcal{A}ut\mathcal{E}_{g,m}|_U \equiv \mathcal{GL}(l,\mathbf{A})|_U := gl(l,\mathbf{A}) \longrightarrow \\ [\mathcal{GL}(n,\mathbf{A})\otimes\mathcal{GL}(l,\mathbf{A})]_U = gl(n,\mathbf{A})\oplus gl(l,\mathbf{A})\end{gathered} \tag{100}$$

2. Perhaps more important however is to observe, again in view of Finkelstein's remarks above, that the constant sheaf $\mathbf{C}\equiv\mathbf{K}$ of complex ($\mathbb{K}=\mathbb{C}$), or even real ($\mathbb{K}=\mathbb{R}$), scalars—which in ADG is by definition embedded into the (common, in the example above) structure sheaf $\mathbf{A}$—like the common (identified) background parameter localization topological space X above, *does not partake into the field dynamics—ie, it does not participate in the dynamical evolution of the fields which is effectuated by the connections which are constant* $\mathbf{K}$*-morphisms.* In contradistinction to Finkelstein however, we maintain that *exactly because the field dynamics 'sees through' both the constant sheaf* $\mathbf{K}$ *and the background* X *(which in turn both remain fixed), they are 'unobservable'*, for as we said throughout the present paper, *the only 'observables'*[469] *in ADG are the fields (connections) themselves.*[470] This of course frees us from the 'responsibility' to look for a quantum description of spacetime structure in order to account for the identifications of X and $\mathbf{K}$ when we tensor-combine gravitational, gauge and matter fields in ADG.[471] For as stressed before, in ADG, unlike the usual spacetime (continuum or discretum) based physical theories (particle or field; relativistic or not; quantum or not), there is no external (to the fields≡connections) variable spacetime (geometry) as such, and the dynamical symmetries of the fields are their own auto-transmutations in $\mathcal{A}ut\mathcal{E}$,[472] where no external (background) spacetime 'parameters' are actively involved.

'self-sustaining' quality of the fields (combining into a unitary field) in ADG.

[468]The same $\mathbf{A}$ for both gravity and matter/gauge fields is assumed here.

[469]That is, '$\mathbf{A}$-*measurable dynamical quantities*'.

[470]To be more precise, the *curvatures* of the fields, which, unlike the connections, are $\mathbf{A}$-morphisms (equivalently, $\otimes_{\mathbf{A}}$-tensors—*viz.* 'geometrical objects' proper) [272].

[471]As we will argue extensively in section 6, the quantization (or the quantum structure) of (the background) spacetime itself should not be an issue at all in a potential application of ADG-theoretic ideas to QG.

[472]To say it again, effectively these are the transformations of the fields' particles ('local aspects of the fields'), whose states are represented by local sections of $\mathcal{E}$, which in turn is regarded as the associated (representation) sheaf of the principal (group) sheaf $\mathcal{A}ut(\mathcal{E})$.

Matter singularities are incorporated into the 'genuinely unitary field' law. After this small detour discussion about the possibility of tensor-combining matter and gauge fields with gravity (accompanied by some quantum undertones) into a 'unitary' ADG-field, we come to the 'real' reason why, *vis-à-vis* singularities, we wished in the first place to bring up the issue of incorporating matter field actions into the Einstein field equations by ADG-means. It has to do with our contention that ADG may help us complete Einstein's unitary (or unified) field theory programme and moreover possibly marry it with the apparently (for Einstein at least) incompatible (with his spacetime continuum based field theory of gravity) quantum theory.[473]

To begin with, as we contended throughout the past trilogy [270, 271, 272],[474] and as we argue repeatedly in the course of the present work, ADG offers the unique opportunity to develop a genuinely unitary field theory, as Einstein had originally envisioned that a genuine completion of his field theory program should be based on the following three accomplishments:[475]

To begin with, as we contended throughout the past trilogy [270, 271, 272],[476] and as it was mentioned in various places above, ADG offers us a unique opportunity to develop a singularity-free, genuinely unitary field theory—one that is not based at all on an external, background spacetime continuum with its inherent singularities; moreover, one that, due to ADG's purely algebraic character, accommodates quantum ideas from the very start.[477] Basically, Einstein had originally envisioned that a genuine completion of his (albeit, spacetime continuum based) field theory program should hinge on the following three accomplishments:

1. First, to develop a unitary field dynamics with sole dynamical variable the '*total*' (unitary) field itself, which satisfies certain (partial) *differential* equations (:physical laws differential geometrically represented).

2. Second, *the envisioned field dynamics (laws) to be free from singularities.*

3. And third, the material particles ('quanta') of the field to be 'explained away' field-theoretically as '*singularities in the total field*', while at the same time, *their dynamical evolution* (in 'time') *to be represented by the differential equations for the unitary field itself and nothing else, nothing more-nothing less.*

[473]This contention of ours has been partly elaborated in [272], and also speculatively, tentatively outlined in [317]. It will be developed in more detail in the last section.

[474]Especially in the last paper [272].

[475]We will return to this issue, with extensive relevant quotations of Einstein, in 7.5.3.

[476]Especially in the last paper [272].

[477]In this respect, the reader has to wait for our remarks in section 6 about the '*inherently quantum*' (self-quantum), '*third quantized*' character of the (abstract) ADG-fields $(\mathcal{E}, \mathcal{D})$ (gravitational, or Yang-Mills, or other).

In summa, a genuinely unified field dynamics should be one of the unitary field 'in-itself' that is singularity-free in the sense that the particle-singularities of the field(s) should be incorporated (or 'absorbed') in the total field-law itself.

Below we give two Einstein quotations from [128] which corroborate the triptych above: the first expresses clearly his anticipation that a field-theoretic completion of GR to a unitary field theory should result in a singularity-free description even of material point-particles, which act as sources of the various radiation force-fields, but from GR's viewpoint they are genuine or 'true' singularities of the gravitational field.[478] In other words, for Einstein one of the primary motivations for formulating a unitary field theory is overcoming the problem of singularities troubling primarily GR—arguably, the spacetime continuum based relativistic field theory *par excellence*:

(Q6.6) "...The essence of this truly involved situation [*ie*, uniting gravity with the other forces of matter][a] can be visualized as follows: *A single material point at rest will be represented by a gravitational field that is everywhere finite and regular, except where the material point is located: there the field has a singularity*[b]...Now it would of course be possible to object: If singularities are permitted at the locations of the material points, what justification is there for forbidding the occurrence of singularities elsewhere? This objection would be justified if the equations of gravitation were to be considered as equations of the total field. [Since this is not the case], however, one will have to say that the field of a material particle will differ from a pure gravitational field the closer one comes to the location of the particle. *If one had the* [unitary][c] *field equations for the total field, one would be compelled to demand that the particles themselves could be represented as solutions of the complete field equations that are free of irregularities everywhere. Only then would the general theory of relativity be a complete theory.*[d]..."

[a]Our addition for completeness.
[b]Our emphasis. For example, this is the case with the inner Schwarzschild singularity at the fixed point-mass source that we resolved earlier.
[c]Our addition for clarity.
[d]Again, our emphasis.

[478]Like the inner Schwarzschild singularity located right at the point-particle that we saw before.

The second quotation from [128] expresses Einstein's scepticism about quantum mechanics—especially, about the apparent 'pseudo-way' in which quantum theory purports to do away with continuous structures when in fact *it still employs the spacetime continuum in order to formulate the dynamics of quantum wave amplitudes as differential equations proper.* Moreover, in an indirect way, the words below put forward Einstein's vision of a unitary field theory that may on the one hand overcome the problem of singularities (in GR) and on the other account the atomistic (quantum) structure of reality:

(Q6.7)

> "...It is my opinion that the contemporary quantum theory represents an optimal formulation of the relationships, given certain fixed basic concepts, which by and large have been taken from classical mechanics. I believe, however, that this theory offers no useful point of departure for future development. This is the point at which my expectation deviates most widely from that of contemporary physicists. *They are convinced that it is impossible to account for the essential aspects of quantum phenomena (apparently discontinuous and temporally not determined changes of the state of a system, simultaneously corpuscular and undulatory qualities of the elementary carriers of energy) by means of a [field][a] theory that describes the real state of things [objects] by continuous functions of space for which differential equations are valid. They are also of the opinion that in this way one cannot understand the atomic structure of matter and radiation. They rather expect that systems of differential equations, which might be considered for such a theory, in any case would have no solutions that would be regular (free from singularities) everywhere in four-dimensional space[b]...*"
>
> [a]Our addition for clarity.
> [b]Our emphasis

From the quotation above, and by 'negation/exclusion' or 'elimination', one could say that Einstein, in contradistinction to his contemporary quantum physicists:[479]

[479]And, it is fair to say, in contrast also to the majority of current theoretical physicists (quantum theorists, of course, included). Let it be also noted here that for most (if not all) contemporary working physicists, Einstein's unified field theory is regarded as being, for all practical (research) purposes, a 'closed and dead' subject, or at best, a dated one of only historical value (except perhaps, but only in a formal and 'peripheral' sense, for higher-dimensional scenarios such as Kaluza-Klein and string theories, that regard themselves as 'natural' continuations, or as outcomes from 'theory evolution', of Einstein's unified field theory ideas [420]). However, in view of the basic

I. Believed that a singularity-free field theory on the spacetime continuum—whose laws are expressed differential geometrically, *ie*, as differential equations—could still be developed. This essentially implied his 'unitary field theory' vision,[480] albeit, one that still abides by the background spacetime continuum (manifold) in which differential equations can actually be formulated (CDG-theoretically).[481]

II. As also mentioned before, he also believed that such a theory could account for the quantum structure of reality, in the sense that the quantum particles of the source or radiation fields will be described by everywhere (in the spacetime continuum) singularity-free (regular) solutions to the total (unitary) field equations.

III. Finally, he criticized the fact that, in truth, quantum theory, in spite of the apparent discontinuity (discreteness) of quantum processes, still tacitly employs the continuum (as it were, 'in disguise') in the form of 'continuous changes of' (*ie*, again differential equations obeyed by) the probability amplitudes for (states of) quantum systems, with those states (wave functions) explicitly being defined on a spacetime continuum (*eg*, Schrödinger's non-relativistic or Dirac's relativistic, or even the Klein-Gordon, wave equations).

Even more tantalizing are the following words taken from three remarkable consecutive paragraphs in [121][482] which show, in order of appearance, an 'oscillation' or 'undecidability' in Einstein's thought about whether to opt for the classical geometrical spacetime continuum and the continuous field theory based on it, or for an algebraic and 'discontinous' description of reality that quantum mechanics appears to mandate,[483] with the third paragraph, which we emphasize due to

didactics of ADG, we feel compelled to look closer to such a possible 'unitary field theory revival', especially *vis-à-vis* the current 'hot' QG issues and problematics that we will encounter in the next section. We feel that we are not just anachronistically 'digging up graves', while, anyway, in this one we are in good company [371, 373, 370, 369, 372]. Indeed, in the last section we will argue strongly for a possible 'revival' (or even 'fulfilment'!) of Einstein's unitary field theory in the light of ADG.

[480] See above.

[481] And in his commitment to the base spacetime continuum *vis-à-vis* field theory Einstein was quite adamant, in the sense that he did not believe that a field theory (and the differential geometric methods effectuating it) could stand on its own two feet without commitment to such a background, as we will see in the last section. However, since, as we have argued throughout the present work, singularities are inherent in the manifold, Einstein's singularity-free unitary field theory vision—one that is vitally based on CDG for its differential geometric implementation—appears to us to be a rather Quixotic attempt.

[482] Which can be found on pages 92 and 93 in article 13, titled '*Physics and Reality*' (reprinted from the *Journal of the Franklin Institute*, **221**, 313 (1936); see also [270]). The entire third paragraph is written in *emphatic* script because of its relevance to the present work.

[483] To the knowledge of the present authors, perhaps the best reference in which the sceptical, ambivalent, almost 'schizophrenic' attitude of Einstein on the one hand towards continuous field theory and its differential geometric constructions on the geometrical spacetime continuum, and on the other, towards the finitistic-algebraic quantum

its relevance here, showing clearly his 'wishful thinking' about a field theory—a total or unitary gravity-*cum*-matter field theory—that could represent particles (quanta) by singularity-free fields:

theory, is [366] (we would like to thank John Stachel for timely sending us this paper). In the concluding section (section 7), we argue in detail how ADG may serve as an appropriate (mathematical) framework in which to bridge the gap between these two apparently opposite (and irreconcilable to Einstein's mind!) modes of description of physical reality—that is to say, how, with the help of ADG, one can do field theory entirely algebraically (*ie*, with finitistic-quantum methods and traits built into the formalism from the very beginning), without at all the use of a background spacetime continuum.

"*...To be sure, it has been pointed out that the introduction of a space-time continuum may be considered as contrary to nature in view of the molecular structure of everything which happens on a small scale. It is maintained that perhaps the success of the Heisenberg method points to a purely algebraic method of description of nature, that is to the elimination of continuous functions from physics. Then, however, we must also give up, by principle, the space-time continuum. It is not unimaginable that human ingenuity will some day find methods which make it possible to proceed along such a path. At the present time, however, such a program looks like an attempt to breathe in empty space.*[a]

There is no doubt that quantum mechanics has seized hold of a beautiful element of truth, and that it will be a test stone for any future theoretical basis, in that it must be deducible as a limiting case from that basis, just as electrostatics is deducible from the Maxwell equations of the electromagnetic field or as thermodynamics is deducible from classical mechanics. However, I do not believe that quantum mechanics will be the starting point in the search for this basis, just as, vice versa, one could not go from thermodynamics (resp. statistical mechanics) to the foundations of mechanics.

(Q6.8) *In view of this situation, it seems to be entirely justifiable seriously to consider the question as to whether the basis of field physics cannot by any means be put into harmony with the facts of the quantum theory. Is this not the only basis which, consistently with today's possibility of mathematical expression, can be adapted to the requirements of the general theory of relativity? The belief, prevailing among the physicists of today, that such an attempt would be hopeless, may have its root in the unjustifiable idea that such a theory should lead, as a first approximation, to the equations of classical mechanics for the motion of corpuscles, or at least to total differential equations. As a matter of fact up to now we have never succeeded in representing corpuscles theoretically by fields free of singularities, and we can, a priori, say nothing about the behavior of such entities. One thing, however, is certain: if a field theory results in a representation of corpuscles free of singularities, then the behavior of these corpuscles with time is determined solely by the differential equations of the total field.*[b]"

[a]Our emphasis.

[b]Again, emphasis is ours.

The concluding apparently paradoxical '*hypothetical certainty*' of Einstein in the last three lines of the quotation above may be rephrased as follows: *should one be able some day to represent field-theoretically particles ('quanta') in a singularity-free manner, then the particle dynamics—as it were, the evolution of those quanta 'in time'—will be already inherent in (or theoretically speaking, be the result of) the field dynamics itself, and there would be no need to assume a priori particles as fundamental theoretical entities on a par with the field concept.*[484] It is precisely in this sense that field theory—Einstein's unitary field theory—aspired to 'explain away' particles and, accordingly, that "*quantum theory could be deducible from that future (unitary field) theoretical basis*". Indeed, some years earlier, in [121], Einstein, upon concluding the section titled '*The Field Concept*' in article 13, '*Physics and Reality*', reexpresses this 'wishful certainty' in a rather categorematic fashion:[485]

(Q6.9) "*...What appears certain to me, however, is that, in the foundations of any consistent field theory, there shall not be, in addition to the concept of field, any concept concerning particles. The whole theory must be based solely on partial differential equations and their singularity-free solutions.*"

Moreover, in contradistinction to the commonly held idea nowadays that Einstein envisioned a (unified) field-theoretic scenario in which the particle-quanta of matter will be 'explained away' as singularities in the (total/unitary) field,[486] he in fact envisioned an even more 'pure', iconoclastic and particle/singularity-free field theory. An excerpt from Clarke *et al.* [88] quotes characteristically Einstein and Rosen from [131] and further adds (also about the Schwarzschild and the other cosmological singularities in GR):

[484] As we will show and argue explicitly in the next section, ADG puts particles (*viz.* local sections of $\mathcal{E}$) side-by-side the algebraic field $\mathcal{D}$ when it actually *defines* a field to be the pair $(\mathcal{E}, \mathcal{D})$. However, it must be stressed here that, in truth, the field $\mathcal{D}$ is *the* fundamental notion in ADG, with $\mathbf{A}$, together with its inherent singularities, introduced by us 'observers' ('measurers' or 'coordinators') in order to localize, 'geometrize', 'coordinatize' the field, thus extract its local, quantum-particle aspects [271, 272, **?**]. In other words, as we will see in the next section when we expose the quantum aspects of ADG-gravity, $\mathbf{A}$ (and in effect $\mathcal{E}$, which is locally $\mathbf{A}^n$) is introduced by us in order to particle-represent the field ($\mathcal{D}$), but this representation of the fields is manifestly singularity-free, as Einstein in (Q?.?) above would have liked it.

[485] Again, the whole excerpt below is written in *emphatic* script due to its significance in the present work.

[486] See Eddington quotation below.

> "...Einstein was fundamentally unhappy with any sort of discontinuity in general relativity. In a paper with Rosen in 1935, Einstein posed the question: '*Is an atomistic theory of matter and electricity conceivable which, while excluding singularities in the field, makes use of no other field variables than those of the gravitational field and those of the electromagnetic field?*' He noted that: '*...writers have occasionally noted the possibility that material particles might be considered as singularities of the field.*[a] *This point of view we cannot accept at all. For a singularity rings so much arbitrariness into the theory that it actually nullifies its own laws. Every field theory, in our opinion, must therefore adhere to the fundamental principle that singularities of the field are to be*
> (Q6.10) *excluded*'[b] [131]. By 'singularities of the field' Einstein meant not only cosmological singularities but also the Schwarzschild 'singularity'..."
>
> [a]Again, see Eddington quotation (Q6.11) next.
>
> [b]This certainly foreshadows Einstein's later maintaining (right at the end of his life!) that "*...It is my opinion that singularities must be excluded. It does not seem reasonable to me to introduce into a continuum theory points (or lines etc.) for which the field equations do not hold...*" (Q2.1). It also indicates his basic conviction that the 'discontinuous' concept of a particle cannot be 'married' to that of the continuous field on the spacetime continuum—for him an unbridgeable conceptual/theoretical divide on which we are going to comment extensively under the prism of ADG-gravity and ADG-field theory in general in subsection 8.5 in the sequel.

We would like to leave aside for the time being (until we pick the discussion up again in the next two sections *vis-à-vis* current QG issues) the detailed arguments supporting our view that ADG meets all the three requirements set by Einstein above for the development of a genuinely unitary field theory, together with quantum-particle traits and a manifest non-commitment to a background spacetime (whether a continuum or a discretum) built into it from the very start. For now we wish to just quote Eddington from [106] about incorporating 'material particle singularities' in the (gravitational) field law itself and then remark on how *ADG meets exactly his words*:[487]

[487]Initially we thought that Eddington's words below were the first to explicitly pitch the idea that material particles could be viewed as '*singularities in the field*', but recently the first author discovered that Michael Faraday had actually pretty much the same idea significantly earlier, as we quote Hermann Weyl from [**?**]: "...not the field should derive its meaning through its association with matter, but, conversely, *particles of matter are ... singularities of the field*" (our emphasis). This quotation first appeared in the first author's recent paper [268].

(Q6.11) "...It is startling to find that the whole of dynamics of material systems is contained in the law of gravitation; at first gravitation seems scarcely relevant in much of our dynamics. But there is a natural explanation. *A particle of matter is a singularity in the gravitational field*,[a] and its mass is the pole-strength of the singularity; consequently *the laws of motion of the singularities must be contained in the field-equations*,[b] just as those of electromagnetic singularities (electrons) are contained in the electromagnetic field-equations..."

[a]Our emphasis.
[b]Again, our emphasis.

Eddington's words above—especially those emphasized—are tailor-cut for what ADG has accomplished here and in the past *vis-à-vis* gravitational (as well as Maxwellian/abelian and Yang-Mills/non-abelian gauge) singularities [259, 260, 262, 273, 274, 275, 272, 264, 265, 267, 268, 269]. Indeed,

ADG shows that one can actually 'absorb' or 'engulf' singularities of any kind in the (judiciously and suitably chosen) structure algebra sheaf $\mathbf{A}$ of generalized arithmetics,[a] while the field (generalized differential=connection) and the law (differential equation) that it defines, still applies (*ie*, holds) in their very presence. Moreover, as we shall see in the next section, from a quantum (particle) vantage, the (local) particle states of the field are represented by the (local) sections of the sheaf $\mathcal{E}$ associated with (*ie*, carrying) the field $\mathcal{D}$.[b] Thus, in view of the fact that $\mathcal{E}$ is by definition locally isomorphic to $\mathbf{A}^n$, the 'particle-singularities' are built into the 'unitary field' $(\mathcal{E}, \mathcal{D})$ of ADG. In a very strong and ostensive sense, *in ADG particles of matter (as well as gauge forces) are singularities in their corresponding fields, with these singularities in no way inhibiting the application (ie, the holding) of the physical field-laws that those fields define as differential equa-*
(R6.4) *tions proper.*[c]

---

[a]'Judiciously and suitably chosen' pertains here to two things as it has already been argued in the present paper: first, the function that contains the singularity to belong to the sheaf $\mathbf{A}$ of generalized coordinate functions that one chooses to employ in the theory, and second, that this $\mathbf{A}$ employed is able to provide one with the basic differential geometric substrate—the 'domain of differentiability' on which one can actually define ∂ (and *in extenso* $\mathcal{D}$) in the manner of (10) and (12); furthermore, possibly without a base manifold ($\mathbf{A} \neq \mathcal{C}^\infty_M$).

[b]Again, technically speaking, and from a Kleinian perspective, $\mathcal{E}$ is the sheaf associated with the self-transmutations (geometrical auto-symmetries) of the field, which comprise the principal sheaf $\mathcal{A}ut\mathcal{E}$. *$\mathcal{E}$ is the carrier space of $\mathcal{D}$* [272].

[c]That is, in contradistinction to Eddington's remarks and to the way gravitational DGSs are normally perceived from the viewpoint of the smooth manifold based CDG, singularities are in no sense obstacles (let alone break-down points) of the fields, no matter what their 'pole strength' (for DGSs for example, no matter what their 'order of differentiability').

6.2 What's in a solution?

It is now widely appreciated that, in view of the PGC, in GR coordinates—the $\mathcal{C}^\infty$-smooth ones of the underlying pointed manifold M—are just '*labels*', or '*names*' of M's points, without any physical significance since they do not actively participate into the actual gravitational dynamics.[488] This is supposed to be the central meaning of the Diff(M)-represented PGC, or equivalently, that the physically 'observable' (dynamical) quantities are $\otimes_{\mathcal{C}^\infty(M)}$-*tensors*—what we called here '*geometrical objects*', '$\mathbf{A}\mathcal{C}^\infty_M$-*functorial*' entities expressed via the homological tensor product functor $\otimes_{\mathbf{A}\equiv\mathcal{C}^\infty_M}$—like for example, in the original formulation of GR, the sole dynamical variable is taken to be the $\mathcal{C}^\infty$-smooth metric $g_{\mu\nu}$.[489] Yet, when a new, particular solution $g_{\mu\nu}$ of the Einstein equations is discovered, or even when an old well known one is invoked for further study or application, it apparently becomes physically important to investigate or make use of its regularity-singularity behaviour on its domain of definition—the differential spacetime manifold M. That is to say, in what region of M the said solution 'holds'—the realm where the gravitational field law appears to be valid.

On the other hand, given the 'fine-tuning-sensitivity' of the non-linear Einstein equations on boundary ('initial') conditions, namely, that given a plethora of various such 'external' prescriptions[490] any point of M can be the *locus* of a singularity for a smooth solution (metric) of the Einstein equations (Q2.?). Moreover, granted that the said singularity will be of a $\mathcal{C}^\infty$-smooth function belonging to the structure sheaf $\mathbf{A}\equiv\mathcal{C}^\infty_M$[491] (that anyway was *a priori* assumed by the theoritician!), the said occurrence of singularities would appear to indicate that *we should actually undermine the physical significance of solutions to the field equations*. In other words, it is the conspiring of the differential manifold assumption to model *physical* spacetime, coupled to the particular boundary conditions for the field partaking in the law of gravity—both of which are 'contingent' features of the theory, externally chosen and prescribed by the theorist)[492]—that lead

[488]See section 2. To be sure, underlying the 'labelness' of points by (smooth) coordinates is the 'monoamphidromous' or 'tautosemous' relation (isomorphism) of Gel'fand duality (spectral theory) identifying M's points with their smooth coordinates in $\mathcal{C}^\infty(M)$ (1).

[489]Thus, in view of the last footnote, even GR, in an implicit way, is fundamentally '*pointless*'. In the next section, in view of Stachel's unfolding of the deeper significance of Einstein's hole argument (and in effect, of the PGC), we will argue that this 'pointlessness' of GR is equivalent to the *dynamical*—as opposed to merely *kinematical*—*individuation of events*; albeit, an individuation not effectuated via the (smooth) spacetime $\mathbf{A}\equiv\mathcal{C}^\infty_M$-metric g as in the usual theory, but via the dynamical field $\mathcal{D}$, which need not *a priori* be *smooth* (*ie*, $\mathbf{A}\neq\mathcal{C}^\infty_M$).

[490]We stress it again that we tacitly assume that boundary/initial conditions, like the various gauge choices in gauge field theory, invariably lie with the 'exosystem' (or 'episystem')—*ie*, with the experimenter (or at least with the theoretician!)—and never with the physical 'endosystem' under experimental/theoretical scrutiny (here, the gravitational field and the law that it obeys) [148].

[491]For after all, to stress it once again, $g_{\mu\nu}$ is supposed to be $\mathcal{C}^\infty(M)$-valued when referred to a (local) $\mathcal{C}^\infty$-chart.

[492]That is, they have nothing to do with the 'physically objective' field of gravity (and the differential law that it

to limiting in one way or another, in the form of singularities, the (entire)[493] physical validity of the law of gravity. As a result, such limitations appear to be physically unreasonable (Q2.1), unbelievable and, ultimately, unacceptable (Q2.?).

In summa, on the one hand the PGC appears to be asking (rhetorically):

physically speaking, '*what's in a name?*' (PGC);

and then the general practice and consensus in GR appears to be asserting

the physical significance—or anyway, of the physical existence (Q2.?)—*of $\mathcal{C}^\infty$-singularities,*

an oxymoron which reflects the deeply disturbing conceptual and technical conflict between the Diff(M)-effectuated PGC of GR and the existence of $\mathcal{C}^\infty$-singularities in GR mentioned in the first section, which in turn results in the (apparently insuperable) difficulty of defining precisely what is a singularity in the classical spacetime manifold based relativistic field theory of gravity (GR) [155]—a difficulty, of course, that is pronounced within the Analytic confines of the differential manifold based CDG (Analysis) [87].[494]

In the light of ADG however, in view of the basic didactics emerging from the complete evasion of the interior Schwarzschild singularity presented above—or better, from the incorporation of that singularity into a suitably chosen structure sheaf of generalized arithmetics (coordinates) while still retaining the essentially algebraic differential geometric mechanism and the physical law of gravity expressed as a differential equation based on that mechanism in the very presence of that singularity—we contend that:

• On the one hand, *the appearance of singularities does not at all reduce the manifold of solutions*[495] (Q2.?)—or to quote the first author from [264]:

(Q6.12) "the exclusion of singularities does not actually reduce the manifold of solutions, since now the same 'singularities' can be incorporated into the solutions..."

• While on the other hand, since we have witnessed the total ADG-theoretic evasion of the base spacetime manifold and the singularities inherent in it (*ie*, in $\mathbf{A} \equiv \mathcal{C}^\infty_M$), we come to ask

obeys), but with our subjective theoretico-experimental tampering with (conditioning of and measuring) it.

[493] That is, over all the base space(time) $X \equiv M$.

[494] That is to say, the difficulty simply betrays the shortcomings and ineptness of the concepts and methods of CDG in dealing with smooth singularities—of course, quite an understandable inability considering the '*vicious circularity*' of the problem at the bottom, namely, that one cannot actually do (C)DG without a base M (manifold monopoly and conservatism), while at the same time singularities are inherent in $\mathcal{C}^\infty_M$; when, as matter of fact, M is nothing but $\mathcal{C}^\infty_M$ (Gel'fand isomorphism).

[495] That is, the realm in which the gravitational field law holds.

what's in a solution?

especially since the gravitational law (*ie*, the field $\mathcal{D}$ defining it) holds in all 'space(time)'—the 'space(time)' which is anyway inherent in the field, in point of fact, in the **A** that *we* choose to represent it, which not only does not actually partake into the law itself (synvariance), but it may also carry singularities of the 'worst' kind.[496]

In toto, we contend that finding a particular solution to the Einstein equations (and subsequently studying its regularity-singularity properties for physics' sake) loses its primary significance in ADG, as we quote the following concluding paragraph from [262]:

[496]Like for example when we choose for structure sheaf Rosinger's algebras of generalized functions hosting spacetime foam dense singularities of all pathological sorts.

"...On the other hand, it is in effect the manifold that creates the 'anomalies', or else 'singularities', of the equations involved, which presumably describe the physical laws [*viz.* fields], the latter being inherent in the world we perceive.[a] So, in agreement with the 'pointless' point of view, we now concentrate our attention on the equation itself, the 'manifold' being just the space (or even '*solution space*') on which the equation has meaning [or holds]. This aspect enables us to consider *equations presenting 'singularities'*, in the ordinary sense of the latter term, insofar as the functions by means of which the equations are expressed (or what amounts to the same, our 'structure algebra sheaf' **A**), provide us with the abstract machinery by which we can still follow the classical procedure within the present abstract framework. As a result, *in spite of the presence of 'singularities', the meaning and the consequences and/or predictions of the equations at issue are all in force, as if no singularities were present at all!*[b] What amounts to the same, to disregard the singularities just means that both the *vector sheaf* (:bundle) and the *field* (:gauge potential) extend across the 'singularity', a fact that classically requires extra, and not always easy(!), technical innovations.[c] So our task would be to give now the appropriate explanation to the predictions that occasionally emerge from the equations considered. *Yet, for this job, to know a particular solution of the equation under consideration is, of course, not always that important!*[d]..." (Q6.13)

[a]This is the PFR in ADG.

[b]Our emphasis.

[c]For example, as we saw in section 2, analytic extension, geodesic completion, Sobolev smearings, topological boundary constructions for the placement of singularities *etc.*

[d]Our emphasis.

These last remarks gain even more weight and significance when one recalls that in ADG the basic—in fact, the sole—gravitational variable is the connection $\mathcal{D}$ and not the metric $g_{\mu\nu}$. Thus apart from the fact that since in ADG there is no background (spacetime) manifold whatsoever its $\mathcal{C}^\omega$- or $\mathcal{C}^\infty$-extensibility are '*non-issues*' in ADG-gravity, at the same time the $\mathcal{C}^\omega$- or $\mathcal{C}^\infty$-extension of a solution-metric across its singularities also becomes a '*non-issue*'. Which brings us to what we think is a subtle point about singularities—especially when DGSs are concerned.

The basic theoretical fallacy: 'judging and blaming the cause by its effects'. We noted earlier when we were discussing Finkelstein's 'resolution' of the exterior singularity of the Schwarzschild solution-metric the apparent violation of the principle of sufficient reason (*ie*, 'time-symmetric laws $\Rightarrow$ time-asymmetric solutions'). Due to it, we there branded singularities as '*differential geometric solution-metric anomalies*'. There is more to this denomination than initially meets the eye. The idea here is that by choosing a particular coordinate system (or 'gauge')—the Eddington-Finkelstein frame—Finkelstein resolves the exterior singularity, but at the cost of lifting ('breaking') the Diff(M)-'symmetry' of the dynamical equations. That is, the solution appears to mandate the choice and the use (by the theorist!) of a special coordinate system in order to manifest the non-pathological character of the outer singularity, and as a result of this choice the PGC appears to be sacrificed. Of course, this 'symmetry-breaking' phenomenon is not unusual, as Jackiw for example points out in [216] when he discusses the significance of anomalies (and of various other symmetry-breaking phenomena) in QFT:

(Q6.14) "...*In classical physics, the principal mechanism for symmetry breaking, realized already within Newtonian mechanics, is through boundary and initial conditions on dynamical equations of motion.*[a] For example, radially symmetric dynamics for planetary motion allows radially nonsymmetric, noncircular orbits with appropriate initial conditions..."

[a]Our emphasis for what we want to point out in the sequel.

And of course one cannot refrain from recalling a similar situation in the context of GR, namely, Penrose's Weyl curvature hypothesis in order to account for the 'observed' cosmological time-asymmetry: while the Einstein equations are time-symmetric, the 'phenomenological' time-asymmetry could be accounted for by very special time-asymmetric initial conditions posed at the Big Bang state of the universe (*ie*, the Weyl curvature tensor—the part of the gravitational field strength which measures the entropic degrees of freedom of the gravitational field—was initially zero).

Similarly, our branding singularities as *differential geometric solution-metric anomalies* may also be attributed, generally speaking, to 'initial conditions', in the following sense: the singularities, which are responsible for lifting the Diff(M)-symmetry, are exactly due to *our a priori* (initial) 'kinematical' assumption (or model) of spacetime as a differential manifold. The operative words here are '*our initial*' in the sense that, as it was also explained earlier in section 3, the assumption by us—the theorists—of spacetime as a smooth manifold is akin to 'initial conditions' in that while it has nothing to do with the field-law itself (being externally prescribed and fixed, like gauge-fixing choices or initial/boundary conditions are), it affects the solutions ('effects') of those laws ('causes').

Moreover, what is lurking in the above is what we regard as a grave 'theoretical fallacy' on the part of the physicist: the anomalies (singularities) of the 'effects' (solutions) of the 'causes' (differential equations representing the field-laws) are often attributed back to the 'causes' themselves—*eg*, to perceive (as one indeed perceives!) of singularities as DGSs (or even as SFSs) and infer that the gravitational field (law) breaks down at them. Plainly, the mistake committed here is that one blames the law (which anyway in ADG is *defined*, as a differential equation, by the field $\mathcal{D}$ itself—the original 'gravitational cause'), instead of the initial, *a priori* posited and externally prescribed (by the theorist) spacetime manifold assumption, for the shortcomings and failings (in the guise of singularities) of the 'effects' (solutions) of these 'causes' (laws). Ultimately, as repeatedly emphasized in the first three sections, one is often tempted to blame the physics (the dynamical field laws) instead of the mathematics (the manifold based CDG), and what is really committed here is an instance of what Finkelstein has recently coined (to the second author) '*the mathetic fallacy*'[497] To stress it again, singularities are faults and blemishes of the manifold based CDG, not of the gravitational field (law) itself. Thus to qualify a bit further the aforesaid denomination, singularities better be thought of as *CDG-based solution-metric anomalies*; they are *mathe(ma)tic fallacies*. All in all, *the manifold is to 'blame' (for the singularities), not the field*, and it is needless to stress it again that that it is *we* that identify physical spacetime with a background differential manifold (arguably, for differentiability's sake).[498]

6.3 Section's Résumé

In this section we 'resolved' (better, completely evaded) in an ADG-theoretic manner the interior Schwarzschild singularity of the gravitational field of a point-particle in two different ways. First, we bypassed it in a finitistic-algebraic fashion issuing from the finitary (:locally finite), causal and quantal version of Lorentzian vacuum Einstein gravity presented in [272]. We regarded the classical pathological point-mass *locus* as a localized, static, spatial point-singularity and used the categorical bicompleteness of $\mathfrak{DT}$ to show that the vacuum Einstein equations (29) of CDG-gravity hold both at the 'discrete' level of finsheaves of qausets as shown in [272], *and* at the 'classical' ('smooth') continuum limit of an inverse/direct system thereof, while no 'infinity' (divergence) of the gravitational field arises at either level whatsoever. The inner Schwarzschild singularity is not a DGS, as the manifold based Analysis (CDG) has so far forced us to think. Second, we bypassed it also in distributional, 'time-extended' sense, by smearing it to a family of spacetime foam dense singularities in the manner of [273, 274, 262, 275] extending along the wristwatch

[497] This fallacy roughly corresponds to the nowadays commonplace positivist position of theoretical physicists that physics *is* (*ie*, it is identified with) the mathematical model used to describe it.

[498] The CDG-conservatism and monopoly mentioned earlier. This is understandable: how else can one apply differential geometry to theoretical physics and gravity in particular?

time-axis of the point-particle. Thus, the singularity should not be thought of as an SFS either, as again the manifold based Analysis has hitherto made us believe. Furthermore, in either evasions no topological boundary constructions were evoked, and the singularity was situated in the manifold's bulk; while also, no geometrical spacetime configurations like a closed trapped surface or a horizon were evoked to delimit the applicability of Einstein's law in the vicinity of the singularity. Finally, we combined the above finitistic evasion *à la* Sorkin, with the 'ultra-infinitistic' distributional one *à la* Rosinger, we defined dense singularity coverings of the singular spacetime, and argued that (29) again holds both at the 'discretum' and at the 'continuum' levels. This only goes to show what we have time and again emphasized in the present paper, namely, that ADG-gravity is genuinely background spacetimeless—whether that background is a 'discontinuum' or a 'continuum'—and manifestly singularities' uninhibited.

7 Evading Singularities ADG-Theoretically: What's in for Quantum Gravity?

As we have repeatedly emphasized in the past trilogy [270, 271, 272], in the recent work [317], and in the present paper so far, spacetime singularities are supposed to be a problem of *classical* gravity (GR) *per se*, long before the quantization of the gravitational field becomes an issue the theorist has to reckon with. However, in the past many theoretical physicists have argued and have aired the hope that a cogent quantum theory of gravity will shed more light on, perhaps it will even resolve completely, the classical problem of gravitational singularities [303]. Recently for example, Perry remarked in [305]:

(Q7.1)

> "...The existence of a singularity [in GR][a] is guaranteed by the singularity theorems of Hawking and Penrose. At a singularity one typically finds that the curvature tensor blows up, and in the context of classical general relativity, one should regard this as being the boundary of spacetime. *In classical theory, one generally regards singularities as being unphysical and an indication that the theory has broken down. It is hoped that a quantum theory of gravity would enable us to describe black holes in a way that would avoid such difficulties*[b]..."
>
> [a]Our addition.
> [b]Our emphasis.

while, very recently [200], Husain and Winkler regard it as imperative that QG resolves in the end the singularities of the classical theory (GR):

(Q7.2) "...It is well known that many solutions of classical general relativity have curvature singularities. The most commonly encountered are the initial and final singularities in cosmological models, and the singularities inside a black hole. *Just as the classical singularity of the Coulomb potential is 'resolved' by quantum theory, it is believed that any candidate theory of quantum gravity must supply a mechanism for curvature singularity resolution*[a]..."

[a]Our emphasis.

At any rate, generally speaking, if QG is approached along QFTheoretic lines, the hope is that in much the same way QED alleviated, or even resolved completely, the unphysical infinities of the classical Maxwellian electrodynamics, so QG should do away with the singularities and their associated infinities of the classical relativistic field theory of gravity (GR). In fact, one can argue that, independently of other motivations for quantizing gravity, one must arrive at QG if anything in order to complete the incomplete, because ridden by singularities, GR. In this regard, from Gibbons' and Shellard's introductory remarks [162] in the volume where Perry's quotation above was found, one can isolate and highlight the following words:

(Q7.3) "...*It is interesting to note that the singularity theorems gave a powerful impetus to the search for a quantum theory of gravity because they show that classical general relativity is an incomplete theory. The obviously attractive direction in which to complete it is by passing to some quantum version.*[a] This argument is quite independent of the many other motivations for quantizing gravity such as providing a consistent marriage between quantum theory and spacetime physics or unifying gravity with the other forces of nature..."

[a]Our emphasis.

In the present section we will discuss how one can incorporate quantum features in ADG-gravity, side-by-side the ADG-theoretic evasion of $\mathcal{C}^\infty$-smooth gravitational singularities, thus be able to address certain currently important issues in QG research.

7.1 Infusing Quantum Ideas into 'ADG-Gravity': Prequantization of Gravity via 'Third Quantization'

Our starting point is to mention the accomplishment along ADG-theoretic lines of a *geometric prequantization of gravity* [271, 272]. To this end, one first observes that since the ADG-notion of field $(\mathcal{E}, \mathcal{D})$ is a primitive, fundamental one, the infusion of quantum ideas into ADG must somehow involve *field*, otherwise known as *second*, *quantization*—albeit, a novel, 'non-standard' kind of field quantization, with an ADG-twist.

But let us take things one at a time. We argued in 3.3.2 earlier that the *background spacetimeless* (whether a continuum or a discretum) and *solely connection-based* (half-order, 'pure gauge' formalism) ADG-theoretic formulation of GR points to the ADG-conception of '*gravity as a gauge field theory of the third kind*'—a 'pure gauge', 'external spacetimeless' theoresis of the autonomous gravitational field 'in-itself'. Closely akin to this 'third conception' of gravity as a gauge theory is the notion of '*third quantization*', which we now come to discuss in more detail.[499]

7.1.1 Bypassing first quantization and abstracting second quantization: third, self, or spacetimeless field quantization

To explain the novel notion of '*third quantization*', as briefly noted above, since it has been amply appreciated ever its inception [259] that ADG refers directly and exclusively to (the algebraic relations between) the 'geometrical objects'—*ie*, physically speaking, the *fields* themselves—that live on 'space(time)', without that external surrogate topological localization background X playing any role whatsoever in its (differential geometric) concepts and constructions, *second* or *field quantization* would *prima facie* appear to be *the* appropriate vehicle via which to infuse quantum ideas in our ADG-gravity.

Indeed, the idea to bypass altogether *first quantization* and tackle directly issues of second quantization by ADG-means has been worked out in the past [263], especially approached via so-called *geometric (pre)quantization* concepts and techniques [259, 260, 261].[500]

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[499] The matters below have also been discussed recently, but more laconically, in [317].

[500] Let it be noted here that in the past it has been argued numerous times (and variously motivated) by many workers in the field of (second) quantization that in order to arrive at field quantization one need not first pass through first quantization [48, 347, 428, 434], in spite of the traditional view to the opposite [165]. For us, as we shall argue below, first quantization, even the continuum based second quantization ideas (QFT), do not seem safe and sound routes to arrive at a quantum theoresis of ADG-gravity, because, generally speaking, we believe that one cannot arrive at a genuinely quantum theoresis of gravity by quantizing 'blindly' the classical, manifold based field theory (GR), much in the same way that one arrives at quantum mechanics by formally (canonically, say) quantizing classical mechanics. A genuinely quantum theoresis of gravity, as we will contend in the sequel, cannot start from classical—especially from spacetime continuum based—paradigms, and merely apply a formal

The upshot of these applications, that we are going to review briefly below, is that ADG not only manages to model key ideas from second and geometric (pre)quantization, such as the quantum particle representation/interpretation of its fields and their (sheaf cohomological) classification (into bosons and fermions) according to their spin [263],[501] but unlike the conventional (Minkowski) manifold based (fiber bundle-theoretic) ideas and techniques of second (QFT) and geometric (pre)quantization [336, 35], it manages to do this without reference to a background 'space(time)', whether the latter may *ab initio* assumed to be a 'continuum' or a 'discretum'. Precisely for this reason we refer to the ADG-based field quantization as *third quantization*. In other words, *third quantization is field quantization without dependence on, reference or commitment to a background (ie, external to the fields) spacetime, whether discrete or continuous*.[502]

Parenthetically we note at this point that this freedom that ADG gives us in doing quantum field theory in a background spacetime (manifold)-free way cannot be overemphasized. Especially *vis-à-vis* the failure (so far) of attempts to arrive at a cogent (*ie*, a conceptually sound and calculationally finite) QG by applying QFTheoretic ideas and techniques to GR (*eg*, the non-renormalizability of gravity when regarded as a perturbative quantum gauge field theory like the other three fundamental forces, or even various deep technical and conceptual problems when the theory is formulated as a non-perturbative, background metric independent canonical loop QG), *third quantum gravity as a gauge theory of the third kind* may prove to be a fruitful route to the 'true' QG. For instance, in a possible 'third quantum' scenario for gravity, and in striking contradistinction to the other external (to the gravitational field itself) spacetime manifold based scenaria for QG (such as loop QG), *we do not expect 'spacetime' (geometry) to be also quantized* (*ie*, we do not regard the 'problem' of the quantum structure of spacetime as being 'important', that is, as being inextricably enmeshed with the problem of QG) simply because *from the start there is no background spacetime in our ADG-theoresis of gravity*, whether this theoresis may be ultimately branded 'classical' or 'quantum'.[503] Let it be also noted *en passant* here that for other

'quantization algorithm' to them. Rather, the description we are seeking for better have quantum traits already built into it from the very start.

[501]An ADG-application that was originally motivated by the, albeit explicitly manifold based, bundle-theoretic musings of Selesnick in [336] (for more on these results, see below).

[502]As we will see shortly, it is perhaps better and more accurate *not* to call the field theory of ADG 'third *quantized*'—and accordingly, the theoretical scheme underlying it 'third *quantization*'—as if there is an underlying '*classical*' counterpart (theory) that is being quantized. Below we will argue that the dynamically autonomous ('autodynamical') ADG-fields $(\mathcal{E}, \mathcal{D})$ are in a deep and subtle sense '*self-quantum*'. That is to say, quantum ideas are built into the ADG-field formalism from the very start, so that ADG-gravity better be called '*third quantum*' or '*self-quantum third gauge field theory*'.

[503]In fact, the *classical–quantum* distinction loses its meaning in ADG-gravity. See remarks in the sequel about spacetime quantization, the questioning in the light of ADG-gravity of the significance of the Planck length, and the *genuinely* background (manifold) independent character of 'third quantum ADG-gravity'.

approaches to QG, the attainment of a quantum description of spacetime structure (*eg*, quantum set theory and quantum topology) is supposed to be prior to—in fact, a necessary stepping stone to—a genuinely quantum theoresis of gravity [142, 148, 145, 150, 202, 203, 209, 210, 211, 340, 149]. Still in further contrast to our spacetimeless 'third gauge' and 'third quantum' ADG-musings on gravity, there are certain theoretical schemes that focus solely on a finitistic and quantum theoresis of spacetime (or anyway, of space) itself [295, 297, 60, 67]. These remarks will be of significance a bit later when we comment on a recent 'resolution' of the interior Schwarzschild singularity by loop QG techniques and results [279, 200], and we compare it with ours herein.

But let us return to the ADG-based geometric (pre)quantization (of gravity). In the general theory,[504] the three most important results (for our exposition here) from the application of the ADG-technology to geometric (pre)quantization and to second quantization [259, 260, 261, 263, 269] are:

1. The representation of (local) quantum-particle states of the ADG-fields by (local) sections of the (associated) vector sheaves $\mathcal{E}$ that enter into the definition of the said fields as pairs $(\mathcal{E}, \mathcal{D})$. In other words, the quantum-particle aspect of the ADG-fields is already built into their very definition (*viz.* they are sections of the $\mathcal{E}$ involved in the $(\mathcal{E}, \mathcal{D})$). Thus, when one talks about an ADG-field, defined as the pair $(\mathcal{E}, \mathcal{D})$, one quite literally understands a '*particle($\mathcal{E}$)-field($\mathcal{D}$) pair*',[505] with the first synthetic ('particle') understood as the local aspect (section) of the vector (sheaf $\mathcal{E}$) representation of the connection field $\mathcal{D}$—the (local) 'carrier' of the field. In a nutshell,[506]

$$\begin{gathered}(\text{particle, field}) \equiv (\mathcal{E}, \mathcal{D}) \Longleftrightarrow \\ \text{local particle representation states of } \mathcal{D} \longleftrightarrow \text{local sections} \longleftrightarrow \text{vector sheaf } \mathcal{E}\end{gathered} \tag{101}$$

with the last equivalence being a theorem in sheaf theory.[507]

Moreover, by virtue of the fact that $\mathcal{E}$ itself is by definition locally (a power of) $\mathbf{A}$, which structure sheaf in turn is the host of all the singularities that may be present in the theory,[508] *in ADG the particles may be regarded as 'singularities in their respective fields'.*

[504] That is, not particularly applied to a physical theory such as Maxwelliam electrodynamics, Yang-Mills theory or gravity.

[505] What in 6.2 next we will coin, from a quantum-theoretic viewpoint, the '*quantum self-dual unitary field*'.

[506] Recalling also the diagram in 3.2.2.

[507] Namely, *a sheaf is its (local) sections.*

[508] Recall, in ADG all singularities are inherent in the generalized arithmetics (coordinates) in $\mathbf{A}$.

2. The (sheaf-cohomological) classification of the aforesaid representation sheaves $\mathcal{E}$ according to the spin of the fields' particles. In short, (local) states of free ('bare') *bosons* are identified with (local) sections of *line sheaves*,[509] while (local) states of free *fermions* are represented by (local) sections of *vector sheaves of rank* $n \geq 1$.[510]

3. The upshot of 1 and 2 is a basic result from the application of ADG-theoretic ideas to geometric prequantization,[511] and it reads: '*every (free) elementary particle is prequantizable; that is, it entails by itself a prequantizing vector sheaf*' [259, 260, 263, 269].

We then complement the result stated above with the following words taken from the first author's [260], which highlight the implications and value of that result *vis-à-vis* geometric, first and second quantization:[512]

(Q7.4) "...One thus avoids, by the preceding conclusion,[a] the so-called '*correspondence principle*', namely, to pass first from the *1st quantization* of the physical system involved, a fact that lies, for that matter, at the basis of *geometric quantization*. Indeed, the latter theory aims basically to *bypass the Hamiltonian mechanics altogether*.[b] Yet, within the same vain of ideas, we also note that *there exist quantum mechanical observables without known classical counterparts*[c]..."

[a]That is, that every free elementary particle is prequantizable.
[b]Here the author cites Selesnick [336] and [259].
[c]Here the author cites [48].

'Third quantum vacuum ADG-gravity' has been preliminarily investigated along geometric prequantization lines and in a finitary, causal and quantal setting in [271, 272]. In view also of the inner Schwarzschild singularity 'resolution' accomplished here both by finitary-algebraic and distributional means, the main relevant results are:

- The quantum-particle interpretation, as the *quanta of dynamical causality*—otherwise coined '*causons*', of the $fcqv$-fields $\mathcal{D}$ defining the finitary, causal and quantal version (88) of the

[509]By definition, *a line sheaf is a vector sheaf of rank* $n = 1$.
[510]The motivating 'forerunners' of this result, albeit ones that are not expressed in abstract sheaf-theoretic terms, but rather in 'classical' (vector) bundle-theoretic ones (over the spacetime manifold), are Selesnick's paper [336] (in fact, the above sheaf-theoretic classification of bare elementary particles according to their spin *à la* ADG has been coined '*the Selesnick Correspondence*' in [259, 260, 263, 269]) and Manin's 'definition' of a *Maxwell field* as '*a connection on a line bundle*' [276, 277].
[511]Especially in conjunction with the ADG-version of *Weil's integrality theorem* [257, 259, 260].
[512]In what follows emphasis is the first author's own.

vacuum Einstein equations for Lorentzian gravity. The third epithet '*quantal*' (q) in the denomination of $\mathcal{D}$ rests precisely on this fact and it suggests that *the dynamical vacuum Einstein equations* (88) *are*, to a certain extent, *already quantum*.[513]

- The sheaf-cohomological classification of the vector finsheaves of qausets involved in $fcqv$-Einstein gravity [271, 272] suggests that *causons are fermionic*,[514] but with the said vector sheaves one may also $\otimes_{\mathbf{A}}$-associate a line sheaf of *bosonic graviton-like states* mediating, by carrying the gravitational force, between causons.[515]

- Regarding 'singularities' and other 'discontinuous' ('non-smooth') phenomena, the ADG-placing of particle-quanta (*viz.* sections of) $\mathcal{E} \overset{\text{loc.}}{\simeq} \mathbf{A}^n$ side-by-side the field $\mathcal{D}$, amounts to integrating (or incorporating) 'particle-singularities'[516] into the Einstein law (29) that the connection field defines (via its curvature). Moreover, exactly because the said law is $\mathbf{A}$-functorial, we materialize Eddington's remarks in (Q?.?), namely: *the laws of motion of the particle-singularities is contained in the field-equations*.[517]

This last remark, coupled to our earlier observation that by geometric prequantization we totally bypass first quantization and go directly to second, field quantization,[518] enables us to address some subtle 'categorical' issues regarding 3rd field quantization *à la* ADG in comparison to 1st and even to 2nd quantization.

[513] We will return to discuss this '*already quantum*' (or even, '*self-quantum*') trait of ADG-gravity (29) in the sequel. The '*fully quantum*' (*ie*, not just 'to a certain extent' as described above) ADG-gravity may be formulated along a sum-over-connection-histories, that is, via a path integral over connection space-type of dynamical scenario, as we will loosely suggest in the sequel. More on this shortly.

[514] This recalls the fermionic (Grassmannian) *chronons* involved in Finkelstein's quantum (causal) net approach to quantum spacetime and gravity [145, 146, 148, 337, 338, 339, 340].

[515] One could speculate here on a possible '*supersymmetric*' scenario that transmutes chronons to gravitons, and *vice versa*. One thing is certain however, with every fermionic causon (representation) state space $\mathcal{E}$, there is always (implicitly) a bosonic (graviton?) line sheaf $\mathcal{L}$ $\otimes_{\mathbf{A}}$-entangled with it, since the following 'absorption' is implicit: $\mathcal{L} \otimes_{\mathbf{A}} \mathcal{E} \simeq \mathcal{E}$.

[516] Recall Einstein's intuition of particles as 'singularities in the field'.

[517] However, unlike Eddington's words that '*the particle's mass is the pole-strength of the singularity*', which seem to indicate that particles are 'stumbling blocks' in the dynamical laws of motion defined by the fields, in our ADG-scheme the $\mathbf{A}$-absorption of the particle-singularities (like we did for the point-mass at the inner Schwarzschild singularity), exactly due to the $\mathbf{A}$-functoriality of dynamics, testifies to the opposite: *particle-singularities are not obstacles in the dynamical evolution of the field, and the latter 'sees through' them.*

[518] In fact, as emphasized before, by ADG-theoretic means we bypass 2nd quantization as well, and have a background spacetimeless, 'purely gauge' (third quantum) field theory of gravity.

7.1.2 The issue of 'dynamical functoriality' versus 'kinematical functoriality'

To get things started, we read from Baez [27]:[519]

[519]We are grateful to Ms Kari Kelly for bringing this pre-print to our attention.

"...There is a famous saying about quantization due to Edward Nelson: '*First quantization is a mystery, but second quantization is a functor!*'[a]...
First quantization is a mystery. It is the attempt to get from a classical description of a physical system to a quantum description of the 'same' system. Now it doesn't seem to be true that God created a classical universe on the first day and then quantized it on the second day. So it's unnatural to try to get from classical to quantum mechanics.[b] Nonetheless we are inclined to do so since we understand classical mechanics better. So we'd like to start with a classical mechanics problem—that is, a phase space and Hamiltonian function on it—and cook up a quantum mechanics problem—that is, a Hilbert space with a Hamiltonian operator on it. It has become clear that there is no utterly general systematic procedure for doing so.
Mathematically, if quantization were 'natural' it would be a *functor* from the category whose objects are symplectic manifolds (=phase spaces) and whose morphisms are symplectic maps (=canonical transformations) to the category whose objects are Hilbert spaces and whose morphisms are unitary operators. *Alas, there is no such nice functor. So quantization is always an ad hoc and problematic thing to attempt.*[c] A lot is known about it, and more isn't. That's why first quantization is a mystery...
...Note that there *is*[d] a functor from the symplectic category to the Hilbert category, namely one assigns to each symplectic manifold X the Hilbert space $L^2(X)$, where one takes L^2 with respect to the Liouville measure. Every symplectic map yields a unitary operator
(Q7.5) in an obvious way. This is called **prequantization**.[e] The problem with it physically is that a one-parameter group of symplectic transformation generated by a positive Hamiltonian is not mapped to a one-parameter group of unitaries with a *positive*[f] generator. So my conjecture is that there is no 'positivity preserving' functor from the symplectic category to the Hilbert category.
Second quantization is the attempt to get from a quantum description of a single-particle system to a quantum description of a many-particle system. Starting from a Hilbert space $\mathcal{H}$ for the single particle system, one forms the symmetric (or antisymmetric) tensor algebra over $\mathcal{H}$ and completes it to form a Hilbert space $\mathcal{K}$, called the bosonic (or fermionic) **Fock space**[g] over $\mathcal{H}$. Any unitary operator on $\mathcal{H}$ gives a unitary operator on $\mathcal{K}$ in an obvious way. *More generally, one has a functor called 'second quantization' from the Hilbert category to itself, which sends each Hilbert space to its Fock space, and each unitary map to an obvious unitary map. This functor* <u>*is*</u>[h] *positivity-preserving*[i]..."

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[a] Our emphasis.

[b] We wholly concur. Moreover, we believe that this is so even when one tries to apply ideas from second quantized (field) theory (QFT) to GR, with the latter regarded of course as a classical field theory. Read on.

[c] Our emphasis.

The first thing to highlight from the words above is that *(geometric) prequantization is a functorial quantization recipe after all*, but it fails when it comes to preserving the 'positivity of dynamical evolution' in the quantum regime (*ie*, a positive Hamiltonian in phase space is not mapped to a positive Hamiltonian operator on $\mathcal{H}$ generating a continuous one-parameter unitary dynamical evolution in it). Thus first quantization fails to be functorial. On the other hand, second quantization is functorial as positive unitaries in the single particle Hilbert space 'lift' rather straightforwardly to positive unitaries in the many quanta Fock space.

From our ADG-perspective, as also mentioned in (Q?.?) above, on the one hand we totally bypass the *non-functorial first quantization* procedure, and at the same time we apply the *functorial (geometric) prequantization* directly to the dynamical ADG-gravity field $(\mathcal{E}, \mathcal{D})$ itself. And, as a result, although *prima facie* we do not abide by the by now standard *categorical-kinematical structures of either classical or quantum mechanics* (*eg*, symplectic manifolds/symplectomorphisms; Hilbert spaces/unitary maps),[520] in ADG the functoriality of prequantization is directly reflected in the dynamical equations (29) for vacuum Einstein gravity, which are $\otimes_{\mathbf{A}}$-functorial expressions, while, as we will argue further below, they are in a deep sense '*already quantum*' (self-quantum; 'quantum in-themselves').[521] *In toto*, we could express this by saying that while in the usual continuum based[522] first and second quantization theories functoriality pertains essentially to the 'preservation' of the kinematical structure as one progressively goes from single-particle classical mechanics, to quantum mechanics (1st quantization), to QFT (2nd quantization), in the *ab initio* field-based and background spacetimeless ADG '3rd quantization' (of gravity) pertains to the functoriality of the 'already quantum' dynamical equations (29). That is why we titled the present sub-subsection '*dynamical versus kinematical functoriality*'.[523]

[520]In any case, in ADG we manifestly do not deal with differential manifold phase spaces of classical systems.

[521]Needless to mention in this respect that the ADG-theoretic geometric prequantization of gravity (in effect, what we earlier called 'third quantization'), not only does not start from a classical single-particle theory and then first quantizes it, it does not start from GR and then applies manifold-based field (second) quantization techniques either. To stress it again, in ADG-gravity we do not start from the classical theory (GR) and then quantize it, but rather we refer directly to the dynamical field of gravity and expose its inherently (pre)quantum character—and hence of the (abstract) dynamical equations (29) that it defines. Let us also note here that the (third quantum) field theory *à la* ADG does not draw any distinction, as it is normally done like in (Q?.?) above, between single-particle (finitely many degrees of freedom) first quantized theory, and many-particle (infinitely many degrees of freedom) second quantized theory (QFT). This is of course closely akin to the fact that the ADG-field theory does not distinguish between a background discretum or continuum spacetime, or *in extenso*, state (configuration or phase) space.

[522]Whether that continuum is the differential spacetime manifold, or the differential configuration or phase space manifold.

[523]This is another manifestation of the theoretical 'phenomenon', already mentioned in section 3 and to be further discussed in the next section in view of Einstein's hole argument testing the PGC of GR, that *in ADG*, contrary to the usual *aufbau* of physical theories so far, *dynamics is prior to kinematics*.

But now we are ready to discuss in greater detail below the '*already quantum*' or '*self-quantum*' nature of the (3rd quantum) ADG-fields mentioned above. Before we do this however, as a general motivational question for developing the external spacetimeless, 3rd or self-quantum (gauge) field theory (of gravity)[524] we recall Isham's words from the introduction of [212]:

(Q7.6) "Attempts to construct a quantum theory of gravity provide many challenges for quantum theory itself. Some of these involve conceptual issues...For example: *is it meaningful to talk about quantum theory in the absence of any background spatio-temporal structure?*[a]..."

[a]Our emphasis.

In view of ADG and the 3rd quantum (gauge) field theory that it supports, our answer to the question above is an emphatically affirmative one: it is meaningful indeed to talk about a background spacetimeless *quantum* ('pure gauge') *field* theory (of gravity). Let us discuss more this '*self-quantum*' character of its autonomous-dynamical ('auto-dynamical') connection (gauge) fields in the manifest absence of an external, background spacetime (whether a continuum or a discretum) .

7.2 A Genuine Field-Particle Duality: Heisenberg's Algebraic Indeterminacy and Bohr's Complementarity Reformulated 'Self-Referentially' in ADG-Theoretic Terms

On very general grounds, and apart from the distinctions between first and second quantization noted above, one may view *quantization as representation*. Wigner and Mackey's seminal work on (unitary) 'group imprimitivity' and 'group quantization' theory testifies to that [419, 251]. For example, it is well established now that in the Minkowski manifold based (2nd quantized) theory of fields (QFT), an elementary particle—the quantum of the corresponding field—can be completely characterized by the *irreducible representations* ('irreps') of the Poincaré group of the external spacetime symmetries. Indeed, irreps are *group characters* and represent the particle's mass and spin (quantum) numbers, which are conserved during the fields' dynamics.[525] By the very definition of irreps, particle states belong to minimal invariant subspaces of the representation (Hilbert) space of the corresponding group. Moreover, apart from the external spacetime symmetries, as it is well

[524]And for the other Yang-Mills forces, including Maxwellian electrodynamics [259, 260, 269].
[525]General Noether's theorem for continuous dynamical symmetries.

known there are internal gauge symmetries whose characters are the so-called 'gauge charges'.[526]

On the other hand, we have repeatedly emphasized thus far that the notion of representation figures prominently in the very definition of the ADG-fields $(\mathcal{E},\mathcal{D})$, and in their 'inherently' (pre)quantized character;[527] for as noted above, the vector sheaf $\mathcal{E}$ is the representation, associated sheaf of (quantum) particle states of (the principal sheaf $\mathcal{A}ut\mathcal{E}$ of the dynamical self-transmutations) of the dynamically autonomous connection field $\mathcal{D}$. Albeit, this 'dynamical autonomy' pertains precisely to ADG's referring directly and solely to the dynamical fields 'in-themselves'—*ie*, without reference to a background spacetime structure. As a result, the 'pure gauge', 'esoteric' Kleinian geometry of the field's particle-quanta is not described by extraneous (*ie*, externally prescribed and fixed) spacetime attributes—*eg*, by the mass and spin characters that quantum fields 'inherit' from the external Minkowski spacetime symmetries in the usual theory (QFT). In other words, the particle-geometry (representation) of the ADG-fields is not effectuated and expressed as usually, via the symmetries of the external spacetime (manifold) on which the relevant fields are normally thought of as being soldered. Keeping in mind this remark, let us discuss further the aforesaid '*self-*' or '*already quantum*' character of the auto-dynamical ADG-fields.

On the face of all this, there is the following 'syllogism': we have the classic and perennial philosophical duality between *state* ('being') and *change of state* ('becoming'). In classical point-particle mechanics, operationally speaking, we have mutually compatible *position-determinations* (determinations of 'stasis' or 'position states') and *momentum-determinations* (determinations of 'kinesis' or 'change-of-position/motion states') in the classical phase space continuum (point manifold) of the particle. In standard (non-relativistic) quantum point-particle mechanics however,[528] these two kinds of acts of determination are fundamentally supposed to be complementary or conjugate (mutually exclusive), with their sharpness (accuracy of determination) being limited by the quantum of action $\hbar$, an operational complementarity that (in natural units $\hbar = 1$) is expressed algebraically by the so-called canonical (equal-time) Heisenberg commutation relations (in the position or coordinate, Schrödinger picture)

$$[\hat{x},\hat{p}] = \imath 1;\ \hat{x} = x,\ \hat{p} = \frac{d}{dx} \tag{102}$$

in which $\hat{x}$ and $\hat{p}$ are (self-adjoint) operators acting on a certain Hilbert space $\mathcal{H}$ of states ($\mathbb{C}$-valued functions) $\psi(x)$ of the quantum particle—the so-called 'carrier' or representation space of these

[526] The usual Standard Model ones being the $U(1)$-electric charge of the abelian electrodynamics, as well as the flavor and colour internal gauge charges conserved in the weak ($SU(2)$) and strong ($SU(3)$) interactions, respectively.

[527] Indeed, in view of the aforementioned geometric prequantization results, quantum-particle traits are built into the formalism from the very start—in fact, from the very definition of ADG-fields as pairs of the sort $(\mathcal{E},\mathcal{D})$.

[528] For example, in the case of a non-relativistic particle moving on the real line $\mathbb{R}$.

complementary acts of determination.[529] Operationally speaking, the algebraic indeterminacy relations above can be read as follows: a sharp act of determination of the position of the quantum results in an uncontrollable, dynamical change (perturbation) in its momentum—as it were, its momentum becomes 'fuzzy' and indeterminate.[530]

In the standard Born-interpretation ψ is then interpreted as a probability amplitude wave-like 'field' obeying in the Schrödinger picture the well known differential (dynamical) and unitary in $\mathcal{H}$ eigenvalue wave equation

$$-\imath\frac{d\psi(x,t)}{dt} = \hat{H}\psi(x,t) = E\psi(x,t) \tag{103}$$

where $\hat{H}$ is the (hermitian) Hamiltonian operator (total energy) of the particle, and t an external time-parameter by (or with respect to) which evolution is 'temporally coordinatized' and 'geometrically ordered' (pictured).[531]

All this is well known and fine. Now, in ADG we have the following abstract or generalized analogue of the standard situation above: we have the *unitary*, background spacetime (manifold) independent—what we refer to as 'spacetimeless'—field $\mathfrak{F}$[532] being represented by the pair

$$\mathfrak{F} := (\mathcal{E}, \mathcal{D}) \tag{104}$$

$\mathfrak{F}$ is what we would like to coin '*quantum self-dual*' (or perhaps more suggestively, '*intrinsically field-and-particle dual*') in view of the following remarks about its very 'definitional' traits:

- On the one hand we have $\mathcal{E}$ built into $\mathfrak{F}$—the 'carrier' or 'representation' space' for the

[529] Let it be noted here that in view of the Stone-Von Neumann theorem, all irreducible unitary representations of the (kinematical) Heisenberg algebra generated by the commutation relations in (102) above are unitarily equivalent.

[530] And *vice versa* about determining momentum sharply and losing information about the quantum-particle's position. In quantum mechanics, '*you win some, you lose some*'.

[531] In a similar fashion, and informally speaking, in the QFT of matter, where quantum fields are quantum systems with an infinite number of degrees of freedom propagating in flat Minkowski space, one follows suit and posits CCRs for the field $\phi(x)$ and its conjugate momentum $\pi(x)$—x now being the 4-coordinates of a point in Minkowski space. The particle interpretation is then established by Fourier analysis of the field into its infinite constituent modes and subsequent interpretation of the (positive and negative frequency) coefficients of that series as creation and annihilation operators of particles (quanta) acting on the fundamental zero-mode state representing the (particle or quantum) vacuum. Moreover, in quantum field theory there is what one usually refers to as the Schwinger duality between the quantum matter sources and their so-called radiation fields (*eg*, in QED one has the electron source of the electromagnetic field)—a duality that is the conceptual analogue of the wave-particle one of non-relativistic quantum mechanics.

[532] Hereafter to be referred to as '*the* $\mathfrak{U}$-*field*'.

connection $\mathcal{D}$ part of $\mathfrak{U}$-field in (104) above.[533] From a geometric prequantization vantage, local sections of $\mathcal{E}$ are interpreted as particle states of the $\mathfrak{U}$-field.[534] At the same time, since by definition $\mathcal{E}$ is locally $\mathbf{A}^n$, and since, as we noted throughout this paper, $\mathbf{A}$ represents our generalized coordinate localizations (:local coordinate determinations='coordinatizations') of the $\mathfrak{U}$-field, the local sections of $\mathcal{E}$ are nothing else but generalized 'local position' or 'coordinate determinations' of the $\mathfrak{U}$-field, which are fittingly interpreted in a particle-kind of way.[535]

For expository completeness, one could add here that the remarks above about the local 'coordinate' or 'particle-position' character of the $\mathcal{E}$ part of the $\mathfrak{U}$-field $\mathfrak{F}$ are supported very precisely in ADG by the following (sheaf-)cohomological result [259, 260]:

(R7.1) every vector sheaf $\mathcal{E}$, which is locally, by definition, a finite power of $\mathbf{A}$, can be completely characterized by a so-called *coordinate 1-cocycle* $(g_{ab}) \in Z^1(\mathcal{U}, \mathcal{GL}(n, \mathbf{A}))$.[a]

[a] Where $\mathcal{U} = (U_a)_{a \in I}$ is a local frame (gauge) of $\mathcal{E}$, as usual.

- On the other hand, the $\mathcal{D}$ component of $\mathfrak{F}$ in (104) above, being a generalized derivative (differential ∂), acts locally on $\mathcal{E}$'s (local) sections to change them—a process representing dynamics in our theory.[536] Of course, in our scheme there is no external agent[537] other than the $\mathfrak{U}$-field itself to effect these changes. Rather, when *we* externally try to coordinatize or localize the $\mathfrak{U}$-field, thus so to speak 'extract' its particle-like attributes[538] by employing $\mathbf{A}$,[539]

[533] As noted numerous times before, in technical jargon, $\mathcal{E}$ may be viewed as the *associated sheaf* of the *'auto-symmetry' group sheaf* $\mathcal{A}ut\mathcal{E}$ of its 'auto-transformations' or 'self-transmutations'.

[534] In fact, as noted before, from a sheaf-theoretic vantage the basic result is that the sheaf $\mathcal{E}$ *is nothing but its sections* [259, 270, 271, 272].

[535] Arguably, a (point-)particle is an ideal, 'ultra-local' spatial (and static!) characterization of a system—as it were, particles are completely localized [on space(time)] aspects of a (quantum) system (see Einstein's quotation next). In a quantum context, one may think for instance of the localized point-like traces that the quantum leaves on the silvered photographic plate in the double-slit experiment.

[536] In much the same way that momentum represents a process of change of position-states. In our scheme, and categorically speaking, $\mathcal{D}$ is a sheaf-morphism—a map or operator acting on and changing the relevant $\mathcal{E}$'s (local) sections.

[537] Or to the same effect, a base spacetime (manifold)—a 'background geometry' so to speak—to which the $\mathfrak{U}$-field is referred and relative to which the dynamics that its $\mathcal{D}$-component defines is 'coordinatized', 'arithmetized', 'quantified' and 'geometrically ordered' (pictured).

[538] Which, as noted above, are represented by the (local) sections of the $\mathcal{E}$ part of $\mathfrak{F}$, which $\mathcal{E}$, in turn, is completely characterized sheaf-cohomologically by the coordinate 1-cocycle g_{ab} (R6.1).

[539] The reader should notice here that it is *us*—the external agents (observers, experimenters, or 'geometers/measurers')—that attempt to coordinatize $\mathfrak{F}$ via $\mathbf{A}$.

the potential part $\mathcal{D}$ of the $\mathfrak{U}$-field 'kicks back (non-linearly) on itself' (*ie*, in a sense, it 'self-interacts', or 'acts within $\mathfrak{F}$') to dynamically change these 'sharp' (localized) position/particle states—$\mathcal{E}$'s (local) sections. Mathematically, one may think of this as being reflected by the fact that $\mathcal{D}$ is not an $\mathbf{A}$-morphism, which in turn means that our (external) geometrical acts of localization (in $\mathbf{A}$) or soldering of the $\mathfrak{U}$-field on $\mathcal{E}$ cannot 'arrest' its $\mathcal{D}$ part[540]—our local gauge acts of pin-pointing, 'arresting' or 'freezing' the $\mathfrak{U}$-field (on $\mathcal{E}$), and, as a result, $\mathcal{D}$ eludes them—*ie*, it effects dynamical changes of particle states (local sections of $\mathcal{E}$).

At this point, and in connection with the foregoing discussion about 'the indeterminacy within $\mathfrak{F}$', it is interesting to bring forth from [126] Einstein's remarks about the essence of Heisenberg's uncertainty relations (102):[541]

(Q7.7) "...On the other hand, it seems to me certain that *we must give up the idea of a complete localization of the particles in a theoretical model.* This seems to me to be the permanent upshot of Heisenberg's principle of uncertainty..."[a]

[a] And in a quite famous joking remark of his, he said: "*The more one chases quanta, the better they hide.*" [120].

Of course, and this is another manifestation of the '*inherent quantumness*'[542] of the $\mathfrak{U}$-field $\mathfrak{F}$ (and *in extenso* of our ADG-theoretic scheme), in a (generalized) sense analogous to Heisenberg's so-called (noncommutative) 'matrix mechanics', almost invariably we effectively work not with $\mathcal{E}$ itself, but with the sheaf $\mathcal{E}nd\mathcal{E} = \mathcal{H}om_{\mathbf{A}}(\mathcal{E}, \mathcal{E})$ of endomorphisms of the particle or quantum (states) itself which, by virtue of the fact that $\mathcal{E}$ is locally $\mathbf{A}^n$, corresponds to the *non-abelian matrix algebra sheaf* $M_n(\mathbf{A})(U)$—the sheaf of (local sections of) algebras of (in general) non-commuting $n \times n$ matrices with entries from $\mathbf{A}$. Accordingly, with respect to the $\mathcal{D}$ aspect of $\mathfrak{F}$, we work with the connection $\mathcal{D}_{\mathcal{E}\otimes_{\mathbf{A}}\mathcal{E}^*}$ on the tensor product sheaf $\mathcal{H}om_{\mathbf{A}}(\mathcal{E}, \mathcal{E}^*) = (\mathcal{E} \otimes_{\mathbf{A}} \mathcal{E})^* = \mathcal{E}^* \otimes_{\mathbf{A}} \mathcal{E}^*$, a connection which is induced by the $\mathbf{A}$-connection $\mathcal{D}$ on $\mathcal{E}$ [259, 260, 272]. In this way, the dynamical changes that $\mathcal{D}$ effects on the local particle (quantum) states (sections) are 'covariant' with the latter's (in general) noncommuting self-transmutations in $\mathcal{E}nd\mathcal{E}$,[543] and the algebraic $\mathcal{D}$ lies at the quantum side of the

[540] As it were, our external attempts to 'dissect' $\mathfrak{F}$ into its constituent parts $\mathcal{E}$ and $\mathcal{D}$, thus restrict it sharply and exclusively to $\mathcal{E}$—*$\mathfrak{F}$'s 'particle picture'*.

[541] in the quotation below emphasis is ours.

[542] See paragraph after the next.

[543] To be precise, the said covariance-proper is captured by the group sheaf $\mathcal{A}ut\mathcal{E} = (\mathcal{E}nd\mathcal{E})^{\bullet}$ of *invertible* endomorphisms of the particle $\mathcal{E}$—the automorphisms of $\mathcal{E}$—which self-transmutations locally correspond to sections living in $(M_n(\mathbf{A}))^{\bullet}(U)$.

quantum/classical divide.[544]

For expository completeness, one could add here that, although the algebraic connection $\mathcal{D}$ manifestly eludes our local $\mathbf{A}$-coordinatizations (equivalently, 'position determinations' or 'quantum particle-localizations') of the $\mathfrak{U}$-field $\mathfrak{F}$ by effecting dynamical changes of the local particle/position states (sections) of $\mathcal{E}$ (something that as noted above is implied by the statement that $\mathcal{D}$ is not an $\mathbf{A}$-morphism), its measurable manifestation—the curvature $R(\mathcal{D})$—is an $\mathbf{A}$-morphism, which in turn, being a closed 2-form, defines a characteristic class $[\mathcal{E}]$ of $\mathcal{E}$s (or equivalently, of 1-cocycles (g_{ab}) determining $\mathcal{E}$) sheaf-cohomologically [259], as follows:

$$\frac{[R]}{2i} = [(g_{ab})] \equiv [\mathcal{E}] \tag{105}$$

En passant we note that, precisely because of this expression, the $\mathfrak{U}$-field $\mathfrak{F}$, represented by the pair $(\mathcal{E}, \mathcal{D})$ as in (104), can be equivalently represented as:

$$\begin{array}{c} \mathfrak{F} := (\mathcal{D}, R(\mathcal{D})), \text{ or :} \\ (\mathcal{D}, \mathcal{E}) \Leftrightarrow (\mathcal{D}, R(\mathcal{D})) \end{array} \tag{106}$$

an equivalence which effectively rests on the *Chern isomorphism*, the following abelian group isomorphism [259, 260]:

$$\mathcal{E} \leftrightarrow g_{ab} \in H^1(X, \mathbf{A}^\bullet) \simeq H^2(X, \mathbb{Z}) \ni R \leftrightarrow \mathcal{E} \tag{107}$$

- All in all, the $\mathfrak{U}$-field $\mathfrak{F}$ has built into it the particles (quanta)—represented by (the local sections of) $\mathcal{E}$, *and* the 'gauge potential'[545] for their dynamical changes—represented by $\mathcal{D}$.[546] The unitary field $\mathfrak{F}$ is self-indeterminate (self-dual or self-complementary[547]) and auto-dynamical.[548] One could then argue that in this sense *the unitary field* $\mathfrak{F}$ *is already 'inherently*

[544] See the generalized ADG-theoretic analogue of Bohr's correspondence principle expressed in the next sub-subsection.

[545] To use a popular gauge-theoretic synonym for the connection.

[546] In analogy with the standard Heisenbergian quantum x–p dichotomy, we could coin the $\mathcal{E}$ part of $\mathfrak{F}$ '*the generalized position/particle picture of the* $\mathfrak{U}$*-field*', while the $\mathcal{D}$ part of $\mathfrak{F}$, '*the generalized momentum/field picture of the* $\mathfrak{U}$*-field*'.

[547] In the sense that the generalized field/momentum aspect $\mathcal{D}$ of the $\mathfrak{U}$-field $\mathfrak{F}$ is complementary to its generalized particle/position aspect $\mathcal{E}$, so that one speaks of $\mathfrak{F} := (\mathcal{E}, \mathcal{D})$ literally as '*particle-field pair*'.

[548] In quantum jargon, one may use the epithet '*coherent*' as a synonym to '*unitary*' for the $\mathfrak{U}$-field $\mathfrak{F}$ in the sense that its dual character is *inseparable* and '*holistic*': that is to say, *one cannot think of the generalized field/momentum*

quantum',[549] thus in no need of the (formal) procedure of 'quantization' to be exercised (in a forced, *ad hoc* fashion by the theorist!) on it. Thus, in a subtle sense, our theory has no 'classical correspondents', and no formal statements of the sort usually encountered in conventional quantum scenarios about pre-existing classical theories, as for example '*the classical theory is recovered at the limit as* $\hbar \longrightarrow 0$', are made.[550]

The last remarks bring us to discuss a generalized ADG-theoretic version of Bohr's *Correspondence Principle (or Limit)* (CP).

7.2.1 A generalized version of the Correspondence Principle in ADG-theoretic terms: the abstract algebra-vs-geometry 'schnitt'

A general expression of Bohr's CP is the following: *while the 'observable' properties of quantum mechanical systems can be represented by non-commuting operators*[551] (the so-called 'q-numbers'), *our measurements of them always 'yield' classical, commuting numbers*[552] (the so-called 'c-numbers'), which are also fundamentally assumed to be *real* ($\mathbb{R}$).[553]

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aspect $\mathcal{D}$ of $\mathfrak{F}$ apart from its generalized particle/position aspect $\mathcal{E}$, and *vice versa*. $\mathfrak{F}$ is not 'fragmentable' or 'dissociable' into its 'constituent' particle ($\mathcal{E}$) and field ($\mathcal{D}$) parts (aspects). Moreover, these two mutually complementary aspects of $\mathfrak{F}$ are autonomous—*ie*, in no way dependent for the 'subsistence' on a pre-existent and external (background) geometrical space(time), and in particular, on a continuum (3rd quantum character of the $\mathfrak{U}$-field $\mathfrak{F}$).

[549] Since the epithet '*quantum*' is already preempted by the usual theory and supported by its technical paraphernalia, we prefer the adjective '*quantal*' [270, 271, 272], also in order to separate our position from the standard theory. 'Quantal' may be thought of as a synonym to the denomination '3rd quantum' that we discussed earlier.

[550] Although we will make loose remarks in this direction when we discuss the (spacetime) continuum-bound appearance of fundamental constants (de)limiting (the range of validity of) spacetime continuum and Calculus-based theories such as SR (c), GR (G) and QM ($\hbar$), and their 'conspiring' to positing a fundamental length in Nature—the so-called Planck length ℓ_P, in section 7.2.

[551] Acting on the aforesaid (complex) Hilbert space $\mathcal{H}$ of states ψ of the quantum system.

[552] And the physics describing the measuring apparatus is *classical physics*.

[553] In turn, this expression of the CP is intimately related to the act of *measurement* and its associated *spontaneous collapse of the wave function* in the usual Copenhagen interpretation of quantum mechanics, whereby, upon measurement of a particular quantum mechanical observable (represented by a certain self-adjoint operator $\mathcal{O}$ in the aforementioned $\mathcal{H}$), the wave function (state) $\psi \in \mathcal{H}$ of the quantum system is supposed to 'collapse' abruptly to one of $\mathcal{O}$'s eigenstates $\psi_n^{\mathcal{O}}$ (with probability $| < \mathcal{O}\psi | \psi_n^{\mathcal{O}} > |^2$), 'yielding' in the process the (real) number o_n—the eigenvalue of $\mathcal{O}$ at $\psi_n^{\mathcal{O}}$ (projection postulate). In principle, quantum theory, in its Copenhagen interpretation at least, says nothing about where or when the collapse of the state vector happens, hence the '*quantum divide*'—the so-called *Heisenberg schnitt*—between q- and c-numbers (or quantum and classical descriptions of the *quantum endosystem* and the *quantum exo- or episystem* respectively [148]) is indeterminate in the theory. In other words, the theory says nothing about the 'emergence of classicality' or 'where one draws the line between classical and quantum mechanical behavior', and the separation between the classical observer ('experimenter') and her measuring apparatus, and the quantum system ('experimentee') is rather arbitrary. In the mathematical formalism, this

As noted in the concluding lines of 6.2, in ADG the autonomous, purely algebraic and background geometrical spacetime (manifold) independent $\mathfrak{U}$-field $\mathfrak{F}$ is '*self-complementary*'—a '*quantum self-duality*' which can be interpreted as an '*inherent quantumness*'. Thus, $\mathfrak{F}$ is *prima facie* in no need of a formal procedure of quantization (and conversely, of a CP), something which would presuppose an already existing classical theory.[554] To be sure, on the mathematical side one could argue that there is a sort of CP between ADG and CDG given by the formal substitution (better, identification)

$$\mathbf{A} \longrightarrow \mathcal{C}^\infty_M \Leftrightarrow \mathbf{A} \equiv \mathcal{C}^\infty_M \tag{108}$$

whereby, *CDG is the particular instance of ADG in case one uses $\mathcal{C}^\infty_M$ as the structure sheaf of generalized coefficients (arithmetics) or coordinates*. In this special case indeed one could maintain that *the smooth field* (with respect to its differential geometric properties) *is sustained by the smooth background locally Euclidean geometric space(time), as in the classical theory (CDG)*.

However, this mathematical correspondence is just a particular manifestation of a deeper physical 'self-correspondence' which is again inherent in the $\mathfrak{U}$-field $\mathfrak{F}$. In turn, this correspondence may be 'categorically' perceived as one between *algebra* and *geometry*. To explain what we mean by the latter, let us refer again to $\mathfrak{F}$ as expressed by the pair $(\mathcal{E}, \mathcal{D})$. $\mathfrak{F}$, its part $\mathcal{D}$ in particular, is a purely algebraic, quantal entity, which in ADG is fundamentally assumed to exist independently of an ambient (background) geometrical space(time) (PFR). To be sure, *geometrization*, or what is the same, localization (of $\mathcal{D}$ and *in extenso* of $\mathfrak{F}$), is achieved as soon as one introduces the *commutative* algebra sheaf $\mathbf{A}$ to coordinatize $\mathcal{E}$—the carrier space of $\mathcal{D}$. The introduction of $\mathbf{A}$ lies on the classical side of the quantum divide: *it is us—the local 'observers' or 'measurers'—who introduce* $\mathbf{A}$ *in order to localize, 'pin-point', or solder* $\mathcal{D}$ *on* $\mathcal{E}$*, which is in turn expressed as a*

arbitrariness and the fuzziness of the transition from the quantum to the classical realm (or *vice versa*) is reflected by such 'loose' and *ad hoc* pseudo-formal mathematical expressions such as '*(canonical) quantization is deformation*' or '*(canonical) quantization corresponds to substituting Poisson brackets by commutators*' ($c \rightarrow q$), and conversely, that *classical mechanics is recovered from quantum mechanics in the 'de-deformation' limit as* $\hbar \longrightarrow 0$' ($q \rightarrow c$). All this is usually subsumed in what is generically known as the *measurement problem* in quantum mechanics. Let it be also noted here *en passant* that, for some researchers, measurement—in its general conception as the process of "*transformation of possibilities* [quantum potentialities generically represented by ψ] *into facts* [measurable spacetime events]"—is the quintessential feature of the quantum, capturing a *fundamental irreversibility or 'time-asymmetry'* supposedly inherent in quantum theory [176]. Similarly, and in the context of QG, Penrose has not only propounded a theory for quantum state reduction due to the gravitational field [299, 301, 302], but also he has gone even further to propose that "*the true quantum gravity is a time-asymmetric theory*"—a proposal basically resting on the Weyl curvature hypothesis supporting time-asymmetric initial conditions for the quantum Universe [300].

[554]We will return to comment more on these issues in the context of QG proper and, as a result, question 'quantization' altogether, in subsections 6.6 and 6.7. below.

finite local power of the **A** *employed.* As far as geometry and space(time)[555] are concerned, they are completely encoded in the abelian **A**—our measurements (coordinatizations or localizations) of 'it all'[556] yielding commutative 'numbers', the local sections of the arithmetics' sheaf **A**, in the process. At the same time, the 'arbitrary' choice[557] and introduction of **A**, and the concomitant (back re)action of the (quantal) field $\mathcal{D}$ to change the local particle states of the quantum represented by the local sections of $\mathcal{E} = \mathbf{A}^n$, is similar to how, upon the perturbing measurement that the classical (exo)system exercises on the quantum (endo)system, the latter's (local) states dynamically change.[558]

A quantum Zeno-type of paradox for the 3rd quantum ADG-fields. Note here the following apparent paradox: while the algebraic **A**-connection field $\mathcal{D}$ 'derives' from **A**,[559] it ultimately elude them (*ie*, our generalized measurements do not 'respect' $\mathcal{D}$ and the latter is not an **A**-tensor—an **A**-sheaf morphism). In a sense, **A** is $\mathcal{D}$'s 'source' (producer), but not its 'sink' (register). Once a **A** 'gives rise' to a $\mathcal{D}$,[560] $\mathcal{D}$ ultimately eludes it. Equivalently, when we introduce (employ) a specific **A** to extract the particle/position-coordinate aspect of the $\mathfrak{U}$-unitary ADG-field $\mathfrak{F} := (\mathcal{E}, \mathcal{D})$—as it were, to localize and geometrically represent $\mathcal{D}$—the latter reacts and effects dynamical changes of local particle/position-states—the local sections of $\mathcal{E}$.

7.3 No Background Geometry to Work with: An Impediment or a Boon to QG?

As alluded to above, the dynamically autonomous ('synvariant'), third gauge ('purely gauge'), third quantum ('self-quantum') and background spacetimeless field $(\mathcal{E}, \mathcal{D})$ of ADG-gravity may prove to be a suitable notion via which to address and tackle certain caustic both structural and conceptual issues in current QG research.

One of these major issues, especially in non-perturbative QGR in its connection based loop QG version [331, 383, 351], is to formulate the theory (*ie*, the quantum dynamics) in a genuinely back-

[555] As noted earlier, in the classical theory (CDG) for instance, the space(time) manifold M is extracted from $\mathbf{A} \equiv \mathcal{C}^\infty_M$ by Gel'fand duality and spatialization.

[556] And recall that in ADG the '*it all*' above essentially corresponds to $\mathcal{D}$, and **A** merely represents our actions towards locating, measuring it—*ie*, the algebraic field $\mathcal{D}$.

[557] The epithet 'arbitrary' here is closely analogous to the completely arbitrary (gauge) choices (and conditions) that the external macroscopic observer makes (and imposes) about what property of the quantum system to measure.

[558] In turn, this is similar to how upon determination of the position (particle aspect) of a quantum system, the act of measurement dynamically disturbs (changes) its momentum, which thus becomes indeterminate or 'quantum fuzzy'.

[559] After all, as discussed in section 3, all DG boils down to **A**!

[560] That is, once *we* use an **A** to geometrically 'capture' (represent) the field $\mathcal{D}$.

ground independent fashion [3]. Roughly, by '*background independence*' it is meant '*background metric independence*'—*ie*, unlike in the usual (mainly perturbative) approaches to QG where one fixes a (usually flat, Minkowski) background metric in order to formulate the quantum dynamics (and expand the relevant quantities about it, as well as to impose meaningful commutation relations among them),[561] here there is no such desire since, anyway, it appears to be begging the question to fix *a priori* (and by fiat!), and moreover to duplicate, the one and only dynamical variable of GR in its original formulation—*ie*, the spacetime geometry (metric).[562] Indeed, Geroch had noticed very early [155] this characteristic feature of GR when he tried, in order to get a better idea of what is a singularity in GR, to compare gravitational singularities with the infinities assailing QFT:

(Q7.8) "...Our intuitive idea of what a singularity should be in Einstein's theory comes from the relatively well-understood infinities which arise in classical field theories, e.g., electrodynamics and hydrodynamics. *Unfortunately, general relativity differs from these theories in one important respect: whereas in other field theories one has a background (Minkowskian) metric to which the field quantities can be referred, in general relativity the 'background metric' is the very* [dynamical][a] *field whose singularities one wishes to describe*[b]..."

[a]Our addition.
[b]Our emphasis.

In the context of QG proper, Baez too, for example, is similarly categorematic in [30]:

(Q7.9) "*The main problem in quantum gravity is that there is no background geometry to work with*[a]..."

[a]Our emphasis.

Let it be stressed here that Ashtekar and coworkers have succeeded over the years in formulating loop QG in an authentically fixed background metric independent way [20], albeit, *a smooth spacetime manifold is still retained in the background* [11, 12]—or else, how could one still use differential geometric ideas and constructions [16] in QG research?[563] We thus read from [11]:

[561]The reader should note that this (flat) background metric dependence is also a feature of the (perturbative) string-theoretic approach to QG.
[562]Let alone that, by fixing the said background metric, one runs the risk of lifting the manifest diffeomorphism invariance of the classical theory (GR). (See below.)
[563]This is another manifestation, now in QG proper, of the CDG-conservatism and monopoly mentioned in section

(Q7.10) "...In this approach, one takes the central lesson of general relativity seriously: *gravity is geometry whence, in a fundamental theory, there should be no background metric.*[a] In quantum gravity, geometry and matter should both be 'born quantum mechanically'. Thus, in contrast to approaches developed by particle physicists, one does not begin with quantum matter on a background geometry and use perturbation theory to incorporate quantum effects of gravity. *There is a* [background/base][b] *manifold but no metric*,[c] or indeed any other physical fields, in the background..."

[a]Our emphasis.
[b]Our addition to make a point.
[c]Our emphasis.

and Ashtekar goes on to stress further:

2 in the context of the singularities of classical gravity (GR). For example, the new connection variables [7] employed in the loop approach to canonical QGR are *smooth* (spin-Lorentzian) connections based on a differential spacetime manifold, let alone that, as noted earlier, the smooth metric is still implicitly present in the theory as it is carried by the smooth tetrad (*vierbein*) variables (1st-order formalism). On the other hand, the (canonical) quantum commutation relations imposed in the theory are genuinely covariant and no fixed (Minkowski) metric is evoked to effectuate them.

(Q7.11)

> "...Although there is no natural unification of dynamics of all interactions in loop quantum gravity, it does provide a kinematical unification. More precisely, in this approach one begins by formulating general relativity in the mathematical language of connections, the basic variables of gauge theories of electro-weak and strong interactions. Thus, now the configuration variables are not metrics as in Wheeler's geometrodynamics, but certain spin connections; the emphasis is shifted from distances and geodesics to holonomies and Wilson loops. Consequently, the basic kinematical structures are the same as those used in gauge theories. *A key difference, however, is that while a background metric is available and crucially used in gauge theories, there are no background fields whatsoever now. This absence is forced on us by the requirement of diffeomorphism invariance (or 'general covariance').*[a]
> This is a key difference and it causes a host of conceptual as well as technical difficulties in the passage to quantum theory. For most of the techniques used in the familiar Minkowskian quantum theories are deeply rooted in the availability of a flat background metric. It is this structure that enables one to single out the vacuum state, perform Fourier transforms to decompose fields canonically into creation and annihilation parts, define masses and spins of particles and carry out regularizations of products of operators. Already when one passes to quantum field theory in curved spacetimes, extra work is needed to construct mathematical structures that can adequately capture underlying physics. *In our case, the situation is much more drastic: there is no background metric whatsoever!*[b] ..."
>
> [a]Our emphasis.
> [b]Our emphasis.

ADG-gravity is genuinely background independent. By contrast, in ADG the theory[564] is not only formulated solely in terns of the gravitational **A**-connection $\mathcal{D}$ without at all the presence of a metric ('purely gauge, half-order formalism'),[565] but also, *a fortiori*, no base differential

[564]That is, the third gauge, third quantum (vacuum) gravitational dynamics (29).

[565]To be sure, as noted earlier in section 3, one (*ie*, *we*, the experimenters/measurers/geometers) can externally (*ie*, by hand) impose an **A**-metric ρ, and then require that *it* be compatible with the gravitational **A**-connection field $\mathcal{D}$, but this ρ has nothing to do with the *physical* gravitational field $\mathcal{D}$ itself. One may think of ρ as an optional,

spacetime manifold is used whatsoever in (the differential geometric formulation of) the theory.[566] In this sense, which is even more 'drastic' than the (still manifold-bound) situation in loop QG,[567] *ADG-gravity is truly background independent.* Of course, it goes without saying that, since singularities are inherent in $\mathcal{C}^{\infty}_{M}$ (*ie*, in the base differential manifold M), loop QG (and of course other continuum based approaches to QG)[568] still has to reckon with them—that is to say, they persist being problems for the theory, hence the theory still aims at 'resolving' them in one way or another.[569]

7.4 Penrose's 'Singularity Manifold': Not an Issue Whether QG Would (let alone Should) Remove Singularities

In the beginning of this section we saw that currently [305, 162, 303] (Q?.?, Q?.?) there is hope from theoretical physicists that QG will (or perhaps *should*) remove the singularities of the classical manifold and CDG-based theory (GR). Joshi in [219], for example, is expressly optimistic (even though he is aware of the manifest absence of a full-fledged QG theory):

(Q7.12) "...Now, if one takes the quantum fluctuations of the spacetime geometry into account, the spacetime singularities or the geodesic incompleteness property of the spacetime is not invariant under such metric perturbations. Quantum fluctuations in spacetime geometry will change the geodesic completeness properties of the spacetime and *it may be possible to envisage a scenario in which, in spite of our not having a full quantum gravity theory as yet, the classical problem of spacetime singularities might be resolved*[a]..."

[a]Our emphasis.

and similarly, Ashtekar also expressed hope for the potential 'resolution' of singularities by QG in the following words taken from [8]:[570]

auxiliary extra structure without immediate physical meaning in ADG-gravity.

[566]That is, it is not necessary in the theory that one assumes up-front $\mathbf{A} \equiv \mathcal{C}^{\infty}_{X}$ as structure sheaf.

[567]See last sentence in (Q?.?) above.

[568]Including string theory.

[569]We will comment on this shortly in connection with a recent 'resolution' of the inner Schwarzschild singularity achieved by loop QG means in [279, 200].

[570]All emphasis below is ours.

(Q7.13) "*...Returning to the singularities of general relativity, the hope is that, in quantum gravity, a similar interference would occur between probability amplitudes for various space-time geometries and prevent the occurrence of infinities...*"

On the other hand, we may consider the following exchange taken from the relatively recent debate between Hawking and Penrose about the 'nature of spacetime' [186]:[571]

(Q7.14) "**Question:** Do you think that quantum gravity removes singularities?

Penrose: *I don't think it can be quite like that...A true theory of quantum gravity should replace our present concept of spacetime at a singularity. It should give a clear-cut way of talking about what we call a singularity in classical theory. It shouldn't be simply a nonsingular spacetime, but something drastically different*[a]*...*"

[a]Our emphasis.

We maintain that Penrose is talking about a drastically different, from the usual notion, '*singularity manifold*' and not merely about one from which singularities are just surgically excised in an *ad hoc* fashion—in his own words, "*simply a nonsingular spacetime manifold*" with singularities removed by 'theoretical fiat'.[572] In other words, a genuine theory of QG should be clear and definitive about the notion and role of singularities in GR—as it were, it should be clear about their origin, their (physical) meaning and their potential utility,[573] while at the same time it should point out precisely to the reasons for their 'anomalous' status in the smooth manifold and CDG-based GR. On the whole, one could say that QG should delimit sharply the latter's domain of consistency and applicability, thus make plain why singularities are (differential geometric) anomalies and in what sense they are physically significant, if they are at all.

Ultimately, and since the sought after QG is supposed to shed more light on, or even remedy completely, the problem of singularities of GR [8, 219], we hope that by the ADG-theoretic means employed herein to evade singularities we have on the one hand *given a clear-cut way of talking about what we call a singularity in the classical theory*,[574] and on the other, formulate (even if indirectly) the notion of a 'singularity manifold'. Indeed, ADG-gravity has highlighted and clarified precisely these two points, namely that,

[571]Again, in Penrose's reply below, emphasis is ours.
[572]Recall Hawking's words, also taken from [186], in (Q?.?).
[573]That is to say, their physical significance and implications, if any.
[574]And it must be emphasized once more that *there is no well defined notion of singularities in the classical theory* (GR) [155, 87]!

1. Singularities, at least in their most clear-cut perception (and 'definition') so far as DGSs, are inherent in $\mathcal{C}^{\infty}_{M}$, and as a result, they are the intrinsic 'faults' of the base (spacetime) manifold-effectuated CDG. They appear to be 'incompleteness' or even 'breakdown' points of Einstein's gravitational field law of GR simply because the latter is modelled differential geometrically (*ie*, as a differential equation) by the smooth manifold based CDG-means, which in turn carry (in the background M) the seeds of their own inapplicability and destruction in the guise of $\mathcal{C}^{\infty}$-singularities.

2. Concerning the notion of 'singularity manifold', from the our ADG-theoretic perspective this is understood as pertaining to a realm which is not just a differential spacetime manifold with its singular points either being surgically excised in an *ad hoc* fashion so as to retain classical differentiability and regular (smooth) laws of physics in the remaining 'effective manifold',[575] or just being pushed to the boundary of its regular points again in order to retain the usual laws in the smooth spacetime 'interior' (bulk) [86, 87], but *one that replaces our present concept of spacetime at a singularity and is drastically different from simply a nonsingular spacetime* (manifold). Indeed, quite on the contrary, in the fundamentally base manifoldless ADG-gravity a 'singularity manifold' can be as pathological-looking (always from the CDG-viewpoint) as Rosinger's spacetime foam dense singularity manifolds we encountered earlier, yet we saw that the gravitational field law defined by $\mathcal{D}$ still holds at their very presence.

3. Thus, *in summa*, in the following way ADG delineates clearly what is a singularity in the classical theory: *a singularity*—at least, a DGS—*is an internal blemish of* M*, ultimately, a genetic shortcoming of CDG, coming from our identifying 'physical spacetime' with a base differential manifold. All singularities are in a strong sense 'coordinate' ones, yet, by being absorbed into* $\mathbf{A}$*, while we are still able to retain (by ADG-means) all the essentially algebraic differential geometric mechanism that is not at all dependent on a geometrical background manifold ('spacetime'), they present no obstruction or breakdown sites to the gravitational dynamics (Einstein equations) defined by the gravitational field* $\mathcal{D}$.[576] GR, as a *physical* theory 'defined' by the gravitational dynamics (Einstein equations), by no means breaks down at singularities, only its mathematical scaffolding—the manifold supported CDG—collapses due to its own faults (singularities). The "*present concept of spacetime at a singularity*" is replaced by the 'holistic', 'unitary' concept of the ADG-field $(\mathcal{E}, \mathcal{D})$ having integrated or incorporated singularities of any kind into $\mathbf{A}$ (or equivalently, into the associated sheaf $\mathcal{E}$ that *we* employ to represent $\mathcal{D}$). Moreover, *prima facie* no (formal process of) 'quantization' (of the classical theory) is evoked to be able to deal with (the) singularities (of the classical)

[575] Thus abide by the aforesaid 'manifold conservatism and monopoly'.
[576] Singularities are 'transparent' to $\mathcal{D}$.

theory, while at the same time ADG-gravity inherently (*ie*, 'by construction' or 'by virtue of its fundamental building blocks'—the fields $(\mathcal{E}, \mathcal{D})$) possesses quantum features. *ADG-gravity is quantum from the very start.*

In view of this 'self-quantumness' of ADG-gravity, in the next subsection we question and criticize many attempts in the past to arrive at a conceptually sound and calculationally consistent (finite) QG not only as a straight-out quantization of GR along QFTheoretic lines while still retaining a background differential manifold (for DG's sake), but also by attempts to quantize directly spacetime itself.[577] Our basic criticism revolves about the following ADG-gravity motivated question: *in view of the fact that an external 'spacetime'*—whether 'continuous' or 'discrete'—*plays absolutely no role and therefore has no physical significance whatsoever in the 3rd gauged, 3rd quantized ADG-gravity, which relies solely on the purely algebraic, 'auto-dynamical' ('synvariant') gravitational* **A***-connection field* $\mathcal{D}$*, what right do we have (or more mildly put, of what relevance is to us) to attempt to quantize spacetime in order to formulate a quantum theory of gravity?* The point here is that ADG-gravity achieves a quantum theoresis of the gravitational field itself, without the involvement of any external '*spacetime paraphernalia*', which are thus not begging for any 'quantization' whatsoever.[578]

As a contrasting warmup to the next subsection, we bring forth some recent words of Baez from [33]:

(Q7.15) "...Here I would like to propose another possibility, namely that *quantum theory will make more sense when regarded as part of a theory of spacetime*[a] ..."

[a]Baez's emphasis.

7.5 Whence Quantization of Spacetime for QG? Trying to Fit an Elephant (GR) Into a Flea's Pyjamas (QFT=SR+QM) in the Presence of a Chimera: the Background Spacetime Continuum

We begin our critique by commenting first on various attempts at '*quantizing the gravitational field-fiber while retaining a base geometrical differential manifold*'.

[577]In general terms, these two attempts may be coined in bundle-theoretic jargon, '*quantization of the fiber*' (while still retaining a base continuum) and '*quantization of the base*', respectively.

[578]Also, in ADG-gravity the traditional classical/quantum distinction inevitably loses its meaning, and so does the formal Correspondence Principle (CP) which typically is thought of as 'undoing quantization'. Accordingly, the Planck length should also be subjected to criticism (see below).

7.5.1 Attempts at quantizing the fiber: arriving at QG by emulating QFT?

Let us first recall some general doubts that Einstein had (as early as prewar times!) about the application of QFTheoretic ideas to GR. By now it has been well established that, for Einstein, the essential, 'defining' so to speak, feature of his unitary field theory research programme would be *some sort of generalization of the PGC of GR*—in any case, the discovery of a logically simple unifying principle underlying both matter and gravity—and certainly not a 'regress' to a fixed background spacetime (geometry), even if quantum principles were to be suitably accommodated, as is the case in the nowadays flat quantum field theories of matter and, *in extenso*, in various attempts at formulating QG abiding by principles and constructions borrowed from QFT on the Minkowski space continuum. Characteristically, Stachel remarks in [368]:

(Q7.16) "...Of course, one could just abandon the dynamical view of the space-time structure, and return to the pre-general-relativistic concept of this structure as a given, non-dynamical one. Indeed, this is the route that (special-)relativistic quantum field theory has taken. I need hardly add that to Einstein such an abandonment represented not progress but a singularly dangerous regression:[a] '*You consider the transition to special relativity as the most essential thought of relativity. I consider the reverse to be correct*'[b]..."

[a]And here Stachel quotes Einstein.
[b]Our emphasis.

In particular, Einstein was rather critical and suspicious towards attempts to *quantize* in one way or another, by applying QFTheoretic concepts and methods (be it on a flat or a curved background spacetime manifold),[579] his relativistic field theory of gravity (GR). Even more generally, he was skeptical about any research endeavor that starts from a classical theory and applies to it in a 'blind', *ad hoc* fashion a formal quantization procedure ('algorithm' or 'recipe') in order to arrive at a quantum description of Nature—as it were, to obtain a formal quantum analogue (or simile!) of a pre-established classical theory.[580] For the attempts to quantize a pre-existing classical field

[579]On a curved spacetime manifold the approach is usually referred to as '*quantum field theory on a curved spacetime*' and is regarded as a stepping stone—a 'semi-classical' or 'semi-quantum' one(!)—to the full QG theory [46, 152].

[580]For example, the nowadays so-called *Quantum General Relativity* (QGR) (or perhaps more fittingly coined, *Quantized General Relativity*) approach to QG may be thought of as the attempt *par excellence* to quantize GR by persistently retaining a classical smooth geometrical background spacetime manifold. Of course, *Canonical Quantum Gravity* is *the* 'canonical' (pun intended) example of such (classically based) endeavors (see Finkelstein quotations below). Bold pioneers of such attempts to 'quantum field theorize' about GR (albeit, in particle physics sort of ways) in order to arrive at QG are Feynman [140] and Weinberg [410].

theory for example, we read from the concluding lines of [372] about Einstein's gut-feeling of '*no-go*' (regarding applications of QFT to GR):

(Q7.17) "*...I do not believe that it will lead to the* [QG][a] *goal if one sets up a classical theory and then 'quantizes' it.*[b] This way was indeed successful in connection with the interpretation of classical mechanics and the interpretation of the quantum facts by modification of the theory on a fundamentally statistical basis. *But I believe that, in attempts to transfer this method to field theories, one will hit upon steadily mounting complications and upon the necessity to multiply the independent assumptions enormously*[c]*...*"

[a] Our addition.
[b] Our emphasis.
[c] Again our emphasis.

Parenthetically, independently of the project of field-quantization (of gravity), and in view of the first sentence in the quotation above, we recall from [272] Landau and Lifshitz's description of the 'paradoxical' character of quantum theory *vis-à-vis* classical mechanics from which it was originally derived (by means of 'quantization'):[581]

(Q7.18) "*...Quantum mechanics occupies a very unusual place among physical theories: it contains classical mechanics as a limiting case, yet at the same time requires this limiting case for its own formulation....*"

To return to Einstein, his warnings for a '*dangerous regression*' pertains precisely to our blind applying the QFT-algorithm to GR in order to quantize gravity as a field theory, albeit, persistently abiding by the spacetime continuum. Arguably, the said '*dangerous regression*' may be attributed to the following '*dangerous analogy*', which is based on the paradigm of the classical (field) theory *par excellence*, GR:

[581] The quotation below also appears in our last paper [272]. Below, emphasis is ours.

(R7.2) *Einstein's equations in the presence of matter, $G_{\mu\nu} = \kappa T_{\mu\nu}$, tempt one to infer that, since the right hand side of the equations—which subsumes the energy-momentum contribution of matter (sources) to (of) the gravitational field—has been successfully quantized by using the concepts and methods of QFT, by 'symmetry' (or analogy) the same must be done to the left hand side—as it were, to 'quantize (field theoretically the spacetime) geometry'.* For, if matter-energy-momentum is quantum—in fact, 'quantized' field theoretically—why not try to quantize spacetime geometry as well?[a] To apply a 'quantum metaphor' to Wheeler's one-line *résumé* of GR "*energy-matter here, tells spacetime geometry how to curve there*": *quantization of energy-matter here, entails quantization of spacetime geometry there.*[b]

[a] Generally speaking, this may be thought of as *the* motivating analogy behind recent attempts at formulating a so-called 'quantum Riemannian geometry' [11, 400, 12].

[b] In our opinion, an even more misleading inference which is usually drawn from the 'dangerous analogy' to the classical theory above is that *one should attempt to somehow quantize spacetime structure itself—ie*, one must give a quantum description of the smooth spacetime manifold of GR (not just of the gravitational field that lives on it), or equivalently, and more mathematically speaking, one could attempt to quantize the manifold based Calculus (CDG) (*eg*, formulate a noncommutative kind of differential geometry [91, 92]). Implicit here is a dissection or dissociation of the gravitational field-fiber from its base spacetime, with a concomitant ascription of an independent (physical?) reality to the latter. We will return to comment on this '*dangerously misled*' above from the perspective of ADG-gravity in the next sub-subsection as well as in the concluding section.

However, Einstein would not 'bite the tempting bait' that GR and QFT presented him. He was caustically critical (albeit, in an uncertain, 'agnostic' manner)[582] of the attempts of his contemporaries at quantum field theorizing 'non-general covariantly' about the generally covariant GR, as we read again from [372]:

[582] This agnosticism of his, originating from his deep doubts that a singularity-free (unitary) field-theoretic description of quantum reality—one that *a fortiori* was free from the shackles of the spacetime continuum, could actually be formulated. (In the next section we will discuss in detail, in the light of ADG and [368], this 'second', 'anti-continuous field theory on the spacetime continuum', facet of Einstein.)

"...Contemporary physicists do not see that it is hopeless to take a theory that is based on an independent rigid space (Lorentz-invariance) and later hope to make it general-relativistic (in some natural way)..."
and
(Q7.19) *"...I have not really studied quantum field theory. This is because I cannot believe that special relativity theory suffices as the basis for a theory of matter, and that one can afterwards make a non-generally relativistic theory into a generally relativistic one. But I am aware of the possibility that this opinion may be erroneous*[a]*..."*

[a]In both quotes above emphasis is ours.

More recently, and in the context of canonical quantization (of gravity), Finkelstein too has aired similar doubts about quantizing (canonically) a classical theory (like GR). In [148] he notes for instance:

"...Canonical quantization works because the Poisson bracket and its commutator replacement have the same meaning...Indeed, commutator relations become Poisson bracket relations as $\hbar \rightarrow 0$...[However,][a] in relativity there are informal understandings about how to correct a non-relativistic theory to make a relativistic one that has more chance of working. After we formulate the relativistic theory we do not need the pre-relativistic one in principle, and can discard it and the relativization rule. They are
(Q7.20) part of history, not part of the theory.
Canonical quantization too is not part of the theory but only a scaffolding to use while repairing the theory and to discard after its work is done. There is no fundamental classical system underlying any quantum system.[b] The quantum commutator relations are more fundamental than the classical Poisson brackets, which are approximate consequences in a suitable classical limit..."

[a]Our addition for text-continuity.
[b]Our emphasis.

and he subsequently gives *physical* reasons for *not* canonically quantizing a classical theory (like gravity *à la* GR), especially when the classical spacetime structure (*ie*, the manifold) supporting the latter is viewed—as it is viewed in *Quantum Relativity* (QR)—as a *macroscopic quantum effect* akin to superconductivity or superfluidity:

(Q7.21) "*...We should not apply canonical quantization to spacetime structure if spacetime is a supercondensate. Canonical quantization is a reasonable way to reconstruct a quantum theory from its classical behavior at high quantum numbers, but it will not recover a quantum theory from the behavior of a low temperature supercondensate.*

For example, one could not discover the helium atom by canonically quantizing the macroscopic two-fluid theory of superfluid helium Nor could one could discover the electron theory of solids by canonically quantizing the field theory of the Josephson potential of a superconductor.[a] Despite their macroscopic nature, such fields are themselves best understood as [order] parameters of coherent quantum modes, not as limits of normal operators (observables) as $\hbar \to 0$.

Instead *we suppose that spacetime structure is already quantized.*[b] Some of the macroscopic variables with which we describe spacetime and gravity today, such as the spacetime metric tensor, already have a quantum nature, like order parameters of a superfluid..."

[a]Our emphasis. In gravity too, "*one would expect to find the graviton by canonically quantizing the Einstein equations of GR no more than one would hope to discover the fine structure of the water molecule by straight-out quantizing the Navier-Stokes equations of hydrodynamics*" (David Finkelstein loosely quoted from [145]). Let it be noted here that to much the same conclusion, but by significantly different motivations and arguments (issuing mainly from thermodynamics), Ted Jacobson was also led in [217].

[b]Our emphasis.

The fundamental 'supposition' (or 'motto') of QR emphasized in the last paragraph in (Q?.?) above—namely, that *spacetime structure is already quantized*—is on the one hand a *conceptual or intuitive meeting point* between our ADG-theoretic perspective on QG and QR, and at the same time on the other, a *technical point of departure* of ADG from QR. This paradoxical situation arises from the fact that while the conceptual meeting point is that both ADG and QR basically assume that *all is quantum*[583], their (technical) ways of implementing this fundamental intuition differ significantly, in the following sense:

[583]For example, we have argued (and will argue more in the sequel) that the ADG-gravitational field $(\mathcal{E}, \mathcal{D})$ is in a strong sense '*already quantum*' ('3rd-quantized').

(R7.3) While ADG fundamentally assumes that '*all is field*'—an inherently dynamical (autodynamical) and already quantum (3rd-quantum and 3rd-gauge gravitational) field—without reference at all to a background spacetime (be it a continuum or a discretum),[a] which thus does not exist at all (in a physical sense)[b] in the theory, QR assumes that '*all is spacetime*' structure (*eg*, topology)—an already quantized spacetime structure (*eg*, quantum topology or quantum geometry).[c]

[a]In fact, *in the algebraic ADG there is no space(time) at all*, or if one wishes to think in such 'geometrical' ('spatio-temporal') terms, *'spacetime geometry' is already inherent in the dynamical gravitational field* (for more on this, see below).

[b]That is to say, as a *dynamical* entity.

[c]At this point we should stress again our preference for the expression '*already quantum*' instead of the '*already quantized*' used in (Q?.?) above, for if *all is quantum ab initio*, presumably it has no need of being 'quantized' (even if the latter word is used 'philologically' to distinguish it all from 'fictitious' non-quantum entities, which anyway do not exist in Nature). All in all, the following question is begging the question: *how does one quantize the quantum?* In any case, in the next sub-subsection we will criticize head-on, with ADG-gravity in hand, any attempt at quantizing spacetime itself.

In the next sub-subsection we criticize the aforesaid 'point of departure' (*ie*, *quantum spacetime structure instead of quantum field*), and then we comment on the 'meeting point' (*all is quantum*), always from an ADG-theoretic vantage.

But before we go on to critically comment on various attempts to quantize the base spacetime itself, let us close the present 'quantization of the field-fiber' sub-subsection by mentioning the currently most plausible and potentially successful of all attempts (so far) at quantizing GR as a field theory; albeit, still manifestly abiding by a base differential manifold. This is the so-called *Quantum General Relativity* (QGR) scenario, and in particular, its approach via *Loop Quantum Gravity* (LQG) and its recent offspring, *Quantum (Riemannian) Geometry* (QRG) [330, 9, 382, 383, 351, 10, 18, 19, 11, 12, 20, 328].[584] Of course, it goes without saying that this is not a place for us to review this approach to QG, but we would like to highlight certain basic conceptual features and results of it so as to set a platform for a brief comparison with ADG-gravity:[585]

[584]This is not meant to be a comprehensive array of references on QGR, LQG and QRG, but the citations in [382, 383, 351, 328] for example constitute pretty much a complete list.

[585]Many of the items in the following list have already been mentioned and discussed in the present paper, as well as in our past trilogy [270, 271, 272] and the more recent paper [317].

- LQG approaches QG as a quantum *gauge theory*, since the basic gravitational variables involved in it are Ashtekar's 'new variables': spin-Lorentzian ($SL(2,\mathbb{C})$-valued) connections [7].[586]

- At the same time, the *vierbein* (comoving tetrad frame) field is also another set of fundamental variables (alongside the connection) so that the formalism is closely akin to Palatini's so-called first-order formalism whereby, as noted earlier, variation of the corresponding action functional with respect to the tetrad field yields the Einstein equations, while variation with respect to the connection field yields the metric compatibility (torsionless) condition for the latter.

- As also stressed earlier, although LQG has managed to arrive at a genuinely background metric independent formulation of (canonical) QGR, still heavily relies on a base differential manifold for its (differential geometric) formulation,[587] with all the well known technical and conceptual problems that come along with this dependence (*eg*, the inner product/quantum measure problem, the problem of time, the problem of observables *etc*).

- On the other hand, since LQG still employs a base differential manifold, the problem of singularities in the classical theory of gravity (GR) still persists[588] and has to be dealt with in one way or another. Indeed, as we shall see shortly, LQG 'resolves' singularities by quantizing the base spacetime itself, via its offspring theory coined QRG. Thus, broadly speaking, LQG belongs to that general category of approaches to QG that maintains (or better, hopes) that a cogent quantum theoresis of gravity will resolve (or even ultimately, do away with) singularities (Q?.?); moreover, this could be achieved by a quantization of spacetime itself—as it were, as we shall see soon, the quantization of the (base) spacetime, which is *a priori* assumed to be a classical smooth continuum, is an 'outcome' (or 'byproduct') of the quantization of the gravitational field (fiber).

The last remarks hint at to the next sub-subsection where we comment on attempts to quantize (or arrive at a quantum theory of) the base spacetime itself. The general philosophy there is that since gravity is usually associated (geometrically) with the structure of spacetime, quantum (or a quantization of) gravity could not be thought of apart from quantum (or a quantization of) spacetime structure.

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[586]Arguably, the original interest in Ashtekar's new gravitational variables was due to the fact that the latter simplify significantly the (Dirac) constraints when gravity (GR) is treated canonically.

[587]This means that the aforesaid connection and tetrad fields are *smooth*.

[588]For after all, as we argued extensively in this paper, the culprit for the $\mathcal{C}^\infty$-smooth gravitational singularities in the manifold based GR is our assumption of spacetime as a differential manifold M. Singularities are inherent in M—as it were, they are innate in $\mathcal{C}^\infty_M$.

7.5.2 Attempts at quantizing the base: the spacetime continuum stands in the way towards quantizing gravity as a field theory, so why not quantize it first?

If a 'direct' quantization—canonical (Hamiltonian), covariant (Lagrangian, action based) or other—of GR, regarded as a continuous field theory on the smooth spacetime continuum, fails for one reason or another, the next tenable position would be to attempt to quantize spacetime structure itself. That is to say, one could suppose for instance that since the 'second quantization algorithm' fails to yield when applied to the field-inhabited fibers themselves, one should perhaps try to search for (or better, ascribe!) quantum traits such as discreteness, noncommutativity *etc* in (to) the classical base spacetime manifold itself—the domain of definition of the said (continuum) fields. In other words:

(R7.4) *If the (gravitational) field-fiber (over the spacetime continuum)*[a] *appears to persistently resist quantization, then why not try to quantize directly the base spacetime manifold itself?*

[a]For instance, in bundle-theoretic terms, one think of the spacetime metric representing the gravitational field as a (local) cross-section of the smooth symmetric tensor bundle over the spacetime manifold M, acting (as a real symmetric bilinear form) on pairs of local sections of the tangent bundle on M (vector fields).

Finkelstein *et al.* for example have recently stated this apparent need for quantizing the spacetime base explicitly and succinctly in [37]:

(Q7.22) "...*Quantizing gravity without quantizing space-time introduces yet a further fragility.*[a] The canonical group of Einstein gravity is much bigger than the classical diffeomorphism group Diff, for it also includes transformations that couple field variables into the space-time coordinates, like de Donder or harmonic coordinates, and Diff does not. In the quantum theory the corresponding space-time coordinates will inherit the non-commutativity of the field variables. It is fragile to postulate that a frame exists in which the space-time coordinates commute with the field variables..."

[a]Our emphasis. 'Fragility' in [37] is a term with technical group-theoretic connotations that we should not be bothered about here. The reader is referred to the paper for its explanation.

The quantization of spacetime itself may also be alternatively motivated as well as be implemented in many different ways issuing from ideas grounded in the so far three main approaches to QG, namely, the Euclidean QG approach, the Ashtekar (LOG and QRG) programme and superstring theory [206]. In the last reference, Isham discusses the issue of whether the basic, essentially classical spacetime continuum based, formalism of quantum mechanics can be suitably modified in a way so as to suit the theoretical possibility that spacetime itself be quantized. Characteristically, we quote him saying:

(Q7.23)

> "...The Euclidean programme, the Ashtekar scheme, and superstring theory all use what are, broadly speaking, standard ideas in quantum theory. In particular, they work with an essentially classical view of space and time—something that, arguably, is a *prerequisite*[a] of the standard quantum formalism. *This raises the important question of whether quantum theory can be adapted to accommodate the idea that spacetime itself (i.e., not just the metric tensor) is subject to quantum effects: surely one of the most intriguing challenges to those working in quantum gravity.*[b]
> (New Section 5.2) **Quantising Space-time.** *Some of the many issues that arise can be seen by contemplating how one might try to quantise spacetime 'itself' by analogy with what is done for—say—the simple harmonic oscillator, or the hydrogen atom. Of course, this may be fundamentally misguided... Nevertheless, it is instructive to think about the types of problem that occur if one does try to actively quantise spacetime itself—if nothing else, it reveals the rather shaky basis of the whole idea of 'quantising' a given classical structure*[c]*...*"
>
> [a] Isham's emphasis.
> [b] Our emphasis.
> [c] Again, our emphasis.

For an instance of a reason why one might attempt to quantize spacetime itself, the appearance of a fundamental, 'minimum' space(time) length—the so-called Planck length ℓ_p (and correspondingly, the Planck time t_P)—when (the physical constants of) SR (c), GR (G) and QM ($\hbar$) are envisioned to be combined in the (still uncharted) QG domain, could be taken as suggesting a natural regularization spacetime continuum cut-off in a covariant, path-integral quantization of gravity for example. Similarly, in view of (R6.?) above, one could also suppose that some sort of *quantum spacetime geometry*, being involved in the left hand side of the 'quantized' Einstein equations, should be formulated [21, 10, 13, 18, 19, 11, 12].[589] With respect to the LQG approach to non-perturbative (canonical) QGR for instance [330, 382, 383, 351], a head-on quantization of

[589] The last two references give a thorough account of the (primarily conceptual) development of this LQG-offshoot approach to QG, QRG.

spacetime has led to discrete expressions ('eigenvalues') for the spatial length, area and volume quantum operators [331, 18, 19, 249, 381], while the quantization of volume also entails indirectly a quantization of time, since the Hamiltonian (constraint), which is supposed to generate the dynamical time-evolution in canonical QGR, can be expressed in terms of the volume operator having a discrete eigen-spectrum [279, 200].[590] Vaas [400] has wrapped-up the basic achievement of LQG and QRG as follows:

(Q7.24) "...This is how quantum geometry revolutionizes our world-view: *Space is quantized like matter!*[a]..."

[a]Our emphasis. For some recent ideas towards a head-on quantization of time, the reader can refer to [60].

and, remarkably, he then continues and stresses the fundamentally relational and *a priori* spacetimeless underpinnings of LQG, which are very close (at least in spirit) to the dynamically 'autonomous', algebraic and background spacetime-free ADG-gravitational fields in which 'spacetime' is inherent:[591]

(Q7.25) "...The question why nothing can be made to fit into half the volume of the smallest unit of space is meaningless in view of such 'space atoms'. It is based on the incorrect assumption of an absolute space in which things fit in... Space and time are not at all fundamental, but rather built up from more basic structures [graph-like entities called 'spin networks'][a]...There is 'nothing' between these graphs. Those entities rest only on themselves, so to speak. '*The spin networks do not exist in the space. Their structures produce the space,*' Smolin stresses. '*And they are nothing but abstractly defined relations...*'[b]..."

[a]Our addition for continuity.
[b]Our emphasis of Smolin's quotation.

while, very recently, Smolin [351] highlighted the main achievement of LQG as follows:

[590]We will comment shortly on how this spacetime quantization result enables one to dynamically extend spacetime past the inner Schwarzschild without encountering any infinity (divergence) of the Ricci tensor. In this *indirect* way—*ie*, via spacetime quantization—LQG claims to 'resolve' the interior Schwarzschild singularity [279, 200], much like we achieved herein—albeit, *directly*, without having to quantize (an *a priori* assumed background differential manifold) spacetime, which anyway does not exist *ab initio* ('by definition') in ADG-gravity.
[591]And its spin-network/spin-foam relatives—see for instance [32, 304, 36].

(Q7.26) "...The result [from the past 17-years' work on loop QG] is a language and dynamical framework for studying the physics of quantum spacetimes, which is completely consistent with the principles of both general relativity and quantum field theory. The picture of quantum spacetime geometry which emerges is to many compelling, independently of the fact that it has been derived from a rigorous quantization of general relativity. The basic structure that emerges is of a new class of quantum gauge field theories, which are background independent, in that no fixed spacetime metric is needed to describe their quantum dynamics. Instead, the geometry of space and time is coded in the degrees of freedom of a gauge field[a]..."

[a]Like ADG-gravity, LQG regards QG as a quantum gauge theory—one that fundamentally hinges on the notion of connection. However, as stressed earlier, in LQG and QRG, unlike in ADG-gravity, although a fixed background metric is not used, a background smooth manifold that supports the said gravitational connection-fields *is*.

Similarly, in Finkelstein's QR there appears to be a fundamental fermionic '*quantum of time*'—the so-called *chronon*. A chronon represents an elementary quantum causal (dynamical) process. An ensemble of chronons and their mutual interactions (interrelations) is then supposed to weave (define) a finitistic ('discrete'), causal and quantal network (relativized and quantized directed graph topology) which replaces the fixed or 'frozen' smooth Lorentzian spacetime manifold of the classical theory (GR) and recovers it as a superconducting phase ('coherent vacuum state'). In turn, the latter corresponds to a macroscopic quantum effect resulting from a coherent Bose-Einstein type of quantum condensation of the said elementary quantum causal processes (chronon Cooper pair-like coherent quantum agglomerations) [145, 146, 147, 148, 150, 337, 338, 339, 340].

There is also another 'discrete' approach to Lorentzian QG, the *causal set (causet) theory* due to Sorkin *et al.* [54, 353, 354, 357, 359, 360, 361, 362]. Causets are locally finite models of spacetime in which what is solely retained from the relativistic Lorentzian continuum of events is the causal or chronological precedence relation between them, while in turn the classical spacetime manifold of GR is regarded as a coarse ('statistical') approximation of the causet, which is thus viewed as the fundamental structure in the quantum deep. Causets as such are not quantum. A classical sequential growth (stochastic) dynamics has already been proposed for them in [321], while a quantum (stochastic) dynamics is currently pursued along quantum measure-theoretic lines [65, 358, 66].[592]

[592]Loosely speaking, here one is looking for a decoherence functional-type of object on the kinematical space of

More on the mathematical side, one could also envision an indirect quantization of the classical smooth base spacetime manifold by quantizing not the continuum itself, but, as it were, the (differential) Calculus (CDG) which is based on it. This was essentially the aim of Connes' celebrated *Noncommutative (Differential) Geometry* (NDG), which, as noted earlier, was achieved primarily by functional-analytic means, while arguably a differential manifold is still somehow retained in the background [91, 222, 92].

On the other hand, in view of all that has been said and achieved in the context of ADG-gravity, we come to ask:

'kinematically consistent' (*ie*, discrete general covariance and Bell causality obeying) causet histories (and their 'observables').

If (the) spacetime (continuum) does not physically exist in the first place, already at the classical level of theoresis of gravity (GR)—that is to say, like the fathers of GR had intuited and warned, '*it corresponds to nothing* [physically] *real, being a mode by which we think, not a condition in which we live—our mental creation so to speak*' (Einstein Q?.?, Eddington Q?.?), and certainly not an element of Physis—*isn't it begging the question to ad hoc (as it were, by 'mathematical fiat') posit its existence*[a] *and then, moreover, try to somehow quantize it on top?*

(R7.5) *All in all, aren't we creating (mathematical) models, subsequently raising them to idols if they prove to be successful,*[b] *while at the same time we forget their 'terrestrial origin' (ie, the fact that they are our own 'fictions' in the first place) and regard them as 'necessary truths',*[c] *only to put an inordinate amount of effort (mental and material resources!) into trying to shoot them down when they actually prove to be problematic and pathological (eg, singularities in QG and the general miscarrying of the spacetime continuum in the quantum deep); and moreover, due to our aforementioned 'forgetfulness', we tend to attribute these pathologies to Physis when the problems simply expose the fact that our creations are of limited applicability and validity, certainly not Nature's own shortcomings and inadequacies?*

[a] In order to actually apply CDG-ideas to Physics.

[b] And the (spacetime) continuum, with the Calculus that comes hand in hand with it, has undoubtedly been the most successful (regarding physical applications) mathematical structure that we have invented so far.

[c] See the Einstein quotation next.

Einstein's words come once again to wake us up to this reality—the '*Golem unreality of spacetime*':[593]

[593] The following quotation concluded our last paper [272].

(Q7.27) "...*Concepts* [like the spacetime continuum][a] *which have proved useful for ordering things easily assume so great an authority over us, that we forget their terrestrial origin and accept them as unalterable facts. They then become labelled as 'conceptual necessities', 'a priori situations', etc. The road of scientific progress is frequently blocked for long periods by such errors. It is therefore not just an idle game to exercise our ability to analyse familiar concepts, and to demonstrate the conditions on which their justification and usefulness depend, and the way in which these developed, little by little*[b]*...*" (1916) [127]

[a]Our addition for making our point clearer.
[b]Our emphasis throughout.

In view of the third gauge ('pure gauge'), third quantum ('self-quantum') field-theoretic underpinnings of ADG-gravity that we have in hand, the critical remarks above about quantizing the base spacetime structure itself acquire further significance. The bottom-line is that ADG not only manages to model key ideas from second and geometric (pre)quantization, such as the quantum particle interpretation/geometric representation ($\mathcal{E}$) of the gravitational field ($\mathcal{D}$) and its (sheaf cohomological) classification as a boson ('graviton': particle mediating the quantum gravitational force; line sheaf of representation) and/or a fermion ('causon': the 'matter'-like quantum of causality acting as a 'source' for the said 'graviton'; vector sheaf of representation), but also unlike the conventional manifold based (fiber bundle-theoretic) ideas and techniques of second (QFT) and geometric (pre)quantization [347, 428, 336, 35], it manages to do this without reference to a background 'space(time)', whether the latter external, 'ambient realm' is taken to be a 'continuum' or a 'discretum'.

This freedom from a base spacetime that ADG gives us about infusing QFTheoretic ideas to QG research cannot be overemphasized. Especially due to the aforesaid faring poorly (so far) of attempts to arrive at a cogent QG by applying QFTheoretic ideas and techniques to GR by retaining a base differential manifold (*eg*, the non-renormalizability of gravity when regarded as a perturbative quantum gauge theory like the other three fundamental forces, or the $\mathrm{Diff}(M)$-associated problems in non-perturbative quantum gauge-theoretic scenaria for canonical QGR such as LQG), the base spacetimeless third quantized, third gauged ADG-gravity may prove to be a fruitful route indeed to QG—a route bypassing directly all the problems that the said approaches encounter due to their assuming a smooth base spacetime continuum. Moreover, and this should be emphasized here, (R7.6) in glaring contradistinction to the other external (to the gravitational field itself) spacetime manifold based (quantum gauge-theoretic) scenaria for QG (such as LQG), *we do not expect 'spacetime' to be also quantized—that is, we do not regard the 'problem' of the quantum structure of spacetime as being 'important', that is, as being inextricably entwined with the problem of QG—simply because from the start there is no background spacetime in our ADG-theoresis of gravity and the ADG-gravitational field is 'already quantum'.*

En passant, let it be also noted here that for other approaches to QG, the attainment of a quantum description of spacetime structure *per se* is supposed to be prior to—in fact, an apparently necessary stepping stone to—a genuinely quantum theoresis of gravity [142, 145, 146, 147, 150, 148, 337, 338, 339, 340, 206, 209, 210, 211, 212, 149, 37]. Still in further contrast to our spacetimeless ADG-musings on gravity, there are certain theoretical schemes that focus solely on a finitistic and quantum theoresis of spacetime (or anyway, of 'space') itself [295, 297, 67].

These critical, 'contrasting' remarks will be of significance in the next sub-subsection when we comment on a recent 'resolution' of the interior Schwarzschild singularity by LQG techniques and results [279, 200], and we compare it with our fundamentally base spacetimeless ADG-theoretic evasion presented above.

7.5.3 A recent 'resolution' of the inner Schwarzschild singularity and its comparison with our ADG-evasion

As noted earlier, all the differential manifold and *in extenso* CDG-based approaches to QG will inevitably have to reckon with the problem of smooth spacetime singularities in the classical theory (GR), simply because singularities are 'inherent' in the background differential manifold employed in GR. Thus, arguably, all these approaches hope (or even expect!) that a cogent quantum theory of gravity will (eventually) 'resolve' (or even ultimately remove!) singularities in some way.

Among these QG approaches, and to that a very promising one(!), is LQG[594] [382, 383, 351]. Notably, within the past half decade, in the context of loop quantum cosmology—the application of LQG ideas and results to quantum cosmology, it has been shown that the initial ('Big Bang') singularity 'predicted' by GR can be indeed 'resolved' or 'bypassed' [51, 52, 13].[595] However, even

[594]Together with string theory, LQG has proven to be so far the most successful approach to (non-perturbative) QG.

[595]Coincidentally, at about the same time as the loop quantum cosmology result above, in the context of the string theory approach to QG, the so-called 'ekpyrotic scenario' also claimed a successful 'passing through' the original cosmological singularity [399, 230, 231].

more significant to the present paper is the following very recent result of Modesto [279], which was also arrived at by LQG means: in a single sentence, *the Schwarzschild black hole singularity of the classical theory (GR) 'disappears' in (L)QG.*[596] In the present sub-subsection we would like to describe briefly this 'disappearance', comment on it and juxtapose it against the 'resolution' (better, 'complete evasion') of the same singularity (*ie*, the interior Schwarzschild one) that was accomplished above by ADG-theoretic means.[597]

Let us first point out again that since, as mentioned before, LQG, although manifestly background metric independent, still expressly employs a base differential (spacetime) manifold for its basic concepts and constructions, the problem of singularities in the classical theory (GR) persists and has to be somehow dealt with in the quantum theory. Without going into any (technical) detail, and quite synoptically, in [279] the inner Schwarzschild singularity is resolved in the following steps:[598]

1. To begin with, one expresses the Ricci scalar curvature, which as noted earlier diverges as $1/r^6$ near the interior ($r = 0$) Schwarzschild singularity, in terms of the spacetime volume.

2. Then, one evokes *the* major result in loop QG, namely, that the said volume is quantized—*ie*, it is promoted to a volume operator having a discrete eigen-spectrum.[599] Thus, near the Schwarzschild black hole, $\mathcal{R}$ is rendered finite and the classical continuum infinities are controlled (in a sense, 'regularized') by quantum theory.

3. Moreover, one can show that the said 'regularization' is not 'kinematical' (*ie*, one that is not *a priori* fixed by hand),[600] but it is a dynamical one, as the Hamiltonian (constraint), which regulates the dynamical, time-evolution in the canonical approach to QGR originally underlying LQG [382, 383], can also be expressed in terms of the volume operator. Thus, as Modesto shows and argues in detail, *the spacetime can be dynamically extended past the inner Schwarzschild singularity, with no infinity involved at all in the process.*

[596]See also [200] for the wider context of black hole singularity resolution by LQG means.

[597]As noted in [317], this comparison should by no means be taken as an attempt to 'downplay' the remarkable indeed result of Modesto, let alone to undermine the significant import and value of LQG as a whole. The aim of the comparison below is simply to highlight the basic differences in general approach, principle and 'attitude' of the two 'resolutions' (and *in extenso* of the two theories, ADG-gravity and LQG), leaving the final word of judgment, critique or praise to the reader and, perhaps more appropriately, to time (*ie*, to future developments and applications of the two theoretical schemata).

[598]The reader is referred to [279] for detailed arguments, calculations and pertinent citations. However, in addition to some of the references therein, we also provide some more relevant references. In this respect, refer also to [200].

[599]This volume-quantization [19] is just one of a series of significant results in Ashtekar's QRG programme, which is an offspring of LQG [11, 12, 382, 383, 351], along with quantization of length [381] and area [18] (see also [331, 328]).

[600]Like for example the spacetime discretizations in lattice QCD.

4. On the other hand, from a differential geometric viewpoint, the upshot of all this is that the said dynamical evolution, which is classically represented by a *differential equation* on the underlying spacetime continuum,[601] is now substituted, in view of the said quantization of spacetime (geometry) in LQG, by a reticular, *difference equation* (discretely parametrized by the coefficients of the physical quantum eigenstates of the volume operator). *In summa*, one can say that *the inner Schwarzschild singularity is resolved thanks to the quantization-cum-discretization of the base spacetime continuum itself.*

Based on the brief description above, our comparison of the two 'resolutions' of the interior Schwarzschild singularity (*ie*, the ADG-gravity one and the LQG one) hinges on two fundamental in our opinion differences:

- **I.** Unlike in the LQG 'resolution' where a quantization of spacetime appears to be necessary, in the ADG 'resolution' this is not so, for the theory is 'inherently background spacetimeless' (*ie*, the theory is indifferent as to whether that background is a 'classical continuum' or a 'quantal discretum' [272, **?**]);[602] and as a consequence of this difference,

- **II.** Unlike the situation in the LQG 'resolution' where the said base spacetime quantization and concomitant discretization appears to mandate the abandoning of the picture of 'gravitational dynamical evolution' as a differential equation proper (and, as a result, the abandonment of differential geometric ideas in the quantum regime), in the ADG 'resolution' all the (essentially algebraic) differential geometric machinery (of the geometrical background spacetime continuum) is retained in full effect (manifestly independently of that background, and *a fortiori*, even if that background is a 'discretum' where differential geometric ideas would traditionally—*ie*, from the CDG-viewpoint of the base differential manifold—seem to fail to apply).

This apparently necessary involvement of the process of quantization-*cum*-discretization of the classical base spacetime continuum in order to render a physical quantity (here for example, the Ricci curvature scalar) finite (thus physically meaningful!), is characteristic of the general intuitive

[601]After all, the Hamiltonian (constraint) in the classical canonical theory (GR) is the generator of temporal diffeomorphisms of the underlying spacetime manifold.

[602]To stress it once again, in ADG-gravity the quest for a quantization of the base spacetime *per se* is essentially 'begging the question': *in the first place, in ADG, what 'spacetime' is one talking about?* Another way to say this, perhaps even more iconoclastically, is that, *from the ADG-viewpoint, gravity* (*ie*, the dynamically 'autonomous' third gauge and third quantum gravitational field) *has nothing to do with 'spacetime', so that a possible quantum theoresis of the former is in no need of a quantum description of the latter.* The ADG-gravitational field is in no need of a background spacetime for its (dynamical) sustenance.

anticipation (and at times 'firm expectation'!) of current theoretical physicists working on QG that, in view of the fact that there is a space-time scale—the so called Planck length-time—arising from combining the fundamental constants of relativity (c from SR and G from GR) and quantum ($\hbar$) theory which are supposed to be consistently merged into the elusive QG, below that scale the classical 'infinitistic' continuum picture of spacetime (and as a result, of the singularities' and infinities' assailed continuous field theory, whether classical or quantum, based on it) should be replaced by something more 'finitistic' and 'quantal'—the '*true quantum spacetime geometry of the genuine QG*', so to speak. Based on the *theoretical* paradigm of ADG-gravity, we would like to challenge in the next subsection this, so popular nowadays, *theoretical* expectation.

But first, as a warmup to the following subsection, let us invoke from [296] some words of Penrose about, on the one hand the invaluable role that the spacetime continuum (:smooth manifold) has played in our formulation of physical theories hitherto, and on the other, the apparent 'need' to scrap it off below Planck length (*ie*, in the QG regime):[603]

[603] In the quotation below all emphasis is Penrose's, unless noted.

"According to present-day theory, all the phenomena of physics take place within the framework of a certain differentiable manifold referred to as the *space-time continuum*. Our familiarity with this idea is such that it is now regarded as almost 'obvious' that space and time should constitute such a structure. However, before discussing the nature of this structure, it is worth examining something of what lies behind this belief. Indeed, there is the definite possibility that some future theory may be found which describes nature more accurately than present theory, but for which the differentiable manifold picture of space-time would not be appropriate. We should not close our minds to such a possibility, but also we should keep in mind the extraordinary range over which the present-day view is such an excellent approximation.

The very accurately 'locally Euclidean' nature of space, and the continuity of time, would, indeed, seem to have supplied the prime motivation, in the first instance, for the rigorous development of the continuum concept. At the time of Zeno, no such rigorous concept of continuum existed, so that the idea of a limit, in space or time, seemed puzzling. It does not seem puzzling to us today, but perhaps we are wrong not to be puzzled! The standard resolution of Zeno's paradoxes refers more to the *mathematical* continuum concept than to the nature of space-time itself. The view of space-time as forming a continuum would imply that a continuous nature would persist, no matter how much a system is magnified. But it is not at all clear that continuous descriptions are really appropriate on a scale small enough that quantum phenomena become important. For example, at a scale of 10^{-33}cm (approximately the radius of an elementary particle), the mere attempt at localization of the position of a particle to that accuracy will,
(Q7.28) as a consequence of the uncertainty principle, imply the probable occurrence of a very large momentum, with the implication that *new* particles are created, some of which may be indistinguishable from the original particle. Thus the concept of 'position' for the original particle becomes obscured. More alarming, moreover, is the picture presented if we allow ourselves to discuss phenomena at a dimension of the order of 10^{-33}cm. At such a dimension, the quantum fluctuations in the curvature of space-time (if both present-day quantum theory and gravitation theory can be accurately extrapolated to this degree) would be large enough to produce alterations in topology. Thus the view of space-time at this dimension would be some kind of chaotic linear superposition of different topologies—a picture in no way resembling a smooth manifold.

Whether or not it is meaningful to talk about the nature of space-time at such dimensions is not at all clear. But if it is *not* meaningful, then we *certainly* cannot refer to space-time as accurately consisting a smooth manifold. On the other hand, it may be argued that the smooth manifold is *adequate* for the discussion of all relevant physical processes. It is my personal view that this cannot *ultimately* be the case. *I do not believe that a real understanding*

In contradistinction to Penrose, our relevant comment here to the emphasized text above, apart from questioning the Planck length (which we will do next), is that, in our ADG-based view, *a better understanding of the nature of elementary particles (:'field quanta') cannot be achieved without the development of a 'better' field theory—one that is not at all dependent on a background spacetime (continuum).*[604] Of course, we agree with Penrose insofar as one abides by the idea that *spacetime is inherent in the dynamical fields* (and their quanta), so that a deeper (or perhaps, a different from the current) understanding of field theory is needed.

7.6 Whence the Planck Length?

To get straight to the point, we wish to open this subsection emphatically, with an apparently 'agnostic' and, as it happens, 'rhetorical' from the viewpoint of ADG-gravity question:

(R7.7) In a purely algebraic ('relational') theory, like ADG-gravity, where no background 'geometrical' spacetime—whether a continuum or a discretum—is involved at all in our (differential geometric) calculations and constructions (*in toto*, in our Calculus), what is the significance (and the use) of having a supposedly fundamental space-time length like ℓ_P-t_P (the Planck length-time)?

To be sure, in the spacetime continuum based QFTs of matter, as well as in various manifold based approaches to QG, the 'utilitarian' or 'pragmatic' attitude towards ℓ_P-t_P is to use it as a *cut-off scale* (with respect to which the continuum is then effectively replaced by a granular structure) in order to regularize the corresponding (ultraviolet divergent) path integrals (for quantum matter and gauge dynamics),[605] or even to come up with a finite value for the black hole horizon's entropy (in QG). The result of course, like in Modesto's LQG-based resolution of the inner Schwarzschild singularity revisited above, is that one is 'forced' to scrap-off the picture of spacetime as a smooth continuum and, inevitably, with it also abandon the applicability of CDG-ideas to QG. We corroborate this by reading from [355]:

[604] In anticipation of 8.5 in the sequel.
[605] Think for instance of lattice QCD.

(Q7.29) "...That matter on the smallest scales sheds its continuous nature is indicated by several features of present-day physics. In particular, *the short-distance 'cut-offs' required (apparently) by both quantum field theory (to 'regularize' the functional integral) and 'quantum gravity' (to render black hole entropy finite) seem ultimately foreign to the notion of differentiable manifold embodied in classical general relativity. Their stubborn presence suggests, rather, that there is a discrete substratum underlying spacetime and accounting naturally for the appearance of a minimal length in the effective theories we now possess*[a]..."

[a]Our emphasis.

Moreover, in more-or-less the same line of thought, and exclusively in the context of QG, we read from [8]:

(Q7.30) "...In quantum gravity, there appear three fundamental constants of nature: Planck's constant, $\hbar$, comes from quantum mechanics; Newton's constant, G, from gravity; and, the velocity of light, c, comes from special relativity. The three constants have physical dimensions. There is a combination of them—[the Planck length ℓ_P]—with the dimension of length. *No other physical theory has a fundamental length built into it...The planck length, on the other hand,*[a] *refers only to universal constants and not parameters specific to a sub-class of physical systems. Therefore, quantum effects of gravity are expected to be significant to all physics around and below the planck-length. The common belief is that our usual picture of space-time as a four dimensional continuum would cease to be a good approximation at the planck length and that the 'microscopic' picture of space-time may be very complicated.*[b] This would have a profound effect on both general relativity and quantum theory..."

[a]As opposed, for example, to the Bohr radius of atomic physics, which involves the mass m and charge e of the electron, as well as the Planck constant $\hbar$, in its expression, and which is thus 'contingent' (due to the particular parameters m and e) only in the realm of specific systems, namely, the atoms.

[b]Our emphasis.

while Ashtekar further adds that:

(Q7.31)

> "...Unification of the principles of quantum mechanics and special relativity has given rise to the quantum theory of fields. An outstanding example of such theories is quantum electrodynamics, the quantum theory of charged particles and electromagnetic fields. In the development of this theory, the focus was on calculations leading to predictions which can be tested against experiments, rather than on issues of mathematical rigor.[a] And the theory has had brilliant successes with experiments...However, in the calculations leading to these predictions, one has to integrate certain expressions over energies of virtual photons which mediate the interaction, and these integrals diverge because the range of integration extends to infinite energies. (The resulting infinities are called ultra-violet divergences...) A systematic procedure—called renormalization—has been invented to subtract out these infinities and to obtain finite answers, and it is these answers that have had experimental success. Thus, the procedure 'works', but seems rather ad-hoc. Now, *integration up to infinitely high energies and momenta corresponds, in the physical-space language, to integration down to infinitely small time and space intervals. Thus, as was emphasized by the founders of renormalization theory, infinities arise because one assumes that the smooth-continuum picture of space-time is valid to arbitrarily small distances. The true structure of space-time is presumably very complicated and the renormalization procedure may be only a convenient trick to get the correct answer without bothering about the details of these complications. Thus, it is only when we have a reasonably good picture of the quantum structure of space-time that we can really understand why the renormalization procedure works*[b]..."
>
> [a]See Dirac's remarks in (Q?.?) and (Q?.?).
>
> [b]Our emphasis. A 'critique' of renormalization in the context of ADG-gravity follows in the next sub-subsection.

Indeed, the Planck length is often invoked to counter various arguments by many researchers, especially those with a bent towards particle physics,[606] who regard gravity's non-renormalizability

[606]That is, those that predominantly believe that one could (or even should!) arrive at a quantum theory of

as a 'fatal blow' to any attempt at arriving at a conceptually consistent, but perhaps more importantly from a practical perspective, a calculationally finite(!), QG. In particular, they doubt a possible formulation of a consistent and finite *non-perturbative* QG, as Ashtekar's words below, taken from [11] which expounds his new QRG theory (in the general context of non-perturbative canonical LQG), corroborate:

gravity by applying concepts and techniques from the continuum based QFT of matter to GR—as it were, like we said earlier, to quantize the field-fiber while leaving the continuous spacetime base intact.

"...In classical gravity, Riemannian geometry provides the appropriate mathematical language to formulate the physical, kinematical notions as well as the final dynamical equations. This role is now taken by quantum Riemannian geometry... In the classical domain, general relativity stands out as the best available theory of gravity, some of whose predictions have been tested to an amazing accuracy, surpassing even the legendary tests of quantum electrodynamics. However, if one applies to general relativity the standard perturbative techniques of quantum field theory, one obtains a 'non-renormalizable' theory, i.e., a theory with uncontrollable infinities..[a]

(Q7.32) ...In the particle physics circles, the answer[b] is often assumed to be in the negative, not because there is concrete evidence against non-perturbative quantum gravity, but because of an analogy to the theory of weak interactions, where non-renormalizability of the initial 'Fermi theory' forced one to replace it by the renormalizable Glashow-Weinberg-Salam theory. *However this analogy overlooks the crucial fact that, in the case of general relativity, there is a qualitatively new element. Perturbative treatments pre-suppose that the spacetime can be assumed to be a continuum at all scales of interest to physics under consideration. Since this is a safe assumption for weak interactions, non-renormalizability was a genuine problem. However, in the gravitational case, the scale of interest is given by the Planck length* ℓ_{Pl} *and there is no physical basis to pre-suppose that the continuum picture should be valid down to that scale.*[c] The failure of the standard perturbative treatments may simply be due to this grossly incorrect assumption and a non-perturbative treatment which correctly incorporates the physical micro-structure of geometry may well be free of these inconsistencies..."

[a] Our emphasis.

[b] To the question posed a bit earlier in that paper by Ashtekar: "*Does quantum general relativity, coupled to suitable matter, exist as a consistent theory non-perturbatively?*"

[c] Our emphasis.

On the other hand, in striking contradistinction to the above, in the two so far quite successful applications of ADG to classical (GR) and QG, spanning the entire 'background spacetime spectrum'—from the '*ultra-continuum*', and thus in a strong sense '*ultra-singular*' and '*ultra-infinite*' [260, 273, 274, 262, 275, 264, 265], to the '*ultra-discretum*', and hence '*non-smooth*' and '*ultra-finite*' [270, 271, 272]—no issue of a forced, by hand as it were, discretization (in order to manage our Calculus, and to secure 'analyticity'—good analytic behavior—of the usual perturbation series expansion) arose whatsoever.[607] In other words, unlike (Q7.?) above, *no fundamental scale such as ℓ_P is invoked at all in order to enable us to apply our concepts and carry out our calculations, since our 'ADG-calculus'—that is ADG's essentially algebraic (relational) differential geometric machinery subsuming the dynamical relations between the fields 'in themselves'—does not depend at all on a background (spacetime) structure, be it 'continuous' ('infinitistic') or 'discrete' ('finitistic').* Moreover, in a finitistic realm, where the continuum based CDG would appear to fail, ADG applies unhindered [272].

In summa, and in view of the genuinely unitary, pure (3rd gauge and 3rd quantum) field-theoretic viewpoint[608] that ADG enables us to maintain and practice, we answer to our opening question (R7.?) above with a another 'rhetorical' question:

Whence ℓ_P?

We firmly believe that this question, far from trying to be 'eccentric' or 'smug', touches on a very subtle and important (for the comprehension and appreciation at least of the spirit of the potential application of ADG to QG) point, namely, that while '*big*' questions and associated arguments, such as whether QG exists non-perturbatively or not, vitally hinge in one way or another on the 'nature' (character) of (a background) spacetime structure (*ie*, whether it is a 'continuum' or a 'discretum', a 'classical' or a 'quantal' structure), *in ADG there is no (background) spacetime at all*, so in a sense such questions become 'irrelevant' and the issues they purport to address are 'non-problems' for ADG-gravity. At the same time, the Planck length becomes an 'obsolete chimera' in our theory, simply reflecting that we have been asking perturbability (analyticity) and renormalizability questions from within (*ie*, from the perspective of) the 'wrong' (mathematical) framework (*ie*, the framework of CDG or Analysis)—which is 'wrong' in the sense that it *a priori* regards as important the 'nature' of the underlying spacetime (manifold) and views the fields

[607]Thus, Ashtekar's words above, that "*perturbative treatments pre-suppose that the spacetime can be assumed to be a continuum at all scales of interest to physics under consideration*", in a strong sense force one to identify the terms '*perturbative techniques*' with '*analytic techniques*'—in other words, *'perturbability' vitally depends on analyticity*, and of course the latter on the assumption that the base spacetime is a smooth (even more so, an *analytic*!) manifold. Then, the crux of Ashtekar's argument above is that while perturbability and renormalizability are legitimate requirements for the electroweak forces since there is no 'natural' minimum spacetime scale (so that in principle all energy-momenta, no matter how large, should be integrated over in the perturbation expansion), for the gravitational force this is not so due to the appearance of the Planck length-time.

[608]As noted repeatedly earlier, a theoretical scheme dealing solely and exclusively with the fields themselves, without reference to an external, ambient spacetime no matter what its 'nature' (*ie*, 'discrete' or 'continuous').

involved as being inextricably tied to (or dependent on) it.[609]

The 'unity' and 'universality' of physical law. The remarks above prompt us, in complete analogy to Einstein's mode of expression (and doubt!) in (Q2.1) about singularities and the 'incompleteness of determination' of field theory that they entail, to maintain in view of ADG's field-realism (or perhaps better, field-solipsism) (PFR):

[609] Shortly, in 7.5.2, we will argue further in support of this point.

(R7.8) It is our opinion that the (external) spacetime scale-dependence of physical law must be excluded from a fundamental candidate theory for QG. It does not seem (physically) reasonable to us to 'dissect' the unitary field by introducing (artificial) 'effective' scales (for an external to the field itself continuum spacetime structure) in order to delimit the range of validity of the law that it obeys (in fact, the law that the field *defines*, as a differential equation, in the first place). Ultimately, scale (like 'spacetime')[a] is introduced and fixed by us observers or experimenters,[b] thus it is a contingent not a fundamental feature of Nature—or anyway, something that could not possibly influence or determine the validity of field dynamics in any way. If anything, such a dependance of the law of gravity on a fundamental spacetime scale shows an *antinomy* of the very term 'physical *law*' (pun intended!) and its supposed universality—*ie*, that the law holds in the whole '*World*', when the latter word however is not understood as 'spacetime' as usual, but as 'field'. The World *is* the dynamical fields that comprise it, nothing else (PFR).

[a]These two words always go together: we talk about *spacetime scales*. At any rate, even if one is talking about the operationally more realistic sounding term like *energy-momentum scale*, quantum theory can always translate the latter (via quantum duality or complementarity) back to time-space scale (in this operational respect, see the Einstein quotation (Q?.?) below). On precisely this quantum duality rests Ashtekar's argument in (Q?.?) above: the higher the energy-momenta (one wishes to integrate over), the smaller the time-space scales one has to dig deeper into (ultraviolet infinities are quantum dual/complementary to infinitesimal scales that the spacetime continuum allows one to reach, even if just theoretically).

[b]From an ADG-perspective, scale, like spacetime, is carried by (is inherent in) **A**.

In other words, we believe in the 'unity' and 'universality' of the (gravitational) field and thus cannot accept that it obeys (defines!) one gravitational law above, say, ℓ_P (where spacetime is a supposed to be a continuum)—the so-called classical gravitational Einstein equations of (GR), and a different one below ℓ_P (where spacetime is intuited to be 'reticular' and 'quantal')—the elusive law of QG, this difference being in turn reflected by the external (to the dynamically autonomous, 'unitary' field) continuum/discretum 'divide' resulting from the '*ad hoc*' (as it were, forced by theoretical fiat) introduction (by us!) of the Planck length. This remark

- on the one hand puts into perspective the nowadays popular anticipation-*cum*-imperative that a cogent quantum theory of gravity *should* have GR as an '*effective theory*' [68] or, what is essentially the same, as a so-called '*low energy limit*'—arguably, the main 'prognosis' of the string-theoretic approach to QG [409, 277].

- and in pretty much the same gist, that QG *should* yield GR at some '*classical correspondence limit*', much in the same way that GR reduces to (non-relativistic) Newtonian gravity as $c \longrightarrow \infty$, while quantum mechanics (or field theory) yields its classical correspondent when $\hbar \longrightarrow 0$,[610]

- and on the other hand, it exposes what we have time and again highlighted in the present paper and throughout our past trilogy [270, 271, 272], namely, that *the* culprit for all the difficulties and anomalies encountered in our differential geometric models and calculations (Calculus!) in physics is *our* assumption that physical space(time) is an external (to the fields), background continuum—a manifold mediating our calculations and at the same time facilitating the geometrical interpretation of our CDG-based constructions and of the physical inferences that we draw based on them.

Of course, it must be made clear here, in order to avoid any misunderstanding, that all these '*fundamental physical constants*' and their combination to ℓ_P are in every sense 'real', as they delimit the range of validity of their corresponding theories—albeit, *our spacetime continuum based and thus CDG-theoretically modelled theories and their defining dynamical laws!*[611]—like SR (c), GR (G), QM ($\hbar$) and QFT (c and $\hbar$), but *it would be a blatantly arbitrary, prima facie uncalled for (and perhaps prove to be 'false' in the long run of QG research) inference, just because our mathematical model of spacetime as a manifold and the CDG-panoply based on it suffer or appear to be problematic and of limited validity in the quantum domain, to posit that Nature herself has a fundamental length or time duration.*[612] To stress it again, since in ADG-gravity (as well as in

[610]Actually, in the case of QFT—the standard denomination of relativistic quantum mechanics of systems with an infinite number of degrees of freedom, the correspondence principle (yielding a non-relativistic classical field theory) is supposed to be represented by the joint application of the formal limits $c \longrightarrow \infty$ and $\hbar \longrightarrow 0$.

[611]That is, theories that both conceptually and technically regard the base spacetime continuum as being on a par with the dynamical laws (defining these theories). In point of fact, the spacetime continuum is viewed as being deeper than and prior to the laws themselves, when one considers that the latter are being modelled after differential equations and that the continuum is assumed *a priori* precisely in order to secure those vital 'conditions of differentiability' (and variation!) of the physically observable quantities engaging in the laws—the fields and their quanta (particles).

[612]Compare this with our basic motto that '*Nature has no singularities*' in sections 1 and 2. This is why we emulated in (Q?.?) above Einstein's words in (Q2.1).

the ADG-formulation of the dynamics of the other fundamental forces, such as the Yang-Mills theories) there is no geometrical background spacetime—be it a discretum or a continuum—but only the fields and their algebraically represented dynamical interrelations (*ie*, the by differential equations modelled physical laws), such *a priori* and arguably *ad hoc* assumptions and resulting practices are foreign to its very essence (or even, mathematically speaking, to its very 'definition'!).

En passant, it must be mentioned here that the supposedly physical importance of the Planck length (and the potential development of a new 'perspective on gravity' at scales below it)[613] was, to the knowledge of these authors, first anticipated by Eddington, who remarks prophetically in [106]:

(Q7.33)

> "...There are three fundamental constants of nature which stand out pre-eminently. The velocity of light, 3.00×10^{10} c.g.s. units; dimensions LT^{-1}. The quantum, 6.55×10^{-27} c.g.s. units; dimensions ML^2T^{-1}. The constant of gravitation, 6.66×10^{-8} c.g.s. units; dimensions $M^{-1}L^3T^{-3}$.
> From these we can construct a fundamental unit of length whose value is 4×10^{-33}cms. There are other natural units of length—the radii of the positive and negative unit electric charges—but these are of an altogether higher order of magnitude.
> With the exception of Osborne Reynold's theory of matter, *no theory has attempted to reach such fine-grainedness. But it is evident that this length must be the key to some essential structure. It may not be an unattainable hope that some day a clearer knowledge of the processes of gravitation may be reached; and the extreme generality and detachment of the relativity theory [from the other theories of matter] may be illuminated by the particular study of a precise mechanism...*"

Also, from an operational viewpoint, equally suggestive are the following remarks of Einstein found in [372]:

[613]A quantum theory of gravity like the one envisioned today?

(Q7.34) "...If one does *not*[a] want to introduce rods and clocks as independent objects into the theory, *then one must have a structural theory in which a fundamental length enters, which then leads to the existence of a solution in which this length occurs, so that there no longer exists a continuous sequence of 'similar' solutions. This is indeed the case in the present quantum theory, but has nothing to do with its basic characteristics.*[b] Any theory which has a universal length in its foundations, and, on the basis of this circumstance, qualitatively distinguished solutions of definite extent, would offer the same thing with respect to the question envisioned here..."

[a]Einstein's emphasis.
[b]Our emphasis.

However, *both Einstein and Eddington* had on the one hand criticized the background spacetime continuum of GR (Q?.?), and on the other they had intuited that a completion of GR (to a unitary field theory so as to include the other matter sources and their radiation force-fields) should consist of the field-dynamics alone, and what's more, they intuited and 'demanded' that the singularities (of matter; the matter particles/quanta) be included in that total field dynamics (Q?.?). To a certain degree, *ADG has achieved both*, without involving any background spacetime at all and, as a result, fundamental spacetime scale either.

We may address and question the issue of a fundamental spacetime scale from another point of view relevant to ADG-gravity. For instance, we may begin with an 'aphorism' and claim that *the notion of 'scale' (or 'measurement gauge'!) has meaning only in an operational context, in fact, it is closely associated with our measurement actions/acts of measurement*. In our scheme, the notion of 'scale' is inextricably tied to the generalized arithmetics **A**—the realm where our acts of measurement 'take values'. A generalized 'gauge' or 'scale' principle is the following:

(R7.9) *The laws of Nature are scale (gauge) independent.*

which in turn, in the case of the spacetime continuum (*ie*, when we take $\mathbf{A} \equiv \mathcal{C}^\infty_M$) would appear to contradict the nowadays widespread view:

- that the law of gravity does not hold (or breaks down!) at singularities (or below Planck length-time scale), or more-or-less equivalently in the continuum based QFTs of matter,
- that there is a natural length-duration spacetime scale[614]—the so-called Planck scale ($\ell_P \approx$

[614]The epithet 'natural' pertaining to the fact that these scales can be expressed in terms of fundamental *physical* constants.

$10^{-35}m$-$t_P \approx 10^{-40}s$)—below which the classical spacetime continuum gives way to a 'discrete' and 'quantal' space(time) structure.

On the other hand, if one wished to retain these numerical physical constants in ADG-gravity, one could indeed do so by regarding them as *global sections of the constant sheaf* $\mathbf{C} \equiv \mathbf{K} := \mathbf{R}$ (of reals), *which is by definition embedded into* $\mathbf{A}$. However, *being elements of* $\mathbf{C}$*, physical constants do not contribute at all to the ADG-gravitational field-dynamics which is effectuated via the ADG-theoretic, essentially algebraic and background independent, differential geometric mechanism.* Equivalently, the ADG-gravitational field $\mathcal{D}$, by virtue of being (by definition) a $\mathbf{C}$-morphism, 'sees through' (via the dynamics that it defines) the fundamental constants. Another way to say this, the physical constants, being *constant*, do not partake into the field dynamics. The ADG-gravitational field dynamics, being also completely background (spacetime) geometry free (which is inherent in $\mathbf{A}$, while the gravitational dynamics is $\mathbf{A}$-functorial as repeatedly emphasized earlier), is also 'blind' or 'indifferent' to various scales that we might employ to gradate or 'tessellate' this fiducial base spacetime.[615]

To stress it once again in order to 'digest' it, physical constants (and their combinations), with values in the real numbers' continuum ($\mathbb{R}$), are 'physical enough' to the extent that they set 'gauges' or 'scales', or even 'natural units of measurement', for delimiting energies (speeds or momenta, actions, energies, or dually, time-durations *etc*) at which various interactions between fields and their quanta (particles) become significant ('observable') and the corresponding (spacetime continuum based field) theories about them are valid (as it were, as effective (field) theories [**?**]), but what 'right' do we have to first combine them to ℓ_P (or t_P)

$$\ell_P = \sqrt{\frac{G\hbar}{c^3}} = 1.6 \times 10^{-33}cm;\ t_P = \sqrt{\frac{G\hbar}{c^5}} = 5.3 \times 10^{-44}s \tag{109}$$

and then infer that the latter combination singles out a 'preferred', 'natural', 'universal' length/time-scale below which (classical continuum spacetime based) field physics fails to yield? Could it instead be that

(R7.10) *it is precisely because of the way we formulate these continuum field theories differential geometrically, by means of (or by mediation in the guise of coordinates—in effect, of our 'measurements' and 'smooth field localizations' on) a background spacetime continuum, that it appears to be necessary to introduce a fundamental length scale in terms of which (and what's more, in order for) our analytic (Calculus based) calculations (to) make sense (eg, to be rendered finite)?*

[615]See Riemann's quotation below.

For one can recall here Riemann's words in [322], which we quote *verbatim* from [265]:

(Q7.35) "Maß bestimungen erfordern eine Unabhängichkeit der Größen vom Ort, die in mehr als einer Weise stattfinden kann." : "*Specifications* [: measurements] *of mass require an independence of quantity from position, which can happen in more than one way.*"

By analogy with ADG's singularity 'absorptions' or 'dissolutions' in **A**, one may conceive here, at least in a metaphorical sense, of the so-called 'perturbation expansion' (of gravitational interactions in the context of perturbative QGR) and the associated 'renormalization technology' inherited from the continuum based QFT, as attempts to '*extend analyticity*' at the *loci* of interaction by *absorbing infinities* in the physical parameters (constants) involved. However, these attempts—all of them based of course on the analytic means of Calculus (CDG)—are doomed to failure, and this failure is usually attributed, physically speaking, to the dimensionality of Newton's constant G involved in the 'minimal length' ℓ_P in (109) above. In turn, this minimal length is supposed to represent a natural 'regularization cut-off' beyond which this 'perturbative analyticity' (renormalization series) cannot be extended further so that the smooth spacetime continuum is supposed to give way to something (which one might still call 'spacetime') more 'discrete', 'quantal' and 'inherently cut-off'—something, of course, that can 'regulate' the uncontrollable infinities of the spacetime continuum.

But GR is simply the relativistic field-theoretic extension of Newtonian gravity, *one that is vitally based on the smooth base continuum* (at least for its differential geometric representation). Thus, the infinite field-redefinitions involved in attempts to renormalize 'field-quantized gravity' are closely analogous to the infinities associated with $\mathcal{C}^\infty$-singularities, with the important difference that in gravity—at least in its original second order formulation in terms of the metric *à la* Einstein (or even in its subsequent tetrad-connection first order formulation *à la* Palatini-Ashtekar), in striking contradistinction to the other (classical or quantum gauge) field theories of matter, *the background geometry (metric) is itself the dynamical variable, so the relevant field quantities involved cannot be referred or 'expanded' with respect to it*. We saw for instance earlier in (Q?.?) how Geroch notes in the introduction of [155] the similarities between the infinities associated with singularities of the gravitational field and the infinities of the other (classical, continuum based) field theories of matter; however, with the important difference that with the former there is no fixed background geometry with to which the gravitational field (*ie*, the metric itself!, in the standard theoresis of GR) can be referred and 'perturbatively' expanded (to remove those infinities). Parenthetically, it must be further noted here, that Geroch goes a bit further and attributes exactly to that fundamental difference between gravity and other classical field theories the difficulty in arriving at a precise definition of singularities in GR:

(Q7.36) "*...In view of the faulty analogy with which we must work,*[a] *it is not surprising that (a) there is no widely accepted definition of a singularity in general relativity, and (b) each of the proposed definitions is subject to some inadequacy...*"

[a]The analogy referred to here being the one between the singularities in GR and the infinities in the (classical and quantum) matter field theories.

This whole rationale is pregnant to a '*general conjecture*' (or 'hunch') that emerges in the light of ADG-gravity: Calculus (Analysis), which is manifestly dependent on a background $\mathcal{C}^\infty$-manifold, cannot be carried through to the ('true') quantum gravity regime,[616] while the incurable infinities that arise are precisely due to our inappropriate application of differential geometric ideas[617] in the quantum deep. In this line of thought, we read from [265] for example:

(Q7.37) "...So we can think here of the famous *'Planck scale'*, that, of course, it is not a matter of analysis/algebra, but rather of the particular manner (*viz.* still classical differential geometry), that it is here undertaken to exploit '*geometry*' in the quantum domain[a]..."

[a]'Geometry' here refers to what one might call '*physical geometry*'—the (structural) analysis directly of the dynamical, essentially algebraic (relational), attributes of fields and their particles (quanta)—and not to what one could call '*mathematical geometry*', which is usually understood as the (structural) analysis of a given, fixed and 'inert' 'space(time)'. Of course, the more subtle distinction is between (differential) geometry understood in a Cartesian (analytic-arithmetic) way—what we will call '*mathematical geometry*' in 7.?, and (differential) geometry seen from a Leibnizian (algebraic-relational) perspective—what we will call '*physical geometry*' in 7.?.

As briefly alluded to above, and in the spirit of the present paper, the following rather 'standard' intuition appears to be *prima facie* 'natural': *at sub-Planckian scales the smooth spacetime continuum gives way to something more finitistic, something 'inherently cut-off'*. Moreover, it is widely hoped that the desired successful 'quantization' of the gravitational field (to a conceptually consistent and calculationally finite quantum theory of gravity) will 'cure', or even evade altogether, not only the non-renormalizable quantum field theoretic infinities of QGR, but the more robust

[616]See quotations of Einstein, Feynman and Isham in the sequel (Q?.?).

[617]'Inappropriate' in the sense that for our differential geometric 'aufbau' we always refer to a background or 'intervening' spacetime (geometry).

ones associated with the $\mathcal{C}^\infty$-smooth singularities. *Isn't such an anticipation 'begging the question', though?* For if we suppose (as many researchers currently do suppose!) that GR and the smooth spacetime manifold supporting it will arise from the 'true' quantum gravity theory at a Bohrean 'macroscopic', 'correspondence' or 'low energies' limit (*ie*, at large scales/weak gravitational field strengths), would not the emergent GR on the differential spacetime manifold automatically carry with it the entire differential geometric anomalies ('singularities') baggage—the burden that the purported quantization of GR so painstakingly tried to get rid of while still retaining the CDG picture of a base spacetime manifold and the smooth fields on it?[618]

(R7.11) On the other hand, if 'space(time)' (and its 'geometry') is inherent in the dynamical fields and their dynamical interrelations, and if we suppose (as we actually do suppose in ADG-gravity) that *everything is field, and that the field is by itself already quantum, as well as that the field represents the actions of (its) quanta without the intervention of any 'space(time) as such' whatsoever*, *why attempt at all to 'save' space(time) by speculating that, if it is not 'continuous', it must then be 'discrete'?* (*ie*, take ℓ_P at 'face value'); moreover, *why at all insist that the 'true' quantum gravity is (somehow) 'quantized GR' and at the same time expect that a successful quantization of GR will alleviate, ultimately remove, singularities and their nonsensical infinities?*[a]

[a]As it recently claims to have achieved in LQG as we discussed earlier, where the quantization of the gravitational field results in the quantization and concomitant discretization of the background continuum, which 'quantum discretization' is in turn employed to 'resolve' the interior Schwarzschild singularity and other cosmological ones [51, 279, 200].

With respect to the first question, *why depend at all on a background 'spacetime' arena—be it a continuous or a discrete realm—when all that there is are fields representing the dynamical interrelations of quanta?*[619] Presumably, finitistic and quantal are the (dynamical) (inter)actions of field-quanta 'in themselves'; *quanta 'see' no ambient spacetime as such*—rather, it is their dynamics—their field-dynamics—that '*defines*' ('creates') physical spacetime (geometry).[620] An-

[618]This in turn seems to place serious doubts on the QGR ('quantum', or better, '*quantized*' GR) program. See next remark.

[619]Previously, in the context of a $\mathcal{C}^\infty$-smooth spacetime manifold, we referred to this tendency as '*CDG or Calculus conservative*'. Here we extend this to a general '*background spacetime conservatism*'.

[620]In this sense we will distinguish between 'physical' and 'mathematical' (spacetime) geometry in subsection 7.?

tonio Machado's verse from his poem '*The Road*' [250] comes to mind:

(Q7.38) "Traveller there are no paths; paths are made by walking."

as well as Wheatley and Kellner-Rogers' remarks from their inspired book [229]:[621]

(Q7.39) "*The future cannot be determined; it can only be experienced as it is occurring. Nature*[a] *doesn't know what it will be until it notices what it has just become.*"

[a]Here, the authors say '*Life*' instead of '*Nature*'.

In toto, all there is are these field-quanta, their existence and 'sustenance' is not conditional on a background space(time), and it is high time, following Leibniz, to devise a Calculus that refers directly to them; while, perhaps more importantly for Physics, base our (differential geometric) calculations on such a (generalized or abstract) Differential Calculus.

In toto, we are tempted to say, in a Leibnizian way, that in ADG fields are '*differential geometric entelechies*' or 'monads', in the sense that they are in no need of reference to an ambient (external or background) space(time) to support themselves;[622] they are irreducible (atomic or '*ur*'),[623] autonomous (differential) geometric (in fact, *purely algebraic*) entities, with the aforesaid 'autonomy' being physically interpreted as '*autodynamicity*': the ADG-gravitational field alone *defines* the (vacuum) gravitational dynamics, without that dynamics being in any way dependent on—or perhaps better, conditioned and constrained by—an external (background) spacetime manifold.[624]

On the face of the above, we come to ask:

later.

[621]In the quotation below, emphasis is ours.

[622]And recall Leibniz [246, 247, 384]: "*Monads are windowless.*".

[623]And recall Einstein: "*The notion of field is a primary, not further reducible one.*" [125].

[624]We have in mind here the usual spacetime diffeomorphism ($\mathrm{Diff}(M)$) constraints that create a host of problems in regarding gravity as a quantum gauge theory [215, 411, 412]. In the next section we devote a whole sub-subsection (7.?.?) to this Aristotelian-Leibnizian character of the third gauge, third quantum ADG-gravitational field.

To what extent is the dimensional 'G' involved in ℓ_P 'bound' to the continuum picture of spacetime on which both the non-relativistic Newtonian gravity and GR fundamentally depend? Then, *mutatis mutandis* for $\hbar$.[a] Isn't that dependence of 'G' on M analogous to the dependence of Bohr's radius on the specific physical systems in focus—*ie*, the material atoms?[b] On the other hand, *what reasons have we got to assume that in the 'true' quantum gravity theory the basic systemic (quantum) variable is 'spacetime', rather than the gravitational field alone—'in-itself', so to speak?*[c] For, *who gave us 'spacetime' in the first place?* Arguably, it is *we* that assumed it in the first place.[d] So, *isn't it possible that the commonly intuited ℓ_P 'cut-off', below which the spacetime manifold—the so-called continuum, is supposed to give way to a 'discrete' spacetime structure—a 'discretum',[e] is due to our inappropri-*
(R7.12) *ate employment of CDG-theoretic ideas to address the problem of the (presumably inherently[f]) quantum gravitational field?*

[a] And one must recall that Planck introduced $\hbar$ in order to regularize the continuous black body radiation spectrum!

[b] We implicitly suppose for a moment, by assuming a 'Devil's Advocate' stance against the problem of quantum gravity, that at sub-Planckian scales the system in focus is spacetime itself.

[c] Of course, the gravitational field conceived now not CDG-theoretically, but in some other way—*eg*, solely as an 'algebraico-categorical' connection $\mathcal{D}$ like in ADG—still retaining though its fundamental 'differentiability' and 'observability' (*ie*, its 'dynamical variability') through $\mathbf{A}$ (*ie*, effectively through $\mathcal{R}(\mathcal{D})$, which is an $\mathbf{A}$-morphism) attribute.

[d] And arguably, we assumed it in the first place in order to be able to do Calculus based on it (manifold and CDG-conservatism and monopoly).

[e] See quotation of Sorkin in (Q7.?) next.

[f] As we presume in the present paper. See also [272].

For example, we read from [55]:

(Q7.40) "...Einstein's General Theory of Relativity has proved to be one of the most successful and enduring theories in physics, and its predictions have been verified in numerous experiments. However, it stands alone amongst field theories in that it is not scale invariant. For example, the differential form of Maxwell's equations, which elegantly describe the electromagnetic field, do not define any intrinsic scale. Conversely, Einstein's field equations, which describe the way that matter curves spacetime, are linked to an apparently arbitrary scale determined by the Newtonian gravitational constant, G...*Numerous attempts have been made to develop a theory of gravitation that is scale invariant, and yet retains the key properties of General Relativity, such as the principle of general covariance*[a]...The concept of a uniquely defined Planck scale is one of the two principal motivations for the pursuit of a quantum theory of gravity, the other being the need for the curvature terms in the gravitational field equation to be quantized in order to be equivalent to the quantized matter fields. *If the Planck factor is removed, we need to ask whether there is still a need for a theory of Quantum Gravity, at least in the form currently being sought. There is no fundamental requirement that the gravitational field should have an inherent quantum structure, and it may well be more reasonable to think of the gravitational field as being quantised as a consequence of the matter fields with which it interacts.*[b] ...Finally, one can speculate that by more clearly defining the distinction between the realms of General Relativity and Quantum Mechanics, we can actually move closer towards constructing a paradigm that unifies the two theories..."

[a] Our emphasis.
[b] Again, emphasis is ours.

Booth's words above are in accord and at the same time in discord with ADG-gravity, in the following sense:

- The background spacetime manifoldless (in fact, spacetimeless altogether) ADG-gravity is 'by definition' (or construction) scale-free, since no external (to the gravitational field $\mathcal{D}$ itself) spacetime is involved whatsoever. Moreover, the PGC of GR is now transferred to the field 'in-itself', thus it has been substituted by synvariance, which is implemented via $\mathcal{A}ut\mathcal{E}$, not $\mathrm{Diff}(M)$. *No spacetime, no fundamental spacetime scale either.*

- Thus, from the ADG-gravitational perspective, a uniquely defined Planck scale is *not* a principal motivation for QG, and a quantization of the gravitational field like one quantizes the matter fields (on the right hand side of Einstein's equations), is not our principal concern.

- Hence, by removing the Planck factor we genuinely question the "*need for a theory of Quantum Gravity, at least in the form currently being sought*", that is to say, with a differential spacetime manifold still present at the background and with a fundamental 'cut-off' scale such as Planck's which is introduced in order to 'tame' our Calculus (perturbatively) at infinitely small scales.

- On the other hand, we do believe that "*the gravitational field should have an inherent quantum structure*" like the other quantum matter fields, but we cannot accept that its quantization is "*a consequence of the matter fields with which it interacts*". For after all, one should still be able to account in a quantum mechanical way for the gravitational field *in vacuo*, while apparently insuperable problematics, both conceptual and technical, arise in various endeavors to do that while maintaining a base manifold [394, 395, 411, 412]

At any rate, it must also be emphasized here that this 'algebraic-discretum' alternative to the smooth background geometrical spacetime continuum is supposed to be a '*real*' and '*radical*' alternative, in the sense that it is expected to retain few features (if any at all) from continuum physics—especially, it is doubtful that it will retain the background geometrical space-time interpretation. Here is a quotation of Einstein, found in [123], that supports precisely this:[625]

(Q7.41)

> "...The alternative continuum-discontinuum seems to me to be a real alternative; i.e., there is no compromise. *By discontinuum theory I understand one in which there are no differential quotients. In such a theory space and time cannot occur, but only numbers and number-fields and rules for the formation of such on the basis of algebraic rules with exclusion of limiting processes.*[a] Which way will prove itself, only success can teach us..."
>
> [a]Our emphasis.

In summa, our thesis *contra* ℓ_P-t_P can be distilled to the following:

[625]See the last subsection 8.?, and especially 8.?.?, for this continuum-discretum 'debate' *vis-à-vis* Einstein.

(R7.13) Both the classical field theory of gravity (GR) and the relativistic quantum theory of matter (QFT) depend on a smooth spacetime manifold—curved for the former, flat for the latter. In attempts to quantize gravity by applying QFTheoretic rules to GR—as it were, to arrive at a 'Quantum or Quantized General Relativity' theory, either canonically or covariantly (via path integrals), perturbatively or not—-the continuum gets in the way in one way or another.[a] To circumvent these continuum-problematics, one evokes the Planck length and argues that in the QG regime there is a fundamental spacetime scale below which the continuum picture ceases being a faithful representation of the 'true' spacetime geometry. In contradistinction, ADG-gravity is a pure gauge field-theoresis of gravity that is fundamentally background spacetimeless—whether this background is 'continuous' or 'discrete'—while at the same time the third gauge, third quantum ADG-gravitational field $\mathcal{D}$, which defines (via its curvature) the (vacuum) Einstein equations (29) is *ab initio*, by the very ADG-formalism, already or self-quantum, as we argue further in the next sub-subsection.

[a]For example, the so-called 'problem of time' or the 'inner product problem' (*ie*, the metric on the physical Hilbert space of states for the quantum gravitational field) in (non-perturbative) canonical QGR, or the functional quantum measure in a sum-over-histories (path integral) type of scenario, or even the non-renormalizability of (perturbative quantum) gravity which simply shows that one cannot continue analyticity down to the infinitely small (when, quite paradoxically, *analyticity is defined by the infinitely small*—*ie*, by the locally Euclidean smooth base continuum!).

7.6.1 The distinctions (epithets) 'classical' and 'quantum' lose their meaning in ADG-gravity: in what sense the ADG-formulated Einstein equations are not already quantum?

The crux of the foregoing arguments is that in (vacuum) ADG-gravity the traditional denominations of gravity as 'classical' and 'quantum' lose their standard meaning in the theory. We maintain that such distinctions are essentially due to the presence of a background spacetime manifold which on the classical side of the quantum divide remains intact, while on the quantum side, although it is initially assumed in order to set up differential geometrically any field-theoretic approach to QG (*eg*, canonical QGR, LQG and QRG) it is eventually supposed to be discretized and quantized, mainly by evoking the minimal Planck scale. By contrast, since in the 'pure field' theoresis of gravity *à la* ADG a base spacetime, continuum or discretum, is not involved at all, a fundamental space-time length-duration has to go, thus, *in extenso*, those two standard epithets to 'gravity' become insignificant. Moreover, since the ADG-gravitational field $(\mathcal{E}, \mathcal{D})$, at least from a geometric prequantization vantage, is 'inherently quantum' (as we argued extensively earlier), in the ADG-approach to QG no issue arises whatsoever of quantizing spacetime itself. *In toto, the nowadays popular quest for a quantization of spacetime structure is begging the question in ADG-gravity*.

This is already reflected in the quantum interpretation carried by the ADG-gravitational vacuum Einstein equations (29), whereby the connection field $\mathcal{D}$ acts, as a sheaf morphism, via its (Ricci) curvature scalar on the local sections of the associated (representation) vector sheaf $\mathcal{E}$, which in turn from a geometric (pre)quantization vantage represents the local quantum particle states of the field. This is the non-linear, 'self-(inter)acting', character of the ADG-gravitational field. Let it be noted here that in a 'fuller' quantum theoresis of the ADG-gravitational dynamics, a non-smooth manifold based path integral over connection space scenario is envisaged to capture the 'complete' dynamics for QG from the purely gauge-theoretic ADG-perspective.[626]

7.6.2 The twilight of a theoretical paradigm-shift in QG

So far we have time and again emphasized that ADG-gravity is fundamentally (background) spacetimeless—in particular, it is background manifoldless. Moreover, the very formulation of ADG-gravity is fundamentally different from the manifold and smooth metric-based one originally due to Einstein. It certainly resembles the recent Ashtekar formulation (AF) essentially regarding (quantum) gravity as a (quantum) gauge theory, but it differs from it as well in at least three, closely related to each other, important ways:

- Like the AF, the basic dynamical variable is the gravitational connection field $\mathcal{D}$, but unlike it, no metric in the guise of the smooth tetrad field of the first-order formalism is involved (ADG-gravity≡half-order formalism).

- In addition, unlike the AF, no external (to the connection) smooth background spacetime manifold is involved: only the algebraic **A**-connection field $\mathcal{D}$ is present. ADG-gravity is genuinely background independent (*ie*, not only background metric, but also differential manifold independent) and *what has been retained from the original formulation is the gravito-inertial interpretation of the theory, not its chrono-geometric one*.

[626] See 7.9.1 in the sequel.

- As a consequence, the theory recognizes no fundamental (minimal) space-time scale in the guise of the Planck length-duration (ℓ_P-t_P) and also it does not regard 'spacetime quantization' as a physically meaningful quest(ion). This puts into perspective the whole QG enterprize (perturbative or not) as no infinity or singularity is present in the theory so that neither ℓ_P would have to be invoked as a regularization 'cut-off' scale in order to render physical quantities finite, nor the 'classical continuum'-to-'quantal discretum' (below ℓ_p) transition is expected to be effectuated by spacetime quantization and concomitant discretization as it is currently maintained in non-perturbative QG scenarios such as LQG [51, 279, 200]. The bottom line is that ADG-gravity is a third gauge, third quantum theory of the gravitational field 'in-itself', with the notion of a geometrical spacetime playing no role in the theory whatsoever.

These three glaring differences are in our opinion pregnant to a significant Kuhnian paradigm-shift [239] in QG research issuing from ADG-gravity, which we itemize in three parts below:

- The first fundamental notion in GR that must be reconsidered and revised in the light of ADG-gravity is that of *spacetime event*. In the manifold based GR, the basic aftermath of Einstein's hole argument as we will extensively describe and comment on in 8.5 is that, basically, a spacetime event is a point p of the base differential manifold M *together* with the smooth gravitational field $g_{\mu\nu}$ defined on it and satisfying the Einstein equations.[627] There is no *spacetime point event-set* interpretation of M without the dynamical gravitational field on it.[628] In contradistinction, in ADG-gravity the dynamics is formulated solely in terms of the gravitational connection field $\mathcal{D}$, not the metric, without any reference to a(n *a priori* posited and fixed, kinematical) background 'geometrical spacetime' structure (arena). Thus, the notion of *space-time event* and of the chrono-geometrical interpretation of the gravitational field $g_{\mu\nu}$ that goes hand in hand with it cannot survive in the theory.

- Related to the above is the issue of causality. Since $g_{\mu\nu}(x)$ represents the local causal structure of spacetime as at every $x \in M$ it stands for the field of *infinitesimal* or *differential* spacetime locality (local causality) [270] which thus becomes a dynamical variable in GR, in ADG-gravity causality too must be revised. In other words, the causal nexus between events, which in the manifold based GR is represented by the smooth variable $g_{\mu\nu}$ obeying the differential equations of Einstein, is replaced by the single notion of connection-field $\mathcal{D}$. This

[627] *A fortiori*, in view of the hole argument, any (smooth) 'coordinate-dressing' $x(p)$ of p has no physical significance in view of the general covariance of the Einstein equations ($\mathrm{Diff}(M)$ implementing the PGC in GR).

[628] This is the Einstein-Stachel '*no field, no spacetime event*' *motto*, as we will see in the next section. Here already one witnesses the germ of the idea that dynamics is essentially prior to kinematics if one recalls that the basic kinematical structure in GR is that of a base differential manifold and a smooth Lorentzian metric on it.

is a Jungian synchronicity-type of coincidence in 'chronological names' (pun intended): the chronological *causal connection* is subsumed under the dynamical *connection field* $\mathcal{D}$. To stress it once again, *in ADG-gravity there are no (dynamical) causal connections between spacetime events, only the background spacetimeless (dynamical) gravitational connection field* $\mathcal{D}$.[629] In ADG-gravity, 'time' (*ie*, chronology and causality) derive from (*ie*, they are inherent in) $\mathcal{D}$ and the dynamics that it defines.

- Now that we have scrapped-off (the background) *spacetime* (manifold) and opted for a choro-chrono-geometrical interpretation-free and purely algebraic gravitational field $\mathcal{D}$, what happens to the by now commonly accepted ideas about *space-time measurements*? One immediately anticipates that measurements of 'spacetime position' (locution) have no longer any place or meaning in the theory. To appreciate how formidable this revision of the classical theoresis of spacetime structure and gravity, and especially of its potentially quantal albeit persistently differential manifold based versions, one may bring forth Weinstein's words from [412] about space-time observables (:measurable dynamical quantities) in non-perturbative canonical QG and how the presence of the Diff(M)-constraint group associated with the background smooth manifold M make the whole enterprize of regarding and treating (canonical) QGR as a quantum gauge theory proper an almost impossible task:

[629]This puts into perspective the 'discrete' causet approach to QG in which the causal connection between 'events' is the sole notion, while dynamics is sought after a functional on the kinematical space of causet histories.

(Q7.42) "...In this paper, I show that general relativity is not a gauge theory at all, in the specific sense that gauge theory has in elementary particle physics. This issue is of crucial importance in attempts to quantize general relativity, because *in quantum theory, the generators of gauge transformations are emphatically not treated as observables, while the generators of spatiotemporal (e.g., Lorentz) transformations are in fact the canonical observables.*[a] Thus the discussion in this paper sheds light on the origin of some of the deep and longstanding difficulties in quantum gravity, including the problem of time, a familiar form of which arises from treating the parametrized time-evolution of canonical general relativity as a gauge transformation[b]..."

[a]Our emphasis.

[b]The 'parametrized time-evolution' referred to above is nothing else but the time-diffeomorphisms generated by the Hamiltonian (constraint) in canonical GR regarded as a constrained gauge system like the other three gauge forces of matter. The main point that Weinstein wishes to make in [412] is precisely that (quantum canonical) GR should not (in fact, it *cannot*) be viewed as a (quantum) gauge theory exactly because the Diff(M) group of the background manifold M is *not* a local gauge group (the structure group of a principal fiber bundle) like in electrodynamics or the Yang-Mills theories of matter.

Thus in the base spacetime manifoldless ADG-gravity there are no spacetime measurements as such and, concomitantly, no spacetime observables either. That we actually do *not* measure the space and time locution of quantum fields and their elementary particles has been convincingly argued in the classic pair of papers by Bohr and Rosenfeld [49, 50]; moreover, here, in the pure gauge field-theoretic context of ADG, we *a fortiori* posit that *there is no space-time 'observables' to measure in the first place.* Having said this, there actually are abstract acts of measurement and localization of the ADG-fields: these are effected precisely when one introduces into the theory the abstract sheaf $\mathbf{A}$ of generalized arithmetics or 'coordinates', which in turn locally 'analyzes' the carrier (representation sheaf) space $\mathcal{E}$ of the field $(\mathcal{E}, \mathcal{D})$ into $\mathbf{A}^n$ and identifies the field's quantum particle states with its local sections. Furthermore, we assume that if there is any (geometrical) space(time) at all in the theory, then it is effectively encoded in $\mathbf{A}$ (Gel'fand duality).[630] On the other hand, since we

[630]In the same way that in the classical theory (CDG) M is recovered from $\mathbf{A} \equiv \mathcal{C}^{\infty}_{M}$ by Gel'fand representation theory. In other words, it is the, external to the gravitational field $\mathcal{D}$, 'measurer' or 'geometer' ('observer') who geometrizes (localizes and measures in 'spacetime' terms) the gravitational field by bringing in her own $\mathbf{A}$.

have only fields and no spacetime (field solipsism), we have *field observables* in the theory. As we have repeatedly emphasized in the present work, as field observables we regard the 'geometrical objects' in the theory. These are the $\otimes_{\mathbf{A}}$-tensors (*ie*, the $\mathbf{A}$-sheaf morphisms), with the curvature $R(\mathcal{D})$ of the connection being the archetypical one. In turn, since the gravitational field dynamics is expressed via the curvature field observable, the dynamics is $\mathbf{A}$-functorial ('synvariant'), and the field 'sees through' our generalized measurements in $\mathbf{A}$ (and *in extenso*, through the 'geometrical spacetime' encoded in $\mathbf{A}$)—in other words, our measurements 'respect' the field dynamics (PFR). As a result, neither our generalized measurement acts are field-perturbing

The Kuhnian gist of ADG: the physical unreality of spacetime. One could say that the central thesis in Kuhn's seminal work [239] is that scientific revolutions happen when a different (from the traditional, 'standard' ones) theoretical paradigm—essentially, a different theory—is adopted by scientists to 'look at the World'. Indeed, theoretical paradigms or 'theories' are thought of by Kuhn as frameworks within the boundaries and from the perspective of which we view what's 'out there'.[631] There is no physical reality (and objective Truth!) out there that we are trying to capture (or even approximate) with our theories. Whatever we can utter about the world we can do so from within the confines of the theoretical paradigm we have adopted. In a deep sense, a theory really determines what we see and the means we devise in order to see 'it'.[632]

In the particular case of ADG and its application to ADG-gravity, all differential geometry and its application to gravity boils down to our choice of $\mathbf{A}$. There is no *a priori* posited (smooth) space(time) geometry 'out there'; only the one that *we* carry in the algebraic baggage $\mathbf{A}$ we have adopted. The significant paradigm shift that ADG and ADG-gravity brings about is that no external smooth space(time) is required in order to do differential geometry (and apply it to gravity); one simply adopts an $\mathbf{A}$ (suitable for tackling the particular physical problem one has in mind, and providing one with the basic d with which one can actually do DG) via which then one 'looks at the world' with 'geometrical eyes'. Of course, the physical objectivity (reality) of the world (*ie*, of the field) is secured in precisely that the physical laws that the fields (*viz.* connections) define are $\mathbf{A}$-functorial (*ie*, 'invariant' under changes of 'point of view' $\mathbf{A}$), which in turn entails the PARD and its expression as a natural transformation, as well as the PFR (field solipsism).

Also quote Feynman from [139]:

[631] In this respect the term 'theory' is in line with the original Greek '$\theta\epsilon\omega\rho\acute{\iota}\alpha$'—way of looking at (and thinking about!) the World.

[632] That is, experiments are planned and experimental apparatuses are designed according to a theory. In a deep sense, theory comes before experiment.

"**Q:** So these infinities have plagued quantum field theory for over a generation. Do you think that a fundamental theory of different particle interactions can still contain these infinities? Or do you think that Dirac was right to say he couldn't believe any theory that contained these infinities?
A: Well, obviously there are no infinities in observation—the mass of the electron is not infinite... Now, [renormalization technology aside,] *it should be possible one day for someone to work out more carefully in a different way a set of equations in which there aren't any infinities and which have the same consequences. I don't mean by inventing a new physics, but rather by reorganizing the statement of what it is you do to make the calculations less awkwardly written. So it's just a matter of mathematical technology in that case...It therefore must be possible to say what the result is without going through the infinities. So I think that those infinities are somehow technical. We're formulating the theories incorrectly when we first write them down*[a]..."
"**Q:** Of course, the really tough problem as far as the infinities are concerned is gravity...How to solve the problem of divergences?
A: ...The question is whether gravity has to be a quantum mechanical theory, like the other quantum mechanical phenomena associated with the other particles. It doesn't seem possible to have the world partly classical and partly quantum mechanical. Therefore, for example, the fact that you can't observe a position and a momentum at the same time with arbitrary accuracy—
(Q7.43) which is what we know from quantum mechanics—should apply to gravity also. We shouldn't be able to use gravitational forces to determine the position and momentum of a particle beyond a certain accuracy, because we'd run into an inconsistency. In trying to modify gravity theory to make it into a quantum theory we discover infinities just like we did in electrodynamics, but which are much more difficult to sweep under the rug. They're much more serious. I don't know how gravity fits in these things, but it has to fit in. It presents a very large number of problems beside the infinities.
In the quantum field theories, there is an energy associated with what we call the vacuum in which everything has settled to the lowest energy; that energy is not zero—according to the theory. Now gravity is supposed to interact with every form of energy and should interact then with this vacuum energy. And therefore, so to speak, a vacuum would have a weight—an equivalent mass energy—and produce a gravitational field. Well it doesn't! The gravitational field produced by the energy in the electromagnetic field in a vacuum—where there's no light, just quiet, nothing—should be enormous, so enormous, it would be obvious. The fact is, it's zero! Or so small that it's completely in disagreement with what we'd expect from the field theory. This problem is sometimes called the cosmological constant problem. It suggests that we're missing something in our formulation of the theory of gravity."

(Q7.43—cont'd)

"It's even possible that the cause of the trouble—the infinities—arises from gravity interacting with its own energy in a vacuum. And we started off wrong because we already know there's something wrong with the idea that gravity should interact with the energy of a vacuum. So I think the first thing we should understand is how to formulate gravity so that it doesn't interact with the energy in a vacuum. Or maybe we need to formulate the field theories so there isn't any energy in a vacuum in the first place. *In other words, there are some mysteries associated with the problem of quantizing gravity which go beyond the infinities. They have to do with the formulation of the theory in the first place.*[a]

"**Q:** There are also some conceptual issues. If you're applying quantum mechanics to gravity, then in a sense you're applying quantum mechanics to space and time..."

[a]Again, emphasis is ours.

There is a plethora of issues raised by Feynman above that we could address under the prism of ADG and ADG-gravity:

1. First thing to note is that Feynman, in view of the unphysical infinities that plague (quantum) field theories of matter and especially (in attempts to quantize) GR, explicitly maintains that for tackling them *no new physics is needed, but rather new mathematical technology.* In his own words, it is as if "*we are formulating the theories incorrectly when we first write them down*". Indeed, that's completely in line with the gist of ADG-gravity: we hold that we do not propose some new physics—as it were, new physical laws—for after all Einstein's equations (29) formally remain the same in our theory and one could say, as Feynman insists, that *the ADG-gravitational equations have the same 'consequences' as in the original, manifold based theory* (GR).[633] However, what radically changes is the theoretical-mathematical framework and technology via which we view and *physically interpret* them 'same' equations.[634] What really changes is the way we arrive at, write and interpret the law of gravity in the first place.

2. The said way on the one hand forces no background geometrical smooth spacetime interpretation of the equations and the underlying mathematical formalism (CDG), so that the theory *ab initio* encounters no problem of unphysical infinities, and on the other, it allows us to interpret them quantum mechanically from the very start. That is, we suggest that

[633]For the dynamical law (Einstein equations) remains formally the same, while the classical theory can be simply recovered by assuming $\mathbf{A} \equiv \mathcal{C}^\infty_X$.

[634]Which from a semantic point of view they are not the same anymore!

when Feynman says that we are formulating gravity incorrectly in the first place, and apart from the 'physical reason' he gives (*ie*, the gravitational interaction with the vacuum energy) infinities-aside, from the perspective of ADG-gravity this corresponds to the 'classical' or traditional way we formulate GR by the CDG-technology, in effect, by assuming a smooth spacetime manifold in the first place. Then our Calculus (mathematical technology) glaringly miscarries with quantum theory (*eg*, the appearance of singularities and meaningless infinities)—*ie*, when one tries to apply quantum ideas to the base manifold and CDG-based relativistic field theory of gravity (GR).

3. On the other hand, the inherently spacetime manifoldless ADG-gravity goes against Feynman's 'pessimistic anticipation' maintaining that any attempt to localize (*ie*, measure the position and momentum) of a particle beyond certain accuracy by using gravitational fields will run into an inconsistency (*eg*, an infinity for a physically measurable physical quantity). That is to say, we do not regard such inconsistencies as *physical* problems, but as shortcomings of the very theoretical-mathematical framework and formalism (the manifold based CDG) within which we formulate the physical theory in the first place. Our disagreement with Feynman becomes even more prominent when he says at the end of the quotation above that "*if you're applying quantum mechanics to gravity, then in a sense you're applying quantum mechanics to space and time*". Under the prism of ADG-gravity, not only we do not apply quantum mechanics to (*ie*, quantize) gravity (as a field theory), but also in the first place there is no space-time in the theory to quantize. ADG-gravity is a background spacetimeless, third gauge and third quantum field theory in which no singularity or infinity is involved at all.

7.6.3 Questions of Renormalizability Questioned and 'Renormalized'

Let us first consider some remarks by Feynman about renormalization found in [137]:

(Q7.44) "*...*[Due to nonsensical infinities],[a] *I don't think we have a completely satisfactory relativistic quantum-mechanical model...Therefore,* I think that the renormalization theory is simply a way to sweep the difficulties of the divergences of electrodynamics under the rug[b]*...*"

[a]Our addition for textual continuity.
[b]Our emphasis.

and attune it with Dirac's well documented not accepting the QFT's infinities, as well as for his optimistic vision that a time will come when a truly relativistic and genuinely infinities'-free quantum mechanics will be developed, as quoted below from [102]:

(Q7.45)

> "...This was all satisfactory so long as one considered only a single particle. There remained, of course, the problem of two or more particles interacting with each other. Then one soon found that there were serious difficulties. Applying the standard rules, all one could say was that the theory did not work. The theory allowed one to set up definite equations. *When one tried to interpret those equations, one found that certain quantities were infinite according to the theory, when according to common sense they should be finite.*[a] That was a very serious difficulty in the theory, a difficulty that still has not been completely resolved.
> *Physicists have been very clever in finding ways of turning a blind eye to terms they prefer not to see in an equation.*[b] They may go on to get useful results, but this procedure is of course very far from the way in which Einstein thought that nature should work. *It seems clear that the present quantum mechanics is not in its final form.*[c] Some further changes will be needed, just about as drastic as the changes made in passing from Bohr's orbit theory to quantum mechanics. *Some day a new quantum mechanics, a relativistic one, will be discovered, in which we will not have these infinities occurring at all*[d]*...*"
>
> [a]Our emphasis.
> [b]Our emphasis.
> [c]Again, our emphasis.
> [d]Once again, our emphasis.

One may start with the observation that singularities are already insuperable problems within the classical theory of gravity (GR based on CDG), as it were, long before the 'quantization' of that theory becomes an issue. Yet, some people, including Ashtekar, Hawking, Joshi and Penrose to name a few, have hoped that a genuine QG will not only shed more light on the problem of singularities, but also it may ultimately resolve them.[635] To that we have the testimony of

[635]Well, the exchange between Hawking and Penrose in (Q7.?) below shows rather that while SWH starts from singularities as being 'given'—hopefully to be 'abolished' by a quantization of the classical theory (GR), RP intuits a drastic revision of "*spacetime at a singularity*"—a *singularity manifold* so to speak—according to which, so to say, "*a true quantum gravity does not quite remove singularities*"; or in other words, *he does not expect that a head-on, 'blind' quantization of the classical theory will resolve the singularity problem* (*ie*, in a sense SWH's question is begging the question!). On the other hand, RP expects such a revision of the concept of spacetime at a singularity to lay bare and clear what is a singularity in GR, thus resolving the puzzle that Geroch puts forth in [155].

recent expectations and remarkable results(!) in Loop Quantum Gravity and Cosmology [51, 279, 200]. *Prima facie*, such hopes could derive for instance from the paradigm of the weaker (than gravitational singularities) but still troublesome infinities assailing QFT, since the quantization of classical field theories enlightened us significantly about the nature of—in QED, for example—the electromagnetic (photon) field right at the electron source, as well as about the (mathematical) nature of quantum fields in general (*eg*, their smeared out, distributional character). Admittedly, if it was not for the QFT infinities, the entire renormalization (group) program, the appearance of anomalies and other physically important issues (*eg*, spontaneous symmetry breaking and the Higgs mechanism, confinement and asymptotic freedom, topological aspects of gauge theories) that were of enormous conceptual, technical and calculational import to our current understanding of the fundamental gauge forces of matter, would simply have not come about [216].

’t Hooft’s vision following Feynman: to replace ‘Analysis’ in QFT by perturbation diagrams and emphasize the importance of (renormalizable) gauge theory (especially in QG). Relevant to the gist of the discussion above would be to recall Feynman’s mistrust of (differential) geometric ideas and associated ‘calculational technology’ (:“*fancy schmanzy differential geometry*”) in (Q4.??), and instead his preference to tackle quantum gravity issues expressly perturbatively, along QFTheoretic lines [136, 140]. That is, he chose to first write down the relational-finitistic (combinatorial) Feynman diagrams for gravity, and then look for a possible geometrical (‘spacetime’) interpretation of the calculational rules and algebra accompanying them; moreover, he insisted that emphasis should be placed on the *gauge-theoretic* character of gravity,[636] rather than on its original (and ‘accidental’!) ‘spacetime-(geo)metric’ one due to Einstein.[637] This attitude, strategy and method foreshadowed and influenced significantly subsequent developments in quantum gauge field theory research in general, as we witness for example in ’t Hooft and Veltman’s introductory remarks in their celebrated paper [393]:

[636]Especially for the quantization of gravity, the formalism that appears be more suitable—albeit more heuristic—(than the canonical) is the path integral one. By the way, this is exactly what we espouse in the present paper-book *vis-à-vis* ADG-gravity and its quantum theoresis (see 7.9.1 below).

[637]Especially for the quantization of gravity, the canonical formalism, which relies heavily on differential geometric (CDG) ideas, appears to be the more geometrically bent (and mathematically/analytically more rigorous than the functional integral) approach.

"With the advent of gauge theories it became necessary to reconsider many well established ideas in quantum field theories. The canonical formalism, formerly regarded as the most conventional and rigorous approach, has been abandoned by many authors. The path-integral concept cannot replace the canonical formalism in defining a theory, since path integrals in four dimensions are meaningless without additional and rather ad hoc renormalization prescriptions.

Whatever approach is used, the result is always that the S-matrix is expressed in terms of a certain set of Feynman diagrams. *Few physicists object nowadays to the idea that diagrams contain more truth than the underlying formalism, and it seems only rational to take the final step and abandon operator formalism and path integrals as instruments of analysis.*[a]

Yet it would be very shortsighted to turn away completely from these methods. Many useful relations have been derived, and many more may be in the future. What must be done is to put them on a solid footing. *The situation must be reversed: diagrams form the basis from which everything must be derived. They define the operational rules,*[b] and tell us when to worry about Schwinger terms, subtractions, and whatever other mythological objects needs to be introduced.

(Q7.46) The development of gauge theories owes much to path integrals and it is tempting to attach more than a heuristic value to path integral derivations. Although we do not rely on path integrals in this paper, one may think of expanding the exponent of the interaction Lagrangian in a Taylor series, so that *the algebra of the Gaussian integrals becomes identical to the scheme of manipulations with Feynman diagrams.*[c] That would leave us with the problems of giving the correct $i\epsilon$ prescription in the propagators, and to find a decent renormalization scheme.

There is another aspect that needs emphasis. *From the outset the canonical operator formalism is not a perturbation theory, while diagrams certainly are perturbative objects. Using diagrams as a starting point seems therefore to be a capitulation in the struggle to go beyond perturbation theory. It is unthinkable to accept as a final goal a perturbation theory, and it is not our purpose to forward such a notion.*[d] On the contrary, it becomes more and more clear that perturbation theory is a very useful device to discover equations and properties that may hold true even if the perturbation expansion fails..."

[a] Our emphasis.
[b] Again, our emphasis.
[c] Emphasis is ours.
[d] Once again, our emphasis.

These words should really be coupled (in order to gain more weight) to some other remarks that 't Hooft made about QG proper a little bit later, in [392]:

(Q7.47)

> "...[In the beginning of the paper, after 't Hooft mentions that in all attempts to quantize gravity one encounters fundamental natural units of length, time and mass (energy) in the form of ℓ_P, t_P and E_P, he continues:][a] But then the theory contains a number of obstacles. First there are the conceptual difficulties: the meaning of space and time in Einstein's general relativity as arbitrary coordinates, is very different from that of space and time in quantum mechanics. *The metric tensor $g_{\mu\nu}$, which used to be always fixed and flat in quantum field theory, now becomes a local dynamical variable.*[b]
> Advances have been made, from different directions,[c] to devise a language to formulate quantum gravity, but then the next problem arises: *the theory contains essential infinities such that a field theorist would say: it is not renormalizable. This problem may be very serious. It may very well imply that there exists no well determined, logical, way to combine gravity with quantum mechanics from first principles. And then one is led to the question: should gravity be quantized at all?*[d]...[to the conclusion of the paper]
> ...Even so, a renormalized perturbation expansion [for gravity][e] would only be a small step forward. At very small distances the gravitational effects must be large, because of the dimension of the gravitational constant, so the expansion would break down at small distances anyhow. *We have the impression that not only a better mathematical analysis is needed, but also new physics. What we learned is that in such a theory the metric tensor might not at all be such a fundamental concept.*[f] In any case, its definition is not unambiguous."

[a]These are the Planck scales for length, time and energy (mass), respectively. Our addition for textual continuity and completeness.

[b]Our emphasis.

[c]Here the author gives a couple of references, which we omit.

[d]Our emphasis again.

[e]Our addition for clarity.

[f]Again, emphasis is ours.

Yet, in spite of all this, we still have Dirac's haunting criticism of this *ad hoc* removal of infinities in QFT by the procedure of renormalization:

(Q7.48) "*Sensible mathematics involves neglecting a quantity when it turns out to be small—not neglecting it just because it is infinitely great and you do not want it.*" [101]

even more so when we try to apply conventional quantum ideas—'quantize' as it were—not just to matter as in QFT, but also to spacetime and gravity itself:

(Q7.49) "*Our present quantum theory is very good provided we do not try to push it too far...We do not try to apply it to particles with very high energies and we do not try to apply it to very small distances. When we do try to push it in these directions we get equations which do not have sensible solutions. We get interactions always leading to infinities...It is because of these difficulties that I feel the foundations of quantum mechanics have not yet been correctly established. Working with the present foundation people have done an awful lot of work in making application in which they can find rules for discarding the infinities but these rules, even though they may lead to results in agreement with observations, are artificial rules, and I just cannot accept that the present foundations are correct.*[a]" [164]

[a]All emphasis is ours.

Quite, for as we argued earlier in this subsection in connection with our background spacetimeless ADG-gravity based doubts about the importance of the fundamental physical constants and their conspiring to setting a supposedly natural scale below which classical spacetime and gravity break down signifying at the same time the onset of quantum gravitational effects whose law still eludes us, from a quantum gauge field-theoretic viewpoint the issue of renormalization (and renormalizability of a gauge field theory) is intimately tied to the assumption of a spacetime continuum, hence it too must be questioned in the light of ADG. But let us first quote again 't Hooft on this from the introduction to [391]:

(Q7.50)

> "...Space and time are continuous. This is how it has to be in all our theories, because it is the only way known to implement the experimentally established fact that we have exact Lorentz invariance. *It is also the reason why we must restrict ourselves to renormalizable quantum field theories for elementary particles.*[a] As a consequence, we can consider unlimited scale transformations and study the behavior of our theories at all scales. This behavior is important and turns out to be highly nontrivial. *The fundamental physical parameters such as masses and coupling constants undergo an effective change if we study a theory at a different length and time scale, even the ones that had been introduced as being dimensionless. The reason for this is that the renormalization procedure that relates these constants to physically observable particle properties depends explicitly on the mass and length scale used.*[b]"
>
> [a] Our emphasis.
> [b] Again, our emphasis.

We may summarize and highlight the main points in all the quotations above regarding how a quantum gauge field theoresis of gravity fares *vis-à-vis* renormalization in the light of our ADG-gravity paradigm, as follows:

- One, like Feynman originally did, may abandon Analysis up-front and attempt to head-on quantize gravity diagrammatically (perturbatively). Then, especially viewing gravity as a gauge theory (like the other renormalizable fundamental quantum gauge forces of matter), sooner or later the procedure of renormalization must be evoked.

- Renormalizability is intimately tied to the assumption of a spacetime continuum [388, 89], which of course shows no *a priori* 'preference' for a fundamental scale (regularization 'cut-off'), while at the same time it is 'responsible' for the singularities and associated infinities in field theory—the *raison d'être et de faire* of renormalization in the first place. Indirectly, in a strong sense Analysis is still to be blamed.

- As a result, a theoretical physicist with a particle physics bent (:QFTheorist) would be entirely content with such a situation if gravity did not prove to be perturbatively non-renormalizable (for one thing, due to the dimensionful Newton constant), something that may be turned around to a 'non-problem' if one wishes to concede that there actually is a minimal space-time length-duration in Nature—an inherent regularization 'cut-off' scale in the elusive QG theory one is after.[638]

- Then one is in principle bound to recognize that QG 'exists' non-perturbatively after all (and the theoretical physicist ceases being a die-hard particle theorist!). Of course, such a 'conversion' of a 'perturbative particle physicist' to a 'non-perturbative quantum gravitist' is not without its technical cost and conceptual concessions: for one thing, the 'renormalization rug' the particle physicist used to sweep the matter gauge field-infinities under is now, in the context of gravity, replaced by the Planck scale, with the concomitant relegation of the elusive 'true' QG law below it.[639] This means that the spacetime continuum, which is invaluable for renormalization, is given up, and also, inevitably, the standard Analysis (CDG) supported by it.

- We appear to have gone in circles: in the beginning we abandoned Analysis to work diagrammatically (relationally-combinatorially) *à la* Feynman. When we realized that the consistency of the rules of playing with these diagrams and the physical sensibility of the interpretation of the results of the diagrammatic method are dependent in one way or another on the spacetime continuum and its (differential/integral) Calculus—in fact, they fundamentally presuppose it[640]—we decided to go 'non-perturbative' and we in a strong sense reinstated the background differential spacetime manifold and its CDG, only to get rid of it by the process of quantization.[641]

In view of all this, we are confident to say that the background spacetimeless ADG may be the "better" (than the continuum based CDG) "mathematical analysis needed" (Q7.?) in QG. In the new mathematical ADG-framework, not only (quantum) gravity is regarded as a (quantum) gauge theory (of the third kind), but also exactly because no external (to the gauge gravitational field itself), background spacetime manifold is involved at all, the (non-)renormalizability of gravity along QFTheoretic lines, as well as the appearance of a minimal spacetime scale are really

[638]See again the Ashtekar quotes (Q7.?) and (Q7.?) above.

[639]Metaphorically speaking, the renormalization rug is now replaced by the black hole horizon-membrane, and only a quantum theory of gravity is supposed to tell us what is going on in its interior, as it were below the Planck scale [386, 387, 389, 391].

[640]For example, in the continuum one can in principle integrate over infinite momenta-energies, or dually, over arbitrarily short distances-time durations.

[641]This is what has happened in LQG (and its affine LQC and QRG theories) for example where, although QG is approached non-perturbatively as a quantum gauge theory in a manifestly background metric independent way, a differential manifold is still retained as a base, only to be quantized in the later stages of the development of the theory, while at the same time the spacetime quantization is used to show how to bypass and 'resolve' the initial cosmological and (Schwarzschild) black hole singularities [51, 279, **?**].

'non-problems'/'non-issues' in the theory. Let alone that no singularities and associated unphysical infinities plague ADG-gravity; hence, like we asked about the Planck length above: '*whence renormalization of gravity?*'

We would like to conclude this subsection by paralleling the way we view ADG's role in QG against 't Hooft's critique of (the mathematics of) (super)string *vis-à-vis* QG from the introduction to [391]—a critique that is similar to his words quoted before (Q7.?) about a better analysis and a new physics in the Planck regime:

(Q7.51)

> "...The same objection [about whether general relativity is genuinely united with quantum mechanics][a] can be brought against the 'superstring' approach to quantizing gravity. The space-time in which the superstring moves is a continuous space-time, and yet we have a distance scale at which a smooth metric becomes meaningless. On the other hand a flat background metric is usually required at ultrashort distances, even in string theories. *It is my conviction that a much more drastic approach is inevitable. Space-time ceases to make sense at distances shorter than the Planck length. Here again I reject a purely mathematical attack, particularly when the math is impressive for its stunning complexity, yet too straightforward to be credible. The point here is also that our problem is not only a mathematical one but more essentially physical as well: what is it precisely that we want to know, and what do we know already?*[b]"
>
> [a]Our addition for textual clarity and continuity.
> [b]Our emphasis.

In certain ways we agree with 't Hooft, while in others we disagree. For starters, it is true that (perturbative) string theory still effectively uses a base spacetime continuum and a smooth background metric, thus we agree with his dissatisfaction about that. But then we disagree with both his invoking of a fundamental scale as a counter-argument against the smooth manifold, and with his doubts that a different mathematical framework and approach to QG will enlighten the problem. At the same time, we share his conviction that the problems of QG are deeply conceptual (not just technical) ones (Q?.?), but also think that a fundamental revision and scrutiny of the basic conceptual aspects of the mathematics (in particular, DG) that we use to formulate (and concomitantly *interpret*!) our physical theories is what is really needed in current QG research: a new mathematical-theoretical framework is pregnant to new ideas and opens new routes of interpretation.

7.7 All is Quantum: 'ADG-Gravity' is 'Inherently Quantum'—a Summary of our Credo (so far)

We would like to use this subsection to summarize in an 'aphoristic' way our theses about ADG-gravity *vis-à-vis* QG. Let us itemize them:

- From the ADG-viewpoint *all is field, and the field is third gauge and third quantum.*

- *The ADG-gravitational field*, represented by a dynamically autonomous, purely algebraic connection $\mathcal{D}$, *is 'already' or 'inherently' quantum, and therefore in no need of a formal procedure of 'quantization'.*

- It follows that *there is no formal distinction in the theory between a 'classical' and a 'quantum' domain.* Strictly speaking, there is no analogue of Bohr's correspondence principle (and the issue of the emergence of classicality) in our ADG-theoresis of gravity.[642]

- Since the theory is fundamentally background spacetimeless (whether the latter may be taken to be a classical continuum or a quantal discretum), *no need arises to quantize spacetime itself.*

- It follows that, from our vantage, *the problem of QG is not fundamentally related to the problem of the quantum structure of space and time.*

- The finitary, causal and quantal gravitational field, as represented à la ADG by a connection on a finsheaf of qausets [270, 271, 272, 317], is already geometrically (pre)quantized. (29), in our view, *represents the quantum Einstein equations.*

- *The so-called classical gravity (GR) pathologies (eg, singularities) as well as the unphysical infinities of the manifold and hence CDG-based quantum gauge field theories of matter are not problems and shortcomings of the physics (ie, of the physical laws), but of the mathematics used to represent and calculate the outcomes (thus interpret as well as draw consequences from) those physical laws—ie, they are anomalies of our background manifold mediated and effectuated Calculus (CDG).*

[642] No 'GR as low energy limit of QG'.

7.8 Sheaf Theory, the 'Narrow' and 'Delicate' Passage from-Local-to-Global-and-Back, and its Potential Import to QG Research

7.8.1 Brief technical generalities

First let us make it clear up-front that the notion of *sheaf* has *by definition* (and by construction!) both *local* and *global* characteristics built in itself. To begin with, and technically speaking for a little while, a sheaf is a *local-topological* structure (or construction), being defined as we saw earlier as a *local homeomorphism* between a base topological space X and the sheaf space Φ [62, 259]. In more detail, and for functional sheaves in particular,[643] one begins by gathering field-information ('field-data') ϕ locally in X—*ie*, 'field-values' for every open subset U in X. This collection of local 'data' about ϕ is then thought of as constituting a so-called *presheaf* (of functions ϕ subjected to restriction maps relative to the lattice of open subsets U of X) [62, 259, 310]. Subsequently, a sheaf is constructed from the said presheaf, by a procedure called *sheafification*, essentially by *collating, gluing, or stitching together*, the aforesaid local information about ϕ[644] into the *global sheaf space* Φ in such a way that one can show that X and Φ are *topologically equivalent, at least locally* (*ie*, they are locally homeomorphic to each other).[645] In fact, one can go as far as to say that

"sheafification is localization" [266].[646]

Concomitantly, a very instrumental notion in sheaf theory, and one that plays a crucial role in the aforementioned stitching up Φ from $\phi|_U$, is that of the *continuous (local) sections* of a sheaf—maps s from a $U \subset X$ to Φ that are themselves local homeomorphisms [259]. In fact, that the notions of a sheaf and of (the collection of) its (local) sections (for all $U \subset X$) are tautosemous is captured by the well known 'slogan-result' in sheaf theory that *a sheaf is its (continuous local) sections*—that is to say, that the (compatible) local information about the field (as encoded in the local sections) is glued together in such a '*coherent*' way so as to comprise the total or global sheaf

[643] That is, sheaves of functions ϕ (on X), such as those to which all physical fields are supposed to belong.

[644] In particular, *compatible* local information about the field—*ie*, ϕ-data or 'values' which agree on the overlap $(U \cap V)$ of the open regions of X $(U, V \subset X)$.

[645] As a matter of fact, functional presheaves '*locally characterized*' are *complete*, which is tantamount to their being sheaves [259, 260, 266].

[646] A very powerful statement indeed, at least *vis-à-vis* physics, since we have argued throughout our past trilogy [270, 271, 272] that, in turn, "*localization is gauging*" and concomitantly "*gauging entails dynamical variability*". Thus, *in toto*, *sheaf theory appears to be the appropriate language in which to formulate dynamical physical theories of a gauge character—ergo* ADG(!), especially if one is faithful to one's intuition (or simply theoretical choice) that the dynamical laws of physics should be represented by *differential* equations proper (*ie*, that some sort of differential geometry should be formulated along sheaf-theoretic lines).

space, which conversely can be analyzed or 'decomposed' at will into the local sections that make it up [259, 266].[647] In this (mathematical) sense, *a sheaf models effectively the transition from local to global, and back*.

7.8.2 From local ('micro'/particle) to global ('macro'/field) and back: the essence of sheaf theory vis-à-vis field theory à la ADG

After these brief technical generalities about how the mathematical concept and structure of a sheaf is suitable to accommodate the transition from local to global and back,[648] we wish to bring forth and discuss recent tendencies of (mathematical) physicists to apply sheaf theory to different research areas of theoretical physics—from QFT and its infinities, to GR and its singularities, and hopefully to QG—exactly thanks to the aforesaid virtue of the notion of sheaf in helping one attain both a worm's (local/particle/'micro') and a bird's (global/field/'macro') eye-view of field theory.

In particular, we will focus on how ADG's conception and definition of a field, as being represented by a pair $(\mathcal{E}, \mathcal{D})$, is tailor-cut for implementing that desired transition from local to global, and conversely, especially in the context of QG. In the process, we will juxtapose, as a *contrapunctus* as it were, certain aspects of Feynman's 'global' (*ie*, entire spacetime histories of a) particle viewpoint of QED *versus* the more 'commonplace' perspective on QED as a local QFT based on local field interactions and differential equations modelling (local) dynamical propagations (in the Minkowski continuum), as concisely described in his Nobel prize address [137].

So for starters, in Feynman's recollection of how he came to develop the spacetime (histories) view of QED [137], we witness two motivations that led him to that development:

- i) the infinite self-energy of the electron: "*The first* [source of difficulties] *was an infinite energy of interaction of the electron with itself*", and

- ii) the infinities associated with the infinite number of degrees of freedom of fields on the Minkowski spacetime continuum: "*The other difficulty came from some infinities which had to do with the infinite numbers of degrees of freedom in the field*".

In view of these difficulties, Feynman was willing to drop the notion of 'local field' (interactions) altogether:

[647] Alternatively, a sheaf may be viewed as a *fibered space*, with the total space Φ being the set-theoretic (disjoint) union (over all the points x of the base topological space X) of its *fibers* or *stalks* $\Phi_{x\in X}$ which, as we noted earlier in this paper, are inhabited by the so-called *germs* of its continuous local sections—the 'ultralocal' elements of a sheaf [259, 310].

[648] To stress it again, the direction '*from-local-to-global*' is the process of gluing (or composition) of local function (field) data that sheafification achieves, while the opposite direction, '*from-global-to-local*' refers to the (dis)section (or analysis) of the total sheaf space into its local (continuous) sections that ultimately constitute it.

(Q7.52) "...And, so I suggested to myself, that electrons cannot act on themselves, they can only act on other electrons. *That means that there is no field at all. You see, if all charges contribute to making a single common field, and if that common field acts back on all the charges, then each charge must act back on itself. Well, that was where the mistake was, there was no field.*[a] It is just that when you shook one charge, another would shake later..."

[a]Our emphasis.

Thus, in a nutshell, Feynman envisaged that direct interactions solely in terms of (a finite number of) quanta (particles), and without the notion of a 'global' field (which, anyway, in a Machian sense, is the quanta that produce/comprise it) would solve the problem of infinities:

(Q7.53) "*...There is no field at all; or if you insist on thinking in terms of ideas like that of a field, this field is always completely determined by the action of the particles that produce it.*[a] You shake this particle, it shakes that one, but if you want to think in a field way, the field, if it's there, would be entirely determined by the matter which generates it, and therefore, the field does not have any independent degrees of freedom and the infinities from the degrees of freedom would be removed..."

[a]Our emphasis.

Furthermore, and perhaps more importantly for our ADG-perspective here, he was ever ready to 'dump' the local/*differential*/Hamiltonian method (of field-theoretic quantization) for a more global/*integral*/Lagrangian one (a covariant action-based quantization—what is commonly known as a 'sum over histories' dynamical scenario)[649]—albeit, a scenario that, almost by definition,[650] still abides by a background spacetime as

[649]A global dynamics that subsumes under an integral—a global object—the contribution of the local dynamical history—the local 'differential behavior' as it were—of every (local) particle comprising the 'global' field.

[650]For, arguably, one cannot speak of 'history' without an *a priori* notion of '(space)time' (paths or trajectories).

(Q7.54)

> "...I would like to emphasize that by this time I was becoming used to a physical point of view different from the more customary one. *In the customary point of view, things are discussed as a function of time in very great detail. For example, you have the field at this moment, a differential equation gives you the field at the next moment and so on; a method, which I shall call the Hamilton method, the time differential method. We have instead a thing that describes the character of the path throughout all space and time. The behavior of nature is determined by saying her whole spacetime path has a certain character...*[a] If you wish to use as variables only the coordinates of particles, then you can talk about the property of the paths—but the path of one particle at a given time is affected by the path of another at a different time. If you try to describe, therefore, things differentially, telling what the present conditions of the particles are, and how these present conditions will affect the future—you see, it is impossible with particles alone, because something the particle did in the past is going to affect the future. Therefore, you need a lot of book-keeping variables to keep track of what the particle did in the past. *These are called field variables.*[b] You will, also, have to tell what the field is at this present moment, if you are to be able to see later what is going to happen. From the overall space-time view of the least action principle, the field disappears as nothing but book-keeping variables insisted on by the Hamiltonian method..."
>
> [a]Our emphasis.
> [b]Our emphasis.

The parallel of the remarks above with ADG's conception of a particle-field pair $(\mathcal{E}, \mathcal{D})$, as well as the potential purely covariant path-integral quantization of such fields, is remarkable indeed, as we briefly explain below:

1. First, let it be stressed that Feynman, one could say, opted for an 'integral' (global) rather than a 'differential' (local) method. This 'global-*versus*-local' dichotomy is *the* quintessential difference between the 'covariant' (path integral, sum-over-histories, Lagrangian action based) and the 'canonical' (differential Hamiltonian based) approaches to quantization in general. Choosing the global method is supposed to capture more naturally the 'non-local' feature of quantum mechanics whereby (quantum field) actions ('wave functions') at a certain

spacetime point (or region) can influence others located at some other region.[651] This is the quintessential aspect of coherent quantum superposition—*the* defining property of quantum systems (*eg*, quantum fields). In the global method all field histories must be taken into account and contribute—in a proportion weighed by the action—to the total path integral quantum field amplitude. On the other hand, the truly local aspects of fields are the (point-) particles, which are localized in spacetime by use of coordinate labels—the basic 'local field variables'. Of course, as Feynman notes above, if one wishes to describe things (:the dynamics) '*differentially*' (:differential geometrically) the coordinates used must be '*differentiable*' ('smooth') functions, so that the presence (and use) of a background spacetime manifold appears to be almost mandatory.

2. Second, the ADG particle-field pair $(\mathcal{E}, \mathcal{D})$ appears to suit well Feynman's description above and in the penultimate quotation, as on the one hand the field (:$\mathcal{D}$) is completely determined by its particles (after all, recall that the carrier/respresentation sheaf $\mathcal{E}$ of $\mathcal{D}$ is completely determined by its local sections, which represent local quantum particle states of the field), and on the other, the local particle information (:sections) is coherently sheaf-theoretically stitched up to comprise the entire sheaf space $\mathcal{E}$ on which $\mathcal{D}$ acts. $\mathcal{D}$, as befits a differential operator, acts locally on $\mathcal{E}$'s sections, but at the same time 'sees' the entire carrier 'particle-space' $\mathcal{E}$. In turn, the quantum dynamics of the field is envisaged to be modelled after a 'global', path integral-type of scenario over the affine space of connection fields (on $\mathcal{E}$), which is the appropriate kinematical space in our ADG-theoresis of gravity.[652] This is how the unitary ADG particle-field pair $(\mathcal{E}, \mathcal{D})$ has both local (particle:=:$\mathcal{E}$) and global (field:=:$\mathcal{D}$) aspects built into it. Of course, as Feynman says, if one wishes to describe the field histories locally and differentially (:differential geometrically) one introduces the coordinate structure sheaf $\mathcal{E}$ of differentiable functions (variables)[653] and the field is relegated solely to a book-keeping device, only keeping track of the (differential) local particle actions.

Sheaf theory in QFT: emphasis on locality. Interestingly enough, in the flat Minkowski manifold based context of algebraic[654] or axiomatic QFT, where the notion of *locality* is of paramount

[651] Always under the condition—an axiom in QFT—of *Einstein locality* (local causality) requiring that the two regions must not be spacelike to each other.

[652] See 7.9.1 below.

[653] And since Feynman arguably had in mind $\mathbf{A} \equiv \mathcal{C}^{\infty}_{X}$, a geometrical spacetime manifold X necessarily got engaged, even if implicitly, into his scheme.

[654] A scheme essentially based on the theory of (noncommutative) Von Neumann or C^*-algebras in which the operator-valued distributions modelling quantum fields are supposed to take their values.

importance,[655] Rudolph Haag in [177] stresses the potential importance of sheaf theory especially concerning its essentially local character. Characteristically, we quote him from that book:[656]

(Q7.55)

> "**Germs.** *We may take it as the central message of Quantum Field Theory that all information characterizing the theory is strictly local i.e. expressed in the structure of the theory in an arbitrarily small neighborhood of a point.*[a] For instance in the traditional approach the theory is characterized by a Lagrangean density. *Since the quantities associated with a point are very singular objects, it is advisable to consider neighborhoods. This means that instead of a fiber bundle one has to work with a sheaf. The needed information consists then of two parts: first the description of the germs, secondly the rules for joining the germs to obtain the theory in a finite region*[b]*...*"
>
> [a]Our emphasis.
> [b]Again, emphasis is ours.

These remarks of Haag are tailor-cut *vis-à-vis* the sheaf-theoretic ADG-field theory and in particular ADG-gravity at least in its finitary-algebraic guise developed in [270, 271, 272], as we explain below:

- First we highlight the remark about the singular character of the pointed spacetime continuum and its 'smearing' or 'blowing-up' by open sets about them. In a sheaf-theoretic setting, one gathers field-information relative to such 'local open gauges' and formulates the field-dynamics locally, as a differential equation, with respect to them.[657]

- The field-law is seen to hold at every such open set and then one stitches up or 'collates' those local field data and build the total field (representation) sheaf space $\mathcal{E}$ on which then one can show that the field law—*eg*, the law of gravity (29)—holds entirely.

- Moreover, in the ADG-setting we can use the bicompleteness of $\mathfrak{DT}$ and actually show that the field-law holds at the limit of infinite refinement (point-localization) of the said open sets covering the 'experimentally actuated' spacetime region X under focus. That is, the field-law

[655]For example, one of the basic axioms of axiomatic QFT is that of *micro-causality*, otherwise known as *Einstein locality* or *the axiom of local causality*.

[656]The reader should note that (s)he will not be able to find this quotation in the 1st 1992 edition of Haag's book. In the 2nd 'expanded' edition, these words can be found on page 326. In this quotation all emphasis is ours.

[657]One should also highlight here Haag's remark that the mathematical structure suitable for doing this is *not* a fiber bundle, but a sheaf.

of gravity holds on (or better, over) each and every point of the underlying background and in its entirety.

This discussion brings us to some other remarks of Stachel [364] about 'the actual practice in GR-research'. Like Haag above, but for different reasons, Stachel insists that the 'appropriate' mathematics, which faithfully represents what general relativists actually do in solving locally the Einstein equations and then extend the solution globally so that the spacetime (manifold) topology is not fixed globally *a priori*, but it rather 'emerges' from stitching up the local gravitational solution-field data, is *not* fiber bundle theory, but *sheaf theory* and in particular *sheaf cohomology*.[658]

Moreover, regarding QG, there is another telling quotation of Stachel, which complements nicely (Q8.?) mentioned in the last footnote and emphasizes the need for developing an approach to QG which takes into account both the locality of GR and the 'globality' of quantum mechanics. What follows are the concluding remarks in [365]:

[658]We will encounter Stachel's pertinent quotation (Q8.?) in the sequel, when we discuss his deeper interpretation and aftermath of Einstein's hole argument.

(Q7.56)

"...The moral I want to draw from this final point may be related to something that Chris Isham talked about. Suppose you take seriously the point of view that there is something fundamentally local about the way general relativity approaches a problem.[a] Then there is another fundamental tension between the basic approaches of general relativity and of quantum mechanics since quantum mechanics, in a deep sense, is fundamentally global in its approach to problems. It doesn't make much sense to talk about the wave function on one patch of space-time, or the sum over all paths on one patch of space-time. In solving a quantum-mechanical problem you have to consider the *whole*[b] manifold from the beginning. The conventional mathematical approach to general relativity, which starts with a manifold, masks this tension. *If we develop a mathematical formulation of general relativity that emphasizes the element of locality from the beginning, it would emphasize this contrast more sharply. Such an emphasis on the tension may be a necessary stage in finding its ultimate solution*[c]*...*"

[a] Indeed, GR is after all a local field theory—the local relativistic field theory *par excellence*, so that Haag's words from the last quotation apply *a fortiori* here.

[b] Stachel's emphasis.

[c] Our emphasis.

Regarding singularities *per se*, we would like to quote Heller from [189] who also notes that in order to cope effectively with all types of gravitational singularities, Geroch's algebraic formulation of GR [157] should be cast sheaf-theoretically:

(Q7.57)

"...To include other type of singularities (also curvature singularities) into the theoretical scheme, *one must change from Einstein algebras to sheaves of Einstein algebras...To allow stronger type of singularities to become part of the theory, we should change from Einstein algebras to sheaves of Einstein algebras...*"[a]

[a] Our emphasis.

And also with Sasin from [193]:

(Q7.58) "*...To cover all types of singularities, one must change to a sheaf of Einstein algebras...*"[a]

[a]Again, our emphasis.

En passant, and in view of these quotes, we would like to ask: physically speaking, what does it mean for '*singularities to be part of the theoretical scheme*'? From the ADG-viewpoint, since the (physical) theory *is* the (dynamical) laws that define it, we could interpret the first remark above as intuiting that *singularities must be integrated or 'absorbed' into the theory; ultimately, into the physical laws (differential equations) that define it.* But this is precisely what ADG-gravity achieves: it 'integrates', 'engulfs' or 'absorbs' the singularity into the **A**-connection field $\mathcal{D}$ which, in turn, 'sees' or 'passes through' them as the law (differential equation) that it defines does not stumble, let alone halt (or break down), at all on them (**A**-functoriality of the ADG-gravitational dynamics).[659]

Sheaf theory in QG: the transition from 'micro' to 'macro'. From what we have said so far about developing a theoretical scheme for QG that goes naturally from local-to-global:

- From Stachel's remarks, it appears that, formally, the tension between GR and QM is the one between local (differential) and global (integral) Calculus methods.
- The upshot of his thoughts is his hunch about the potential import of sheaf theory and sheaf cohomology in order to relieve that tension.
- Then comes Feynman's opting for a global, (path) *integral*-type of quantization of any field theory, instead of a local (Hamiltonian) *differential* one; *as well as his questioning the local field concept altogether*—recall, in his view, the field is relegated to a local book-keeping device for the entire history-actions of particle-quanta.
- Then also along came Haag who, in glaring contradistinction to Feynman, insisted that the quintessence of (Q)FT is that the theory is strictly local, thus, like Stachel, he propounded a sheaf-theoretic formulation of algebraic QFT.
- Finally, Heller *et al.* too pitched the idea of casting sheaf-theoretically the algebraic formulation of GR *à la* Geroch in order to deal effectively with all sorts of gravitational singularities.

All in all, the idea of using sheaf-theoretic methods in GR, QFT and, ultimately, QG has been brewing in the mind of theoretical physicists for quite some time lately, but let it be stressed here

[659]See Eddington quotation above about integrating singularities into the dynamical equations.

that this idea still revolves in one way or another about a base (spacetime) manifold—*ie*, that the sheaves envisaged are still soldered on a differential manifold; they are *smooth sheaves*. Stachel in particular has regarded this manifestation of the 'manifold monopoly' in sheaf theory as an obstacle to the further application of sheaf theory to QG, as we recall from (Q?.?) his words that "*...As far as I know, no one has followed up on this suggestion,*[660] *and my own recent efforts have been stymied by the circumstance that all treatments of sheaf theory that I know assume an underlying manifold...*".

However, to the mind of these authors, Selesnick's remarks below, from a private communication with the second author [341], emphasize in the best way the passing from local to global with the help from sheaf (and topos) theory *vis-à-vis* QG:[661]

(Q7.59) "*...One of the primary technical hurdles which must be overcome by any theory that purports to account, on the basis of microscopic quantum principles, for macroscopic effects (such as the large-scale structure of what appears to us as space-time, ie, gravity) is the handling of the transition from 'localness' to 'globalness'. In the 'classical' world this kind of maneuver has been traditionally effected either measure-theoretically—by evaluating largely mythical integrals, for instance—or geometrically, through the use of sheaf theory, which, surprisingly, has a close relation to topos theory. The failure of integration methods in traditional approaches to quantum gravity may be ascribed in large measure to the inappropriateness of maintaining a manifold—a 'classical' object—as a model for space-time, while performing quantum operations everywhere else. If we give up this classical manifold and replace it by a quantal structure, then the already considerable problem of mediating between local and global (or micro and macro) is compounded with problems arising from the appearance of subtle effects like quantum entanglement, and more generally by the problems arising from the non-objective nature of quantum 'reality'. Although there is a rich and now highly developed mathematical theory of 'noncommutative geometry' (which has had considerable success in application to traditional quantum field theories), a concomitant noncommutative sheaf theory seems to have been slow in coming...*"

[660] That is, to use sheaf theory and sheaf cohomology in GR and QG.

[661] Below, all emphasis is ours.

We agree with most of Selesnick's remarks above, especially with the one about the currently mathematically ill-defined path integrals abounding in theoretical physics[662], as well as with his remarks that we perform all sorts of quantization procedures on the object-fibers (*eg*, fields) living on a spacetime manifold while leaving this classical continuum background intact. On the other hand, as has been repeatedly commented on throughout the present paper-book, we would not go as far as Selesnick goes and maintain that what is actually needed is a quantization of that classical background, let alone that one has to resort to noncommutative mathematics (*eg*, noncommutative geometry, sheaf and topos theory) in order to tackle the problem of QG. Rather, we hold that *no background geometrical spacetime, and in particular a base manifold, is needed at all in an entirely algebraic (sheaf-theoretic) approach to QG regarded as a quantum gauge theory.* Of course, we have ADG-gravity in mind.

But let us now proceed and address from the perspective of ADG the nowadays caustic QG issue of black hole information-loss. The following discussion will again bring forth the broad character, versatility and potential import of ADG-ideas in QG research.

7.8.3 Field information lost in a black hole and found in ADG: the brighter side of black holes

By now it must have become clear that in ADG, and especially in ADG-gravity, singularities are not regarded as being sites past which the dynamical gravitational field law (Einstein equations) cannot be continued, let alone as *loci* in (or at the boundary of) 'physical spacetime' (whatever that means in ADG-gravity before the field dynamics is set up) where the Einstein equations break down like the spacetime manifold and, *in extenso*, the CDG-based approaches to classical and quantum gravity have hitherto made us believe. Expressly, for example, we cannot (and we do not!) think of background geometrical spacetime configurations, such as a closed trapped surface and its ensuing black hole horizon, to be formed by the *physical* process of gravitational collapse for instance, for one thing simply because there is no background spacetime whatsoever (*ie*, a 'continuous' or 'discrete' base/medium on/in which fields dynamically propagate) in the theory to begin with. This puts into perspective, question, and to a large extent doubt, the viability in our theoresis of the by now standard and thought of as being basic concepts, technical tools and general working philosophy of black hole theory, in which every black hole is normally supposed to conceal in its core, as it were hidden behind its horizon, a spacetime singularity.

In view of the above, in this sub-subsection we wish to comment in particular on the nowadays caustic and to a certain degree controversial issue of '*black hole information, or quantum coherence, loss*'. In a nutshell, very roughly and simplified, the original scenario due to Hawking [183] had

[662]For it is true that we do not have yet a well developed integration theory over infinite-dimensional function spaces as envisaged by Feynman.

it that a quantum (particle), initially prepared by an external (to the black hole) experimenter (:'observer') in a pure state, dynamically evolves past the black hole horizon (:'falls into the black hole') and is eventually thermally radiated (:'spat out') in a mixed state (described by the external observer by a density matrix)—an account that appears to violate the linear evolution law of quantum mechanics thus corresponding to a loss of quantum coherence (a sort of 'quantum superposition breakdown'). An obvious 'explanation', viewing the black hole interior as a 'black box'-type of 'region of no pure return', would be an epistemic uncertainty on the part of the external observer[663] to the effect that she, in a hidden variables kind of way, averages over the unobservable particle states beyond the horizon—*ie*, she effectively 'thermalizes' (or 'thermodynamicizes') the system. In other words, the quantum, once past the horizon, is regarded as being coupled to a heat bath in the black hole interior, from which it is subsequently emitted in a thermal (mixed) state [387, 389], with the concomitant breaking of quantum coherence and the loss of information.

Of course, one could immediately observe that Hawking's arguments, which are closely akin to his black hole instability and associated evaporation results, are glaringly semi-classical (:QFT on a black hole background environment), so that again all our shortcomings should be attributed to our not having (as yet!) a cogent QG to describe the physical laws in the black hole interior.[664] Metaphorically speaking, information loss is essentially attributed, once again, to our epistemic uncertainty and theoretical 'inadequacy' in the realm past the black hole horizon. In other words, currently only a quantum theory of gravity is expected to describe the gravitational field and what happens to matter and their radiation fields right at a black hole singularity.[665] In a similar vain, it also follows that QG is expected to unveil the quantum (microscopic) origin of black hole thermodynamics, whose laws (especially the famous second, Bekenstein-Hawking entropy one) essentially subsume our ignorance (arising from the coarse thermodynamic description) about the

[663] For instance, she does not have a clue about the dynamical evolution laws past the horizon's opaque veil, so that any sort of predictability by her is lifted.

[664] As it happens, we do not have a quantum theory of black holes, or what essentially amounts to the same, a theory of quantum (microscopic, of Planck size and mass) black holes where quantum gravitational effects are expected to reign supreme [387]; however, a QFTheoretic (scattering matrix) 'approximation' to such a theory, describing what the external observer—the one outside the horizon—sees as in-going (to the back hole) and out-coming (from the black hole) states of quanta, has already been worked out [389].

[665] And let it be noted here, as Susskind remarks in [378], that while for the external observer quantum linearity and coherence appears to be violated, for an in-falling observer (and particle-quanta), one might argue, there could be a "*brick wall*" at the horizon that ping-pongs back outside the in-falling quantum matter in a pure state. This alternative however would violate the EP of GR, since we know that the horizon is a 'soft', flat, semi-permeable membrane. Thus there's no way out of the impasse: for the external observer QM is violated, while for an in-falling one, GR. It has been suggested by numerous people in the past that to arrive at the 'true' QG (some of) the fundamental principles of both GR and QM should be somehow modified. Penrose, for example, remarks in [303]: "*Is this* [*ie*, the QFTheoretic] *the right way to think about quantum gravity, or should we be looking for some more even-handed marriage, with some give on both sides?*".

black hole's 'true' microscopic degrees of freedom and their quantum microstates.

From all this heat (and debate) that the issue of black hole information loss has generated over the years, we wish to isolate an apofthegma and its corollary from a recent paper by Leonard Susskind [378] in which he tries to come to grips and resolve this apparently paradoxical—or in his own words, "*weird*"—scenario:

(Q7.60)

> **Apofthegma:** "...I can now state the principle of Horizon Complementarity. All it says is that *no observer ever sees a violation of the laws of nature.*[a] More specifically it says that to an observer that never crosses the horizon, the horizon behaves like a complex system which can absorb, thermalize, and re-emit all information that falls on it. No information is ever lost. In essence, the world on the outside of the horizon is a closed system..."
>
> **Corollary:** "...It is clear from Horizon Complementarity that *a revision is needed in the way we think about information being localized in spacetime.*[b] In both classical relativity and quantum field theory the spacetime location of an event is invariant, that is, independent of the observer. Nothing in the theory prepares us for the kind of weirdness I described above..."
>
> [a]Our emphasis.
> [b]Our emphasis.

The ADG-gravity answer to black hole coherence loss 'paradox' involves again a sort of 'cutting the Gordian knot'-type of reasoning similar in spirit to the interpretation of the ADG-theoretic resolution of the inner Schwarzschild singularity we gave above:

1. For starters, on general grounds and as noted earlier, on the face of the background spacetime manifoldless ADG-gravity we do not accept (because it never arises in our theoresis!) that there is a black hole event horizon surrounding a spacetime singularity beyond which, and in the immediate vicinity of the singularity, the so-called 'classical' Einstein equations cannot be continued and, ultimately, do not hold ('break down'). It follows that we do not view the horizon as demarcating not only a physical, but also a 'theoretical' boundary outside of which one law of gravity—the so-called 'classical' one of Einstein (GR)—holds, but that another one—the still to be discovered QG law—is in force in its interior.[666] As a 'counter-scene' to our comments here, let us bring forth from the introduction of [389] the following words:

[666]This is completely analogous to our earlier critique of the Planck length/time as denoting a fundamental scale in Nature below which the Einstein field law of gravity (and the spacetime continuum supporting it at least differential geometrically) breaks down, and especially it applies to quantum black holes of radius of around the Planck length.

(Q7.61) "There is quite a bit of controversy (and confusion) regarding the nature of physical law governing a black hole. Some of the difficulties have their origin in the deceptively clean picture given by the 'classical' (here this means 'non-quantum mechanical') solutions of Einstein's equations of gravity in the case of gravitational collapse. The metric tensor describing the fabric of space-time appears to be smooth and well-behaved in the vicinity of a region we call the 'horizon', a surface beyond which there are space-time points from which no information can reach the outside world..."

Here we once again promulgate the unity and universality of the physical law of gravity—*ie*, Einstein's law *à la* ADG-gravity (29) holds everywhere and the gravitational field 'sees' neither a background spacetime or horizon, nor even a spacetime singularity to that matter (since it is still in force over it, as we saw in Schwarzschild's case). In a peripheral, slanted sense, this unity accords with Susskind's apofthegma above that '*no observer ever sees a violation of the laws of nature*'.[667]

2. About the closely related to the above 'field-information conservation' and 'spacetime localization' mentioned in Susskind's corollary, the spacetimeless, dynamically autonomous, unitary ADG-gravitational field $(\mathcal{E}, \mathcal{D})$ defining (29), is not 'leaking' information to its 'environment',[668] while, as we emphasized earlier, the sheaf-theoretic character of the ADG-gravitational law secures that local field information is coherently stitched up to comprise the global field law. To stress it once again, (29) holds on the total sheaf space $\mathcal{E}$ over the whole surrogate localization background X, without the latter, in contradistinction to Susskind's corollary above, being interpreted as 'spacetime' (so that one can speak of 'spacetime localization' proper). Again, in a peripheral and slanted sense, the unitary and autonomous $(\mathcal{E}, \mathcal{D})$ may be regarded as a(n information-wise) 'hermetically' closed (and 'sealed' from an external/background to it 'spacetime environment') dynamical system.

3. In continuation of the point made above, the geometrically (pre)quantized ADG-gravitational field $(\mathcal{E}, \mathcal{D})$—with $\mathcal{E}$ now regarded as the associated/representation (Hilbert) sheaf of local

[667]The sense is slanted, because on the one hand the relativity of the ADG-gravitational field, effectuated via $\mathcal{A}ut\mathcal{E}$ concerns it and it alone, without reference to an external observer/measurer/experimenter localized (with coordinate origin at a point) in an external (to the ADG-gravitational field) spacetime, and on the other, even when an external observer brings along with her an $\mathbf{A}$ to coordinatize (or 'measure') and geometrically represent the field, the $\mathbf{A}$-functoriality of the law that the latter defines secures that the field 'sees' no background spacetime (horizon), or even singularity (all of which are built into $\mathbf{A}$).

[668]If one can call that the background, surrogate localization (topological) space X, which does not partake into the gravitational dynamics.

quantum (particle) states of the (principal sheaf $\mathcal{A}ut\mathcal{E}$ of dynamical auto-transformations of the) field—may be viewed as a closed *quantum* system, and (29) as its quantum evolution law. Again, there is no information leaking to the ambient spacetime environment, since *ab initio* there is no such milieu in the theory. The (pre)quantized ADG-gravitational dynamics (29), although manifestly non-linear (as befits gravity), it still respects locally coherent quantum superpositions (between the local quantum particle states of the field—*ie*, the local sections of $\mathcal{E}$), reminiscent of the EP in GR whereby locally the (curved) spacetime continuum is isomorphic to the (flat) Minkowski vector space [270]. *A fortiori*, here, since no external (to the field) smooth spacetime continuum is involved, thus no $\mathrm{Diff}(M)$ either, there is no "*possible breakdown of the superposition principle*" [411, 412] and the concomitant lifting of quantum coherence. *In toto*, the ADG-gravitational field is a self-quantum, coherent auto-dynamical system.

4. The last thing that we would like to comment on, which also nicely wraps up the whole discussion above about black hole information loss, is 't Hooft's proposal to regard all ultra-small (of localization radius smaller than the Planck length) and super-heavy (with mass larger than the Planck mass) elementary particles as microscopic (quantum) black holes [387, 389]. To this end, let us first quote him extensively once again from [389]:

"...But what if the hole is small, so that classical observers are too bulky to enter? Or let us ask a question that is probably equivalent to this: suppose one keeps track of *all*[a] possible states a black hole can be in, is it then possible to describe the hole in terms of pure quantum states alone? Will the very tiny black holes evolve according to conventional evolution equations in quantum physics or is the loss of information a fundamental new feature, even for them?

There is a big problem with any theory in which the loss of information is accepted as a fundamental item. This is the fact that all effective laws become fuzzy[b]...

[Then 't Hooft gives an example in quantum mechanics where a system dynamically evolves from pure to mixed states. He then continues...]

In our example we clearly see what the remedy is. *The extra uncertainty had nothing to do with quantum mechanics; the Hamiltonian was not yet known because of our incomplete knowledge of the laws of physics... Returning with this wisdom to the black hole, what knowledge was incomplete? Here I think one has a situation that is common to all macroscopic systems: because of large number of quantum mechanical states it was hopelessly difficult to follow the evolution of just one such state precisely. One was forced to apply thermodynamics. The outcome of our calculations with black holes got the form of thermodynamic expressions because of the impossibility, in practice, to follow in detail the evolution of any particular quantum state.*[c]

But this does mean that our basic understanding of black holes at present is incomplete...If we want to understand how a black hole behaves when it reaches the Planck mass, we expect thermodynamic expressions to break down.

The importance of a good quantum mechanical description is that it would enable us to link black holes with ordinary particles. The Planck region may well be populated by a lot of different types of fundamental particles. Their 'high energy limit' will probably consist of particles small and heavy enough to possess a horizon and thus be indistinguishable from black holes...

(Q7.62) The reason why black holes should be used as a starting point in a theory of elementary particles is that anything that is tiny and heavy enough to be considered an entry in the spectrum of ultra heavy elementary particles (beyond the Planck mass) must be essentially a black hole.

Black holes are defined as solutions of the classical, i.e. unquantized, Einstein equations of General relativity. This implies that we only know how to describe them reliably when they are considerably bigger than the Planck length and heavier than the Planck mass. What was discovered by Hawking in 1975 is that these objects radiate and therefore must decrease in size. It is obvious that they will sooner or later enter the domain that we presently do not understand.[d]

...[For the study of microscopic (quantum) black holes,][e] what we require is first of all some quantum mechanically pure evolution

Again, the whole issue revolves in one way or another about the problem of 'spacetime localization' as also Susskind mentioned above: the thing is that the general opinion nowadays is that one cannot localize in space-time a quantum particle with accuracy higher (:uncertainty smaller) than the Planck length-time without creating a black hole (singularity). Equivalently, in the Planck regime, where quantum gravitational effects are expected to be significant, all elementary particles can be modelled after black holes; hence a quantum theory for black holes appears to be mandated and come hand in hand with a quantum theory of gravity. Moreover, the theoretical threshold or divide (alongside the Planck length) appears to be the horizon, outside of which the non-quantum GR (and the picture of spacetime as a classical smooth continuum) holds galore, but inside of which it appears to break down.

Concerning ADG-gravity, (29) may be regarded as a (pre)quantum evolution equation for the ADG-gravitational field; while a fully quantum gravitational propagator, as noted earlier, is currently expected to be modelled after a path integral-type of dynamics over the $\mathsf{A}/\mathcal{A}ut\mathcal{E}$-moduli space of ADG-gravitational connections—a dynamics which in turn will not encounter any hindrance from a spacetime horizon or a singularity concealed behind it.[669] *In summa*, let it be emphasized here that in a strong sense the entire present 'paper-book' comes precisely as an 'antithesis' to 't Hooft's thesis above.

7.9 Genuinely Quantum and Purely Gauge Vacuum Einstein Gravity

As noted in section 4, the idea to view Einstein gravity (GR) as a gauge theory, based predominantly on the notion of (affine, non-metric) connection and less on the metric as in the original pseudo-Riemannian geometry based theory of Einstein, dates back to the pioneering work of Weyl, Eddington and Cartan.[670] *Vis-à-vis* QG, this idea seems to be quite a natural one if one wishes to (as many theoretical physicists do indeed aspire to) 'unify' gravity with the other three fundamental forces of Nature, since the latter have proven to be gauge forces *par excellence*.

Just three years before Ashtekar's ground-breaking discovery of the new (spin-Lorentzian connection) variables for classical and quantum gravity [342, 7]—a discovery that imparted tremendous impetus to the quest for formulating GR as a gauge theory proper, hence it revived the interest in approaching QG like one approaches the other three quantum gauge theories of matter—Ivanenko and Sardanashvily made some very telling, almost prophetic (at least for Ashtekar's work), remarks in their well known Physics Report [215]:

[669]See sub-subsection 7.9.1 in the sequel.

[670]The first two also wanted to unite gravity with electromagnetism—the first attempts at a unified field theory.

> "...These and some other difficulties[a] of the GR picture of gravity motivate one's attempts to reformulate gravitational theory from non-conventional standpoints extending the framework of Einstein's GR.
>
> But why gauge gravity? Can the gauge treatment of gravity really solve the above-mentioned problems? Beforehand nobody knows. But today many of these problems seem to be put in the shadow of the urgent goal of the gravity unification with the elementary particle world. Just this goal stimulated by the grand unification program in contemporary particle physics puts the gauge version in the forefront of modern gravitation research.
>
> Today, gauge theory provides the theoretical base of all modern unification attempts in particle physics. It has become clear that weak and electromagnetic interactions can be successfully unified by the Weinberg-Salam gauge model, and there is strong evidence that strong interaction is also mediated by gauge particles or gluons within the framework of chromodynamics. In field theory gauge potentials become a standard tool for describing interactions with very different symmetries. And apparently the single gap in the modern gauge picture still remains gauging the external or space-time symmetries of fields and particles, that includes the gauge gravity also.
>
> Moreover, gauge theory using the mathematical formalism of fiber bundles realizes bin fact the known program of the 1920s to build the geometric unified picture of various interactions. And it is strange enough that gravitation theory, being the first example of field geometrization, has still not any recognized gauge version. Although the first gauge treatment of gravity was suggested immediately after the gauge theory birth itself [...][b]
>
> The main dilemma which during 25 years has been confronting the
> (Q7.63) establishment of the gauge gravitation theory, is that gauge potentials represent connections on fiber bundles, while gravitational fields in GR are only metric or tetrad (vierbein) fields.
>
> Connections as fundamental quantities appeared together with the metric in Weyl's and Eddington's generalizations of GR on gravity with nonmetricity and torsion,[c] and in this quality were again recognized by Einstein in his last scientific paper [...].[d] But even in the gauge gravitation theory connections cannot at all substitute the metric, because there are no groups, whose gauging would lead to the purely gravitational part of space-time connections (Christoffel symbols or Fock-Ivanenko spinorial coefficients). To separate such gravitational components from gauge fields, e.g., of the Lorentz group, metric or tetrad fields have to be introduced.
>
> At the present time the gravitation theory is viewed actually as the affine-metric theory possessing two independent potentials, namely, metric and connection, and just this constitutes the peculiarity of the gauge approach to the gravitation theory in comparison with the gauge models of the Yang-Mills type for internal symmetries..."

[a]The authors here have just mentioned a few problems encountered

It is fair to say that to this day, even after Ashtekar's new connection variables formulation of gravity three years after the quotation above, all the obstacles and problems stressed above regarding the (quantum) gauge theoresis of gravity, although alleviated, still remain in effect. Of course, as noted earlier, the new spin-connection variables, by simplifying the gravitational constraints, tremendously facilitated the canonical quantization of gravity regarded as a constrained gauge system[671] (*à la* Dirac-Bergmann), and it achieved the formulation of canonical QGR in a genuinely background metric independent fashion. However, Ashtekar's first order formalism

- on the one hand it still is in essence an affine-metric theory in the sense of Ivanenko and Sardanashvily, as it regards the spacetime metric (in the guise of the *vierbein* field) as a gravitational dynamical variable on a par with the affine connection. That is, the Ashtekar scheme is *not* a purely gauge (*ie*, solely connection based) formulation of gravity like the half-order (third gauge) formalism of ADG-gravity is,

- and on the other hand, it still employs a background (spacetime) differential manifold, thus also, inevitably, it involves the four (primary or first-class) $\mathrm{Diff}(M)$-external spacetime constraints of GR (which anyway it aims to simplify in the first place).

Moreover, as noted earlier and in [317], it is precisely the *non-gauge* character of the $\mathrm{Diff}(M)$ group of the external spacetime manifold that prevents one from formulating gravity as a gauge theory like the other three fundamental forces [411, 412]. This 'non-gauge character' of $\mathrm{Diff}(M)$ means that it is 'wrong' (and certainly misleading) to think of the $\mathrm{Diff}(M)$-invariances of the (external spacetime manifold based) theory of gravity (GR) as gravitational gauge symmetries proper. For plainly, technically speaking at least, the diffeomorphisms in $\mathrm{Diff}(M)$ are (by definition) the (external) 'symmetries' (automorphisms) of the (external) $\mathcal{C}^\infty$-smooth base spacetime manifold (*ie*, $\mathrm{Aut}(M) \equiv \mathrm{Diff}(M)$), *not elements of the automorphism group of a principal fiber bundle* as in the other three gauge theories of matter. In other words, diffeomorphisms are *external* not *internal* symmetries. Indeed, as repeatedly mentioned throughout this paper, especially *vis-à-vis* the problem of (canonical/Hamiltonian or covariant/Lagrangian) quantization of gravity, regarding $\mathrm{Diff}(M)$ as gravity's gauge symmetry group proper leads to a number of formidable problems, such as the so-called *problem of time* and the *inner product/quantum measure problem.*[672]

[671] For the general theory of quantization of gauge systems with constraints, see [195].

[672] Let alone that even in the classical theory (GR) $\mathrm{Diff}(M)$ causes problems in defining precisely what is a singularity in the theory (recall that in the present paper we 'defined' singularities as solution-metric differential geometric anomalies) as well as, by underlying the PGC, it apparently halted for a couple of years (till its significance was further appreciated) the development of the theory at its very birth. This refers to Einstein's hole argument, which we are going to tackle under the prism of ADG-gravity in the next section.

Of particular interest to us here is *the problem of defining physically meaningful observables* in a possible (canonical) quantization of (vacuum) Einstein gravity. This problem too is due to the external spacetime Diff(M)-invariances of the classical theory (GR), as the following excerpt from [412] corroborates:

(Q7.64)

> "...The distinction between the two sorts of invariance[a] is absolutely vital in the context of quantum theory. We understand how to quantize gauge theories. A primary aspect of this is to represent some subset of the classical gauge-invariant quantities (classical observables) as self-adjoint operators (quantum observables) obeying relevant commutation relations. However, *it is not at all clear what it means to quantize a diffeomorphism-invariant theory. One can attempt to turn the diffeomorphism-invariant quantities into observables—this is effectively what one is doing when one formally treats the diffeomorphism group as a gauge group.*[b] However, in addition to leading to the problem of time in canonical gravity [...][c] and possibly a breakdown of the superposition principle [...],[d] *this approach leaves one with virtually no known observables whatsoever for vacuum gravity in the compact case*[e] [...][f]..."
>
> [a]Here Weinstein refers to the distinction noted above, namely, that while the gauge symmetries are internal and can be organized into a principal fiber bundle, the Diff(M)-invariances are external and cannot, thus they should not be regarded as gauge symmetries proper.
>
> [b]Our emphasis.
>
> [c]Here Weinstein gives the following references: [204, 238, 411], as well as another one of his not included at the back.
>
> [d]Reference given here in the original is omitted.
>
> [e]Our emphasis.
>
> [f]Here Weinstein refers to the papers by Torre [394, 395].

Indeed, similarly, in the 'bottom-up', discrete approach to Lorentzian QG called causet theory [361], Sorkin has for a long time been categorematic about and rather critical of any attempt at defining physically meaningful observables in the presence of the Diff(M) implementing the PGC in the classical theory (GR) as it is especially done in the canonical QGR scenario, unless of course one is willing to sacrifice general covariance altogether:[673]

[673]Let it be noted here that in causet theory there is still a 'discrete' analogue of the Diff(M)-modelled PGC of the smooth continuum based 'top-down' approaches to QG (such as the canonical QGR and its associated LQG

"...**What are the 'observables' of quantum gravity?**

(Q7.65) *Just as in the continuum the demand of diffeomorphism invariance makes it harder to formulate meaningful statements,*[a] so also for causets the demand of discrete general covariance has the same consequence, bringing with it the risk that, even if we succeeded in characterizing the covariant questions in abstract formal terms, we might never know what they meant in a physically useful way. I believe that a similar issue will arise in every approach to quantum gravity, discrete or continuous (*unless of course general covariance is renounced*)[b]..." [361]

[a]Think, for example, of the statement that light slows down when passing near the sun. (Sorkin's own footnote.) Emphasis above is ours.

[b]In the context of canonical quantum gravity, this issue is called 'the problem of time'. There, covariance means commuting with the constraints, and the problem is how to interpret quantities which do so in any recognizable spacetime language. (Again, Sorkin's own footnote. The last sentence in this footnote is omitted here.) Emphasis in the parenthesis above is ours.

In the following quotation, taken from [65],[674] Sorkin *et al.* go a bit further and note:

scenario), which is called '*causet-vertex labelling independence*'—much in the same way that in GR the gravitational dynamics (Einstein equations) is indifferent to our smooth coordinate labelling of the points of the underlying manifold (see quotation after the one next).

[674]We will provide a more complete quotation in (Q8.?) in the sequel, when, in the context of our analysis of Einstein's hole argument from the perspective of ADG-gravity, we will criticize not only the usual smooth continuum based Diff(M)-implemented general covariance, but also 'discrete' general covariance like the one proposed in Sorkin *et al.*'s causet scenario.

"...After all, labels in this discrete setting are the analogs of coordinates in the continuum, and the first lesson of general relativity is precisely that such arbitrary identifiers must be regarded as physically meaningless: the elements of spacetime—or of the causet—have individuality only to the extent that they acquire it from the pattern of their relations to the other elements. It is therefore natural to introduce a principle of 'discrete general covariance' according to which 'the labels are physically meaningless'.

(Q7.66) But why have labels at all then? For causets, the reason is that we don't know otherwise how to formulate the idea of sequential growth, or the condition thereon of Bell causality, which plays a crucial role in deriving the dynamics. *Ideally perhaps, one would formulate the theory so that labels never entered, but so far no one knows how to do this—anymore than one knows how to formulate general relativity without introducing extra gauge degrees of freedom that then have to be cancelled against the diffeomorphism invariance.*[a]

[a]Our emphasis.

In striking contrast to all of the points raised in the quotes above, in ADG-vacuum Einstein gravity there is no external (to the gravitational field) background spacetime manifold, and hence no Diff(M) either. All there is is the ADG-gravitational field $(\mathcal{E},\mathcal{D})$ 'in-itself', and the vacuum gravitational dynamics that it defines (29). Since the connection is the sole dynamical variable in our theoresis of gravity (half-order formalism), without any 'metric-commitment',[675] ADG-gravity is a purely gauge theory; moreover, with quantum traits built into the formalism from the very start—*ie*, from the very definition of the ADG-gravitational field as the pair $(\mathcal{E},\mathcal{D})$. However, like Sorkin posits above, in ADG-gravity *covariance is inevitably replaced by synvariance—ie*, one considers the (principal) group sheaf $\mathcal{A}ut\mathcal{E}$ of dynamical self-transmutations ('auto-transformations') of the ADG-gravitational field $(\mathcal{E},\mathcal{D})$ 'in-itself', so that no need to 'gauge' (localize) the external (Lorentz) spacetime symmetries, which anyway do not exist (since the external spacetime does not exist in the first place!), arises whatsoever. In this light, on the one hand Ivanenko and Sardanashvily's remarks that "*...apparently the single gap in the modern gauge picture still remains gauging the external or space-time symmetries of fields and particles, that includes the gauge gravity also...but even in the gauge gravitation theory connections cannot at all substitute the metric, because there are no groups, whose gauging would lead to the purely gravitational part of space-time*

[675]That is, without the smooth spacetime metric g (second-order formalism), or the tetrad e (first-order formalism) being involved at all in the theory.

connections (Christoffel symbols or Fock-Ivanenko spinorial coefficients).[676] *To separate such gravitational components from gauge fields, e.g., of the Lorentz group, metric or tetrad fields have to be introduced...*" are rendered obsolete, and on the other, Sorkin's position about general covariance that "*...Ideally perhaps, one would formulate the theory so that labels never entered, but so far no one knows how to do this—anymore than one knows how to formulate general relativity without introducing extra gauge degrees of freedom that then have to be cancelled against the diffeomorphism invariance...*" is simply overcome since, apart from the ADG-gravitational connection $\mathcal{D}$, there is no other dynamical (gauge) degree of freedom involved that has to vanish when operated on by the diffeomorphism constraints (which anyway do not exist in ADG-gravity, because *ab initio* there is no background manifold).

In particular, regarding the issue of 'observables' (especially in the context of QG), since in the inherently (third) quantum vacuum ADG-gravity the sole dynamical variable is the connection $\mathcal{D}$, the only '*measurable* dynamical variable'—what is commonly understood by the term '*observable*'—is the curvature of the connection $R(\mathcal{D})$. The latter, on the one hand is measurable because it is **A**-valued,[677] and on the other, exactly because it is an **A**-morphism by the very definition of the ADG-gravitational field as the pair $(\mathcal{E}, \mathcal{D})$, it remains (gauge) invariant under the action of the principal group sheaf $\mathcal{A}ut\mathcal{E}$ (of dynamical self-transmutations) of the field on the associated (quantum particle states' representation) sheaf $\mathcal{E}$ of the field. Equivalently, and categorically speaking, the dynamical (differential) equations (29) in which $\mathcal{R}(\mathcal{D})$ partakes (in fact, it defines!), are **A**-functorial ('synvariance'). All in all, in ADG-gravity the sole observable proper is the most important '*geometrical object*' (*ie*, an **A**-tensor) concomitant of the sole dynamical variable in the theory—the connection field $\mathcal{D}$: $R(\mathcal{D})$.

From a more general 'theory-*aufbau*' perspective, this accords with Einstein's advice to Heisenberg[678] that, roughly, *not only a physical theory cannot be built solely on observable entities, but also, quite on the contrary, that it should determine what is observable in the first place* [187]. In ADG-gravity too the theory fundamentally hinges on the notion of connection (:field) $\mathcal{D}$, which is an unobservable entity since it is not a 'geometrical object' (*ie*, it is not an **A**-morphism or **A**-tensor). However, the curvature field $R(\mathcal{D})$ that the algebraic connection field $\mathcal{D}$ determines, via which the vacuum gravitational dynamics is in turn expressed (29), is an observable proper.

---

[676] The ADG **A**-connections are *not* (interpreted as) '*space-time connections*', as they are purely gauge fields and no external to them background spacetime (manifold) is involved.

[677] With **A** the structure sheaf of generalized coordinates or arithmetics in the theory.

[678] When Heisenberg asked for his opinion whether quantum theory, much in the way that Einstein had developed the theory of relativity (especially the operational SR), should be based solely on observable entities.

7.9.1 Path integral quantization of ADG-gravity

We envisage a genuinely covariant, background manifold independent and inherently singularities' and infinities' free quantum gravitational dynamics. Since the relevant physical kinematical configuration space in ADG-gravity is the moduli space $\mathsf{A}/\mathcal{A}ut\mathcal{E}$ of the affine space A of gravitational connections modulo the 'internal' gauge transformations in $\mathcal{A}ut\mathcal{E}$, the said quantum dynamics can be modelled after an abstract functional ('path') integral involving a generalized, *Radon-type of measure* on $\mathsf{A}/\mathcal{A}ut\mathcal{E}$. The abstract *differential* and *integral* calculus on A, that has been developed in gory detail in [259] and [269] respectively, will be of great import here.

Since the ADG-technology is expressly base manifold-free, the main feature of this quantum dynamics is that *it is manifestly background independent*, with 'background independence' here meaning, as emphasized numerous times before, not only 'background *metric* independence' as for example in the Loop Quantum Gravity (LQG) approach to non-perturbative canonical QGR [20, 382, 383, 351, 3], but also 'base *differential spacetime manifold M* independence' [270, 271, 272, 317]. By the background M independence of ADG-gravity we can totally bypass formidable obstacles in coming up with a suitable generalized measure to model the envisaged dynamics, which are due to Diff(M) [28, 29, 31]. For as we repeatedly stressed above, in the purely gauge ADG-gravity, Diff(M) is replaced by $\mathcal{A}ut\mathcal{E}$ (and covariance by synvariance!) [272, 317].

7.10 Section's Résumé

Again, we summarize this long section by itemizing its basic tenets:

1. Much in the same way that the external, background spacetimeless ADG-gravity may be regarded as a gauge theory of the third kind—a genuinely, purely gauge theory, with gauge group of the field's auto-symmetries organized in the principal sheaf $\mathcal{A}ut\mathcal{E}$—it can also be regarded as a third quantized (or better, third or self-quantum) field theory. This is the basic result from the geometric prequantization of gravity: $\mathcal{E}$ is the associated (representation) sheaf of $\mathcal{A}ut\mathcal{E}$ and its (local) sections may be thought of as the quantum-particle/generalized position states of the field $\mathcal{D}$. In turn, $\mathcal{D}$ acts on them to (dynamically) change them, as (29) depicts, and in this action we witness the quantum field-particle duality, which reflects a generalized version of Heisenberg's indeterminacy relation: by employing $\mathbf{A}$ to localize (geometrically represent) the field $\mathcal{D}$ (as acting on its associated representation sheaf $\mathcal{E}$, which by definition is locally isomorphic to $\mathbf{A}^n$), the latter acts on the carrier space's ($\mathcal{E}$'s) sections to change them. In other words, the ADG-gravitational field $(\mathcal{E}, \mathcal{D})$ is 'quantum self-dual' (self-quantized), something that can be easily proven sheaf-cohomologically.

2. With the self-quantum ADG-gravity we have in our hands a genuinely *background independent* and quantal theoresis of the gravitational field, with background independence meaning here not only background metric independence (since anyway no metric is involved in our purely gauge theory), but also background differential manifold independence. In this vain, no issue arises of quantizing the external (to the gravitational field) base spacetime manifold itself and, for example, some recent 'resolutions' of the inner Schwarzschild singularity using LQG means in which the singularity is resolved by evoking some discretization-quantization of the space-time continuum results from QRG, are quite irrelevant/redundant from our ADG-gravity viewpoint. In a nutshell, we do not expect a quantum theory of gravity to remove singularities, in the same way that we do not believe that a conceptually cogent and calculationally finite QG will be arrived at as a quantization (along QFTheoretic lines) of the classical theory (GR).

3. From the point made above it sort of follows that we cannot believe on the (physical existence of the) Planck space-time length either: that is, we do not believe in it either technically (calculationally)—as a cut-off length introduced by fiat in order to regularize divergent analytic expressions (*eg*, perturbation series expansions) and manage/control our continuum based 'field-Calculus'—or more importantly, conceptually as supposedly demarcating a 'theoretical boundary' above which the classical field theory (law) of gravity on the spacetime continuum holds (GR), but below which another (set of elusive) law(s) holds in the so-called QG regime, and all this because our continuum based Analysis miscarries (or anyway, it proves to be problematic, of limited applicability and validity in the quantum domain). *In toto*, ADG-gravity recognizes no fundamental (space-time) scale (or length-duration) as limiting the dynamical field-law of gravity, since there is no spacetime (continuum) in the theory to begin with (let alone that ADG-gravity does not recognize as conceptually sound any formal procedure of quantization of a pre-existing so-called 'classical' theory as GR).

4. A fully quantum dynamics for gravity is currently anticipated to be modelled after a genuinely covariant, functional or 'path' integral-type of scenario over the affine moduli space $\mathsf{A}/\mathcal{A}ut\mathcal{E}$ of purely gauge-equivalent gravitational $\mathbf{A}$-connections $\mathcal{D}$ [269], with the epithets 'genuinely' and 'purely' indicating our authentically gauge version of the PGC of GR being effectuated by the dynamical self-transmutations ('auto-symmetries') of the gravitational field dwelling in $\mathcal{A}ut\mathcal{E}$, and expressly not in $\mathrm{Diff}(M)$ since no external (to the field itself), base spacetime manifold M is involved in the theory. This absence will significantly enhance and facilitate our search for a suitable Radon-type of measure on the said moduli space to model our 'sum-over-connection-histories' dynamics, and it will not stumble and falter on M and its 'symmetry' group $\mathrm{Diff}(M)$ [16, 17, 28, 29, 31].

8 Physico-Philosophical Finale: 'ADG-Gravity', GR, Singularities, QG and the Euclidean vs the Cartesian Conception of Differential Geometry

In this concluding section of the present 'paper-book' we give a physico-philosophical account of the potential import and aftermath of ADG-gravity *vis-à-vis* classical gravity (GR), its $\mathcal{C}^\infty$-smooth spacetime singularities and QG, as well as we discuss the wider significance of ADG for (future) developments in applied (predominantly to the problem of QG) differential geometry.

8.1 The Einstein-Feynman-Isham 'No-Go' of CDG in the Quantum Deep

Below there are three quotations, in chronological order, by Einstein, Feynman and Isham respectively, about the miscarrying of the spacetime continuum (manifold) and, *in extenso*, about the inadequacy of the usual differential calculus (CDG) based on it in the quantum regime, and in particular, in QG research. Our comments following each quotation will depict a progressive refinement in the meaning of the words (and attitude!) of those people about the inapplicability of the manifold and its Calculus (Analysis) in our attempts to capture the elusive '*true quantum theory of gravity*'.

But before we present these, let us recall from [322] certain 'prophetic' remarks of Riemann about (the status and validity of Euclidean) 'geometry in the infinitely small':

Riemann

(Q8.1) "...*It is upon the exactness with which we follow phenomena into the infinitely small that our knowledge of their causal relations essentially depends... It seems that the empirical notions on which the metrical determinations of space are founded, the notions of a solid body and of a ray of light, cease to be valid for the infinitely small. We are therefore quite at liberty to suppose that the metric relations of space in the infinitely small do not conform to the hypotheses of (Euclidean) geometry...The question of the validity of the hypotheses of geometry in the infinitely small is bound up with the question of the ground of the metric relations of space*[a]..."

[a] Our emphasis throughout.

Brief Comments: Without reading too much in them, there is both quantum and relativity anticipations in Riemann's words above. The first sentence has more of a quantum flavor, as the exactness with which we can follow (the causal relations between) phenomena—*ie*, the precision of our determination (localization or measurement) of events (and the establishment of their causal nexus)—in the very small is (we would say nowadays) limited by the quantum of action (quantum indeterminacy). The rest of the quotation reminds one of relativity ideas, since the empirical notions (and implicitly, the operationalist or instrumentalist means) used to found (and determine) the 'metric relations of space'—in Euclidean geometry, these are the 'optical' straight lines of light (and their incidence relations), as well as the (relations between) rigid (imponderable) bodies—are intuited to perhaps 'miscarry' in the infinitely small. *In toto*, it is speculated that Euclidean geometry is not the 'true' theory of space in the very small;[679] moreover, in retrospect, by identifying the term '*geometry in the infinitely small*' with '*infinitesimal geometry*'—what we would nowadays call *differential geometry* or *infinitesimal calculus*—by a small stretch of the imagination we can maintain that *Riemann questioned the validity of the differential calculus in the very small* ('quantum deep'?). Which brings us to the main triptych of quotations by Einstein, Feynman and Isham.[680]

First comes a quotation of Einstein taken from [365],[681] which at the end is also augmented by the conclusion of a later quotation of his found in [125]:[682]

[679]These ideas are not that surprising bearing in mind that Riemann was Gauss' student so that he must have been exposed to (the novel back then) non-Euclidean geometry ideas. Also, let it be mentioned here that, apart from his mentioning 'causal relations' in the first sentence, Riemann is primarily concerned with a 'static' theory of 'pure space' ('being'), without bringing in arguments about 'time' ('dynamics' or 'becoming') *per se*, so that the geometry that he envisions is not a physical (dynamical) 'space-*time*' one as in Einstein's theory of relativity (*eg*, no clocks are mentioned). One could say that for Riemann 'space' is an objective, 'real substance' out there, whose metrical relations (attributes) *we* determine.

[680]These three quotations can be also found right before the introduction to [272], but there we made no significant comments on them.

[681]This is a more extended (Q?.?).

[682]These are the concluding words in (Q?.?). As usual, *emphasized* text in quotations is written in *emphasis* script, while in square brackets and roman script are our additions for clarity and completeness of expression.

Einstein

"'...you have correctly grasped the drawback that the continuum brings. If the molecular view of matter is the correct (appropriate) one; *ie*, if a part of the universe is to be represented by a finite number of points, then the continuum of the present theory contains too great a manifold of possibilities. I also believe that this 'too great' is responsible for the fact that our present means of description miscarry with quantum theory. *The problem seems to me how one can formulate statements about a discontinuum without calling upon a continuum space-time as an aid; the latter should be*
(Q8.2) *banned from theory as a supplementary construction not justified by the essence of the problem—a construction which corresponds to nothing real. But we still lack the mathematical structure unfortunately.*[a] How much have I already plagued myself in this way of the manifold!...", and...
"...This ['discrete' ('discretum' based) or 'discontinuous' scenario] *does not seem to be in accordance with a continuum theory,* and must lead to an attempt to find a purely algebraic theory for the description of reality. But nobody knows how to obtain the basis of such a theory.*"*

[a]Our emphasis.

Brief Comments: Einstein here appears to be in favor of a 'discontinuous' ('discrete' or finitistic) description[683] of physical reality in the quantum deep, instead of a manifold based, continuous (field) one, since the spacetime continuum (and the field theory based on it) "*miscarries with quantum theory and...it corresponds to nothing real*". Moreover, he intuits that *the desired 'discontinuum' based theory should be formulated by purely algebraic means*, but unfortunately, at that time, theorizing along such finitistic-algebraic lines was just 'wishful thinking', lacking a sound (mathematical) basis (foundation).[684]

We now come to some similar remarks by Feynman in [138] *contra* the continuum which, furthermore, mention the words *geometry* and *infinities* (coming from singularities).

[683]That is, one based on a 'discontinuum' or a 'discretum'.

[684]In 8.5.? we will return to comment further, in the light of ADG, about Einstein propounding a 'genuinely discrete', "*purely algebraic physics*" [368] *contra* the traditional (and more commonly associated with the name of Einstein) geometric spacetime continuum (manifold) based field-physics.

Feynman

(Q8.3) "*...the theory that space is continuous is wrong, because we get...infinities* [viz. 'singularities'] *and other similar difficulties* ...[while] *the simple ideas of geometry, extended down to infinitely small, are wrong*[a]*...*"

[a]Our emphasis.

Brief Comments: Feynman, like Einstein, is clearly against the continuum, which miscarries with quantum theory (Q?.?)—the description of physical reality at the very small. For him, this miscarrying is exemplified by the infertility of "*the simple ideas of geometry*" when applied to the *infinitely* small,[685] this 'wrongness of geometrical concepts' being manifested by singularities and their associated physically nonsensical infinities.[686] Of course, one can subsume this '*application of ideas of geometry to infinitely small*' by the more widely used term '*infinitesimal geometry*', or better, '*infinitesimal calculus*', or even, '*infinitesimal analysis*'[687]—all synonyms of *CDG*.

But *explicitly*, the terms 'classical *differential* geometry', the '*Planck scale*' and '*quantum gravity*' appear in the following remarkable quotation of Isham [203]:

Isham

(Q8.4) "*...at the Planck-length scale, differential geometry is simply incompatible with quantum theory...*[so that] *one will not be able to use differential geometry in the true quantum-gravity theory*[a]*...*"

[a]Our emphasis.

Brief Comments: Isham is straight-out categorematic: *CDG (and the classical $\mathcal{C}^\infty$-smooth manifold model of spacetime supporting its constructions) does not go hand in hand with the quantum, and it will therefore be of no import to QG research.* On the other hand, and this is one of the basic theses of the present paper, from an ADG-theoretic point of view it is not exactly that differential geometric ideas cannot be used in the quantum regime—as if the intrinsic differential geometric mechanism (which in its essence is of an algebraic nature as we have amply argued throughout this paper) fails in one way or another when applied to the realm of QG—but rather that when

[685]One could say, at infinitely small 'scales' or 'distances'.

[686]That is, for Feynman, what "*corresponds to nothing real*" about the geometrical spacetime continuum that Einstein mentioned in (Q8.?), is the latter's singularities and their unphysical infinities—precisely our thesis here.

[687]And it is fitting to recall here Leibniz's perception of a *Geometric Calculus* as a relational (algebraic) *Ars Combinatoria*—a *combinatorial art*, as communicated to de l'Hôpital: "*the secret of Analysis lies* [precisely] *in an apt combination of symbols"* [265]. On the one hand, this appears to corroborate the aforesaid 'purely algebraic physics' propounded by Einstein, and on the other, of course, it was subsequently vindicated by Feynman's own combinatory-diagrammatic method of describing quantum dynamical propagations and interactions.

that mechanism is geometrically effectuated or implemented (represented) by the (mediation in the guise of the smooth coordinates of the) background $\mathcal{C}^\infty$-smooth spacetime manifold as in CDG, then all the said problems crop up and are insurmountable (within the confines of, *ie*, with the concepts and the methods of, the manifold based CDG). Thus, to pronounce this subtle but crucial from the ADG-perspective difference, we maintain that the second part of Isham's quotation above should also carry the adjective '*classical*' in front of '*differential geometry*', and read: '*one will not be able to use differential geometry*' (or equivalently, a geometrical base differential spacetime manifold) '*in the true quantum-gravity theory*'. *In toto*, the aforesaid subtle distinction hinges on the physical non-existence of a background smooth spacetime manifold, not of the inapplicability of the (essentially algebraic) mechanism of differential geometry.

8.2 Euclid and Leibniz versus Descartes and Newton vis-à-vis DG

In this subsection we would like to comment briefly on a common 'misconception' of the term '*Euclidean geometry*' and the misleading view of DG in general that this misconception subsequently led to.

It is fair to say that *geometry in Ancient Greece*, which found its apotheosis in Euclid's axiomatics [134], *was of a purely relational character*. That is, for Euclid for example, geometry pertained to the study of 'incidence', 'parallel', 'congruence', 'contiguity', 'tangent', 'inscription', 'circumscription' *etc* <u>relations</u> *between the various 'geometrical objects'* such as points, lines, triangles, circles *etc*. *In toto*, Euclid did geometry entirely *relationally*, by using solely the 'geometrical objects' themselves, without the 'intervention' of any 'space' (*eg*, without considering the 'space' in which those objects were 'embedded' so to speak).[688]

It was Descartes', revolutionary for his time, idea to introduce numbers (coordinates) to represent ('coordinatize') the said 'geometrical objects' that '*arithmetized*' Euclid's relational geometry. Objects thus became parts of the coordinatized 'Cartesian space' (*eg*, the Cartesian plane to which the aforementioned circles, triangles *etc* belonged) and their relations became 'algebraic' equations between their coordinates.[689] This marked the birth of the so-called *Analytic Geometry* in which Euclidean geometry was in a sense algebraicized.[690] *En passant*, let it be noted here that, more

[688] The second author recalls a telling exchange with David Finkelstein, over breakfast, during the 5th Quantum Structures conference in Cesena (Italy), in which David told him (as a mild critique of the fact that in ancient Greece *geometry* not *algebra* prospered, while it appears that for the description of the quantum deep nowadays, algebra is needed not geometry), quite philologically and picturesquely, that "*for the ancient Greeks, geometry was drawing and comparing circles and triangles on sandy beaches*"—the operative word here being '*comparing*', not of course '*drawing on sandy beaches*'.

[689] In fact, the objects themselves became algebraic equations—*eg*, the planar circle of radius r is nothing but the set of point-*loci* in the Cartesian plane whose coordinates (x, y) satisfy the polynomial equation $x^2 + y^2 = r^2$.

[690] Thus, analytic geometry may be understood as the 'procedure' towards an 'algebraic analysis of geometry'.

than two centuries after the cartesian coordinatization-*cum*-arithmetization-*cum*-algebraicization of geometry, Descartes' idea was still considered to be a bold and radical one, as we read for example from Hermann Weyl's [413]:

(Q8.5) "*... The introduction of numbers as coordinates* [in the Cartesian fashion][a] *was an act of violence*[b]*...*"

[a]Our addition.

[b]Our emphasis. This Weyl quote appears also in a quotation of Shiing-Shen Chern given next.

while Cassirer [76], antithetically, praises on epistemological grounds the revolutionary Cartesian arithmetization of geometry:

Later on, by a process of abstraction, this procedure gave rise to *Algebraic Geometry* where, very loosely, one extracts 'geometrical information' (*eg*, 'space') from algebra(ic structures)—*eg*, spectral theory of polynomial rings and other algebraic varieties—and one focuses on solutions of *algebraic* equations (*eg*, the circle mentioned above is the 'geometrical object'—*ie*, the 'geometry'—corresponding to the solution of the *algebraic* equation $x^2+y^2-r^2=0$).

(Q8.6) "...[First, Cassirer praises Descartes' method of analytic geometry, quoting him:] '*The sciences in their present condition are masked and will only appear in full beauty when we remove their masks; whoever surveys the chain of the sciences will find them no more difficult to hold in mind than the series of numbers*'[a] [Then, he comments on this quotation:] This is the goal of the philosophical method: to conceive all its objects with the same strictness of systematic connection as the system of numbers. From the standpoint of the exact sciences in the time of Descartes, this is the only manifold which is built up from a self-created beginning according to immanent laws, and thus can conceal within itself no question in principle insoluble for thought. The demand that spatial forms be represented as forms of number and be wholly expressed in the latter, may appear strange when regarded from the standpoint of the Cartesian ontology; for in this, 'extension' signifies the true substance of the external objects and is thus an original and irreducible condition of being. But here the analysis of being must be subordinated to the analysis of knowledge. *We can only bring space to exact intelligibility by giving it the same logical character as hitherto belonged only to number. Number is not understood here as a mere technical instrument of measurement. Its deeper value consists in that it alone is completely fulfilled the supreme methodological postulate, which first makes knowledge knowledge. The conversion of spatial concepts into numerical concepts thus raises all geometrical enquiry to a new intellectual level*[b]..."

[a]Cassirer's own emphasis.
[b]Our emphasis.

Now, in our view, when it comes to *differential* geometry, there is an analogous dichotomy of views about Calculus or Infinitesimal Analysis[691] that may be traced back to the very creators of differential calculus, namely, Leibniz and Newton. The first, like we saw in (Q2.?), envisioned a *purely relational* ('combinatorial') Geometric Calculus in which "*one operates directly on the geometrical elements,*[692] *without the mediation of coordinates*"; or what amounts to the same, *one refers to the (algebraic) relations between the 'geometrical objects' per se, without the mediation (or in the presence) of any (background) 'space'*. By contrast, Newton developed differential

[691]Or, to emulate the term Analytic Geometry above, '*Infinitesimal Analytic Geometry*'.
[692]What we call here 'geometrical objects'.

calculus in the manifest presence of a background 'space' (assisting the geometrical picturization/representation of his differential calculus' concepts and constructions)[693] to what is presently identified—via modern technical terms and constructions—with the manifold based CDG. In complete analogy to the Euclidean-*vs*-Cartesian conception of geometry described above, we have here the Leibnizian-*vs*-Newtonian conception of *differential* geometry.

In fact, this is more than just an analogy if one considers that the 'space' in, on and via which we nowadays develop CDG (Calculus) *à la* Newton is a *locally Euclidean* (*ie*, locally $\mathbb{R}^n$) one—a (differential) manifold M, which is in turn identical to the algebra $\mathcal{C}^\infty(M)$ of its points' coordinates (Gel'fand duality). This is a (local) 'arithmetization' ('coordinatization') of the points of that ambient 'space' identical to Descartes' embedding and coordinatization of Euclid's geometrical objects in the cartesian plane ($\mathbb{R}^2$).

(R8.1) Alas, we have been misled by the term 'locally *Euclidean*', which should better be called 'locally *Cartesian*' and hence *the usual manifold based CDG should better be thought of as a Cartesian-Newtonian conception of DG*. In contradistinction, *ADG is a Euclidean-Leibnizian conception and practice of DG*, as there is no background, locally Cartesian 'space' (:manifold) that intervenes in our calculations (in our Calculus!) in the guise of coordinates, while the differential geometric machinery of the theory refers to and derives from the (algebraic) relations between the 'geometrical objects in-themselves'—the connection fields.

8.2.1 The Promethean 'haute couture' of modern differential geometry

At this point, keeping the remarks made above in mind, it would be nice to dwell a bit on the following telling quotation of Shiing-Shen Chern taken from [81]:[694]

[693] Again, think of the derivative (or differential) of a function at a point in the latter's domain, as the slope of the tangent line to the graph-curve of the function at that point.

[694] The quotation that follows is broken up into five 'paragraphs'—marked (I)–(V)—on which we comment separately afterwards.

"...It was Descartes who in the seventeenth century revolutionized geometry by using coordinates. Quoting Hermann Weyl, '*The introduction of numbers as coordinates was an act of violence*' [413].[a] From now on, paraphrasing Weyl, figure and number, like angel and devil, fight for the soul of every geometer **(I)**...
...General coordinates need only the property that they can be identified with points; i.e., there is a one-to-one correspondence between points and their coordinates—their origin and meaning are inessential. **(II)**
If you find it difficult to accept general coordinates, you will be in good company. It took Einstein seven years to pass from his special relativity in 1908 to his general relativity in 1915. He explained the long delay in the following words: '*Why were another seven years required for the construction of the general theory of relativity? The main reason lies in the fact that it is not so easy to free oneself from the idea that coordinates must have an immediate metrical meaning*'.[b] **(III)**
(Q8.7) After being served by coordinates in the study of geometry, we now wish to be free from their bond. This leads us to the fundamental notion of a *manifold*.[c] A manifold is described locally by coordinates, but the latter are subject to arbitrary transformations. In other words, it is a space with transient or relative coordinates (principle of relativity). **(IV)**
I would compare the concept with the introduction of clothing to human life. It was a historical event of the utmost importance that human beings began to clothe themselves. No less significant was the ability of human beings to change their clothing. If geometry is the human body and coordinates are clothing, then the evolution of geometry has the following comparison.

Synthetic geometry	Naked man
Coordinate geometry	Primitive man
Manifolds	Modern man **(V)**..."

[a] Quotation (Q8.?) above.
[b] Quote (Q?.?) given earlier in section 3, when we were exploring the deeper meaning of (the smooth) coordinates (of the base manifold) in GR.
[c] Chern's own emphasis.

- **(I)** It is an interesting philological remark that Weyl made, that is, to identify the Euclidean figure—what we called earlier the '*inherently geometrical object*' (or equivalently, the 'geometrical object in-itself')—with an *angel*, while the Cartesian number—the coordinates introduced 'by hand' or 'by fiat' (*ie*, by "*an act of violence*") by Descartes to represent arithmetically those genuinely geometrical figures of Euclid—with the *devil*. Like the *Promethean fire*, the Cartesian arithmetization (and concomitant algebraicization) of the purely relational Euclidean geometry, was a double-edged sword, "*a boon and at the same time a curse brought by a man, from the Gods, to Man*" [1]. Descartes brought Euclid's geometry 'down to earth'—as it were, from Olympus to Athens—for after all, as Aeschylus says in *Prometheus Bound* [1]:

(Q8.8) "*...Number* [like fire] *is the most remarkable of human inventions...*"[695]

- **(II)** Like we emphasized in sections 2 and 3, in the case of the (locally) Cartesian—what is 'misleadingly' called today (locally) Euclidean—coordinate patches of a differential manifold M,[696] this 1–1 correspondence ('identification') between the geometrical (Euclidean) points of M with their smooth coordinates in $\mathcal{C}^\infty(M)$ is what we referred to as Gel'fand duality (spectral theory) in section 2, and represented it by the amphidromous arrows in (1). To something more significant, in ADG the '*origin and meaning*' of a structure sheaf $\mathbf{A}$ chosen to deal with a differential geometric situation (problem) '*is inessential*', insofar that $\mathbf{A}$ furnishes one with the essentially algebraic differential mechanism—*ie*, it effectively provides one with a d—with which one can do *differential* geometry in the first place.[697]

- **(III)** We commented extensively on this in section 3. The upshot was that although (the real smooth) coordinates (of the manifold) do not have an immediate metrical meaning, on the one hand they are the elements where the components of the metric take values (thus the metric qualifies as being a smooth $\otimes_{\mathcal{C}^\infty_M}$-tensor field in GR), and more importantly on the other, they are the very vital ('kinematical') preconditions for setting up (formulating) GR differential geometrically, since a differential manifold is nothing else but $\mathcal{C}^\infty_M$.

- **(IV)** That coordinates do not have an immediate metrical meaning in GR essentially means that they do not have any dynamical (physical) significance in the theory, as they do not partake into the gravitational dynamics (Einstein equations). This is the meaning of the

[695] "*Αριθμόν έξοχον σοφισμάτων*". Our [...]-addition and emphasis.

[696] When we speak of (locally) Cartesian coordinates, we do not just mean *orthocanonical* coordinate systems as the epithet *Cartesian* has come to signify in modern (analytic) geometry. We mean general coordinate (numerical) identifiers (labels) of the (Euclidean) points of a manifold (*ie*, more in line with the general sense of 'coordinates' due to Gauss and his student Riemann). But the idea is the Cartesian identification of a point of an n-dimensional (real) manifold, with an (arbitrary) n-tuple of (real) numbers.

[697] See also (III) and (IV) next.

PGC of GR—*ie*, that the Einstein equations are generally covariant (form-invariant under general coordinate transformations). Thus, the biunique (1–1 correspondence) smooth coordinate identifiers (labels) of the points of M are not physically significant, and hence GR is fundamentally pointless—or to emulate Chern, coordinates (and the manifold's points that they correspond to) are '*dynamically transient*'. Of course, as emphasized in sections 2 and 3, the PGC and, what Chern also refers to above, the principle of relativity (PR), are essentially supported by the PFR, namely, to paraphrase again Einstein from (Q2.?), '*what has the dynamical field of gravity to do with the coordinate systems that we propose to describe it?*'. Recall again Aeschylus from above: number,[698] a remarkable invention as it may be, it is still our concept, and no matter how successful our descriptions of Nature have been hitherto based on it,[699] soon its 'human limitations' are reached.[700] Perhaps it is in this particular light, namely, that we impose our (undoubtedly ingenious mathematical) inventions, concepts and constructions to Nature—or equivalently, that we "*forget their terrestrial origin and accept them as unalterable facts, which then become labelled as 'conceptual necessities', 'a priori situations', etc*"[701] (Q?.?)—that Weyl's remark in (Q8.?) should be understood: *it was a violent act*[702] *against Nature to introduce numbers as coordinates and, in extenso, to identify in physics physical spacetime (geometry) with that of a locally Cartesian space(time).* However, in line with the quintessential didactics of ancient Greek tragedy, sooner or later such '*anthropic hubris*' is 'punished' by Nature[703]—*ie*, at least our (mathematical) theories of Her crash (or breakdown) at the limit of their applicability and validity.[704]

(R8.2) *It is always relieving to see Nature 'outsmarting' us in the end.*

- **(V)** Chern's account of the historical development of geometry and its comparison with the 'progress' that humans made in clothing themselves, is quite interesting in its own right: at the first, more 'primitive' so to say level, he puts synthetic geometry, which is more-or-less as we described the (Euclidean) geometry in Ancient Greece above—*ie*, relational and

[698]Here, the locally Cartesian coordinate-identifiers of the points of a manifold.

[699]And in general, regardless of the enormous success in both pure and applied mathematics (theoretical physics) that the concept of a locally Cartesian space (:a manifold) and the CDG-framework based on it has enjoyed last century.

[700]In the case of the background manifold, the singularities of GR, and the general miscarrying of the manifold based CDG in the quantum (especially in the QG) domain.

[701]In view of Aeschylus' quotation (Q8.?) above, our comments about the locally Cartesian character of a manifold, as well as Einstein's words here about the man-made nature of *our* concepts, it is a quite remarkable coincidence that a manifold was called a *Man*-ifold (pun intended).

[702]Albeit, a *revolutionary* one, as Chern says in paragraph I of the quotation above.

[703]By the Gods in ancient Greek tragedy.

[704]Like the CDG-based GR *contra* singularities.

coordinate-free. The synthetic (or the Euclidean) geometer is compared to 'naked man', as he lays bare (he studies) his naked body (:geometry=the 'geometrical objects'—points, lines, surfaces—and their relations) as it is ('in-themselves'). At the next level he puts the Cartesian introduction of coordinates, which marks the commencement of Analytic Geometry—the arithmetization of geometry. Indeed, coordinates may be likened to the clothes in which we dress ourselves: the (points of the) 'geometrical objects' of Euclid are 'labelled' by numbers and they satisfy algebraic equations. At the last level we encounter the modern (differential) geometer studying the (differential) geometry of manifolds—(smooth) 'spaces' which are locally labelled by coordinates in the Cartesian fashion, but with these coordinates having, as Chern says, only 'relative' or 'transient' significance since they are 'variable' (*ie*, liable to general/arbitrary coordinate transformations). Of course, it would hardly be an exaggeration to claim that geometers arrived at the notion of *manifold* in order to be able to do *differential* geometry (CDG or Analysis).

Below is *our* (slightly and suitably modified) version of Chern's table in (Q8.?) above:

Euclidean-Synthetic Geometry	Naked man
Cartesian Analytic Geometry	Primitive man
Locally Cartesian (Differential) Geometry on Manifolds	Modern man
Background Manifoldless (Differential) Geometry (ADG)	Future man: 'back to nakedness'

with the last entry indicating the current tendency (in differential geometry via ADG) to free ourselves from a background 'space(time)' (:manifold). It should be noted here that as theoretical physics research in GR gave tremendous impetus to mathematicians (geometers) to develop the *differential* geometry of smooth manifolds (CDG), so nowadays QG research has given us numerous reasons and motivations to 'throw away' the background (spacetime) manifold and, if possible, still develop differential geometry, entirely algebraically (categorically), in its absence—*eg*, like in ADG.

In this line of thought, we cannot refrain from quoting the following pertinent words from the prologue to the Russian edition of the first author's book [259]:

(Q8.9) "*.....This is an unexpected help from ADG now when the necessity has grown to study manifolds with singularities and even to remove the underlying space (for example, 'spacetime') and proceed to a direct description of the structures on this manifold, which may be important for many branches of contemporary mathematical and theoretical physics*[a]*...*"

[a]Our emphasis.

Indeed, ADG scraps off the underlying manifold and proceeds directly to study (purely algebraico-categorically, *ie*, sheaf-theoretically) the (relations between the) '(differential) geometrical objects' that live on that 'space', which 'space', in turn, does not contribute at all to ADG's inherently algebraic differential geometric mechanism. Thus viewed,

(R8.3) *ADG is a Euclidean-Leibnizian way of doing DG*

and the current (differential) geometer, after trying all his clothes on, frees himself completely from coordinates and 'space', and goes back to nakedness—as it were, 'back to the future man'. But in the next subsubsection, let us dwell a bit on how and in what sense the 'ADGeometer' actually manages not to be flashed and dazed by the glittering and luxurious clothes of Analysis, and goes back to the bare basics (ie, without his fashionable background geometrical manifold clothes).

In this gist, let us close this discussion with some remarkable, from the Leibnizian ADG-vantage, words of Ernst Cassirer from [76] about 'infinitesimal analysis' (*ie*, DG or Calculus), which follow immediately after his appraisal of Descartes' method of arithmetizing geometry quoted above(Q8.?):

(Q8.10) "...Criticism of these methods [of Descartes' in the field of infinitesimal geometry][a] begins with Leibniz and is brought to a first conclusion with his founding the analysis of position. It is charged that analysis is not able to establish the universal principle of order, upon which it prides itself, within the field to be ordered, but that it is obliged to have recourse to a point of view external to the object considered. The reference of a spatial figure to arbitrarily chosen coordinates introduces an element of subjective caprice into the determination; the conceptual character of the form is not established on the basis of properties purely within itself, but is expressed by an accidental relation, which may be different according to the choice of the assumed system of reference. Whether from among all the various equations which can be applied, according to this process, in the expression of a spatial figure, the relatively simplest is chosen depends upon the individual skills of the calculator, and thus upon an element which the strict progress of method seeks to exclude. *If this defect is to be avoided, a procedure must be found which is equal to the analytic methods in conceptual rigor but which accomplishes the rationalization wholly within the field of infinitesimal geometry and of pure space. The fundamental spatial forms are to be grasped as they are 'in themselves' and understood in their own laws without translation into abstract numerical relations*[b]..."

[a]Our addition for textual clarity and continuity.
[b]Our emphasis.

Cassirer hit the nail on the head *vis-à-vis* the Euclidean-Leibnizian ADG: what he refers to as '*fundamental spatial forms*' are the Leibnizian '*geometrical objects*' we saw in the Bourbakis' quotation (Q2.?), that, in turn, correspond to the 'algebraic' (:relational) and Euclidean (in the sense we gave to this epithet above) ADG-fields (*viz*, connections $\mathcal{D}$),[705] which are, to paraphrase Cassirer, "*grasped* (by the ADG theory) *'in themselves' and understood in their own laws without translation into abstract numerical relations*". That is to say, to stress it once more, *the ADG-fields are autonomous, self-sustaining (differential) geometric entities in no need of Cartesian arithmetization; or equivalently, in no need of the intervention of 'space' in the guise of (numerical) coordinates.*

[705]And their 'purely geometrical' concomitants (*eg*, curvatures $R(\mathcal{D})$).

8.2.2 Freeing DG from the glittering trappings and the golden shackles of Analysis

As we have been arguing back in section 4, CDG—the differential Calculus or Analysis on smooth manifolds—is a topologico-algebraic affair, since basically, in order to differentiate (*ie*, to define a differential d), one must be able to take differences and form products (algebraic structure), as well as take limits thereof (topological structure). In the guise of the topological algebra $\mathcal{C}^\infty(M)$ (or equivalently, the structure sheaf $\mathcal{C}^\infty_M$), the differential manifold M underlying CDG is a rich enough structure to accommodate the usual notion of differentiability (*ie*, furnish us with a d with which we can actually do DG via M) and in this sense the underlying 'space(time)' mediates (or intervenes in) our usual differential geometric calculations—our Differential Calculus.

On the other hand, as it was also highlighted in section 4, *ADG's principal virtue is that it provides a purely algebraic framework for doing DG, without commitment to or dependence on an underlying 'space'* (and at that, a differential manifold one!)—as it were, its principal achievement is '*the complete algebraicization of Calculus*'. In other words, in view of the topologico-algebraic character of d noted above, in ADG we underplay and 'atrophize' the 'spatial', topological (:geometric) aspect of Analysis—*ie*, the fact that CDG has its basis on a base[706] ('smooth') space (:manifold)—and we pronounce its purely algebraic aspect. Indeed, as we argued many times in the present paper, the arbitrary base topological space X on which the algebra and differential module sheaves of interest are soldered plays no role in the *'inherently' algebraic* differential geometric mechanism of ADG, which, as it happens, derives from the stalk of the said sheaves—*ie*, from the algebraic structures dwelling in the relevant sheaf spaces, not in the base space. For anyway, the basic differential operators in ADG (*ie*, the connections ∂ and $\mathcal{D}$) are defined as *sheaf morphisms*, so *by definition* the base topological space X plays essentially no role in the ADG-notion of 'differentiability'.[707]

In the light of the remarks above, the usual Cartesian (Analytic) conception of (differential) geometry,[708] brings to mind George Birkhoff's words in [45]:

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[706]Pun intended.

[707]Another way to say this is that, as *Algebraic* Geometry is defined by *algebraic* equations (as mentioned a couple of footnotes before), so *Differential* Geometry is defined by *differential* equations, which on the one hand in CDG are made possible by the intervention of a background 'space' (:geometrical representation of the 'evolution' or the 'change-relations' that the differential equations stand for), but on the other, in ADG they are equations between sheaf morphisms—*ie*, they are relations between the geometrical objects themselves—without a mediating (geometrical representation) 'space' whatsoever. Shortly we will come to comment on this 'end of geometrical picturization' in ADG, and its potential significance in theoretical physics.

[708]Recall from a couple of footnotes above our calling the usual Differential Calculus '*Infinitesimal Analytic Geometry*'.

(Q8.11) "...[There is this] *disturbing secret fear that geometry may ultimately turn out to be no more than the glittering intuitional trappings of analysis*[a]..."

[a]Our emphasis.

which can be suitably modified a bit here to the following nightmarish feeling—a 'Kafka Castle' [220] type of foreboding—about DG:

(R8.4) There is this *disturbing secret fear* that *Differential* Geometry may ultimately turn out to be no more than *the glittering intuitional trappings of Infinitesimal Analysis*.

But what are we really trapped by in our yearning to do DG? The answer we suggest is: *by* (the underlying) *'space'*.

(R8.5) Insofar as Analysis (:the usual Differential Calculus) is understood as the (apparently inextricable, at least from the classical, manifold based viewpoint) enmeshing of Algebra ('relations') and Topology ('space'), what has so far 'trapped' our differential geometric endeavors is (the base) 'space' (:manifold), and *the fear is that DG will turn out to be no more than CDG* (*ie*, Infinitesimal Analysis on Manifolds).

In contradistinction, *ADG, by suppressing the underlying topological ('spatial') and at the same time by pronouncing the overlying algebraic ('relational') aspect of DG,*[709] *comes to dispel that fear that Birkhoff mentions above and free us* (*ie*, our DG) *from*, to use a phrase by Chris Isham [203], *'the golden shackles' of the geometrical manifold based CDG.*

The reader should not underestimate here the significance of the epithets '*glittering*' (to trappings) and '*golden*' (to shackles) used by Birkhoff and Isham, respectively. For there is no doubt that the CDG of smooth manifolds has been of great import to both pure and applied mathematics (in particular, in theoretical and applied physics); nevertheless, its 'negative spell' and shortcomings has been felt in both disciplines. In particular, it has been felt very early on by Einstein *vis-à-vis* the singularities of GR (which are due to the base spacetime continuum), and the 'inherently finitistic-algebraic' nature of the quantum. We thus recall again, from (Q?.?), Einstein's words about a 'discontinuum based theory' as opposed to the (spacetime) continuum based CDG of GR:

[709]The terms 'underlying' and 'overlying' here are meant to refer to the base (topological) and the (algebra inhabited) sheaf spaces involved in ADG, respectively. ADG 'dissects' the topological from the algebraic aspect of differentiability, it 'downplays' the former and 'overplays' the latter.

(Q8.12) "*...By discontinuum theory I understand one in which there are no differential quotients. In such a theory space and time cannot occur, but only numbers and number-fields and rules for the formation of such on the basis of algebraic rules with exclusion of limiting processes*[a]*...*"

[a]Our emphasis.

These words are remarkable indeed, for they show that Einstein had not only deep physical intuition, but also penetrating mathematical vision as well. He advocated "*a purely algebraic theory for the description of reality*" (Q?.?) not based on the spacetime continuum, thus inevitably a theory not founded on CDG. To his mind, 'differential quotients', essentially the basic process of taking derivatives (:the basic structural procedure in doing DG!), could not occur since limiting processes had to be excluded—in effect, since topology (*ie*, the background 'space') had to be excluded.[710] Thus Einstein separated (in his mind at least) the topological from the algebraic aspects of DG, scrapped the first and opted for the second; however, since in the CDG of manifolds the algebra is apparently inextricably entwined with the (continuum) topology, dropping the latter meant for him the complete abandonment of differential geometry[711] (in the quantum regime). By contrast, ADG does not at all drop differential geometry: it simply abandons the base continuum[712] with its infinitary limit processes (and retains a general, in principle arbitrary, topological space on the background), while at the same time it emphasizes the algebraic character of DG—what we called earlier DG's inherently algebraic differential geometric mechanism.

(R8.6) All in all, we can now say, to parallel Leibniz's remarks to de l'Hôpital,[a] that '*the secrets* (or perhaps better, the virtues!) *of Analysis lie in the algebra that it employs, while its pitfalls* (or better, shortcomings, especially in the quantum deep) *in the intervening 'geometry' (or topology, or even 'space(time)') that it also employs, with its infinitistic, limiting procedures'...*"

[a]See footnote ? above.

In the light of the remarks above, it is fitting to make the following important, *vis-à-vis* the singularities and their associated unphysical infinities of the spacetime manifold grounded physical theories,[713] general observation in the form of an 'aphorism':

[710]For there is no notion of limit (:convergence) without topology.
[711]And of the spacetime interpretation that goes hand in hand with the background geometrical continuum.
[712]And its geometrical spacetime interpretation.
[713]GR and QFTs of matter included.

(R8.7) *In algebra* (*ie*, in the 'relational' aspect of DG) *there is no 'infinity' involved whatsoever. Infinities* (in DG) *arise when the background topological continuum* (*ie*, the base 'spatial' aspect of DG), *with its infinitary limit processes, gets involved* (in our differential geometric calculations—our Differential Calculus), *like in the spacetime continuum based CDG.*[a]

[a]Which manifold founded CDG is *the* mathematical framework within (or *the* mathematical language in) which both GR and the QFTs of matter are formulated.

8.3 What is Differential Geometry?—ADG's Unifying Platform (based on a theme by Shing-Shen Chern)

Motivated by the discussion in the previous subsection, in the present one we would like to recapitulate the basic tenets of ADG under the prism of the main stages in the historical development of geometry—placing particular emphasis on issues concerning *differential* geometry *per se*—as perceived and (in the order) presented by Shing-Shen Chern in his paper "*What is geometry?*" [82]. What will emerge from this summary-via-comparison is that ADG judiciously selects and combines, for the (axiomatic/abstract) development of an entirely algebraic (:sheaf-theoretic) and base $\mathcal{C}^\infty$-smooth manifold-free *differential* geometry, certain key ideas and central notions in the developmental history of geometry, as well as it revises or totally evades others pertaining to *differential* geometry proper. To this end, the discussion in the previous subsection will come in handy.

First of all, it goes without saying of course that this is no place and no time[714] for us to attempt at a 'definition' (or concise description) of such a broad mathematical subject as DG. We will rather present certain ideas, mainly motivated and supported by lessons we have learned from developing and applying ADG to theoretical/mathematical physics—especially to classical and QG—which could help one outline and sketch, albeit coarsely, quintessential in our opinion features that a predominantly physically motivated (mathematical) theory, such as ADG, ought to possess in order to qualify as being 'differential geometric' proper. In this endeavor we will also set the stage for certain important distinctions between 'physical' and 'mathematical' space and geometry *à la* Peter Bergmann that we are going to draw shortly.[715] What is going to emerge from these distinctions is that what ought to qualify as (D)G proper is in our view fundamentally conditioned by—in fact, inextricably tied to—*physical* considerations. For if we have learned

[714]For there is no *space-time* in the first place! (pun intended)
[715]In subsection 8.4 next.

anything at all from Einstein's GR is that geometry, far from being a crystalline-rigid structure *a priori* fixed by the theoretician (better, the mathematician!) and being detached from natural processes (of dynamical propagation and interaction, is a flexible, conditioned by and subject to *physical dynamics*, enterprize [143, 144, 308].

After these anticipatory remarks about '*the physicality of geometry*', the starting point of our present discussion about what is or what 'ought' to qualify in our ADG-based view as differential geometry is the rather basic observation, already touched on briefly in footnote ? above, that as *algebraic* geometry essentially began as the study of the (solution) properties and associated (solution) spaces of *algebraic* (*eg*, polynomial) equations, so similarly *differential* geometry may be generally perceived as being primarily concerned with the study of *differential equations* (DEs) and their *solution spaces.* One can gain more insight into and understand the 'naturalness' of this broad, 'mock definition' of DG by dissecting the term into its two constituent words—*differential* and *geometry*—in the following way:

Differential d: the condition for setting up the DE; the origin of the entire DG set-up. We commence with an 'aphorism', namely that

(R8.8) *there can be conceived no differential equation and in extenso, by our pseudo-definition above, no DG, without first having some sort of 'differential' (operator)* $\mathtt{d}$.

The question that follows then is *where does this 'differential' come from*? Since $\mathtt{d}$—in effect, the concept of derivative—is a *topologico-algebraic* notion as repeatedly stressed before, while, *a fortiori*, the operation of differentiation is a *local* process, the usual CDG of (finite-dimensional) manifolds M secures a (mathematical) milieu—a 'space'—in which a $\mathtt{d}$ can be snugly accommodated, as follows:

- Since by definition an n-manifold is *locally* isomorphic to $\mathbb{R}^n$, with the latter coming equipped with the usual topological and differential structure of the real line $\mathbb{R}$ (and the Cartesian products thereof), $\mathtt{d}$ is well secured; but perhaps more importantly, from a functional (and from our ADG-theoretic) viewpoint,

- Since (a real smooth manifold) M itself is equivalent to ${}^{(\mathbb{R})}\mathcal{C}^\infty(M)$ by Gel'fand duality and its associated spectral theory (1), the latter, which is *the* archetypical example of a (non-normable) *topological algebra*, secures a place for the topologico-algebraic $\mathtt{d}$, with the *local* nature of $\mathtt{d}$ being in turn satisfied by the sheaf-theoretic localization of ${}^{(\mathbb{R})}\mathcal{C}^\infty(M)$—as the structure *sheaf* ${}^{(\mathbb{R})}\mathcal{C}^\infty_M$—over M (simply regarded as a topological space).[716]

[716] For, after all, the notion of sheaf is, by definition, a *local-topological* one, being defined as a *local homeomorphism*.

- All in all, we recall from [272], albeit slightly modified, the following diagram summarizing the '*raison d'être et de faire*' of CDG:

  Background Geometry (BG) M

  $$\text{CDG} \equiv \mathcal{C}^{\infty}\text{–Manifolds } M \equiv \mathcal{C}^{\infty}_{M}(\equiv \text{BG}) \xrightarrow{(a)} \text{Tangent bundles} \xrightarrow{(b)}$$
  $$\text{Smooth Vector Fields}(\equiv \text{Derivations}) \xrightarrow{(c)} \text{Differential Equations}$$

  in the following sense: *the smooth manifold was 'made' (or 'invented') for the tangent bundle* (a)*, which in turn was made for the vector fields* (b)*, which were finally made for the differential equations* (c); hence, by 'arrow-transitivity', the diagram above vindicates our 'mock definition' of DG, that is to say, that *the manifold supported CDG was ultimately 'invented' for the DEs.* To summarize, in CDG the base manifold M, as a background geometrical space, 'mediates' our Calculus—*ie*, the smooth coordinates $\mathcal{C}^{\infty}(M)$ play a crucial role in our setting up DEs—as it is the vital pre-condition for securing a $\mathtt{d}$ in terms of which DEs can be formulated in the first place.

In glaring contrast, as it was also observed in [272], in the purely algebraic (:sheaf-theoretic) ADG, no smooth base manifold is used at all in order to 'mediate our calculations', ultimately, to set up DEs. In fact, *ADG refers directly to the algebraic fields* (*viz.* connections) *themselves, which define directly DEs without dependence on a background geometrical manifold M to provide us with a differential, which differential (or the generalized differential—ie, the connection $\mathcal{D}$) in turn comes from (structure sheaves of) algebra(s)*:

No Background Geometry (BG) M

$$\text{ADG} \equiv \text{'Algebra'} \xrightarrow{(a')} \text{Fields } (\mathcal{E}, \mathcal{D}) \xrightarrow{(b')} \text{Differential Equations}$$

which, again by arrow-transitivity, can be read as follows: *ADG refers in an algebraico-categorical way directly to the dynamical fields—represented by pairs such as '$(\mathcal{E}, \mathcal{D})$'—and to the differential equations (physical laws) that they define, without the intervention (neither conceptually nor technically) of any notion of (background geometric manifold) space(time) M, or equivalently, independently of any intervening smooth coordinates $\mathcal{C}^{\infty}(M)$.* In other words, *ADG deals directly with the differential equations (the laws of physics), which now are 'categorical equations' between sheaf morphisms—the* **A***-connections $\mathcal{D}$ acting on the (local) sections of the associated vector (representation) sheaves $\mathcal{E}$ under consideration.*

'Geometry' and 'space': the result from solving the DE; the 'secondary effect' of the DE; 'solution space'. The foregoing discussion can be distilled to the following: the entire DG enterprize revolves in one way or another about the notion of differential $\mathtt{d}$ which one wants to possess in order to actually do DG—*ie*, write down (and hopefully solve!) DEs in the first place. In the usual theory (CDG), $\mathtt{d}$ is secured by the character of the underlying (background) 'space' (:manifold) employed at the very basis of the theory. In this sense, in CDG (or Analysis) 'geometry' (the base geometrical M) precedes $\mathtt{d}$ as well as the DEs that can be formulated with it. Indeed, Chern in [82] has pondered on the 'mysterious' origin of $\mathtt{d}$:

(Q8.13) "*...A mystery is the role of differentiation. The analytic method is most effective when the functions involved are smooth. Hence I wish to quote a philosophical question posed by Clifford Taubes: Do humans really take derivatives? Can they tell the difference?*[a]..."

[a]We are going to answer to this question, from a physical point of view, in the next subsection when we talk about '*physical (differential) geometry*'.

which is of course settled (*ie*, $\mathtt{d}$ is not that mysterious after all!) within the realm of the smooth manifold based CDG (Analysis).

In striking contradistinction however, in ADG no base M is employed to furnish us with a differential (or a generalized one, $\mathcal{D}$).[717] The latter is obtained in the theory abstractly (categorically) from algebra, not (point set-theoretically) from a background geometrical 'space'. The DEs in ADG are equations between sheaf morphisms (the $\mathtt{d}$s involved in the theory). In ADG, at the origin (or basis) of DG is algebra (loosely speaking, a 'relational' structure), which furnishes us with an algebraico-categorical $\mathtt{d}$, in terms of which DEs can then be written down. The terms 'geometry' or 'space' then pertain to the '*solution space*'—the realm where the relevant DEs actually hold, much like we said some footnotes before about *algebraic* geometry and we gave the example of the circle as the 'solution space' of the *algebraic* equation $x^2 + y^2 = r^2$. This is exactly how we think ADG-theoretically, for instance, about the differential (vacuum) Einstein equations (29): there, the 'geometrical object' in the theory—namely, the curvature $\mathcal{R}(\mathcal{D})$ of the algebraic gravitational connection field $\mathcal{D}$—satisfies (29); moreover, it does so on the entire 'carrier' (or representation) sheaf space $\mathcal{E}$.[718] This observation lies at the heart of the ADG-conception of DG and more importantly of its general viewing 'geometry' (or 'space') as the result ('solution space') of algebra—*ie*, of the differential equation defined by the algebraic $\mathtt{d}$ (or equivalently, $\mathcal{D}$). In turn, in the domain of physics, this reflects our general attitude about the priority of dynamics over

[717]Which in turn means, *à la* Chern, that *differentiation still remains a 'mystery'*.
[718]Which itself comes from algebra, as by definition it is (locally) of the form $\mathbf{A}^n$.

kinematics and about the higher significance of the field law itself than of its solution that we saw earlier, as well as about what actually constitutes the '*physical geometry*' (or '*physical space(time)*') that we will see in the next subsection.

But before we proceed to the next subsection where we argue, inspired by Peter Bergmann, in favor of a 'physical geometry' instead of a 'geometrical physics', let us borrow from the conclusion of [82] the six "*major developments in the history of geometry*" and their corresponding protagonists, and juxtapose them against the basic features of ADG, which we claim combines basic features from all the basic 'epochs' in the development of geometry and especially of *differential* geometry. The following table, summarizing this juxtaposition, is 'self-explanatory' in view of what has been said above:

Chern	**ADG**
Axioms (Euclid)	ADG is an *axiomatic*, Euclidean (*relational*) conception of *differential* geometry.
Coordinates (Descartes, Fermat)	Structure algebra sheaf $\mathbf{A}$ of generalized arithmetics ('coordinates'); all differential geometry boils down to it—*ie*, to *algebra*.
Calculus (Newton, Leibniz)	ADG is more *Leibnizian* (*relational-algebraic*) than *Newtonian* (*geometrical-analytic*) $\Rightarrow$ do DG with the geometrical objects (fields) themselves, without at all the intervention of Cartesian $\mathcal{C}^\infty$-coordinates$\equiv$spacetime manifold M.
Groups (Klein, Lie)	Geometrical objects (fields and their quanta) are totally characterized by their transformation (symmetry) groups $\Rightarrow$ *Synvariance$\equiv$Covariance with respect to the field itself, without reference to a background smooth spacetime manifold M*; work with $\mathcal{A}ut\mathcal{E}$, not with $\mathrm{Aut}(M) \equiv \mathrm{Diff}(M)$.
Manifolds (Riemann)	A $\mathcal{C}^\infty$-manifold (CDG) is a particular instance of ADG; when $\mathbf{A} \equiv \mathcal{C}^\infty_M$, and it is not a basic structure for doing DG, but merely the classical one.
Fiber bundles (Elie Cartan, Whitney)	Vector and algebra (associated or representation) sheaves (of $\mathcal{A}ut\mathcal{E}$).

8.3.1 Modern comparisons: ADG versus Noncommutative and Synthetic Differential Geometry

With the table above in mind, and since we have briefly alluded earlier to algebraic geometry, perhaps it would be fruitful at this point to throw in some fleeting remarks about ADG *vis-à-vis* the general notion of a sheaf and its historical development, as well as about the potential relationship between ADG and the other two more 'popular' nowadays—because they have been around and worked out longer, as well as because so far they have been more widely applied (to theoretical physics—in particular, to spacetime and gravity)—theories of DG: Connes' *Noncommutative Differential Geometry* (NDG) [91, 222] and Kock-Lawvere's *Synthetic Differential Geometry* (SDG) [232, 243].

Sheaves in algebraic geometry: the notion of 'space', 'symmetry' and their quantum descendants. The origin of the notion of *sheaf* goes as far back as Weierstraas [167], when Analysis was taking its first 'toddler' steps. Indeed, sheaf theory became the 'bread and butter' of subsequent investigations in the field of Complex Analysis.[719] Two of the most commonly used definitions of a sheaf are due to Leray, as *a complete* (functional) *presheaf*, and subsequently due to Lazard, as *a local homeomorphism* [259].

Sheaf theory underlies virtually all modern developments in the field of *algebraic* geometry [178, 344, 132] whose historical development goes hand in hand with the evolution of the notion of 'space', its 'geometry' (*eg*, topology), and their 'extraction' or 'derivation' from various algebraic structures and varieties [75].[720]

Undoubtedly, one of the turning and high points in the development of sheaf theory is its 'categorification'—*ie*, the infusion of homological algebra (category-theoretic) ideas (and jargon!) into sheaf theory. Eilenberg and MacLane's inspired category theory provided the basic platform on which to found fundamental sheaf-theoretic notions[721] Perhaps, the apotheosis of the category-theoretic perspective on sheaf theory *visà-vis* algebraic geometry is the notion of a topos [253],

[719] One has just to recall the great import of sheaf theory into such central Complex Analysis' subjects as Riemann surfaces, Kähler-Stein manifolds, holomorphic (vector) bundles *etc*, with which one usually associates the names of Oka, Stein, Henri Cartan, Grauert, Remmert, Leray and many others (no references provided).

[720] Here one has just to recall such basic algebraic geometric notions as ringed spaces, schemes and their various spectral topologies, as well as the names of the pioneers in this field such as the 'Bourbakis' [343]—the group of French mathematicians which includes among others Schwartz, Weil, Dieudonné, Chevalley, Cartan, Serre, Grothendieck—as well as Gel'fand, Zariski, Swan, Shafarevich, Hartshorne, Manin, and many more.

[721] Here one has just to recall how basic algebraic geometric notions were formulated categorically, as for example presheaves as contravariant functors, the process of sheafification as a functor, (global) sections as functors, spectral functoriality, geometric morphisms, and many more.

originally due to Grothendieck.[722] Subsequently, the notion of topos was further refined, distilled and abstracted by Lawvere and Tierney so that it almost detached itself from algebraic geometry proper and made contact with logic [163, 240][723]

Of great importance in algebraic geometry are *noncommutative* algebraic varieties (rings, modules, algebras)—the object of study of the so-called noncommutative algebraic geometry [404, 401]. Here the purely mathematical interest focuses around defining 'noncommutative spaces' and their 'noncommutative geometry' (*eg*, 'noncommutative topology' [58]).[724] This is supposed to make contact with quantum physics (and possibly/hopefully with QG), in the sense that the proposed noncommutative spaces could possibly represent 'quantum space' (or even, 'quantum spacetime') [283, 284, 402]. In noncommutative (algebraic) geometry too categorical language and techniques abound [403], while of crucial importance in this area is arriving at some 'natural' notion of 'noncommutative sheaves' [57].[725] Noncommutative sheaves then are also envisaged to inherit the epithet 'quantum', the structures supposedly having quantum physical significance [95]. Finally, the interest naturally arises of grouping these noncommutative sheaves into a 'noncommutative topos' structure [56], whose quantum interpretation (as a 'quantum topos') could provide us with a mathematical universe in which the logical and the geometrical properties of quantum 'systems' (such as the spatial topology, the space*time* topology, or the causal structure) are intimately entwined and mutually affecting.

Sheaves in differential geometry. Although algebraic geometry is teeming with sheaves and in its domain sheaf theory prospers, sheaves have not figured 'seriously' in *differential* geometry proper, at least not until the advent of ADG; or before that, they have been indirectly and to a lesser extent involved, via topos theory, in SDG.[726] This may be partly explained by the 'dominance of Calculus'—the enormous success that smooth manifolds and fiber bundles over them (the basic

[722] Just recall Grothendieck's revolutionary abstraction, via categorification, of topology (*eg*, Grothendieck sites), the categorical definition of sheaves and general fibered spaces over such abstract 'pointless' topologies (suitable for generalized sheaf-cohomology theories), and the concomitant organization of those sheaves into 'larger' categorical universes—the celebrated Grothendieck topoi.

[723] In fact, it is more fair to say that topos theory 'unified' (the sheaf theory based algebraic) geometry with logic, especially after it was realized that every topos has an internal 'language' (or logic), which, unlike the standard Boolean (classical) one of the topos **Set** of 'constant sets', it is intuitionistic (in this sense one speaks for example of the topos $\mathbf{Shv}(X)$ of sheaves of sets over a topological space X as being 'a universe of variable sets') [244, 253].

[724] In the case of topology for example, interest lies in the possible definition of a spectrum for a noncommutative algebra [281].

[725] The epithet 'natural' here meaning, for example, that the construction of noncommutative sheaves over the aforesaid 'noncommutative (spectral) topologies' should be functorial.

[726] Here one could perhaps add the categorical, Grothendieck-style of perspective on abstract differential structures, such as the so-called *differential modules* (D-modules) [224] as well as the versatile (and exotic) cohomology theories based on them [228, 225] (the second author wishes to thank Goro Kato for introducing him to this beautiful

entities of CDG) have enjoyed in both pure and applied mathematics (theoretical physics) [196].[727] However, as we have repeatedly noted in the present paper, the manifold based CDG "*simply miscarries with quantum theory*" (Einstein), so that one "*will not be able to use it in the true QG theory*" (CDG).

Concerning QG, one could say that the road of applying sheaf theory 'bifurcates': on the one hand we encounter vigorous research activity focusing on noncommutative (quantum) space(time)s and their sheaves, being essentially motivated by (noncommutative) *algebraic* geometry as noted above; while on the other, we have ADG, which is a direct application of sheaf theory to *differential* geometry *per se*, and *vis-à-vis* QG applications, it does not address at all the structure of (the background) 'space(time)' (*eg*, whether it is 'discrete' or 'continuous'; 'commutative'/'classical' or 'noncommutative'/'quantal'), but rather it refers directly to the 'geometrical objects' that live on that (in principle arbitrary and not participating to the DG mechanism itself) surrogate (sheaf-theoretic localization) background.[728]

However, there is on the one hand Connes' NDG, where noncommutative algebras abound in a differential geometric context (but no sheaf theory *per se* appears to have been seriously used so far), and Kock-Lawvere's SDG, which revises CDG at a 'logical level', by employing topos theory (which as noted above, has a close relation to sheaf theory, since for one thing, the 'canonical' example of a topos having a non-classical, non-Boolean, intuitionistic-type of internal logic, is the topos $\mathbf{Shv}(X)$ of sheaves of structureless sets over an in principle arbitrary topological space X). Which brings us to the comparison between ADG *contra* NDG and SDG that we initially set out to comment on in this sub-subsection.

- **ADG versus NDG:** The crucial observation which may enable one to relate straightforwardly ADG with NDG is that, in a nutshell, the latter may be 'reduced' to (*ie*, it may be viewed as a particular 'restriction'/case of) the former when one assumes a *non-abelian algebra sheaf* as *structure sheaf* $\mathbf{A}$ of generalized coordinates or 'arithmetics' in the theory, as well as (sheaves of) modules of 'differential form'-like objects over it. Moreover, as we already emphasized back in 4.2.2, in ADG there are 'naturally' occurring NDG-related ideas and structures, principally in the guise of what we called 'noncommutative Kleinian (esoteric) geometry' of the ADG-fields $(\mathcal{E}, \mathcal{D})$.[729]

mathematics).

[727]Another instance of the 'differential manifold-conservatism' and the 'CDG-monopoly' we talked about in the early sections of this paper.

[728]In a nutshell, as repeatedly noted before, from the ADG-theoretic perspective on gravity, the issue of a quantum description ('quantization') of space(time) structure itself is a 'non-issue', since 'spacetime' (especially the smooth continuum of GR) does not (physically) exist.

[729]See remarks in footnote ?? before.

Regarding spacetime singularities however, and following some critical remarks about NDG in [273, 274], it must be noted that, while with the passage to noncommutative structure algebra sheaves enables one to deal with stronger singularities than in the abelian case—especially in the classical (CDG) case corresponding to assuming $\mathcal{C}^\infty_X$ for structure sheaf (*ie*, the base space(time) X is a differential manifold), NDG still falls far short of encompassing the complete range of singularities encountered even in Schwartz's linear distribution theory. Moreover, the only type of derivation (differential operator) defined in NDG is a commutator with a fixed operator, hence it is quite a restricted notion of differential operation even within the category of Banach algebras. Thus, it is even less capable of dealing with with singularities on arbitrary closed nowhere dense sets (of finite-dimensional Euclidean or locally Euclidean—*ie*, manifold-type of—spaces) [273], let alone with the so-called 'spacetime foam dense singularities' like the ones teeming Rosinger's abelian differential algebras of generalized functions, which have been successfully treated ADG-theoretically in [274] and used earlier here in connection with the ADG-evasion of the inner Schwarzschild singularity.

On the whole, one could say that NDG, *à la* Connes, may be perceived as an attempt to 'quantize Calculus' by functional analytic means, while the notion of manifold is still manifestly retained at the background. On the other hand, the whole algebraico-categorical, being sheaf-theoretic, machinery of ADG has been developed independently of any notion of manifold, a notion which, *vis-à-vis* the problem of QG, has been repeatedly criticized by many workers in the field of QG. Moreover, ADG, being free of any manifold concept, is able to cope with problems pertaining to 'singularities' by applying methods of non-linear PDEs (*eg*, Rosinger's 'differential algebras of generalized functions' [324, 325, 327]). Finally, it is also worth noting here the conceptual simplicity of the machinery of ADG.[730]

- **ADG versus SDG:** From a technical perspective, a preliminary step one could take in order to initiate a fruitful comparison between ADG and SDG, both abstractly (mathematically) and with respect to (physical) applications to QG, is to show somehow that the category of differential triads is in fact an (elementary) *topos* [253]—a so-called 'cartesian closed category'. This is only 'natural' and feasible since $\mathfrak{DT}$ is *bicomplete*—*ie*, it closes under (finite)

[730] That is, passing to noncommutative structure sheaves may only complicate (mathematically) issues (*ie*, pragmatically, if we can do all DG using commutative algebras, why resort to noncommutative ones? [285])—let alone that it is not at all clear, physically speaking, that the use of noncommutative coordinates has anything to do with the problem of QG *per se* (again, what does 'quantum spacetime' have to do with QG?). Of course, it would be just short-sighted not to acknowledge the manifold applications that NDG has enjoyed in fundamental physics in recent times—from the study of gauge theories of the standard model as well as spacetime and gravity [79, 80, 92], to the so-called D-branes in open string theory in the presence of a background B-field (no references given). Noncommutative geometry has also been suggested as a UV cut-off in QFT and it has been recently appreciated that noncommutative Yang-Mills theory is exactly solvable (again, no references provided).

categorical (inverse) limits and (direct) colimits, it has *canonical subobjects*—*ie*, a canonical subobject classifier $\mathbf{\Omega}$ can be defined in $\mathfrak{DT}$, it possesses canonical *products and coproducts*, as well as an *exponential structure* [289, 290, 291, 292, 293]. In particular, in connection with finitary, causal and quantal ADG-gravity, the immediate vision is to organize the finsheaves of qausets in [272], on which a locally finite, causal, quantum and singularity-free[731] version of (vacuum) Lorentzian gravity holds, into a topos in which to address deep logico-geometrical issues in QG [253]. This potential application of ADG has been variously anticipated in the past [311, 312, 313, 314, 315, 317], and it is currently being under intense development [316].

From a purely mathematical perspective, under the prism of topos theory, it would be particularly interesting to see how can one carry out the basic ADG-theoretic constructions *internally* in the $\mathfrak{DT}$-topos by using the internal, intuitionistic-type of language (logic) of the topos [240, 253].[732]

Ever more importantly for a purely mathematical comparison between ADG and SDG, having delimited the topos-theoretic (logical) background underlying both ADG and SDG, one can then compare the notion of *connection*—arguably, the key concept that actually qualifies both theories as being *differential* geometries proper—as this concept appears in a categorical guise in both theories [233, 232, 243, 259, 260, 269, 405, 406, 408].

Finally, one may try for an ambitious, 'synthetic juxtaposition' of the so far various and numerous physical applications of ADG [259, 262, 273, 274, 270, 271, 272, 261, 263, 265, 316, 269, 264, 267, 268] and SDG [171, 172, 173, 175, 174, 169, 70, 207, 71, 69, 72, 208] (as well as topos theory in general [398, 226, 227]), to GR, quantum theory, and their desirable 'unison', QG. Even with regard to this modern potential 'unification' of the purely algebraico-categorical, sheaf-theoretic ADG and the similar topos-based SDG, we have (even if indirectly, but quite prophetically) Einstein's 'approval', for we recall from the conclusion of [368][733] the following telling Fraenkel–Einstein exchange:

[731]This paper.

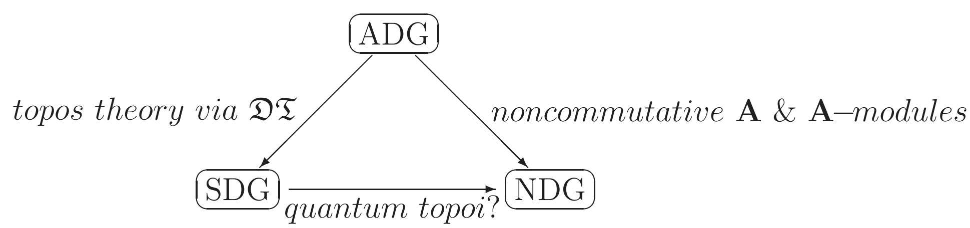
[732]For example, by stepping into the constructive world of the topos, one could bypass the 'problem' (because $\mathbf{A}$-functoriality-violating) of defining *derivations* in ADG (Chris Mulvey in private communication with the second author). *En passant*, the reader will have already noticed that no notion of '*tangent vector field*' is involved in ADG—*ie*, no maps in $Der: \mathbf{A} \longrightarrow \mathbf{A}$ are defined, as in the classical geometrical manifold based theory (CDG). Loosely, this can be justified by the fact that in the purely algebraic ADG the (classical) geometrical notion of 'tangent space' to the (arbitrary) simply topological base space X involved in the theory, has essentially no meaning, but more importantly, no physical significance since X itself plays no role in the (gravitational) equations defined as differential equations proper via the derivation-free ADG-machinery.

[733]With the original citation being [151].

(Q8.14) "...In December 1951 I had the privilege of talking to Professor Einstein and describing the recent controversies between the (neo-)intuitionists and their 'formalistic' and 'logicistic' antagonists; *I pointed out that the first attitude would mean a kind of atomistic theory of functions, comparable to the atomistic structure of matter and energy. Einstein showed a lively interest in the subject and pointed out that to the physicist such a theory would seem by far preferable to the classical theory of continuity. I objected by stressing the main difficulty, namely, the fact that the procedures of mathematical analysis, e.g., of differential equations, are based on the assumption of mathematical continuity, while a modification sufficient to cover an intuitionistic-discrete medium cannot easily be imagined. Einstein did not share this pessimism and urged mathematicians to try to develop suitable new methods not based on continuity*[a]*...*"

[a]Our emphasis.

- *In toto*, pictorially, but sketchily, the relation which we envision between ADG *contra* SDG and NDG can be cast as follows:



(110)

with the potential relation between SDG and NDG being possibly effectuated, via ideas issuing from noncommutative algebraic geometry (as well as its related noncommutative operator algebra and functional analysis), by the aforementioned notion of noncommutative ('quantum') topology, the noncommutative sheaves based on it, and the ('quantum') topoi thereof. For example, the quest for a quantum topos may follow the route via C^*-quantales—structures that may well qualify as noncommutative ('quantum') topologies proper [281, 283, 284],[734] as the following 'analogy' depicts:

[734]'Originating', one could say, by a desire to find a noncommutative version of the Gel'fand-Naimark representation theorem for abelian C^*-algebras, thus arrive at the noncommutative analogue of the *spectrum* (of a non-abelia C^*-algebra)—a topological space 'inherent' in the algebras involved, which is a generalized, classical one (a so-called

$$\frac{\text{(sheaves over) locales}}{\text{(sheaves over) quantales}} = \frac{\text{topoi}}{?} \qquad (111)$$

with the elusive (abstract) noncommutative ('quantum') topos structure being expected to replace the question-mark in equation (111) above, as well as in the diagram (110) just before it.[735] And one has to recall again here the central role that (noncommutative) C^*-algebras play in NDG [91, 47].

8.3.2 ADG versus 'Quantizing on a Category'

The discussion above about the potential links and close affinities between the categorico-algebraic ADG with NDG and SDG in the general light of homological algebra (category and topos-theoretic) ideas, brings us to another mathematical approach to QG to which we would like to relate ADG. This is Isham's very recent '*Quantizing on a Category*' (QC) general mathematical scheme [209, 210, 211, 212]. The principal goal of QC is to quantize systems with configuration (or more generally, history) spaces consisting of 'points' having 'internal structure'. The main motivation behind this approach is the apparent failure of applying the conventional quantization concepts and techniques to 'systems', such as causets or space(time) topologies for instance, whose configuration (or general history) spaces are far from being point-like differential manifolds consisting of structureless points. Isham's approach basically hinges on two innovations: first to regard the relevant entities as objects in a category, and then view the categorical arrows as analogues of momentum ('derivation maps') in the usual (manifold based) theories. Although this approach includes the standard manifold based quantization recipes, it goes much further by making possible the quantization of systems whose 'state' spaces are not smooth continua (*eg*, causets or finitary topological spaces).

There appear to be close connections, both conceptually and technically, between QC and ADG—affinities which one would be tempted to explore further. *Prima facie*, and generally speaking, both schemes concentrate on evading in one way or another the pointed differential manifold—be it the configuration space of some physical system, or the background spacetime arena of classical or quantum (field) physics, and they both employ 'pointless', categorico-algebraic methods. Both focus on an abstract (categorical) representation of the notion of derivative (or derivation)—QC, by abstracting from the usual continuum based notion of vector field (derivation), arrives at the categorical notion of an 'arrow field' which may be thought of as a map or morphism (:section of certain non-functional presheaves and their associated representation Hilbert

locale [253]) in the case of commutative C^*-algebras [283, 282].

[735] Jim Lambek, Chris Mulvey, Steve Selesnick and Freddy Van Oystaeyen in private communication with the second author.

presheaves involved in QC) which respects the internal structure of the system-objects that one focuses on (*eg*, topological spaces or causets); while in ADG, the notion of derivative (:differential d) is abstracted and generalized to that of a connection field $\mathcal{D}$, defined categorically as a sheaf morphism, on a sheaf of suitably algebraized structures (*eg*, causets or finitary topologies [318, 319, 309, 310, 270, 271, 272, 317]).

On the other hand however, in 'philosophical' aims and general outlook, ADG and QC differ: the second aims at quantizing (*ie*, developing quantum theories of) non-continuum models of 'space' (*eg*, finitary topologies) or 'space*time*' (*eg*, causets), while ADG is indifferent to the character of any background space(time) and it develops a field theory with quantum traits built into the formalism from the very start, hence in no need of (a formal process of) quantization. Nevertheless, the technical and conceptual affinities between ADG and QC are so strong that it is well worth attempting a marriage of the two.

8.4 'Physical Geometry' or 'Geometrical Physics'? (ADG-Theoretic Variations on a Theme by Peter Bergmann)

We would like to remark under the prism of ADG, and ADG-gravity in particular, on a fundamental difference in our opinion between what Peter Bergmann called in [41] '*physical geometry*' and '*geometrical physics*'. Let us open these remarks with the following 'duality' between algebra and geometry originally intuited by Sophie Germain already back in the late 17th century [154]:

(Q8.15) *"L' algèbre n' est qu' une géométrie écrite, la géométrie n' est qu' une algèbre figurée."* [a]

[a] *"Algebra is nothing but a 'written geometry',* [while] *geometry is nothing but a 'figured algebra'"*.

Loosely and intuitively speaking, from the vantage of the applied ADG to theoretical physics, '*physical Geometry*'—what Sophie Germain calls '*algèbre figurée*'—is the geometry 'deriving' from the ('dynamical') relations between the geometric fields themselves, which relations can in turn conveniently be (mathematically) grounded in algebra. In this sense, physical geometry is Euclidean[736] and ADG appears to be well suited to represent it mathematically. The word 'deriving' above is meant to suggest that *as 'physical geometry' we regard the 'geometry' or 'structure' of the 'solution space' of the dynamical field equations, which are mathematically expressed algebraically* (*ie*, 'relationally'—here by sheaf-theoretic means). This is related to what we posited earlier, namely, that '*kinematics (geometry) comes after or is the result (outcome) of dynamics*'.[737] In

[736] See comments earlier on our ADG-based perception of what is commonly referred to as Euclidean geometry.

[737] On a par is the statement that '*geometrical spacetime is the effect of, or is inherent in, the algebraic field*'.

other words, much in the same way of our Wheeler-type of 'principle'-*motto* that *no theory is a physical theory unless it is a dynamical theory*, we here hold that *no geometry is a physical geometry unless it is the result of a dynamics*. Most of this has already been anticipated by the first author in the recent paper [268].

On the other hand, at least insofar as differential geometric considerations are concerned, thus far '*geometrical physics*' has been exercised (implemented) by CDG-theoretic means. CDG is a theoretical framework which, almost by definition, gives a central operative role and, as a result, a physical significance, to an *a priori* fixed, kinematical-geometrical background smooth spacetime manifold, as well as to the analytic, Cartesian-Newtonian way in which (smooth) coordinates ('point-like space(time) locations') mediate or intervene in our calculations (Differential Calculus) so as to mislead one into thinking that the most important 'supplier' of the differential geometric mechanism and technotropy is the *underlying* pointed spacetime manifold, rather than, in a Euclidean-Leibnizian fashion, the (algebraic relations between the) '*overlying*' geometro-physical objects themselves (*ie*, the fields and their particle-quanta). In this sense, 'geometrical physics' is part, or an aspect, of the more general term 'mathematical physics'.[738]

Below, Peter Bergmann [42] makes it precise what a '*Physical Geometry*', in contradistinction to a '*Geometric Physics*', really ought to be:

[738]Which should be contrasted against the term '*physical mathematics*' that we prefer (see last subsection 8.6).

> "...In concluding, permit me to come back to my original question: to what extent do geometric considerations play a role in these theories? I believe that the answer is largely heuristic. *Any physical theory derives its justification from physical motives, and how well that theory helps us understand the physical universe.*[a] The judgment as to desirable aspects of the invariance group of a theory must ultimately rest on the physical decision as to the equivalence of apparently distinct descriptions of a physical situation (choice of coordinate frame, choice of gauge frame, etc). But given a conceptual framework, geometric realizations, and geometric imagery will often prove a powerful tool in elaborating the theory...
>
> (Q8.16) ...Such transformations might point in a direction in which *fields get unified under an invariance group in which the space-time manifold no longer plays a pre-eminent role. A world point would derive its identity from its dynamic environment, or it might possess no identity at all.*[b]
>
> Whether such a theory can still be called geometric is a question of terminology, not of principle. *Certainly, physicists will go in directions suggested to them by physical considerations, not by a mathematical discipline, and that is what geometry is.*[c] Considerations of an aesthetic nature are important, to be sure; for that we have the testimony of such giants as Einstein and Dirac. *But physicists must not lose touch with its foundation, soil as it were: we must above all remain sensitive to what Nature in its subtlety attempts to tell us*[d]."
>
> [a]Our emphasis.
> [b]Again, our emphasis.
> [c]Emphasis is ours.
> [d]Our emphasis.

In the penultimate paragraph of the quotation above, and in connection with the dynamical (not kinematical) character of (smooth) coordinates—the 'dynamicalization' of coordinates—propounded in [374], we should mention here ADG's thesis that *'space(time)' is inherent in the field*, which appears to turn upside-down the hitherto conception in physics that the kinematics—the 'possibility space'—of a theory should be delimited (defined) prior to the dynamics—the 'actuality space'.[739] And let it be stressed here that for us, in contradistinction to Bergmann, *this*

[739]This is our basic thesis, multiply emphasized earlier in this paper-book, that '*dynamics (algebra) comes before*

(*ie*, that algebra/dynamical relations comes before geometry/kinematical spacetime) *is indeed a question of fundamental principle, not just of terminology!* All this is already subsumed under the basic ADG-field pair $(\mathcal{E}, \mathcal{D})$: the dynamical Law is (defined by) $\mathcal{D}$, while the (geometrical) expression (:geometrical representation) of the Law is R—one that in turn delimits its 'solution space' (*ie*, the entire sheaf space $\mathcal{E}$ on which the law holds, which is completely described by its sections—the quantum particle states of the field).[740]

Of course, in line with Bergmann's words above, in ADG-gravity not only "*the space-time manifold no longer plays a pre-eminent role*", but in fact *it plays no role at all*. Furthermore, the categorico-algebraic ADG-formalism is fundamentally pointless, thus no question arises as to *where from does a world point derive its identity*. Spacetime itself (and at that, not necessarily a continuum, or even a discretum!) derives (its identity) from the dynamical fields: in this sense we have claimed throughout this paper that *'background geometrical spacetime' is inherent in the 'geometrical objects' (:the dynamical physical fields) that comprise 'it'*.[741]

And finally, about the last paragraph in (Q8.16) above: we totally agree with Bergmann, and in a wider sense, that a physical theory is *not* its underlying mathematical formalism,[742] needless to say that a physical theory does *not* progress solely from input(s) from an advancing mathematical discipline.[743] In other words, the primary considerations of a physicist are *physical* ones, not mathematical. On the other hand, again if we abide by the Wheelerian-type of principle that *a physical theory is defined by, and is nothing but, the dynamical laws that it purports to describe and aspires to interpret*, coupled to the initial choice-*cum*-'assumption' (of the theoretical physicist) to mathematically model those laws differential geometrically (*ie*, as differential equations) for whatever intuitive reason she may be inspired and motivated by,[744] the mathematical apparatus (:theory of DG) appears to be intimately entwined with the physical theory. Choosing the standard CDG for that purpose inevitably brings into play the 'metaphysical' geometrical background manifold,

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kinematics (geometry)'.

[740]Recall again that, sheaf-cohomologically speaking, $\mathcal{E}$ is completely characterized by $R(\mathcal{D})$ (*eg*, Weil's integrality).

[741]As repeatedly noted before, in ADG-gravity and ADG-field theory in general, we cease talking about '*world points*', or equivalently about '*spacetime events*', and we speak solely of '*world fields*'. *In extenso*, we do not talk about '*spacetime geometry*', but exclusively about '*field geometry*', with the noun 'geometry' pertaining to the (structure of the) 'solution space' of the dynamical field-law—the realm where the field law holds. Fittingly, ADG was originally coined the (differential) '*geometry of vector sheaves*' [259], it being understood in a generic way that that the vector sheaves $\mathcal{E}$ involved—the representation or carrier spaces for the connection fields $\mathcal{D}$—are the universes where the field laws such as (29) hold.

[742]This accords with our earlier maintaining that 'physical spacetime' (whatever that means) is traditionally modelled after a smooth manifold (:locally Euclidean space) just because the original mathematical formalism underlying GR happened to be the CDG-based (pseudo)-Riemannian geometry.

[743]However, see also counterpoints to this position *à la* Dirac and Faddeev raised in 8.6 below.

[744]For example, for the sake of (infinitesimal) locality.

which is almost inevitably interpreted physically as 'spacetime'. In other words, we are inclined to hold that the differential spacetime manifold (:a background locally Euclidean space) was forced on physicists by a mathematical discipline (CDG), *not* by physical considerations proper. However, if the physicist chooses ADG, one leaves out *ab initio* the metaphysical[745] mathematical extra-baggage of the geometrical base manifold (and, inevitably, its pseudo-physical interpretation as 'spacetime'), and she directly refers to/focuses on the dynamical fields (:connections) and the laws (:differential equations) that they define. And, to emulate Bergmann's expression above, *this is what 'physical (differential) geometry'* (:the geometry defined by the laws defining a physical theory) *is all about*.

8.4.1 The end of 'geometrical picturization' in theoretical physics: breaking Sophie Germain's 'symmetry-duality' between algebra and geometry in favor of the first

By now it must have become clear to the reader that the 'iconoclastic' stance against (differential) geometry that we assume in view of the ADG-theoretical paradigm and its physical applications—ADG-field theory in general and ADG-gravity in particular—is the following: the principal 'icon' that has been 'religiously' venerated by theoretical physicists is that of a background geometrical space-time—be it a continuum or a discretum.[746]—and it is this 'icon' (or better, '*idol*') that we would like to 'cut'[747] (or better, tear down) in fundamental physics. Undoubtedly, geometrical picturization in physics, through the medium and 'lens' of an ambient spacetime environment intervening (via coordinates/Cartesian arithmetics) in our physical models and calculations with them, has proven to be invaluable in the past, but we feel it is high-time we left it behind, as a dated intuitive/heuristic method of tackling issues especially in the quantum deep where the

[745]And in view of singularities, '*paraphysical*'!

[746]In Riemann's words [322], a continuous or a discrete manifold. Arguably, at least in field physics, the continuum has dominated the discretum, although traditionally the concept of a (material) particle is supposed to carry within it some fundamental 'discreteness' and 'pointedness' (one usually speaks of 'point-particles'). On the other hand, the (dynamical) evolution of point-particles is normally taken to be 'continuous' (*ie*, differential equations in a space-time continuum).

[747]In Greek, '*εικονοκλάζω*' (:verb), from which the noun 'iconoclasm' derives, is a compound word consisting of the noun '*εκόνα*' (:icon, picture) and the verb '*κλάζω*' (:ancient Greek meaning 'to cut' or 'to separate', and then 'to compare and classify' things—think of the English word 'class' and the Greek word '*κλάσμα*' meaning 'cut part' or 'section', and in mathematics 'fraction' or 'proportion'). According to Webster's Encyclopedic Unabridged Dictionary of the English Language, an '*iconoclast*' (I kon′a klast′, noun) is: (i) a breaker or destroyer of images, especially those set up for religious veneration, and/or (ii) one who attacks cherished beliefs, traditional institutions, *etc.*, as being based on error or superstition. Historically, in Byzantium (723-843AC), '*iconoclasm*' or '*iconomachy*' was the polemic current against '*iconolatry*'—the worshipping of Christian icons (predominantly in churches). The 'canonical' example of an iconoclast scientist is Galileo and Darwin.

usual differential geometric means of the manifold based CDG appear to be inadequate and at best problematic.

In this stance, again we are not alone. Einstein too, although he relied heavily on geometrical picturization (heuristically/intuitively) and on the spacetime continuum based Riemannian geometry (technically) in developing GR and the theory was a great success—an enormous single-handed achievement, had subsequently expressed scepticism about 'geometric reasoning' and the concomitant 'geometrization of physics' (especially *vis-à-vis* the *aufbau* of a unified field theory!), as Bergmann recalls below:

> "...*What is geometry?*[a] I suspect that there is no answer to this question that will satisfy everybody...[b] Depending on the properties ascribed to a new model for space-time, its structures might lend themselves to interpretations that are reminiscent of fields known to physicists. *How does such 'geometrization' contribute to unification? Einstein stated repeatedly that he did not consider geometrization of physics a foremost or even a meaningful objective, and I believe that his comments remain valid today. What really counts is not a geometric formulation or picturization but a real fusing of the mathematical structures intended to represent physical fields.*[c] "
>
> [a]Our emphasis. Bergmann's question recalls Chern's '*What is Geometry?*' in [82, 81] that we commented on in the previous subsection. Let it be stressed here however that the first was rather trying to address the question of what we wish to coin '*physical geometry*' (and *in extenso* of '*physical space(time)*'), while the second attempted to answer to that question by giving an outline of the development of what we would like
> (Q8.17) to call '*mathematical geometry*' (and *in extenso* of '*mathematical space*'). What we wish to draw here is a clear distinction between physical and mathematical geometry and, *in extenso*, space(time). The 'geometry' that the mathematician is interested in—the 'mathematical geometry' and 'space', is very different from the 'physical geometry' that the physicist is interested in, although as it has been repeatedly noted before, in the theoretical physicist's mind—which has been greatly influenced by mathematics—there is a conflation (and resulting confusion) of the two terms, at least insofar as the background differential manifold is interpreted as 'physical spacetime' with the concomitant use of CDG-concepts and methods in field theory, classical or quantum.
>
> [b]Then Bergmann attempts to loosely define (mathematical) geometry as "*any kind of mathematical structure that begins with the construction of a set of points that satisfies the minimal properties of continuity that justifying one in speaking of a space...*" (something that recalls the mathematical notion of a *topological space*—the minimal structure that a space can have apart of course from being a bare/structureless set), and he then gives examples of properties of and structures on such geometric spaces "*that have been used by physicists in their pursuits and endeavors to understand nature*".
>
> [c]Our emphasis again.

In toto, we believe that ADG-field theory, by down-playing and atrophizing geometrical picturization (by means of a geometrical background spacetime) while at the same by concentrating on the algebraic (dynamical) relations between the physical fields themselves (without recourse to a metaphysical background), has contributed significantly towards materializing Einstein's meaningful objective above, namely, *to fuse the mathematical structures intended to represent physical fields under the general and unifying notion of connection* $\mathcal{D}$ (:generalized differential)—the principal notion with which one can actually do (differential) geometry after all.

8.5 Einstein's 'Organic Theory' Approached via a Rhetorical (for ADG) Question: Can One Do Field Theory Without the Spacetime Continuum?

"...Adhering to the continuum originates with me not in a prejudice, but arises out of the fact that I have been unable to think up anything organic to take its place..."

The starting point for the following discussion is Einstein's confessionary words in (Q2.?), together with his wish to find a purely algebraic theory for the description of reality (Q2.?, Q?.?). In (Q?.?), '*organic*' is understood here in the context of ADG as meaning, on the one hand simply on pragmatic grounds, *a theory that works as well as the spacetime continuum (manifold) based field theory*, and on the other, as *a 'self-sustaining' theory, which is 'autonomous', without its 'paracytic' dependence on extraneous, auxiliary structures* (*eg*, 'spacetime'[748])—structures that are vital for its (mathematical) subsistence, consistency and operativeness. Such autonomous theoretical entities were the Leibnizian monads, which ("$\epsilon\nu\tau\epsilon\lambda\epsilon\chi\epsilon\iota\alpha\nu$ $\tau\epsilon$ $\kappa\alpha\acute{\iota}$ $\alpha\upsilon\tau\alpha\rho\kappa\epsilon\iota\alpha\nu$ $\epsilon\chi o\upsilon\sigma\iota\nu$")[749] [246, 247, 384].

8.5.1 Reconciling the "better known Einstein" with Stachel's "other Einstein"

First, as it has already been mentioned and analyzed to some extent in [272] and as it has been transparent in various quotations above, we shall state it up-front that Einstein's 'dissatisfaction' with 'continuous' field theory on the spacetime continuum was essentially due to two reasons:

1. The singularities and associated unphysical infinities plaguing his GR on the smooth spacetime manifold—the spacetime continuum.

2. The (successes of the) finitistic and algebraic quantum theory (of matter).

The basic 'thesis' in [368] is that there were two (facets of) Einstein(s):

[748] This external 'spacetime' be it a continuum or a discretum. (See the 7.5.1 next for this.

[749] Translation from the original Greek: "*Monads have entelechy (ie, an 'inner perfection', a 'built-in end') and are self-sufficient/sustaining (ie, they are 'autonomous').*"

1. The more well known Einstein, advocating a 'continuous' field theory on the spacetime continuum (SR, GR, unitary field theories for gravity and electromagnetism)—`What does the 'completion of GR' consist in?`[750] Let us call that facet of Einstein '*Geometric Continuum Einstein*' (GCE).

The basic argument of Einstein is that *if the spacetime continuum is to be renounced as being fundamental, then a field theory (based on it) must also go.* We borrow two quotations of his from [368])[751] that corroborate this inextricable dependence of field theory on the spacetime continuum and the equivalent 'implication-by-negation' '*no field theory⇒no spacetime continuum*':

(Q8.18) "...I consider it entirely possible that physics cannot be based upon the field concept, that is on continuous structures. Then *nothing*[a] will remain of my whole castle in the air including the theory of gravitation, but also nothing of contemporary physics[b]..."

[a]Our emphasis.

[b]Implying at the same time that theoretical physics till his time was essentially '*continuous field physics on the spacetime continuum*'.

Especially for GR, as Einstein notes above, which is the field theory *par excellence*, field theory and the spacetime continuum appear to go hand in hand, and it is almost unimaginable that one can separate them (*ie*, retain one and abandon the other). Stachel puts it succinctly in [368] as the apparently inevitable shortcoming in implementing the PGC of GR if one throws away the spacetime continuum and, as a result, the field-theoretic outlook as well:

[750]From the Einstein's autobiography [128]: a completion consists in i) *evading singularities*, and ii) *describing the atomistic/quantum structure of reality.*

[751]See there for exact references.

(Q8.19) "...Suppose one believes that future physics will continue to employ the space-time continuum as a fundamental element, as Einstein often did. *Then, if further progress in physics is to incorporate the dynamical view of space-time inherent in the general theory, it is hard to see how it can escape from the field viewpoint. How else could one express general covariance?*[a]...
...The important point to emphasize is that this was undoubtedly Einstein's final viewpoint about the progress of physics.*If the space-time continuum is to remain a fundamental element of future physics, some sort of generally-covariant field-theoretic generalization of the general theory of relativity is required. As mentioned above, he devoted almost forty years of his life to the search for such a generalized theory.*[b]
I am sure you caught the caveat in my last remark: '*If the space-time continuum is to remain a fundamental part of future physics*'[c] This caveat brings me, finally, to the subject of this paper: 'The Other Einstein'.[d]"

[a]Our emphasis.
[b]Our emphasis.
[c]Again, our emphasis.
[d]What we coin below, '*Algebraic Finitistic Einstein*' (AFE).

But before we go on to describe Einstein's other facet, the AFE, let us again borrow from [368] an Einstein quote that further supports the aforesaid '*no spacetime manifold, no field theory*' motto of the GCE:

(Q8.20) "...I must confess that I was not able to find a way to explain the atomistic character of nature. My opinion is that if the objective description through the field as an elementary concept is not possible, then one has to find the possibility to avoid the continuum (together with space and time) altogether. *But I have not the slightest idea what kind of elementary concepts could be used in such a theory*[a]..."

[a]Our emphasis.

We also encounter in Einstein's autobiographical sketch, as recalled in [288]:

(Q8.21) "...It appears dubious whether a field theory can account for the atomistic structure of matter and radiation as well as of quantum phenomena. Most physicists will reply with a convinced '*No*',[a] since they believe that the quantum problem has been solved in principle by other means. However that may be, Lessing's comforting words stay with us: 'The aspiration to truth is more precious than its possession'..."

[a]Einstein's emphasis.

Ergo, Einstein did not have a (mathematical) theory in his hands that would enable him to do field theory without using the background spacetime manifold, a theory which, consequently, would do away or evade singularities as well as address 'finitistic quantum questions'.[752]

2. The lesser known (and much overlooked in our opinion!) Einstein, advocating a "*purely algebraic physics*" [368], possibly based on a 'discretum'—anyway, one without ties to a base spacetime continuum. Let us call that facet of Einstein '*Algebraic Finitistic Einstein*' (AFE).

3. For Einstein, this continuum/discretum' schism is a fundamental, *non-bridgeable* one—a strictly mutually exclusive theoretical 'binary alternative' which basically rests on the fact that, for him, it was almost unimaginable that one could do field theory—in fact, do *differential geometry* solely with those fields!—on, say, a reticular ('discrete') background space(time), let alone one teeming with singularities and supporting quantum constructions which are *prima facie* incompatible with continuous field descriptions on the spacetime continuum. The unabridgeable continuum-discretum divide, and more importantly, the essential difference between the two being that while the continuum supports *differential geometric* constructions, the discontinuum does not, is explicitly stated in the following remarkable words by Einstein [123]:

[752]Of course, we have ADG [259], and its various applications so far in classical and quantum gravity [273, 274, 275, 262, 270, 271, 272], and herein. Remarkably for the concluding sentence in (Q8.20) above, *ADG's elementary (fundamental) concept is again that of a field (viz. connection); albeit, one that is not at all based on a spacetime continuum*, while the basic means employed are *entirely algebraic* (:sheaf-theoretic).

(Q8.21) "*...The alternative continuum-discontinuum seems to me to be a real alternative; i.e., there is no compromise. By discontinuum theory I understand one in which there are no differential quotients. In such a theory space and time cannot occur, but only numbers and number-fields and rules for the formation of such on the basis of algebraic rules with exclusion of limiting processes.*[a] Which way will prove itself, only success can teach us..."

[a]Our emphasis.

4. ADG offers a sheaf-theoretic way of doing a differential geometry based field theory independently of the 'nature' or character of the background space(time)—*ie*, whether it is a continuum or a discretum—but one that deals directly with the dynamical fields—what we have called throughout the present paper-book, the '*geometrical objects*'—'in-themselves' (*ie*, autonomously).

Stachel's observation (in accordance with GCE) is the following: field (and even particle) theory cannot be thought of independently of the (background) spacetime continuum, so that if one does away with one concept, the other must go as well. However, the continuum supports differential (and differentiable!) physical quantities (fields), and laws expressed as differential equations between them. Moreover (from the bottom of the first page of [368]), in the context of GR, in order to be able to formulate the PGC, the continuum postulate appears to be necessary—in other words, one cannot express general covariance without a smooth spacetime manifold. *Or can (s)he?*, we would like to ask rhetorically in view of ADG-field theory and ADG-gravity in particular.

At the same time, he argues (again in accordance with Einstein, but now with AFE), if we drop the continuum, the next tenable position is for a 'discontinuum'—what we here refer to as a 'discretum', not being able to support differential equations though, but only relations between discontinuous, discrete quantities.[753]

[753]Here, the reader should go back and read the telling excerpt from a dialogue between Einstein and Fränkel in (Q8.14) about the apparently unbridgeable 'continuum'/'discontinuum' (also found at the end of [368]). Especially telling there are Fränkel's position that "*the procedures of mathematical analysis, e.g., of differential equations, are based on the assumption of mathematical continuity*"—in effect, on the assumption of a background continuum (:manifold). In 8.3 earlier, we expressed ADG's 'counterpoint' to this point by Fränkel, which is another instance of what we have here coined 'CDG and manifold conservatism and monopoly'.

8.5.2 Not a question whether it is a continuum or a discretum; for ‘it’ is not!

Following our remarks in subsection 7.6 before, the Planck space-time scale ℓ_P-t_P appears to be a ‘catchall’ for everyone working in QG.[754] However, in a theoretical/mathematical scenario for differential geometry, such as ADG, in which a base space(time)—be it a ‘continuous’ manifold/continuum or a ‘discrete’ space/discontinuum or ‘discretum’) plays no role whatsoever in the ‘intrinsic’ or ‘inherent’ so to speak, and essentially algebraic, differential geometric mechanism, as well as in our calculations (Calculus!) based on it, *prima facie there appears to be no Planck length-time issue at all.*

That is to say, to stress it again, the operative role that ℓ_P-t_P plays in the usual continuum based theories as a ‘natural cut-off’ below which the manifold is supposed to give way to something fundamentally reticular or granular/finitistic[755]—a ‘discretum’ from the point of view of which the usual spacetime continuum and the GR based on it are purported to be coarse, ‘effective theories’ (Q?.?), is in a strong sense rendered obsolete, and there is no need at all to evoke it.

In other words, from a differential geometric (but not classical analytic in the usual Cartesian–(locally) Euclidean sense of the smooth manifold[756]) perspective, it is not a question whether ‘*it*’ (*ie*, ‘spacetime’) is a continuum or a ‘discretum’, for ‘*it’ is not*—*ie,* it plays no role in our ADG-based ‘Calculus’, whose mechanism/machinery derives from the algebraic structure—*ie*, the (dynamical) relations between the ‘geometrical objects’ themselves that live on ‘space(time)’, without at all the intervention in the classical Euclidean–Cartesian–analytic sense of the latter into the said inherently algebraic differential geometric mechanism in the guise of coordinates, as in the classical, $\mathcal{C}^\infty$-manifold-based theory.

All in all, we maintain that ℓ_P-t_P enters our physics, because of the way (the manifold way, which is the only way we know so far of doing differential geometry![757]) we try to implement differential geometric ideas to the physics of the very small, while, we hold, the very notions of ‘very small’ and ‘very large’ (scale) lose their meaning when viewed from the ADG-perspective. For, in any case, as Sorkin mentions in [357],

[754] We owe this graphic characterization to Chris Isham (private e-correspondence with the second author).

[755] That something still perceived though as ‘space(time)’, even if in some ‘oblique’, ‘slanted’ way. Such for instance is the primitive conception of ‘spacetime’ at sub-Planckian scales advocated by causet theory [54].

[756] See 7.? above.

[757] Hence the CDG-conservative attitude mentioned throughout the present paper-book.

(Q8.22) "*...Historically you could say that Quantum Theory deals with the very small and General Relativity with the very large, but the essence of the distinction is not really one of size. Rather, 'the quantum of action' is in general important whenever no more than a few degrees of freedom are excited,*[a] *while gravity—or in other words General Relativity—is important whenever a large enough amount of energy is compressed into a small enough space. More specifically, gravity is important when the ratio Gm/rc^2 is of order unity, where m is the total mass-energy, r is the radius of the region into which it has been compressed, and G and c are respectively the gravitational constant and the speed of light...*"

[a]This generally, but by no means always, means that only a few particles are involved. (Sorkin's original footnote.)

which, of course, behooves us to analyze the role the physical constants play in these theories, and their eventual contribution into the ℓ_P-t_P below which quantum gravitational effects are supposed to become important.

To make the point stronger here, we would like quote a passage from a four year-old now report of a referee concerning a research project proposal of the second author whose main objective was to apply the ADG-formalism so as to on the one hand develop a finitary, causal and quantal Lorentzian gravity [270, 271, 272], and on the other tackle the problem of spacetime singularities (in classical GR), as done in [317] and of course herein. (S)he first summarized the project:

(Q8.23) "...The candidate proposes broadly three topics of study, all following naturally out of his previous work. First he aims to develop a finitary or 'discrete' analysis of the spin-connection formulation of general relativity, which would blend elements from the approaches of Ashtekar, Finkelstein and Sorkin. He would then apply this new formalism to the problem of quantum gravity. In this work, he would use heavily the 'ADG' formalism of Mallios. His second project would apply the ADG technology in an attempt to 'resolve' the singularities of the classical Einstein equations, such as that residing at the center of a black hole...[and (s)he then wrote in parenthesis]...*This is somewhat at odds with the first project, since, if successful, it would to some extent remove the need for any postulate of spacetime discreteness*[a]*...*"

[a]Our emphasis.

As if the 'resolution' of the black hole singularity and associated unphysical (curvature) infinity was actually the result of quantization and spacetime discretization.[758]

For anyway, to stress it once again in the present work (as we have also time and again done it throughout our past tetralogy [270, 271, 272, 317]), in ADG the base topological space (on which the vector, algebra, and **A**-module sheaves in focus are soldered), be it a continuum or a discretum, plays absolutely no role in the inherently algebraic differential geometric mechanism, which derives as it were 'from the stalk' (ie, from the 'geometrical objects'—physically speaking, the dynamical physical fields 'in-themselves'). Ultimately, it is not a question whether it is a continuum or a discontinuum, because it plays no epicurical role in the said differential geometric mechanism—or more importantly, in the physical dynamics expressed as (differential) equations between the relevant sheaf morphisms representing the fields (*viz.* connections) and their curvatures (**A**-functoriality and **A**-natural transformation PARD of the dynamics). In a philological sense, in ADG the generic and in principle arbitrary base (topological) space X recalls a bit Archimedes' famous dictum "*Give me somewhere to stand, and I shall move the Earth*"[759] [4],[760] in the sense that once X has been chosen and specified as a 'surrogate' background in order to (sheaf-theoretically) localize and solder the 'geometrical objects' (*ie*, the fields $\mathcal{D}$, their auto-symmetries' principal sheaves $\mathcal{A}ut\mathcal{E}$, and their associated representation quantum-particle sheaf spaces $\mathcal{E}$),[761] it is then for all practical-calculational (:differential geometric-differential Calculus) purposes effectively discarded, as it does not partake at all in the sheaf-morphism expressed differential equations modelling the field-dynamics.[762] This has already been anticipated and discussed to a great extent in the first author's [265, 268, 267].

Last but not least pertinent here is the remark, following from Finkelstein's telling words in the introduction of [142]:

[758]As it has been repeatedly emphasized throughout the present paper-book, and recently in [317], in ADG-gravity, at least as it concerns QG, no *a priori* spacetime discretization and quantization is evoked to evade the gravitational singularities, such as the Schwarzschild one, normally thought of as lying at the center of a black hole.

[759]In the original ancient Greek: "Δός *μοί πᾶ στῶ καί τάν γᾶν κινήσω*".

[760]See also Simplicious [348].

[761]Archimedes' "*somewhere to stand*".

[762]Archimedes' "*moving the earth*".

(Q8.24) "...Until we find a satisfactory theory of space-time structure, we shall be beset by the dilemma of the discrete versus the continuous, the idea already posed by Riemann, in much the following terms:
(a) A discrete manifold has finite properties, whereas a continuous manifold does not. *Natural quantities are to be finite.*[a] The world must be discrete.
(b) A discrete manifold possesses natural internal metrical structure, whereas a continuous manifold must have its metrical structure imposed from without. *Natural law is to be unified.*[b] The world must be discrete.
(c) A continuous manifold has continuous symmetries, whereas a discrete manifold does not. *Nature possesses continuous symmetries.*[c] The world must be continuous
The third argument is especially serious for rotational and Lorentz symmetry, which are much more serious to counterfeit than translational symmetry. Subgroups can be found as dense as desired in the translation group that are not everywhere dense, but I do not think that they exist for the rotation or Lorentz groups.
Since Riemann a new approach to this dilemma has become available. The same question about matter asked for two millennia—Is it continuous or is it discrete?—has at last been answered in this century: No. Matter is made neither of discrete objects nor [of continuous][d] waves but of quanta. In most familiar terms, a quantum is an object whose coordinates form a noncommutative algebra...A quantum manifold is a third possibility for space-time too. This possibility would pass us cleanly between the horns of Riemann's dilemma:
(a′) A quantum manifold, like a discrete one, *has better convergence than a continuous manifold*[e]—remember Planck and the black body.
(b′) A quantum manifold, like a discrete one, *is born with internal structure, and is even more unified, being coherent.*[f]
(c′) A quantum manifold, like a continuous one, *possesses continuous symmetries.*[g]..."

[a]Our emphasis.
[b]Again, our emphasis.
[c]Emphasis is ours.
[d]Our addition for clarity or completeness.
[e]Our emphasis.
[f]Once again, our emphasis.
[g]Our emphasis again.

that

(R8.9) *In the same way that the quantum passes through the horns of Riemann's discretum-vs-continuum dilemma by being neither ('continuous') wave (field) nor ('discrete') particle—paradoxically, as it were,* neither and at the same time both*!—so here the ADG-field is indifferent to whether the background spacetime is 'discrete' (particle-like) or 'continuous' (field-like)—it is a coherent combination of field ($\mathcal{D}$) and particle ($\mathcal{E}$) qualities. In this sense too, as we argued before, the ADG-field is 'in-itself' already quantum—a '*self-*', or '*third quantized*' entity and a fortiori prima facie in need of quantization.*

However, there is a lot more in Finkelstein's remarks in (Q8.24) above that we can comment on in the light of the 'autodynamical', 'self-quantum' and 'background spacetime indifferent' (*ie*, whether the geometrical base spacetime is 'continuous' or 'discrete') ADG-field theory that we hereby propound. Let us itemize our comments:

- First of all, it goes with little saying that our ADG-gravity efforts are *not* beset in any way by Riemann's continuous-*versus*-discrete spacetime dilemma. *In the background spacetimeless ADG-gravity, the question whether spacetime is a discretum or a continuum is begging the question, for there is no 'spacetime' external to the gravitational field itself.*

- From (a), we isolate the remark, which accords with our basic *credo* here, that *there are no infinities (or singularities) in (the laws of) Nature.*

- From (b), we highlight our basic thesis here that the ADG-gravitational field is dynamically autonomous, in no need of an external (background) spacetime structure (whether continuous or discrete) to be prescribed from without. The ADG-gravitational field $(\mathcal{E}, \mathcal{D})$ is 'unified' (what we called 'unitary') and 'quantum coherent' (what we coined 'third quantum').

- From (c) above and its ensuing remarks (about Lorentz and rotational Lie symmetries), which appear to favor the argument for a spacetime continuum picture, we fundamentally disagree with the position that "*Nature possesses continuous symmetries*"—in point of fact, with the position that Nature possesses any symmetry at all. In the same way that we do not hold that the notion of (an external to the fields) geometrical spacetime—whether discrete or continuous—is physically meaningful, we also maintain, now from a Kleinian perspective, that its symmetry group (whether discrete or continuous, respectively) is of no physical significance either.[763] Since we have repeatedly argued throughout this work that '*all is field*' (to the extent that if there is any 'geometrical spacetime'—what is usually called 'spacetime geometry'—in our ADG-theoresis, it is inherent in the dynamical fields defining the laws of Nature), if there is any 'symmetry' at all in our scheme, it is encoded in the principal sheaf $\mathcal{A}ut\mathcal{E}$ of dynamical autosymmetries of the ADG particle-field pair $(\mathcal{E}, \mathcal{D})$.[764]

- Then Finkelstein draws from the analogy between spacetime and matter[765] and directs the quest towards the development of a '*quantum manifold*'—one that is neither discrete nor continuous and at the same time both ((a′–c′).[766]

- However, to stress it once more, since in ADG-gravity (the structure of) spacetime is not an issue at all, the quest for its 'true' quantum structure (*eg*, arriving at it via some process of quantization, or even developing from scratch a full-fledged 'quantum theory of spacetime') does not arise either.[767]

The last remarks about the background geometrical spacetime continuum-*contra*-discontinuum indifference or 'blindness' of ADG-field theory, as well as its associated unitary 'third quantum coherence' and 'unitarity' *vis-à-vis* field and particle attributes of the ADG-field $(\mathcal{E}, \mathcal{D})$, brings

[763] To put it in another, more general, way: *'symmetry' goes hand in hand with 'measurement', that is to say, like 'space(time)' and its 'geometry', it lies with us—the external to the fields themselves—'observers', 'measurers' and 'symmetry perceivers'*.

[764] The intimate connection between 'symmetry' and geometry can be seen in our scheme in the fact that as soon as one prescribes (externally to the field $\mathcal{D}$ of course!) a structure sheaf $\mathbf{A}$ of generalized arithmetics ('coordinates'), which in turn may carry inherently in it a 'discrete' or a 'continuous' spectral spacetime geometry (Gel'fand duality) and represents $\mathcal{E}$ locally as $\mathbf{A}^n$, the corresponding principal sheaf $\mathcal{A}ut\mathcal{E}$ of the dynamical field autosymmetries becomes (again locally) the group sheaf $M_n(\mathbf{A}(U))^\bullet$ (n being the rank of the associated sheaf $\mathcal{E}$).

[765] An analogy already present in the usual 'action-reaction' way the (non-vacuum) Einstein field equations are physically interpreted, namely, that matter (fields) on the right hand side curve spacetime geometry on the left. This, as noted earlier, 'misleads' one into thinking that since the fields on the right hand side are quantum or quantized (QFT), so must be the spacetime geometry on the left—arguably, *the* analogy motivating and guiding most of the current approaches to QG.

[766] Interestingly enough, with his remark that "*a quantum is an object whose coordinates form a noncommutative algebra*", in a way he prophetically anticipated Connes' noncommutative (differential) geometry, and in footnote 2 he also refers to the pioneering work of Snyder [352] as (to his knowledge) the first to suggest that space-time is a quantum space—a 'quantum manifold' defined as a space whose points are labelled by noncommuting coordinates.

[767] At the same time, to emphasize it once again, if there is any 'noncommutative geometry' involved in ADG-gravity, it must be sought in the non-abelian Klein group $M_n(\mathbf{A}(U))^\bullet$ of (local) dynamical auto-symmetries of the coherent and autonomous ADG-gravitational particle-field pair $(\mathcal{E}, \mathcal{D})$ as well as of the dynamical equations (29) that it defines.

us to comment on the later Einstein's[768] 'schizophrenic' and ambivalent attitude towards the geometrical spacetime continuum and the continuous field theory based on it on the one hand, and on the other, the possibility of developing a purely algebraic theory for the description of the world in the quantum deep, one that is *a fortiori* based on a fundamental discretum (with the concomitant loss of the spacetime interpretation in the inherently reticular quantum domain).

8.5.3 ADG-theoretic marriage of the 'two Einsteins': a possible completion of GR to a genuinely unitary field theory

Einstein's dissatisfaction with the spacetime continuum based field theory granted, he apparently turned a blind eye to his 'second nature' urging him to look for a purely algebraic and finitistic method for the description of reality due to the singularities of GR and the finitistic-algebraic paradigm of quantum mechanics, and insisted till the end of his life on looking for a unified (better, 'unitary') field theory (of the electromagnetic with the gravitational forces)[769] which could on the one hand evade the problem of singularities, and on the other 'explain' the atomistic character of matter and radiation, thus evading altogether quantum theory.

Below are two quotations from [128]: the first expresses clearly his anticipation that a field-theoretic completion of GR to a unitary field theory should result in a singularity-free description even of material point particles, which act as sources of the various radiation force-fields, but from GR's viewpoint they are genuine or 'true' singularities of the (gravitational) radiation field. In other words, *one of the primary motivations for formulating a unitary field theory is overcoming the problem of singularities troubling primarily GR*—arguably, the background geometrical spacetime continuum based field theory *par excellence*:

[768]As later Einstein we would consider Einstein's work after the accomplishment of GR—at least as Einstein's ideas and work after 1920, which were predominantly focused around his unitary (nowadays called unified) field theory vision.

[769]Back then the two known elementary/fundamental forces of Nature.

(Q8.25) "...The essence of this truly involved situation [*ie*, uniting gravity with the other forces of matter—*ie*, electromagnetism][a] can be visualized as follows: *A single material point at rest will be represented by a gravitational field that is everywhere finite and regular, except where the material point is located: there the field has a singularity*[b]*...* Now it would of course be possible to object: *If singularities are permitted at the locations of the material points, what justification is there for forbidding the occurrence of singularities elsewhere?*[c] This objection would be justified if the equations of gravitation were to be considered as equations of the total field. [Since this is not the case], however, one will have to say that the field of a material particle will differ from a pure gravitational field the closer one comes to the location of the particle. *If one had the* [unitary][d] *field equations for the total field, one would be compelled to demand that the particles themselves could be represented as solutions of the complete field equations that are free of irregularities everywhere. Only then would the general theory of relativity be a complete theory....*"

[a]Our addition.

[b]Our emphasis. This is the Schwarzschild singularity at the fixed point mass that we 'resolved' earlier, and in [317], by ADG-means.

[c]Our emphasis again.

[d]Again, our addition.

The second quotation from [128] expresses Einstein's well documented scepticism about quantum mechanics—especially about the 'pseudo' way in which quantum theory purports to do away with continuous structures when in fact it still employs the spacetime continuum in order to formulate the dynamics of quantum wave amplitudes (*eg*, quantum fields) as *differential equations*—while, in an indirect way, it puts forward his support of a unitary field theory that may on the one hand overcome the problem of singularities in GR, and on the other 'explain' the atomistic structure of reality:

(Q8.26)

> "...It is my opinion that the contemporary quantum theory represents an optimal formulation of the relationships, given certain fixed basic concepts, which by and large have been taken from classical mechanics. I believe, however, that this theory offers no useful point of departure for future development. This is the point at which my expectation deviates most widely from that of contemporary physicists. *They are convinced that it is impossible to account for the essential aspects of quantum phenomena* (apparently discontinuous and temporally not determined changes of the state of a system, simultaneously corpuscular and undulatory qualities of the elementary carriers of energy) *by means of a theory that describes the real state of things [objects] by continuous functions of space for which differential equations are valid.*[a] They are also of the opinion that in this way one cannot understand the atomic structure of matter and radiation. They rather expect that systems of differential equations, which might be considered for such a theory, in any case would have no solutions that would be regular (free from singularities) everywhere in four-dimensional space. *Above everything else, however, they believe that the apparently discontinuous character of elementary processes can be described only by means of an essentially statistical theory, in which the discontinuous changes of the systems are accounted for by continuous changes of the probabilities of the possible states*[b]..."
>
> [a]Our emphasis.
> [b]Our emphasis again.

From the quotation above, and 'by negation/exclusion', one could say that Einstein, in contradistinction to his contemporary quantum physicists:[770]

- Believed that a singularity-free field theory on the spacetime continuum—whose laws are expressed differential geometrically, *ie*, as differential equations—could still be developed. This essentially implied his 'unitary field theory' vision,[771] albeit, one that still abides by the spacetime continuum (manifold) in which differential equations can be formulated.

- As also noted before, he also believed that such a theory could account for the quantum structure of reality, in the sense that the quanta of the source or radiation fields will be

[770]And, it is fair to say, in contrast also to the majority of current quantum theorists.
[771]See above.

described by everywhere (in the spacetime continuum) singularity-free (*ie*, regular) solutions to the total (*ie*, unitary) field equations (if we had them).

- Finally, he maintained that, in truth, quantum theory, in spite of the apparent discontinuity of quantum processes, still tacitly employs the continuum (as it were, 'in disguise') in the form of 'continuous changes of' (*ie*, again differential equations obeyed by) the probability amplitudes for (states of) quantum systems, with those 'field-states' (wave functions) too being defined on a spacetime continuum (*eg*, Schrödinger's non-relativistic or Dirac's relativistic wave equations).

Even more forceful and telling are the following words taken from three remarkable consecutive paragraphs in [121][772] which show, in order of appearance, an 'oscillation', 'ambivalence', or 'indecisiveness' in Einstein's thought about whether to opt for the background geometrical spacetime continuum/field theory or for an algebraic/discontinous description of reality *à la* quantum mechanics, with the third paragraph showing clearly his 'wishful thinking' about a field theory that could represent particles (quanta) by singularity-free fields:

[772]Which can be found on pages 92 and 93 in article 13, titled '*Physics and Reality*' (reprinted from the *Journal of the Franklin Institute*, **221**, 313 (1936); see [270]). The entire third paragraph is written in *emphatic* script for emphasis.

"*...To be sure, it has been pointed out that the introduction of a space-time continuum may be considered as contrary to nature in view of the molecular structure of everything which happens on a small scale. It is maintained that perhaps the success of the Heisenberg method points to a purely algebraical method of description of nature, that is to the elimination of continuous functions from physics. Then, however, we must also give up, by principle, the space-time continuum.*[a] It is not unimaginable that human ingenuity will some day find methods which make it possible to proceed along such a path. *At the present time, however, such a program looks like an attempt to breathe in empty space.*[b] There is no doubt that quantum mechanics has seized hold of a beautiful element of truth, and that it will be a test stone for any future theoretical basis, in that it must be deducible as a limiting case from that basis, just as electrostatics is deducible from the Maxwell equations of the electromagnetic field or as thermodynamics is deducible from classical mechanics. However, I do not believe that quantum mechanics will be the starting point in the search for this basis, just as, vice versa, one could not go from thermodynamics (resp. statistical mechanics) to the foundations of mechanics.

(Q8.27) *In view of this situation, it seems to be entirely justifiable seriously to consider the question as to whether the basis of field physics cannot by any means be put into harmony with the facts of the quantum theory. Is this not the only basis which, consistently with today's possibility of mathematical expression, can be adapted to the requirements of the general theory of relativity? The belief, prevailing among the physicists of today, that such an attempt would be hopeless, may have its root in the unjustifiable idea that such a theory should lead, as a first approximation, to the equations of classical mechanics for the motion of corpuscles, or at least to total differential equations. As a matter of fact up to now we have never succeeded in representing corpuscles theoretically by fields free of singularities, and we can, a priori, say nothing about the behavior of such entities. One thing, however, is certain: if a field theory results in a representation of corpuscles free of singularities, then the behavior of these corpuscles with time is determined solely by the differential equations of the field.*[c]"

[a] Our emphasis.
[b] Our emphasis again.
[c] All emphasis is ours.

The concluding 'hypothetical certainty' of Einstein in the last three lines of the quotation above may be rephrased as follows: *should one be able some day to represent field-theoretically particles ('quanta') in a singularity-free manner, then the particle dynamics—as it were, the evolution of those quanta in time—will be already inherent in (or theoretically speaking, be the result of) the field dynamics itself, and there would be no need to assume a priori particles as fundamental theoretical entities side-by-side the field concept.* It is precisely in this sense that field theory—Einstein's unitary field theory—aspired to 'explain away' particles and, accordingly, that "*quantum theory could be deducible from that future (unitary field) theoretical basis*". Indeed, earlier in [121], Einstein, upon concluding the section titled '*The Field Concept*' in article 13, '*Physics and Reality*', reexpresses this certainty in quite a categorematic fashion:[773]

(Q8.28) "*...What appears certain to me, however, is that, in the foundations of any consistent field theory, there shall not be, in addition to the concept of field, any concept concerning particles. The whole theory must be based solely on partial differential equations* [for the fields alone][a] *and their singularity-free solutions.*"

[a]Our addition.

8.5.4 What (mathematical) theory was Einstein looking for in order to materialize his unitary field theory vision?

We have seen in many quotations before Einstein's 'agnosticism' and 'scepticism' about a possible future theory that can on the one hand accommodate both his unitary field theory life-long grand project and the purely algebraic and finitistic quantum. Those doubts undoubtedly originate from his conviction that a field theory necessarily depended on the geometrical background spacetime continuum, while the latter apparently gravely miscarried with the combinatory-algebraic quantum theory.

Especially *vis-à-vis* differential geometry, it is worth recalling here Pais' remarks in the concluding paragraph of [288]:

(Q8.29) "...Yet as his life drew to a close, occasional doubts on his [unitary field theory][a] vision arose in his mind. In the early fifties he once said to me, in essence: '*I am not sure that differential geometry is the framework for further progress, but if it is, I believe I am on the right track*'[b]..."

[a]Our addition.
[b]Our emphasis and underlining.

[773]Again, the whole excerpt below is written in *emphasis* script for emphasis(!)

And one should immediately contrast this against Einstein's own remarks in (Q8.?), but especially, against Feynman's and Isham's quotations in (Q8.?) and (Q8.?), respectively.

A hypothetical theoretical scenario for a singularity-free, purely algebraic field theory in the quantum domain and its potential ADG-theoretic realization. In view of the remarks above, we would like to suggest that ADG, ADG-field theory, and ADG-gravity in particular (as well as the ADG-theoretic formulation of the other gauge, Yang-Mills forces [259, 260, 269]), is 'tailor-cut' for realizing his unitary field vision, for the reasons we itemize below:

- First of all, ADG is a purely *algebraic* and essentially *categorical* theory, as it relies on sheaf theory and sheaf cohomology.

- The central notion in ADG is that of a *field* (*viz.* connection), hence the theory is (mathematically speaking) *differential geometric*; albeit, in contradistinction to the CDG and thus background spacetime manifold based current classical and quantum field theories, it does not base its concepts and technotropy on a base differential manifold—in fact, on any background 'space(time)' whatsoever, be it a 'classical continuum' or a 'quantal discretum'.

- The ADG connection fields allow one to write total differential field equations for the dynamical laws of Nature, equations that *a fortiori* are *free from singularities*.

- In contradistinction to Einstein's expectations however, the ADG particle-field pairs $(\mathcal{E}, \mathcal{D})$ are not forced in any way to sacrifice the important particle aspect of fields.[774] On the contrary, it places it (*ie*, $\mathcal{E}$, whose local sections represent particle states of the field) side-by-side the field (:$\mathcal{D}$) concept, without any singularity arising in the theory.

- Finally, in contrast to Einstein's reservations about the applicability of differential geometric ideas in the quantum domain, as repeatedly explained earlier, the ADG-fields are 'inherently quantum' (:third quantized).

8.5.5 No potholes in ADG's path: evading the whole of Einstein's hole argument (based on a theme by John Stachel)

We have emphasized numerous times throughout this paper, as well as in the past trilogy [270, 271, 272], the totally base spacetime manifold free and purely algebraic (categorical) way in which ADG is able to formulate classical gravity (GR) and how it goes some way in capturing quantum

[774] After all, in the quantum domain the field-particle duality (also commonly known as Bohr's Complementarity) is an important aspect of the (physical interpretation of the) theory.

features of that elusive QG theory. With respect to the latter aim in particular, we have argued, primarily in [272], how this background manifold independence can evade directly and entirely certain caustic problems that the smooth spacetime manifold presents in various attempts to quantize, either canonically ('Hamiltonianly') or covariantly ('Lagrangianly'), GR; namely, *the inner product problem* and *the problem of time*, both of which essentially involve the diffeomorphism group $\mathrm{Diff}(M)$ of the base spacetime manifold M, which group in turn implements mathematically the PGC of GR.

In this sub-subsection we will see, in the classical context of GR, how the said manifold independence of ADG enables us to evade directly and completely a gedanken scenario originally proposed in 1913 by Einstein and Grossmann in order to put to a test the PGC of GR, usually referred to as *Einstein's hole argument* (EHA) [129]. We do not intend to give here a detailed account of EHA since many thorough expositions of it exist in the literature—[197, 286, 364, 366, 367, 371, 369], to mention a few—but we will rather discuss its deeper meaning, its far reaching (conceptual) consequences and aftermath as elaborated by Stachel in [366, 367, 369]. We will show how that deeper significance of the EHA is captured precisely by the '*spacetime manifold free and solely gravitational field and its automorphisms (synvariance)*[775]' picture of GR that ADG affords. Thus, in the process of our elaborations on the EHA and its consequences under the prism of ADG, we will not just stop at the straightforward and quite obvious cutting the Gordian knot-type evasion-result '*no background (smooth) spacetime manifold and consequently no* $\mathrm{Diff}(M)$*, no EHA*' that ADG achieves,[776] but we will show how the EHA's deeper significance, as unfolded by Stachel, is already included in (as it were, it is a particular instance of) the general reversal of the traditional priority of kinematics (spacetime geometry) over dynamics (algebra) that ADG enables us to maintain not only in the context of (classical) gravity (GR), but also with respect to the other (quantum) gauge forces (Yang-Mills theory). All in all, the arguments about the EHA to be given below vindicate the basic consequence of the ADG-theoretic perspective on gravity (and, *in extenso*, on the other gauge theories),[777] namely, that *the algebraic/dynamical gravitational field* (as represented by a connection $\mathcal{D}$ on a suitable vector sheaf $\mathcal{E}$) *is (physically) prior to the geometrical/kinematical spacetime manifold*, or more laconically put, that *dynamics (algebra) is prior to kinematics (geometry)*.

To recapitulate, the way in which the EHA is (physically) interpreted by Stachel allows us, as we shall see below,

- (a) not only to argue that the background differential spacetime manifold free ADG, the

[775] As represented by the algebraic connection variable $\mathcal{D}$.

[776] We provided arguments of such a kind for the direct ADG-theoretic evasion of the inner product problem and the problem of time in the manifold based approaches to QG, whether canonical or covariant, in [272]—*ie*, '*no base spacetime manifold and, as a result, no* $\mathrm{Diff}(M)$*, no inner product problem or problem of time*'.

[777] For anyway, from an ADG-theoretic perspective, GR is another kind of gauge theory [270, 271, 272].

'synvariant' notion of gravitational field dynamics that the latter supports, as well as the (smooth) metric-free (purely gauge-theoretic) formulation of GR solely in terms of the algebraic **A**-connection $\mathcal{D}$ that goes hand in hand with 'synvariance', are able to bypass in one-go the whole hole argument,[778] but also, and perhaps more importantly,

- (b) to present a concrete example of the radical '*dynamics before kinematics*' conceptual reversal that ADG is pregnant to, as we discussed earlier in 3.2.3.

The bottom-line of the following presentation of the EHA as expounded by Stachel will be that the smooth coordinates of the points of M are not what distinguishes and individuates them as *physical events* proper, but that the dynamical field $g_{\mu\nu}$ of GR, as a *solution* to the dynamical field equations (Einstein equations), assumes that individuating role. Alas, in ADG not only there is no *a priori* posited background geometrical spacetime manifold, let alone a smooth Lorentzian metric on it (which is normally assumed to represent the gravitational field), but also the sole dynamical gravitational variable is the entirely algebraic, $\mathcal{C}^\infty$-smoothness unrelated **A**-connection $\mathcal{D}$ on the vector sheaf $\mathcal{E}$ one chooses to employ as the carrier (or particle state) space of the gravitational field $\mathcal{D}$. But without further ado or thinking ahead, let us first expose the EHA and summarize Stachel's physical interpretation of it.

Brief exposition of the EHA à la Stachel. Thus, let us first recapitulate the EHA as distilled and expressed in modern (differential geometric) language by Stachel [366]. So first, in accordance with the original formulation of GR by Einstein, the 'structural conditions' for the expression of the HA are those set by the basic kinematics of GR which involves a four-dimensional differential manifold M (with its points being coordinatized by $\mathcal{C}^\infty(M)$) and a smooth metric tensor field $g_{\mu\nu}$ of Lorentzian signature.[779]

The next assumption of the EHA is to consider a subset $\mathcal{O}$ of M which is devoid of any matter—an empty region in M representing the '*hole*'—while matter fields are supposed to be present outside (and at the boundary of) this hole, that is, in its complement $M-\mathcal{O}$. Then, Einstein assumed that (the points of) M (are) is initially charted by a coordinate system $\mathcal{X}=(x_\mu)$ relative to which the energy-momentum stress tensor T assumes values $T_{\mu\nu}^{M-\mathcal{O}}$ in $M-\mathcal{O}$ and it vanishes identically in the empty hole ($T_{\mu\nu}^{\mathcal{O}}=0$), while the metric g takes on values $g_{\mu\nu}$ and satisfies the generally covariant Einstein field equations on the whole M.

Now, apply a smooth coordinate transformation $f:\ \mathcal{X}\longrightarrow\mathcal{X}'$ (or diffeomorphism $f:\ M\longrightarrow M$) of the following kind:

[778] Pun intended.

[779] Actually, as we shall see below, the *a priori* fixation of the Lorentzian manifold kinematics of GR—'*a priori*' here essentially means '*before the dynamical equations for the gravitational metric are solved*'—is exactly what the EHA, as interpreted and generalized by Stachel, comes to question.

- f is the identity on $M - \mathcal{O}$ ($\mathcal{X}_{M-\mathcal{O}} = \mathcal{X}'_{M-\mathcal{O}}$),

- while inside the hole $\mathcal{O}$, $\mathcal{X}_{\mathcal{O}} \neq \mathcal{X}'_{\mathcal{O}}$.

It follows that with respect to the new coordinates $\mathcal{X}'$, $T'^{\mathcal{O}}_{\mu\nu} = 0$ and $T'^{M-\mathcal{O}}_{\mu\nu} = T^{M-\mathcal{O}}_{\mu\nu}$—*ie*, $T_{\mu\nu} = T'_{\mu\nu}$ everywhere on M. On the other hand, while $g'_{\mu\nu} = g_{\mu\nu}$ identically on $M - \mathcal{O}$, $g'^{\mathcal{O}}_{\mu\nu} \neq g^{\mathcal{O}}_{\mu\nu}$ in general, although g' too is supposed to satisfy the same set of generally covariant field equations. Moreover, if we include the coordinated point arguments of these fields—*ie*, write $g_{\mu\nu}(x)$ relative to $\mathcal{X}$ and $g'_{\mu\nu}(x')$ relative to $\mathcal{X}'$—Einstein intuited the following apparent contradiction:

- while $g'_{\mu\nu}(x')$ and $g_{\mu\nu}(x)$ correspond to the *same* gravitational field,[780]

- the f-pushed forward $g' = f^*(g)$,[781] *but still referred to the initial coordinate system* $\mathcal{X}$ (*ie*, write $g'(x)$ by suppressing indices), also satisfies the Einstein equations on M *but does not correspond to the same gravitational field as* $g(x)$ (or equivalently, as $g'(x')$) *even though the two fields agree on the matter-filled* $M - \mathcal{O}$ *and on the boundary* $\partial\mathcal{O}$ *of* $\mathcal{O}$.

Einstein went ahead and interpreted this contradiction as the following apparent *violation of causality* coming from a mathematical non-uniqueness of solutions on the entire spacetime M of the field equations: *generally covariant field equations for gravity determine two distinct gravitational fields* (*ie*, solutions $g(x)$ or $g'(x')$, and $g'(x)$) *in the presence of the same matter sources.* Thus he abandoned the assumption of generally covariant field equations for gravity and required that further constraints on the coordinates and their transformations should be imposed within the hole $\mathcal{O}$ as well as in going to it from the 'matter-plenum' region $M - \mathcal{O}$.

At this point it must be emphasized that there have been rather simplistic analyses and interpretations of the EHA, one of them, [197], regarding it as an 'anomaly' in Einstein's thought and work in the sense that Einstein, 'blinded' by his life-long commitment to causality, overlooked the rather trivial fact that although the values of the components of a tensor such as the metric g change under a coordinate transformation (*ie*, $g'_{\mu\nu} \neq g_{\mu\nu}$), the tensor itself remains the same—*ie*, it represents the same gravitational field. Characteristically, we quote Bannesh Hoffmann:

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[780] After all, the Einstein equations are generally covariant.
[781] Stachel calls g' '*the metric g dragged along f*'.

> "...*It is clear from both the argument pertaining to the unchanging T_{ab} and the argument pertaining to the changing g_{ab} that Einstein was saying that if the components of a tensor change under a coordinate transformation, the tensor itself changes. But in fact, just the opposite is the case. Precisely because g_{ab} is a tensor, the g_{ab} and $\bar{g}_{ab}$ actually represent the same field even though $\bar{g}_{ab} \neq g_{ab}$. In general, we should expect the components of a tensor to change when the coordinates are changed. That is the nature of tensors. Indeed, it is the first thing one learns in studying the tensor calculus...How could Einstein have made so elementary an error?...I suggest that it was because Einstein had long been profoundly concerned about causality and determinism...*" (Q8.30)

Stachel, however, in [366] would not settle for such simplistic interpretations of the EHA:

> "...In trying to interpret these passages,[a] then, I have proceeded on the assumption that he was trying to express something nontrivial...Thus, I am led to reject one common interpretation of the 'hole' argument, which assumes that Einstein did not realize that the transformation of the components of the metric tensor under a coordinate transformation results in a re-description of the *same*[b] gravitational field in a *different*[c] coordinate system...So Einstein's point is *not*[d] that $G^{'}(x^{'})$ and $G(x)$ are different.[e] His point is that $G^{'}(x)$ and $G(x)$—same x—are different [quoting then Einstein and Grossmann from [130]]: '$G^{'}(x)$ *also describes a gravitational field with respect to K^f which however does not correspond to the actual (i.e., originally given) gravitational field*'[g]..." (Q8.31)

[a] Immediately preceding this quotation were two translated passages from [129, 130].

[b] Stachel's emphasis.

[c] Stachel's emphasis.

[d] Stachel's emphasis.

[e] Stachel, following Einstein and Grossmann in [130], uses the symbol 'G' for the totality of the components $g_{\mu\nu}$ of the metric tensor g.

[f] Our coordinate frame $\mathcal{X}$ above.

[g] Our emphasis.

For, after all, Einstein well understood that $g(x)$ and $g^{'}(x^{'})$ represent the same gravitational field, but was baffled that $g^{'}(x)$ and $g(x)$—*both referred to $\mathcal{X}$*—appeared to be different.

Stachel, after a careful semantic analysis of the EHA in [366], was led to the following subtle 'conclusion': the question whether $g(x)$ and $g^{'}(x)$ are *physically* equivalent (*ie*, whether they represent the same gravitational field) or not is not a purely mathematical one as long as one does not regard coordinates as *physically individuating* the points of M—that is, as long as one does not regard coordinates as the structure that qualifies the points of M to *physical events* proper. Stated in a positive way, if one maintains *a priori* (like Einstein did in the HA) that coordinates physically individuate the points of M—*ie*, that the assignment-labelling $M \ni p \mapsto x(p)$ physically distinguishes, as events, the points of M, then one can indeed maintain that $g^{'}(x)$ is physically distinct from (inequivalent to) $g(x)$. However, if one does not think of coordinates as physically individuating structures on the spacetime manifold, the original $g(x)$ and the f^*-dragged metric $g^{'}(x)$—again, *both referred to the same coordinate system*—are physically indistinguishable (equivalent). In other words, *the hole in Einstein's hole argument lies with the argument*[782]—*ie*, with the coordinate x in the argument of g.

Einstein, Stachel argues convincingly in [366], perhaps because he, still as late as 1913, had not shed completely the idea that coordinates have a direct metrical significance (Q2.7), abandoned "*with a heavy heart*" the PGC of GR precisely because he thought of coordinates as physically individuating the points of M as events proper. Furthermore, Stachel asserts that

(Q8.32) "...the physical equivalence or nonequivalence of these two fields is *not*[a] a purely mathematical question. Or, more accurately, it does not *become*[b] one unless and until one introduces the additional, *nonmathematical*[c] assumption or postulate that, in regions where no matter is present, the points of a manifold are physically individuated only by the properties that they inherit from the metric field..."

[a]Stachel's emphasis.
[b]Stachel's emphasis again.
[c]Or better, '*physically minded*'. Again, Stachel's emphasis.

In other words, in *vacuo* (*ie*, in the absence of matter), *it is the dynamical field itself*—in GR, the metric $g_{\mu\nu}$ *satisfying the Einstein equations—and not the a priori prescribed (kinematical) coordinates, that is physically individuating the points of M as spacetime events proper*.

At the same time, Stachel makes it clear that a mathematician would in a sense be 'justified' in regarding the coordinate labelling of the of the points of M as *mathematically* individuating structures since for example, as we have repeatedly noted throughout the present paper, M (initially regarded as a structureless set) inherits its topological and differential structure from $\mathcal{C}^\infty(M)$. Indicatively, we quote him from [366]:

[782]Double pun intended.

(Q8.33) "...One might be tempted to assert that coordinate-labelling systems provide such individuating fields for the points of the manifold. And so they do for *mathematical*[a] properties, such as the differential-topological structure of the manifold. But the point is (no pun intended!) that no mathematical coordinate system is *physically*[b] distinguished per se; and without such distinction there is no justification for physically identifying the points of a manifold—which are homogeneous anyway—as physical events in space-time. Thus, the mathematician will always correctly regard the original and the dragged-along fields as distinct from each other. But the physicist (or indeed anyone applying differentiable manifolds) must examine the question in a different light; and the answer will depend upon the means available for making extra-mathematical distinctions between the points of the manifold—or, more concisely upon the presence of individuating structures..."

[a] Stachel's emphasis.
[b] Stachel's emphasis.

Let it be added here that this *mathematically individuating* role that coordinates play for the points of a smooth manifold has also been advocated by Auyang. In [25], in accounting about Gauss and Riemann's ground-breaking work in infinitesimal (differential) geometry, she remarks about the role of coordinates (in the former's theory of surfaces):

(Q8.34) "...The Gaussian coordinates [arbitrary curvilinear coordinates][a] consistently assign to each point on a surface a unique pair of ordered numbers, and refrain from going further. *The Gaussian coordinates individuate, but neither relate nor measure.*[b] *This is the idea utilized in differential geometry, where the bare differentiable manifold is just a system of identifiable points*[c]*...*"

[a] Our addition.
[b] Auyang's emphasis.
[c] Our emphasis.

Of course, in a strong sense this is the very physical essence of the PGC of GR; namely, that *in GR, points are not sacrosanct, that no coordinate system is physically preferred, and that coordinates are not physically distinguishable ('observable' so to say entities partaking into the gravitational dynamics).* Quite on the contrary, as also mentioned in the first section of the present paper in the categorical language of ADG, the 'observable' geometrical object (or the 'measurable dynamical entity') in GR is an $\otimes_{\mathbf{A}}$-tensor (or equivalently, an $\mathbf{A}$-morphism),[783] such as the metric or the Riemann curvature tensors—objects that 'see through' the base spacetime manifold's points and our $\mathcal{C}^\infty$-smooth coordinatizations of them. Plainly then, Stachel's 'hunch' that Einstein's support of the HA and his tentative abandonment of the PGC of GR until 1915 is 'evidence' that as late as 1913 he was still not convinced of the metrical insignificance of coordinates. For since the metric is the sole dynamical variable in his formulation of GR, one could say that still by 1913–14 he was not sure about the aforesaid *dynamical insignificance of coordinates*—the essence of the PGC[784] maintaining that "*systems of differential equations* [about the gravitational field $g_{\mu\nu}$] *are generally covariant—i.e., they remain the same with respect to arbitrary substitutions of the* x_ν" [130].[785] But Einstein, already in 1916 [122], reinstated the PGC in GR 'at the expense of' the EHA and its apparent implication that the smooth coordinates of the spacetime continuum have a direct metrical, hence dynamical, meaning:

(Q8.35) "*...The laws of nature are to be expressed as equations which hold good for all systems of coordinates, that is, are co-variant with respect to arbitrary susbstitutions whatever (generally co-variant)...*[a]"

[a]Einstein's own emphasis.

All in all, since the smooth coordinates of (the points of) the spacetime manifold M are not dynamical entities in GR, and since as we mentioned in section 1 *à la* Wheeler, no theory is a *physical* theory unless it is a dynamical theory,[786] in GR the points of M, or their smooth coordinate labels in $\mathcal{C}^\infty(M)$, are non-physical (mathematical) structures, and the physical-*versus*-mathematical individuation distinction that Stachel draws as the deep aftermath of the EHA highlight precisely this.

Furthermore, in a remarkable passage from [366], Stachel gives the crux of his arguments about the whole of the EHA:

[783]In the case of a base differential manifold M, $\mathbf{A} \equiv \mathcal{C}^\infty_M$.

[784]Which, as he notes in [130] "*is the principle of relativity in its most far reaching sense*"—see our elaborations about this point in section 1.

[785]See Stachel's translation of the original paper written in German in [366].

[786]That is to say, a physical theory *is* (defined by) its dynamics.

(Q8.36) "...More generally, spatio-temporal individuation of the points of the manifold in a general-relativistic model is possible only after specification of a particular metric field, i.e., *only after the field equations of the theory (which constitute its dynamical problem) have been solved.*[a] Once this is done, the points of the manifold-with-metric become full-fledged physical events, endowed with gravitational as well as spatio-temporal properties..."

[a]Our emphasis.

as well as its 'negative' statement taken from [364]:

(Q8.37) "...In any theory in which the metrical structure is given *a priori*, the physical identity of points of the space-time manifold is indeed established independently of any dynamical considerations. But, in general relativity the metrical structure forms part of the set of dynamical variables, which must be determined [that is to say, *solved*!] before the points of spacetime have any physical properties..."

These words strike a very sensitive chord in the present paper. To begin with, they perfectly accord with what Peter Bergmann—the advocate *par excellence* for the '*pointlessness of GR*'—says in (Q?.?) above:

(Q8.38) "*...A world point would derive its identity from its dynamic environment, or it might possess no identity at all*[a]*...*"

[a]Our emphasis.

Moreover, they corroborate—in fact, they are a particular instance of—the central didagma of ADG, namely, that, physically speaking, *dynamics is prior to kinematics*. Here, in the context of GR, the (mathematical) kinematics of GR positing up-front M as a pointed differential manifold *a priori* assumed to support a smooth Lorentzian metric[787] does not physically qualify (cannot be physically interpreted) as a gravitational *spacetime of events* unless first its dynamical equations (for the dynamical metric field) have been prescribed (and possibly, solved!).

Stachel succinctly resumes all this to the following 'aphorism', which he calls "*Einstein's criterion*"—generally speaking, the aftermath of the EHA-*contra*-the PGC of GR:

(Q8.39)
- "*No metric, no anything.*[a]"

[a]Stachel's own emphasis.

[787]See (Q?.?) from [359] above.

a motto which we can readily generalize ADG-theoretically to:

(R8.10)

- *No (dynamical) field, no anything*, or equivalently, to
- *No dynamics, no spacetime*, or even more abstractly, to
- *No dynamics, no kinematics; (spacetime) geometry is the result of algebra (ie, of the gravitational dynamics)—as it were, the 'solution space(time)' of the physical law corresponding to the dynamics, the realm where the field equations hold (ie, where they are valid).*[a]

[a]In this respect, we may also recall Stachel's own generalization in [374] (and in [373]) of the "*no metric, no anything*" motto above the "*dynamic individuation of fundamental entities*".

Of course, this ADG-generalization of the main aftermath or deeper significance of the EHA as uncovered by Stachel rests precisely on its direct and complete evasion, as described in the following steps:

1. For the ADG-theoretic formulation of the (vacuum) Einstein equations of GR no base differential (*ie*, $\mathcal{C}^\infty$-smooth) spacetime manifold is used whatsoever [262, 272]; therefore, the question (or problem in the original manifold based GR-formulation of the EHA) of implementing the PGC via $\mathrm{Diff}(M)$ becomes immediately a 'non-issue' ('non-problem').

2. Moreover, since in the ADG-theoretic formulation of (vacuum) Einstein gravity the sole dynamical variable is not a smooth metric $g_{\mu\nu}$ (on a locally Euclidean background manifold), but an algebraic $\mathbf{A}$-connection $\mathcal{D}$ (on a suitable vector sheaf $\mathcal{E}$, which is locally isomorphic to $\mathbf{A}^n$), the original Einstein-Grossmann formulation of the EHA in terms of $g_{\mu\nu}$ is rendered practically irrelevant or obsolete.

3. All in all, since for ADG, physically speaking, '*all (and there) is (only the) field*' $(\mathcal{D}, \mathcal{E})$, without the (*a priori*, even mathematically speaking!) existence of an external, base spacetime manifold, on the one hand the two points above, from a Leibnitzian-Kleinian (relational/algebraic) sense, point to the generalization of the PGC of GR involving $\mathrm{Diff}(M)$ to the *Principle of Field Synvariance* (PFS) involving solely $\mathcal{A}ut(\mathcal{E})$, and on the other, they automatically entail that the gravitational field (*viz.* the $\mathbf{A}$-connection $\mathcal{D}$) and the algebraically (:sheaf-theoretically) implemented dynamical law (differential equation—here, the vacuum Einstein equations) that it defines is manifestly prior the physical ('spacetime') geometry—the solution space of the field equations that the field itself is pregnant to. To emphasize it again, *first comes dynamics (algebra), then kinematics (geometry); dynamics (algebra) is the 'cause' of kinematics (spacetime geometry)*; hence, *no dynamics, no spacetime.*

Even 'discrete general covariance' questioned ADG-theoretically. In 7.5.2 above we argued that, from an ADG-theoretic perspective, it makes no difference whether the base 'space(time)' is assumed to be a continuum or a discretum, for *the theory is intrinsically background spacetimeless*. To further support this claim, we wish to comment in the light of ADG on certain remarks of Brightwell *et al.* in [65] about the search for a 'discrete' analogue of the PGC of GR in the finitistic-relational theoretical setting of the causet approach to QG, as well as about how the so-called problem of time—which is closely related to the PGC of GR via Diff(M)—appears from a discrete vantage. If anything, these comments will be of value to the exposition here, because we have already worked out a finitary, causal and quantal version of vacuum Einstein gravity by ADG-theoretic means [272]. Moreover, in the latter paper we maintained, as we did above, that our genuinely background independent theory evades in one-go hard, both conceptually and technically, problems in (the still manifold based, both canonical or covariant, approaches to) QG such as *the inner product problem* and *the problem of time*. Here, we argue on top that this evasion is total, in the sense that the said problems are bypassed not (only) because ADG is base spacetime manifold independent, *but because it is background spacetimeless period*, whether this background is continuous or reticular.

Thus, we first quote Brightwell *et al.* In [65], upon discussing how to implement the PGC of GR in the discrete realm of causets, the authors remark:[788]

[788]The quotation below is sectioned into three parts. Our comments following it are itemized according to this three-fold partition.

"...**(1)** After all, labels in this discrete setting are the analogs of coordinates in the continuum, and the first lesson of general relativity is precisely that such arbitrary identifiers must be regarded as physically meaningless: the elements of spacetime—or of the causet—have individuality only to the extent that they acquire it from the pattern of their relations to the other elements. It is therefore natural to introduce a principle of 'discrete general covariance' according to which 'the labels are physically meaningless'.
(2) But why have labels at all then? For causets, the reason is that we don't know otherwise how to formulate the idea of sequential growth, or the condition thereon of Bell causality, which plays a crucial role in deriving the dynamics. *Ideally perhaps, one would formulate the theory so that labels never entered, but so far no one knows how to do this—anymore than one knows how to formulate general relativity without introducing extra gauge degrees of freedom that then have to be cancelled against the diffeomorphism invariance.*[a]
(Q8.40) **(3)** Given the dynamics as we *can*[b] formulate it, *discrete general covariance plays a double role.*[c] On one hand it serves to limit the possible choices of the *transition probabilities*[d] in such a way that the labels drop out of certain 'net probabilities', a condition made precise in [...].[e] *This is meant to be the analog of requiring the gravitational action-integral S to be invariant under diffeomorphisms (whence, in virtue of the further assumption of locality, it must be the integral of a local scalar concomitant of the metric).* On the other hand, general covariance limits the *questions*[f] one can meaningfully ask about the causet (*cf.* Einstein's 'hole argument'). *It is this second limitation that is related to the 'problem of time', and it is only this aspect of discrete general covariance that I am addressing in the present talk.*[g] **(3)**"

[a]Our emphasis.
[b]Brightwell *et al.*'s emphasis.
[c]Our emphasis.
[d]Brightwell *et al.*'s emphasis.
[e]Reference omitted. The reader is referred to [65] for this.
[f]Their emphasis.
[g]Our emphasis.

(1) First, in our ADG-theoretic perspective on gravity, coordinates (or coordinate-labelling of spacetime points) play no role in the actual gravitational dynamics.[789] In our ADG-based scheme, the gravitational dynamics is effectuated (represented) via **A**-sheaf morphisms—in particular, via the curvature of the connection [272]. As a result, the supposed 'coordinate **A**-individuation of the elements of the background structure (space)'—in ADG, the base topological space X—plays no role in the dynamics, which is in turn expressed purely relationally (algebraically) solely in terms of the **A**-connection (field) $\mathcal{D}$ (on the 'carrier sheaf space' $\mathcal{E}$). Thus, as we saw earlier, the PGC in our theory, implemented via $\mathcal{A}ut\mathcal{E}$, concerns only the field $(\mathcal{E}, \mathcal{D})$ 'in-itself', while the generalized coordinate **A**-labelling (or what amounts to the same, the background space(time) dependence[790]) practically disappears (*ie*, it is not involved at all in the gravitational dynamics—Einstein's equations).

(2) Second, to the question "*why have labels at all then?*", our ADG-based reply is simply that *we assume* **A** *in order to get* $\mathcal{D}$;[791] *but once we have been supplied with a connection,* **A** *disappears from our calculations—the differential equations*[792] *for gravity that we can set up ADG-theoretically.* Plainly then, we have a formulation of GR as a *pure gauge theory* (*ie*, solely in terms of the gravitational connection $\mathcal{D}$) manifestly without introducing any "*extra gauge degrees of freedom*[793] *which have to be cancelled against the diffeomorphism invariance*", which anyway does not exist in our manifold-free theory.[794]

(3) And third, about the double role that GC plays in the gravitational dynamics "*as we can formulate it*" ADG-theoretically:

- (i) The gravitational action integral S—in our theory, a functional only of the connection $\mathcal{D}$ [272]—is manifestly $\mathcal{A}ut\mathcal{E}$-invariant, since $\mathcal{D}$ itself is a manifestly local entity (variable), and S is thus the integral of the Ricci scalar curvature $\otimes_{\mathbf{A}}$-tensor $\mathcal{R}(\mathcal{D})$ (equivalently, **A**-sheaf morphism) of the connection $\mathcal{D}$, *not of the metric.*[795]

- (ii) On the other hand, GC, as formulated purely algebraically by ADG-theoretic means, and in a background spacetime (whether 'continuous' or 'discontinuous') independent way, solely

[789] As noted many times before, in this sense ($\mathcal{C}^\infty$-)coordinates (or equivalently, the points of M), are physically meaningless.

[790] We tacitly abide by the assumption that space(time) is inherent in **A** (*eg*, the way that M is the spectrum of $\mathcal{C}^\infty(M)$ by Gel'fand duality).

[791] After all, *all DG boils down to* **A** (3.2), and $\mathcal{D}$ is viewed ADG-theoretically as nothing else but a generalized derivative operator ∂.

[792] Which are equations between **A**-morphisms ($\otimes_{\mathbf{A}}$-tensors).

[793] Again, the only dynamical gauge variable in our scheme is $\mathcal{D}$ [272].

[794] Again, no background M, no $\mathrm{Diff}(M)$ implementing the PGC in the classical theory [272].

[795] Recall again, in our manifold-free formulation of GR, the gravitational variable is the connection, not the metric [272].

in terms of the self-transmutations of the gravitational field $(\mathcal{E}, \mathcal{D})$ itself in $\mathcal{A}ut\mathcal{E}$, *does not limit at all the meaningful questions one can ask about this background* (whether the latter is a continuum/manifold or a discretum/causet, say), *simply because the latter does not physically exist at all in our theory—ie, it plays no role whatsoever in the gravitational dynamics as formulated ADG-theoretically (in fact, algebraico-categorically) as equations between sheaf morphisms.* In other words, and as a critique of the continuum (manifold) viewpoint, as it was emphasized in [272] and repeatedly noted in connection with the EHA above,

no M, no Diff(M), *no EHA, and no problem of time.*

ADG simply cuts the 'Gordian knot' of the background spacetime manifold, thus it evades automatically all the problems in classical (*eg*, PGC effectuated via Diff(M), EHA) and quantum gravity (problem of time, inner product problem) that go hand in hand with the assumption of a background smooth spacetime continuum.

Bridging "the outstanding gap between our creed and our deed" in GR. Stachel [364], based on the aforedescribed aftermath of the deeper meaning of the EHA, went even further and noted a discrepancy or discord in the way we mathematically lay out up-front the Lorentzian manifold kinematics of GR—interpreting it *a priori* as a spacetime of physical events—and the actual way that physicists go about and solve the Einstein equations. Characteristically, we quote him from [364]:

(Q8.41) "...The standard mathematical formulation of the general theory [of relativity] starts with a bare differentiable manifold, the points of which are interpreted as physical events, and then imposes various fields on this manifold. In particular, a second-rank symmetric covariant tensor field is interpreted as the space-time metrical structure interrelating these events, and as the gravitational potentials. While this way of introducing the space-time structure is quite adequate in the case of non-generally covariant theories, in which this structure is given *a priori*, *it does not do full justice to Einstein's relational concept of space-time, or to the actual practice of relativists in formulating and solving problems*[a]... The standard treatment does not correctly describe the actual practice of relativists in solving the field equations...I need hardly remind the audience that we do not start out with a fixed manifold topology, but solve the field equations on a generic patch.[b] Once such a solution has been found, the problem of finding its maximal *global*[c] extension is a major one. It involves a lot of hard work, which generally includes the formulation of criteria for an acceptable extension in order to arrive at a unique answer. Questions such as maximal analytic extension, geodesic completeness (timelike and/or null), and the adjunction of ideal boundary points at infinity (null and/or spatial) have generated some of the most interesting research on general relativity in recent years...[d] Yet we persist in formalizing what we do as if the choice of manifold topology were made at the outset. It appears to me unwise to allow this gap between official ideology and daily practice. Not only does it mislead students and researchers in other fields attempting to understand the nature of our work; *the gap may conceal important new insights into the nature of the theory. The correct mathematical formulation of a problem often suggests further and unexpected avenues of progress*[e]*...*"

[a]Our emphasis.

[b]That is to say, *initially, we determine the field locally.*

[c]Our emphasis.

[d]Indeed, in particular in research on the analysis of $\mathcal{C}^\infty$-smooth spacetime singularities [87], as we saw in section 2.

[e]Our emphasis.

Then, quite remarkably, right at the end of [364] Stachel goes on and drops a very telling for us hint regarding the use of the 'right' mathematics for dealing with the aforesaid problem of going from local-to-global solutions of the dynamical Einstein equations for gravity rather than freezing *ab* *initio* (*ie*, before actually solving the field equations) the (global) spacetime manifold topology:[796]

(Q8.42) "...However, the fibre bundle approach clearly does not solve the second problem discussed in the previous section.[a] The topology of the base manifold is given *a priori*, so that a different fibre bundle must be introduced *a posteriori* for each topologically distinct class of solutions. The process of finding the global topology cannot be formalized within the fibre bundle approach. *It appears that sheaf cohomology theory is the appropriate mathematical theory for dealing with the problem of going from local to global solutions*[b]... Let me close by appealing to the mathematically more sophisticated members of our group to work on closing this outstanding gap between our creed and our deed."

[a]That is, not fixing up-front the global topology of the manifold, and globalizing a local solution to the Einstein equations—*in toto*, (analytically) extending a local solution (a local region where the law holds) to a global one (the total spacetime manifold where the law is valid).

[b]Our emphasis.

Needless to say that *sheaf cohomology lies at the very heart of ADG* [259, 260, 271] and the issue of the law being valid both locally and globally is settled by the very nature of the sheaf-theoretic methods of ADG—in effect, *this is precisely what the vacuum Einstein equations that $\mathcal{D}$ satisfies on $\mathcal{E}$ means*[797]—but what's perhaps even more important, in ADG all this is achieved manifestly without an external, base spacetime manifold.

Individuation is distinction/discrimination; ultimately, difference: dynamical individuation of 'spacetime events'. From the foregoing ADG-theoretic generalization of the deeper significance of the EHA it follows that, in a subtle sense, it is the dynamical field $\mathcal{D}$ alone and 'in itself' that physically individuates spacetime events.[798] In other words, the gravitational connection field $\mathcal{D}$ is the sole physically individuating structure in the 'world'—a 'world' physically featureless (and meaningless) when devoid of $\mathcal{D}$—ultimately, when devoid of dynamics (that $\mathcal{D}$ defines). To emphasize this (dynamical) *field solipsism* once more:

(R8.11) *All* (*ie*, the 'world') *is* (the dynamical) *field.*

[796]The reader should refer to our earlier discussion in 6.8 about the issue of going sheaf-theoretically from-local-to-global (solutions) in GR *à la* ADG.

[797]Again, see 6.8.

[798]If there are any such things at all, for in ADG the field itself substitutes the spacetime (events) of GR, being the sole dynamical (thus physically significant!) entity in the theory.

Furthermore, in connection with our remarks earlier in section 3 (3.3) about the connection, rather than the traditional/original metric, ADG-theoretic formulation of GR and therefore the essentially gauge-theoretic character of gravity, we would like to stress here on the one hand the *mathematical* significance of the notion of connection and its fundamental role in setting up a theory of *differential* geometry,[799] and on the other, its role as a *physically* individuating structure.

With respect to the first aspect of $\mathcal{D}$, we bring forth from [414] Weyl's words about the importance of the notion of (affine) connection (and of the parallel transport of vectors that it effects) in Riemannian—more generally, in differential—geometry:[800]

(Q8.43) "...The later work of Levi-Civita, Hessenberg, and the author, shows quite plainly that the fundamental conception on which the development of Riemann's geometry must be based if it is to be in agreement with nature, is that of the infinitesimal parallel displacement of a vector...[Hence,] *a truly infinitesimal geometry must recognize only the principle of the transference of a length from one point to another infinitely near to the first.*[a]"

[a]Weyl's own emphasis.

While, for the important role that the connection plays as a physically individuating structure, we bring forth from (Q?.? Stachel's remarks about Eddington's following up Weyl's affine connection based (unified field) theory: [801]

[799]Let us stress it once again, ADG's original motivation was to formulate an entirely algberaic notion of connection, and its starting point so to speak was the first author's observation that the usual differential ∂ is another sort of connection [259, 260].

[800]As noted in section 3, Weyl was the founder of gauge theory in attempts to unite gravitodynamics (Einstein's equations) with electrodynamics (Maxwell's equations).

[801]Eddington who, as noted in section 3, attempted to further Weyl's (affine) connection based gauge theory.

(Q8.44) "...Noting the very special nature of Weyl's generalization, Eddington started by assuming that *there was no a priori connection between the metric and initially arbitrary symmetric connection.*[a] He observed that the curvature or Riemann tensor and its contraction, usually referred to today as the Ricci tensor, may be formed from the affine connection without any metric. This Ricci tensor, however, will in general not be symmetric, even though the connection is. *What has this to do with physics?*[b] Eddington noted that an affine connection enables us to compare tensors at neighboring points (in particular, to say when two neighboring vectors are parallel). He regarded the possibility of such a comparison between quantities at neighboring points in space-time as the minimum element necessary to do any physics: '*For if there were no comparability of relations, even the most closely adjacent, the continuum would be divested of even the rudiments of structure and nothing in nature would resemble anything else*'[c]..."[d]

[a]Our emphasis. Just like in ADG, where $\mathcal{D}$ is primary, but $g_{\mu\nu}$ of secondary importance.

[b]Again, our emphasis.

[c]Our emphasis.

[d]Quotation from [107].

In other words, *no individuation means no difference—no $\mathcal{D}$—ultimately, no dynamics.*

The vicious circle of the smooth coordinates: an important ADG-theoretic clarification.

8.5.6 "I have an answer; can somebody please tell me the question?"—Woody Allen's joke: a 'principle' for QG?

Elaborate a bit on the following telling remarks of Stachel from [365]:

(Q8.45) "...But the moral [of the hole story] is that if you are trying to formulate a quantum theory of general relativity (or of any generally covariant theory), you must always bear in mind that you do *not*[a] have a manifold of events to start with, before finding a solution to the quantized equations for the metric field. This raises the following question: suppose one *had*[b] a consistent quantum formalism for general relativity (we should only be so lucky!), and had found a solution to the quantized field equations; what could one do with it? If you have a wave function of the universe, for example, how do you interpret it physically? Usually, one interprets the quantum formalism in a background space-time, so that such a question may be answered as follows: You put an apparatus *here* to create a particle *now*, and then you put an apparatus *there* to detect it *then*; one can then use quantum probability amplitudes (such as wave functions) to compute transition probabilities between two such events. But in general relativity you have no a priori *here* and *now* or *there* and *then.*[c] What 'here and now' and 'there and then' signify are among other things that *one must solve the equations in order to determine.*[d] I am reminded of Gertrude Stein's last words. She roused herself from a coma to ask worriedly: '*What is the answer?*' Then she smiled and asked: '*What is the question?*' and lapsed into her final coma. *In a sense, in matters of quantum gravity one must know the answer before one knows the question.*[e]"

[a]Stachel's emphasis.
[b]Stachel's emphasis.
[c]All Stachel's emphasis.
[d]Our emphasis.
[e]Our emphasis.

Ergo, Woody Allen's joke at the heading of this sub-subsection. Physical propositions involving geometric spacetime locutions (*eg*, this or that event: 'here and now' or 'there and then') are essentially about the answer (solution/geometry/kinematics), not about the question (dynamical equations/algebra/dynamics). Gertrude Stein's, or more strikingly, Woody Allen's apparently oxymoronic joke is very fitting here, because the way we attempt to dress a generally covariant theory (such as GR) with the standard formalism and interpretation of QM is paradoxical—literally speaking, *we are begging the question*. We are posing kinematical questions to an essentially (and solely) dynamical theory!

8.5.7 Field axiomatics and field realism: field over spacetime (point events) and an abstract conception of 'causal nexus'

This is one aftermath of the EHA and especially of its ADG-theoretic generalization, which is represented by the statement '*dynamics over kinematics*'. A point manifold M does not have the 'right' to qualify physically as spacetime (and its points as events) unless the dynamics (Einstein equations) is prescribed and solved on it.

In ADG, and in particular in the ADG-theoretic perspective on gravity, the sole fundamental, irreducible, *ur* as it were, notion is that of a *field*, which is represented by the pair $\mathcal{E}, \mathcal{D})$. The field determines—in fact, *defines*—the dynamics (differential equation). There is no *a priori* spacetime in ADG. If there is any spacetime (geometry) at all, then that is inherent in the field—it is the realm where the field holds, where the field equations are valid (satisfied).

8.5.8 Field over matter: generalizing Mach's principle in the light of ADG

In the first section we referred to ADG as being a *Leibnizian-Machian* theory (R2.??). Throughout the paper we 'justified' the first epithet, '*Leibnizian*', on various physico-mathematical grounds—primarily, in that ADG is a purely relational, algebraic Calculus (*ie*, theory of DG), in the development and expression of which a background point-manifold (*ie*, a locally Euclidean space)—or what amounts to the same, the smooth coordinates thereof—plays no role whatsoever. Rather, ADG is concerned solely with the 'geometrical objects in-themselves' and with their algebraic (inter)relations, without an external, 'intervening' space(time) being needed at all for the mediation and (differential) geometrical realization (picturization) of these relations (Q?.?). But how about the second epithet, '*Machian*'? As seen in (Q?.?), this epithet bears almost the same meaning as the one we gave in the light of ADG to the term 'Leibnizian'; albeit, unlike Penrose's remarks pertaining to the theory of spin-networks, our ADG-based interpretation of that term is in the context of DG proper.

Here, however, we would like to give a slightly 'slanted' interpretation to the epithet 'Machian'—one that is more in line with the 'traditional' meaning of Mach's principle, as it was originally conceived and used by Einstein in the *aufbau* of GR. Furthermore, under the general philosophical prism of ADG, we are going to provide a generalization of Mach's principle to the effect that we will contend that '*field is prior to matter*', thus slightly distorting both the *geometry-matter symmetry* of GR (Einstein's equations), as well as the *field-particle duality* of quantum theory. The ultimate apofthegma will be that Mach's principle, viewed from an ADG-theoretic vantage, is completely analogous to the ADG-theoretic generalization of the aftermath of EHA, which, as we argued above, can be concisely stated as '*dynamics (algebra, the field) over kinematics (geometry, spacetime)*', with the only difference being that, here, the statement concerns matter (particles):

'*field over matter*'.

Usually, Mach's principle in GR is thought of as a 'heuristic explanation' of the origin of inertia: loosely speaking, the inertial properties of a material body are determined 'immediately, at a distance' by all the other masses in the Universe—by the 'global', total distribution of matter in the Universe, so to say.[802] We read, for example, from [114]:

(Q8.46) "...[Since Galileo and Newton] no one thought of giving up the concept of space, for it appeared indispensable in the eminently satisfactory whole system of natural science. *Mach*, in the nineteenth century, *was the only one who thought seriously of an elimination of the concept of space, in that he sought to replace it by the notion of the totality of the instantaneous distances between all material bodies. He made this attempt in order to arrive at a satisfactory understanding of inertia*[a]..."

[a]Our emphasis.

At the same time on the other hand, Mach's principle may be perceived as a 'relational' principle originally adopted by Einstein in order to 'balance' or 'causalize' the dynamical equations for gravity: matter is the source ('cause') of gravity, and the spacetime-geometrical left hand side of Einstein's equations is determined ('balanced') by the matter-energy distribution on the right hand side. As Wheeler has time and again put this 'action-reaction'-type of 'symmetry' between gravity and matter in GR [278]:

(Q8.47) "Space acts on matter, telling it how to move. In turn, matter reacts back on space, telling it how to curve..."

In fact, foreshadowed by Einstein's interpretation of Mach's principle as *an attempt to eliminate space*, an 'extreme' expression of Mach's principle may be taken as holding that *it is the (relations between) material bodies that determine the (geometrical) structure of spacetime (gravity); spacetime (structure) devoid of matter is a purely (mathematical) artifact*. Characteristically, we quote Torretti from [**?**] regarding the crucial role that Mach's principle (MP) played in building GR:

[802]See Pauli's quotation (Q?.?) in 7.5.9 next.

(Q8.48) "...The philosophical component of GR lies mainly with Mach's Principle...The insight which Mach's Principle is supposed to express certainly guided Einstein during the long strenuous search that led to the formulation of General Relativity. In traditional philosophical language that insight can be stated thus: *Absolute space and absolute time—as well as absolute spacetime, without matter, are physically inviable mathematical contraptions*[a]... On the other hand, [in Newtonian gravity][b] *the presence of matter makes no difference in the structure of spacetime. According to Einstein, such lopsidedness in the mutual relationship between two physical entities is utterly at variance with everything we know of nature.*[c] Such was the insight that Einstein sought to embody in the Mach Principle of 1918. This can be rephrased as follows: *The linear connection, or rather, the spacetime metric on which it depends in a relativistic theory, is fully determined by the distribution of matter...However, unless one understands the 'distribution of matter' in a Pickwickian sense, the field equations do not agree with the above version of Mach's Principle, since they also have solutions if the stress-energy tensor is identically zero.*[d] Later in life Einstein rejected the 1918 Mach Principle. In 1954 he wrote to Felix Pirani: '*One shouldn't talk any longer of Mach's principle, in my opinion. It arose at a time when one thought that ponderable bodies were the only physical reality and that in a theory all elements that are not fully determined by them should be conscientiously avoided. I am quite aware of the fact that for a long time, I, too, was influenced by this fixed idea.*'[e]..."

[a]Our emphasis.
[b]Our addition.
[c]Our emphasis.
[d]Our emphasis again.
[e]Once again, our emphasis.

Indeed, one can contend that Einstein, although originally influenced by Mach's principle in raising up GR, he eventually abandoned it in view of the existence of Einstein equations in *vacuo*—in other words, *the existence of a gravitational field in the absence of matter.*[803]

Recall Pauli [294]:

[803]Again, see Pauli's quotation (Q?.?) in 8.?.? next.

(Q8.49) "...In the subsequent development of the general theory of relativity a problem cropped up which could not be finally settled. Ernst Mach had suggested that inertia might be traced back entirely to the action of distant masses. If this principle of Mach's were correct, Einstein's G-field would have to vanish if all matter were removed. In setting up his theory Einstein was probably guided by this principle and regarded it as correct. But it has not been possible to deduce it from the equations of the theory. *It seems to be inherent in the nature of the field concept that while the field is influenced by the distribution of mass, it nevertheless continues to exist as an independent reality even when all masses are removed. What the ultimate solution will be we do not know*[a]..."

[a]Our emphasis.

8.5.9 Field solipsism versus structural realism vis-à-vis QG

According to Stachel [373], there are three broad interpretations of *structural realism*:

1. The Pythagorean interpretation according to which *only structures exist*, while our perceptions of the World are illusory, together with a concomitant straight-out renunciation of the reality (*ie*, the existence) of matter.

2. The Platonic scenario according to which there is an unstructured reality (void) out there, including non-dynamical (inert) matter, on which *structures are imposed from without*—as it were, to give the World shape, form and dynamism.

3. The Aristotelian theoresis, that Stachel espouses, which constitutes the philosophical background against which his ideas on QG are presented in [373],[804] according to which—and we quote Stachel:

(Q8.50) "...The world is composed of entities of various kinds, and *it is inherent in their nature to be structured in various ways. It is impossible to separate these entities from their essential structural properties and relations. Matter is inherently propertied, structured and dynamic (Aristotle)*[a]..."

[a]Our emphasis.

[804]See also below.

We, on the other hand, supported by ADG, have taken in this paper the philosophical stance of *field solipsism* or *pure field realism*, which can be expressed in a way that combines (and at the same time renounces) traits of all the three positions above,[805] especially once the primitive notion of *field* (in its ADG-theoretic guise) substitutes that of *structure*, as follows:

[805]Although, on the whole, like Stachel, we tend to abide more by the Aristotelian interpretation of structural realism rather than by its Pythagorean or Platonic antagonists.

Fields are real—nothing else really exists in the (physical) World.[a] Nothing physically real (significant) would remain if all fields were removed from the World.[b] The World is composed solely and entirely of fields[c] which are inherently dynamic and structured, with their essential dynamism and structural relations being inextricably entwined (*ie*, inseparable from each other).[d] Moreover, as noted before, spacetime (geometry) and matter are inherent in the (dynamical) fields—to use an Einsteinian expression (Q2.?): *both spacetime (geometry) and matter are structural qualities of the dynamical fields, not having an independent existence of their own.*

At the same time, we are inclined to depart from Stachel's main argument and position in [373] that "*the fundamental physical entities,*"—in our theoresis, the ADG-fields—"*while they have 'quiddity' (a basic nature), they lack 'haeccity' (inherent individuality)*". While we agree with Stachel that the ADG-fields cannot (and should not) be thought of 'individually'—*ie*, apart from the dynamical relations and processes that they engage into, as it were, apart from the dynamical laws that they define (possibly also engaging into dynamical interactions with other fields),[e] we do not go as far as to claim that "*all properties that do not inhere in their nature are relational*". For this is exactly the point, namely, that the ADG-
(R8.12) fields are 'autochthonous', 'auto-dynamical', 'dynamically holistic' ('unitary' or 'integral') algebraic entities, with all dynamical relations (physical laws) inherent in them (since the fields define them in the first place). *There is no 'predicative relations' (properties) other than the dynamical ones defined by the fields.* In other words, there is no structure (or structural realism) apart from dynamical one.[f]

[a] This is in accordance with the Pythagorean position 1 above [373].

[b] This is in accordance with the Platonic position 2 above [373]. *Vis-à-vis* the notion of 'spacetime' in particular, no 'spacetime' whatsoever remains if all fields are removed ('switched off') from the world. For 'spacetime', if such a 'thing' ('concept') exists even theoretically speaking (not actually, *ie*, physically!), is inherent in the fields.

[c] To use an Aristotelian term, *the (physical) World is a 'field-plenum'* [5].

[d] In our scheme, dynamics (*ie*, the field) 'derives' from structural-algebraic relations. This is in accordance with the Aristotelian position 3 above [373].

[e] Here, the word 'other' surely assumes already some individuation!

[f] This is part and parcel of our fundamental *motto* that dynamics comes before kinematics. Structure, understood as kinematical structure (*eg*, ambient, *a priori* posited and fixed, spacetime geometry), is a consequence (result) of the (field) dynamics. In this respect, our philosophical stance again 'structural (field) realism' is even more Aristotelian (and, from the point of view of Differential Calculus, Leibnizian) than Stachel's, as we shall also elaborate in further extent in the last section (8.5.10).

Of course, *our perceptions* of the fields' inherent or innate structure and dynamism[806] are effectuated (triggered) by our measurements/observations[807] of them—that is to say, their intrinsic structure is brought out by our measurements/coordinatizations of them (conveniently encoded in the **A** that we employ); it is expressed or represented (by us) as '*geometry*'.[808] Coordinatizations (*ie*, the introduction of generalized/abstract coordinates **A** by us 'measurers' or 'geometers' external to those real fields) may be thought of as Platonic (in Stachel's sense described above) attempts at individuating, dissecting the fields and imparting (or forcing) structure (*eg*, 'spacetime geometry') on them from without.

In particular, the inherent *dynamical structure* of the fields is expressed as *differential geometry*, with the fields (*viz.* connections) being represented ADG-theoretically, via (our measurements in) **A**, as pairs $(\mathcal{E}, \mathcal{D})$. The introduction of **A** (by us)[809] may be regarded as an attempt to 'individuate' the holistic fields—as it were, they are attempts to localize (locally measure in our 'local laboratories'/local gauges) and extract from their 'global', all-pervading nature, their local (particle/'position') aspects (properties). However, the structure (geometry) encoded in $\mathcal{E}$[810] is inseparable—in a quantum sense, *coherent*—from the dynamical change represented by $\mathcal{D}$, and this coherence (or 'unitarity') may be interpreted quantum mechanically as complementarity (and operationally, as self-indeterminacy).[811] All in all, for us, *differential* ($\mathcal{D}$) *geometry* ($\mathcal{E}$) is synonymous to *dynamical* ($\mathcal{D}$) *structure* ($\mathcal{E}$)—a structure analyzed by means of the **A** that *we employ* to 'coordinatize' or 'measure' the dynamical fields, to quantify ('arithmetize', in a Cartesian sense) our perceptions of them so to speak, and thus extract their local particle traits ('properties').[812] Only in such a 'slanted' quantum-theoretic sense, *our field structural realism is 'observer' (and observation) dependent*.[813] So, on the whole, there are perceptions,[814] and the dynamical

[806]What *in toto* one might refer to as '*dynamical structure*' and, *in extenso*, the philosophical position underlying it as '*dynamical structural realism*' (or even as '*real structural dynamism*').

[807]What one might call '*quantified perceptions*'.

[808]'*Geometry*' here being broadly understood ('defined') as '*the structural analysis of some space*' [272, 374].

[809]And, as a result, the whole 'differential geometric enterprize' (:theory of differential geometry) that *we* engage into (:develop) in order to study and describe those physically real and external to us fields. For after all, *differential geometry is our activity, not Nature's. There is no differential geometry in Nature.*

[810]Recall, $\mathcal{E}$ is (locally) of the form $\mathbf{A}^n$ (n a finite positive integer).

[811]See subsection 6.2 above about the innate (global) field-(local) particle duality of the ADG-fields $(\mathcal{E}, \mathcal{D})$. The ADG-fields are coherent (inseparable), self-quantum entities, and thus '*individual*' (*ie*, in a literal sense, they cannot be divided into their field $\mathcal{D}$ and particle $\mathcal{E}$ aspects).

[812]Recall again, the local sections of $\mathcal{E}$ represent, from a geometric (pre)quantization standpoint, the local quantum-particle states of the field.

[813]And exactly because of these 'extraneous interventions' by us 'observers' or 'measurers of the fields' (geometers!), such a realism is not Platonic (*ie*, not strictly detached), but it is in accord with the observer dependent physical reality that quantum theory advocates.

[814]In contradistinction to the Pythagorean position 1 above [373].

structure inherent in the fields is perceived/expressed ('quantified'/'Cartesianly arithmetized' or 'geometrized', and brought out) from without—in point of fact, from us 'observers'.[815]

What GR structure(s) to quantize? Then, against the philosophical background of the Aristotelian interpretation of structural realism, Stachel goes on and addresses the question of what structures of GR one 'should' quantize as an attempt to arrive at a quantum theory of gravity. In what follows we quote the whole of section 6 in [373] where this question is posed, as we wish to respond to it from our ADG-theoretic '*field solipsistic*' vantage.[816]

[815]Partly, as the Platonic perspective 2 on structural realism maintains [373], but unlike it, these interventions are not as invasive, as the conventional quantum theory has it, so as to maintain a strict 'observer dependent reality'. For one thing, the dynamics that the fields define are **A**-covariant or **A**-functorial (:'synvariant'), so that the dynamical laws see through our attempts of coordinatization and individuation of the fields. *In summa*, there are dynamically autochthonous fields out there, and we simply (differential) geometrize them in order to represent them mathematically. Mathematics is ours, physics Nature's.

[816]In the following excerpt we have Greek letter-ordered the paragraphs of the original text so as to facilitate our analysis of them in the sequel.

> α. There are a number of space-time structures that occur in the general theory of relativity. The *chrono-geometry* is represented mathematically by a *pseudo-metric tensor field* on a four-dimensional manifold. The *inertio-gravitational field* is represented by a *symmetric affine connection* on this manifold. Then there are *compatibility conditions* between the previous two structures (the covariant derivative of the metric with respect to the connection must vanish).
> β. It is possible to start from the metric field and derive from it the unique connection (the Christoffel symbols) that automatically satisfies the compatibility conditions. This is what was first done historically, and still done in most textbooks.
> γ. It is also possible to treat *metric and connection* as *initially independent*, and then allow the compatibility conditions to emerge, along with the field equations, from a Palatini-type variational principle.
> δ. Each of these methods may be combined with a *tetrad formalism* for the metric, combined with various mathematical representations of the connection, e.g., connection forms, tetrad components of the connection.
> ϵ. But it is possible to abstract from the (four-) volume-defining property of the metric, resulting in a *conformal structure* on the manifold. (Q8.51) This is all that is needed to represent mathematically the *causal structure* of space-time. It is also possible to abstract from the preferred parametrization of geodesics associated with the connection, resulting in a *projective structure* on the manifold. This represents mathematically the class of *preferred (time-like) paths* in space-time. Compatibility conditions between the causal and projective structures can be defined, which guarantee the existence of a corresponding compatible metric and connection (Ehlers, Pirani and Schild).[a]
> ζ. *Just as it was so important in the historical development of quantum mechanics to choose an appropriate formulation of classical mechanics, to which to apply some quantization technique, it may well be the case that one or the other of these formulations of general relativity will be more helpful in solving the quantum gravity puzzle in one or more of the various ways in which it has been–or will be–posed.*[b]
>
> [a] Here Stachel most probably refers to the celebrated paper [112] and subsequent follow-up work [113].
> [b] Our emphasis.

In what follows we itemize our reply, to the aforesaid question '*What Structures* (from GR) *to Quantize?*' that Stachel puts as heading to the last section (6) of [373], according to our letter-ordering of its paragraphs above. Our reply below will recall various important issues that have been discussed in many places throughout the present paper-book, and may be regarded as a *résumé* of key conceptual issues in classical and quantum gravity that have been addressed, highlighted and resolved under the prism of ADG herein:

- α First of all, *in the ADG-theoretic formulation of GR, there is no (background) spacetime structure at all—ie*, a base realm on which g is soldered to measure for instance spacetime intervals, or to characterize smooth paths (supposedly followed by material particles) as being timelike, null, or spacelike (chrono-geometry). That is, in ADG's perspective on gravity there is, physically speaking,[817] no chrono-geometry whatsoever, represented by a pseudometric tensor field on a four dimensional spacetime manifold, since, to be blatant, *there is no spacetime manifold 'out there' to begin with.* To emphasize it again, *GR à la ADG is fundamentally spacetimeless, while the traditional geometrical picturization and associated 'geometrical reasoning'*[818] *by means (or with the help) of a base spacetime manifold is of little (conceptual) import in our purely algebraic (relational) theory.* On the other hand, *in ADG the inertio-gravitational field—the only physical (dynamical) variable in the theory—is represented by the* $\mathbf{A}$*-connection* $\mathcal{D}$.[819] Then, only when $\mathbf{A}$ is introduced (by us—the external 'episystem' of 'observers' or 'measurers') and concomitantly the locally $\mathbf{A}^n$ vector sheaf $\mathcal{E}$ is adjoined to the purely algebraic gravitational field $\mathcal{D}$ as its *carrier* (or representation/geometrization) *space*,[820] can the $\mathbf{A}$-metric ρ be employed on $\mathcal{E}$ and (voluntarily) be made compatible with $\mathcal{D}$ (if we wish) [272]. Otherwise, it (*ie*, the metric) has no *a priori* status (interpretation) as a measurement device on (a) spacetime (manifold), since *a fortiori* there is no spacetime manifold to start with.[821] To stress it once more, the gravitational field $\mathcal{D}$ alone exists 'out there' independently of us (field solipsism), while we capture it and realize/represent it (differential) geometrically by our generalized arithmetics ('measurements') in $\mathbf{A}$, bringing along the (our) $\mathbf{A}$-metric ρ, which may be made compatible with it. $\mathcal{D}$ *is primary and fundamental*, ρ *secondary and contingent*.[822]

[817]And by '*physically speaking*', we are talking about a structure that has absolutely no *dynamical* role in the theory—*ie*, it is not a dynamical variable ('observable') in the theory.

[818]In terms of points, lines, surfaces *etc*, as usual.

[819]Moreover, exactly because $\mathcal{D}$ (unlike the $\mathbf{A}$-metric ρ, or its curvature $R(\mathcal{D})$) is not a tensor (in our language, an $\mathbf{A}$-morphism), 'resists' decomposition into its inertial (flat) ∂ and gravitational $\mathcal{A}$ parts, unless of course we—the external 'episystem' in the whole scenario—evoke a local (coordinate) gauge and 'unnaturally' split it into $\partial + \mathcal{A}$.

[820]Recall, in ADG, by the term *field* we refer to the pair $(\mathcal{E}, \mathcal{D})$.

[821]To say it once more, ADG-gravity is a purely (third) gauge formulation of gravity, based solely on the notion of connection—"*the gravito-inertial field*" itself according to Stachel—without *a priori* reference to a smooth background spacetime manifold or to a smooth metric based on it. Thus, also no "*chrono-geometric*" structure and interpretation is *ab initio* evoked in ADG-gravity.

[822]Depending on *our* choice of $\mathbf{A}$.

- β That the metric determines a unique connection (and vice versa) is in our view a historical (mathematical) accident of the manifold based Riemannian geometry—however, it is an identification or equivalence (of the metric with the connection) that masks the physical significance of the latter ($\mathcal{D}$) and the insignificance of the former (g). That is to say, that the gravitational field came to be identified with g is also in our view a theoretical accident, with the mathematics (*ie*, the CDG-based Riemannian geometry) to blame for this 'misrepresentation' and 'misinterpretation'.[823] Rather, the metric-represented gravitational field conceals the fundamental gauge-theoretic character of gravity.[824] The recent tendencies to pronounce the importance of the notion of connection in GR, thus view gravity as another gauge theory [342, 7, 9, 10, 11] are certainly important, albeit they are still bound by the "*golden shackles of the manifold*" and of the CDG based on it [203], while at the same time the (smooth) metric is still present in the theory in the guise of the (smooth) comoving tetrad (*vierbein*) field.[825]

- γ In ADG it is not an issue whether connection and metric are independent or not: $\mathcal{D}$ *is primary, while g secondary*, not the other way round. However, it is interesting indeed that the metric compatibility condition for the connection[826] derives, along with the field equations, from a Palatini-type of variational principle [272].

- δ The tetrad (*vierbein*) formalism for the metric, together with the $sl(2, \mathbb{C})$-valued 1-form representation of the gauge potential part $\mathcal{A}$ of the self-dual (spin-Lorentzian) connection, is what is essentially involved in the Ashtekar 'new variables' approach to classical and quantum gravity [272].

- ϵ Here Stachel notes another metric-*versus*-connection 'duality'. Since the spacetime metric field, at least locally, delimits the causal structure of the world,[827] one may base one's the-

[823]In our theoresis, we are talking about *the connection compatibility of the metric*, not about *the metric compatibility of the connection*, as usually (*ie*, in the smooth pseudo-Riemannian geometry based original formulation of GR by Einstein, as well as what is "*still done in most textbooks*" nowadays.

[824]Recall Feynman's words in (Q?.?).

[825]This is to the effect, as emphasized earlier, that while Ashtekar's formalism and its associated quantization scenario (LQG and canonical QGR, as well as its offshoot QRG) treat non-perturbative quantum gravity as a quantum gauge theory in a manifestly fixed background *metric* (geometry) independent way, they are still not able to formulate QG in a background *differential manifold* independent way. For after all, the connections, tetrad fields *etc* used there are *smooth*, and the differential geometric methods are those of the manifold based CDG.

[826]That is, a torsionless connection.

[827]That is to say, locally at every spacetime point, according to the EP, $g_{\mu\nu}$ reduces to the Minkowskian $\eta_{\mu\nu}$, which delimits the local causal structure at each event, *ie*, which events in the immediate neighborhood of each event causally influence (causal past) and are influenced by (causal future) it. In turn, GR may be thought of as the dynamical theory of the field of local causality [270].

oresis of gravity on causality alone (which itself determines the conformal structure of the world).[828] On the other hand, one could work directly with the causal (null and timelike) geodesics determined by the connection (resulting in what Stachel calls above the 'projective' structure of the world), and then try to bring together the conformal and the projective structures. In glaring contradistinction, such an endeavor would be meaningless in the purely algebraic ADG-gravity, since the theory is fundamentally base spacetimeless, while such 'classical' (Aristotelian and Kantian) categories of (physical) thought, such as space(-time), events and causality, are all replaced by the notion of field (*viz.* algebraic connection), which is a fundamental, "*primary, not further reducible*" [125] notion in the theory.

- ζ In this respect we fundamentally and expressly diverge in opinion from Stachel: as noted numerously in the present paper-book, we do not expect QG to arise as the result of quantizing the (structures of the) classical theory (GR), so that the question of choosing the most appropriate (for quantization) formulation of GR in order to arrive at a cogent QG is from the ADG-viewpoint 'begging the question'. This basic difference in 'working philosophy' conceals a host of novel ideas in ADG-gravity not encountered in any QG approach so far; and to name a few: background smooth manifold independence,[829] QG as a dynamically autonomous third-quantum gauge field theory of the third kind, absence of a fundamental space-time length in Nature, no (perturbative non-)renormalizability, no (black hole) information loss, no spacetime quantization, and no Correspondence Principle.[830]

Field solipsism over structural realism. Field-solipsism: `All is field, and the field is (inherently) quantum` $\Longrightarrow$ `All is quantum` (Finkelstein [148]).

- In Wittgenstein's Tractarian sense [421], '*solipsism is pure realism*', as follows:[831]

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[828] Since conversely, as it is well known, causality (mathematically represented by a partial order relation) determines nine out of ten components of the Lorentzian metric of GR, the tenth being the spacetime volume element (*ie*, the conformal determinant of $g_{\mu\nu}$). Indeed, some 'bottom-up', 'discrete' approaches to QG, such as Sorkin *et al.*'s causet theory that Stachel also mentions in [373], capitalize on causality as a fundamental notion.

[829] In fact, background spacetime independence altogether: whether this background is 'continuous' or 'discrete'.

[830] For example, no 'emergence of classicality', or viewing GR as an 'effective theory'.

[831] In the quotation below, all emphasis is Wittgenstein's unless noted otherwise.

(Q8.52) "...*The limits of my language* means the limits of my world...Logic fills the world: the limits of the world are also its limits...What solipsism *means* is quite correct, only it cannot be *said*, but it shows itself...The world and life are one. I am my world (the microcosm). The thinking, presenting subject; there is no such thing...The subject does not belong to the world, but it is a limit of the world...From nothing in *the field of sight* can it be concluded that it is seen from an eye...This is connected with the fact that no part of our experience is also a priori. Everything we see could also be otherwise. Everything we can describe at all could also be otherwise. There is no order of things a priori. *Here we see that solipsism strictly carried out coincides with pure realism.*[a] The I in solipsism shrinks to an extensionless point and there remains the reality co-ordinated with it..."

[a]Our emphasis.

8.5.10 The ADG-fields as Aristotelian-Leibnizian 'entelechies': 'autodynamical monads' (with windows)

Recall Aristotle's conception of potentia and 'autodynamics' from [306] (compare it with our ADG-conception of 'autonomous', 'unitary fields'):

(Q8.53) "...[Aristotle says that] the [Platonic] Form or essence is *in*[a] the thing, not, as Plato said, prior and external to it. *For Aristotle, all movement or change means the realization (or 'actualization') of some of the potentialities inherent in the essence of a thing*[b]...Accordingly, the essence, which embraces all the potentialities of a thing,, is something like its internal source of change or motion...'*Nature*', he writes in the Metaphysics,[c] *belongs also to the same class as potentiality; for it is a principle of movement inherent in the thing itself*[d]..."

[a]Popper's emphasis.
[b]Our emphasis.
[c]See [6].
[d]Again, our emphasis.

8.6 Bird's-Eye-View: Mathematical Physics or Physical Mathematics?, and What 'It' Should Be (ADG-Theoretic Musings and 'Post-Anticipations' of Paul Dirac and Ludwig Faddeev)

We argued above, based on the main didactics of ADG-gravity in the light of some 'directives' given by Peter Bergmann (inspired by Einstein), in favor of a 'physical geometry' than of a 'geometrical physics'. In a nutshell, we posited that 'geometry'—in particular *differential* geometry—and 'space(time)' is of interest to the physicist, not *a priori*—as it were, as a pure mathematical discipline offering the convenience of 'geometrical picturization' (:representation) of physical situations and processes, but insofar as it is the 'result' or the 'outcome' of physical laws (:dynamical field laws)—processes which, in turn, *a priori* know no 'geometry' (:'spacetime'), but quite on the contrary, they 'give birth to it'.[832] Accordingly, we argued for an algebraization (:'relationization'), rather of a geometrization, of physics, which finds its 'natural habitat' (at least regarding the application of differential geometric ideas to theoretical physics) in ADG.

Here we would like to extend this 'philosophy' in favor of a 'physical mathematics' rather than the most commonly used term 'mathematical physics'. This extension finds us in accord with the thesis taken recently by Ludwig Faddeev in [135] *vis-à-vis* what mathematical physics should (or anyway, ought to) be. Briefly, Faddeev maintains that we should finally break away from the classical developmental route followed so far by theoretical physics, which can be resumed by the cycle: 'experiments⟶predictions⟶mathematical elaborations/formulations⟶further experiments *etc.*', and implore all our mathematical resources to plough deeper into 'physical reality', leaving experiments (and experimentalists!) to 'catch up' with the new mathematics (and with theoreticians!), not the other way round.

Especially, we would like to borrow from [135] some prophetic remarks in this line of thought by Dirac [100]:[833]

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[832] That is, the laws (:the fields defining them) are the causes of 'space(time)', and conversely, 'physical geometry' the effect of these dynamical field laws. On this hinges the aforenoted reversal of the traditional priority of kinematics over dynamics: indeed, we posited in the light of ADG that, in a deep sense, *dynamics comes before kinematics.*

[833] Once again, we split the quote below into two parts and we comment on them separately after.

(Q8.54)

"...The steady progress of physics requires for its theoretical foundation a mathematics that gets continually more advanced. This is only natural and to be expected. What, however, was not expected by the scientific workers of the last century was the particular form that the line of advancement of the mathematics would take, namely, it was expected that the mathematics would get more complicated, but would rest on a permanent basis of axioms and definitions, *while actually the modern physical developments have required a mathematics that continually shifts its foundation and gets more abstract...It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalization of the axioms at the base of mathematics rather than with logical development of any one mathematical scheme on a fixed foundation.*[a] **(I)**

There are at present fundamental problems in theoretical physics awaiting solution [...][b]the solution of which problems will presumably require a more drastic revision of our fundamental concepts than any that have gone before. Quite likely these changes will be so great that it will be beyond the power of human intelligence to get the necessary new ideas by direct attempt to formulate the experimental data in mathematical terms. The theoretical worker in the future will therefore have to proceed in a more indirect way. *The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalise the mathematical formalism that forms the existing basis of theoretical physics, and* after[c] *each success in this direction, to try to interpret the new mathematical features in terms of physical entities*[d]... **(II)**"

[a]Our emphasis.

[b]Dirac here mentions a couple of outstanding mathematical physics problems of his times. We have omitted them.

[c]Dirac's own emphasis.

[d]Our emphasis throughout.

- **(I)** The words from this paragraph to be highlighted with ADG in mind are: '*a mathematics that gets more abstract*' and '*advance in physics is to be associated with a continual process of abstraction* [leading to a] *modification and generalization of the axioms at the base of math-*

ematics'. Indeed, the axiomatic ADG essentially involves an abstraction of the fundamental notions of modern differential geometry (*eg*, connection), resulting in a modification and generalization of the latter's basic axioms [259, 260, 269]. And it is precisely this abstract and generalized character of ADG that makes us hope that its application could advance significantly (theoretical) physics (and in particular, classical and quantum gravity research).

- **(II)** In this paragraph, apart from breaking from the traditional circle mentioned above (*ie*, Dirac's anticipation that '*new ideas* [won't come] *by direct attempts to formulate the experimental data in mathematical terms*'), what should be highlighted is on the one hand Dirac's prompting us '*to generalise the mathematical formalism that forms the existing basis of theoretical physics*', and on the other, '*to try to interpret the new mathematical features in terms of physical entities*'. Again, ADG comes to fulfill Dirac's vision, since *the* (or at least the bigger part of the) mathematics that lies at the heart of current theoretical physics—namely, (the formalism of) *differential geometry* (*ie*, CDG on smooth manifolds)—is abstracted and generalized, while after this generalization has been achieved, the physical application and interpretation (of ADG's novel concepts and features) has been carried out (especially in the theoretical physics field of classical and quantum gravity research). We believe that this is '*a powerful method of advance*' indeed.

But perhaps it is more important to conclude this 'paper-book' by stressing once again that ADG is not so much a *new* theory of DG—the main '*mathematical formalism that forms the existing basis of theoretical physics*'—but a theoretical framework that abstracts, generalizes, revises and recasts the existing CDG by isolating and capitalizing on its fundamental, *essentially algebraic* ('relational' in a Leibnizian sense) features, which are not dependent at all on a background smooth geometrical 'space(time)' (:manifold). In a way, from the novel viewpoint of ADG, we see 'old' and 'stale' problems (*eg*, the $\mathcal{C}^\infty$-singularities of the manifold and CDG based GR) with 'new' and 'fresh' eyes. Schopenhauer's words from [335] immediately spring to mind:

(Q8.55) "*...Thus, the task is not so much to see what no one has yet seen, but to think what nobody yet has thought about that which everybody sees*[a]*...*"

[a] All emphasis is ours.

8.7 Section's Résumé

As we did with the previous sections, we may wrap-up this final section by itemizing and highlighting its main theses:

1. First, we recall *à la* Isham (Q8.?) that the general consensus until today is that one could not use CDG, and *in extenso* a smooth base manifold, in attempts to arrive at the true QG theory—*eg*, by applying the continuum based QFTheoretic technology (and the smooth fiber bundle mathematical panoply of modern gauge theories) to GR. Arguably, the strongest reasons for this 'no-go' attitude is on the one hand the singularities of GR, which are supposed 'classical' issues one has to reckon with before the actual quantization of the gravitational is evoked,[834] and on the other the irremovable (non-renormalizable) infinities that (perturbative) QFTheoretic approaches to QG are assailed by.

2. By virtue of the above, we argued, motivated by ADG and ADG-gravity, that such a drastic measure as a complete abandonment of differential geometric ideas in the quantum regime is not really necessary, as long, of course, as a base (spacetime) manifold—*the* culprit for the aforesaid (differential geometric and physical) diseases—is truly abandoned. *Ergo*, scrap CDG and invite ADG in the quantum deep. In this line of thought, we argued that in ADG no infinitary (limit) processes occur as in the standard Analysis (Differential Calculus) on manifolds, because the role of a background space(time) is atrophized (and topology is 'suppressed'), while algebra is being emphasized and capitalized on. And there is no infinity in algebra! Furthermore, we argued *à la* Einstein and Bergmann, that from the purely algebraico-categorical (relational-pointless) perspective of ADG, one is not inclined anymore to 'geometrize physics' and 'picturize' (via some pre-existing, mathematically posited by fiat, underlying 'spacetime') one's constructions. Rather, 'physical spacetime geometry', if anyone still wishes to think in such 'classical picturesque' terms, is inherent in the fields—or better, it is the result ('solution space') of the dynamical laws that the algebraic (relational) fields define (differential geometrically speaking, as differential equations). Here we have another example of the novel ADG-gravity *motto* that '*dynamics (algebra) comes before kinematics (geometry)*'.

3. We then suggested that ADG could qualify as a strong candidate for the (mathematical) 'organic' theory that Einstein was searching for in order to materialize his unitary field theory *vis-à-vis* either the singularities of his geometrical manifold based GR and the quantum-particle aspects of the other field theories of matter. Here we encountered a plethora of issues and questions that Einstein raised and we answered ADG-theoretically. Most characteristic of all was his 'ambivalence' and associated agnosticism about a 'discrete-algebraic' non-field-theoretic physics which is not based on a (spacetime) continuum *versus* a 'continuous-(differential) geometric' one that is manifestly based on a background geometrical (spacetime) continuum. We argued that ADG-gravity (and the general viewpoint of ADG facing Yang-

[834] Although it is characteristically expected nowadays that QG will relieve GR from its pathological singularities.

Mills and the other gauge forces of matter) passes through the horns of Einstein's dilemma most characteristically by formulating a background spacetimeless field theory—one that is indifferent as to whether an external (to the fields themselves) spacetime is a 'quantal-discretum' or a 'classical continuum'. A characteristic example of such an ADG-theoretic bypass of a major problem that Einstein encountered during the *aufbau* of GR, regarded as a relativistic field theory of the gravitational field on the spacetime continuum, is his (and Grossmann's) hole argument which was originally proposed in order to test the viability of the PGC, which is usually modelled after $\mathrm{Diff}(M)$. Precisely due to the background M independence of ADG-gravity, this becomes a 'non-argument' in our theoresis, while Stachel's deep interpretation of the conceptual consequences of the EHA was essentially seen to be another manifestation of one of the pillar-aftermaths of ADG-gravity noted above, namely that dynamics (the field) is prior to kinematics ('spacetime geometry'), or equivalently, that spacetime is inherent in the dynamical field.

4. We then arrived at a host of arguments and theses based on this 'field-over-spacetime' priority, such as 'field over spacetime events and the causal nexus between them', 'field over matter' in a Machean sense, as well as 'field solipsism and autodynamicity' in an Aristotelian-Leibnizian 'windowfull' sense, to name a few.

5. Finally, as an extension and generalization of Bergmann's 'physicalization of geometry' (as opposed to 'geometrization of physics') mentioned above, we maintained that there is no 'mathematical physics' as such, but only 'physical mathematics'; while, based on some telling remarks of Dirac as recalled by Faddeev, we put forth that theoretical physics research—and especially QG—is in need of more abstract and general(ized) mathematical concepts and structures, so that when it comes to applications of differential geometric ideas in QG, ADG is well qualified to be identified with such a framework.

9 Appendix: Glossary for ADG-Gravity

Below is a list, in alphabetical order, of certain *novel* basic concepts, ideas and principles, that have been mentioned in this work, in the past trilogy [270, 271, 272], as well as in the recent paper [317]. These central notions, some of them carrying rather standard meaning and connotation,[835] were

[835] Indeed, some of the notions appearing in the list below have well established meaning and connotations in theoretical physics' nomenclature, jargon and general literature. Their new 'definitions' given here in no way aim at confusing things (and if they do so, we sincerely apologize in advance for our inability to come up with different names/terms). Rather, they are presented on the one hand just to challenge some of those traditional terms, and on the other simply for the reader to have ready at his disposal a concise explanation of certain new and/or 'exotic'-

born during the development of *ADG-gravity* and we feel that they are important for understanding it better.

1. **ADG-gravitational field:** This pertains to the pair $(\mathcal{E}, \mathcal{D})$, where $\mathcal{D}$ is the gravitational connection field proper (*sdb* 6), and $\mathcal{E}$ its (local quantum-particle) associated (representation) vector sheaf (*sdb* 4).

2. **Algebraicity—relationalism:** This pertains to the Leibnizian-Machean trait of ADG, and *in extenso* of ADG-gravity, that all differential geometry, and as a result the ADG-gravitational dynamics (Einstein equations) that are represented differential geometrically as differential equations proper, refers directly to (in fact, it derives solely from) the algebraic relations between the 'geometrical objects' (*ie*, the *physical fields*; *sdb* 18) that live on a surrogate base (spacetime) scaffolding, without that background playing any role whatsoever (in the guise of coordinates) in that differential geometric mechanism (*sdb* 10) nor in the said autonomous field-dynamics (*sdb* 5), which in turn is formulated just via that essentially algebraic (:sheaf-theoretic) mechanism.

3. **Analytic (smooth and continuous) manifold extension:** From an ADG-vantage, the usual notion of analytic ($\mathcal{C}^\omega$), smooth ($\mathcal{C}^\infty$), or just continuous ($\mathcal{C}^0$) extensibility (of a manifold M) simply pertains to changing ('enlarging') the structure sheaf $\mathbf{A}$ (*sdb* 17) of analytic, smooth, or continuous functions that one assumes up-front to chart (label or coordinatize) M's points, it being implicitly assumed here that one invariably starts with M as a structureless point-set and then one 'dresses' it with a topological ($\mathcal{C}^0$), differential ($C^i;\ i = 1 \ldots \infty$), or analytic ($\mathcal{C}^\omega$) structure (sheaf) thus qualify it as a topological, differential, or analytic manifold, respectively. From a differential geometric viewpoint, this hinges on the fact that a differential manifold M is nothing but the (algebra or structure sheaf of) differentiable functions (in $\mathcal{C}^\infty(M)$ or $\mathcal{C}^\infty_M$) on it (Gel'fand duality; *sdb* 23), so that when one wishes to extend it, one should simply enlarge one's ('space' or algebra of) differentiable functions.

4. **Associated sheaf:** This is the vector sheaf $\mathcal{E}$ of representation of the connection field $\mathcal{D}$ *sdb* 6). It is the carrier (or action) space of $\mathcal{D}$ and its (local) sections represent (local) particle quantum states of the said field (*sdb* 12). Technically speaking, $\mathcal{E}$ is the sheaf associated with the principal sheaf $\mathcal{A}ut\mathcal{E}$ (*sdb* 26) of self-transmutations (dynamical Kleinian 'auto'-symmetries; *sdb* 5, 21) of the (quantum-particle states of the) field.

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sounding concepts that she encounters during the reading of this work. Since these terms are randomly dispersed and occur repeatedly throughout this voluminous work, we believe that the compact concentration of them into a glossary will significantly facilitate her reading. Finally, let us mention that some of the 'definitions' in the list are interdependent, so the acronym '*sdb*' (followed by a number x from 1 to 42) written in parenthesis after the appearance of a not-yet-defined term means '*see definition (of that yet undefined term in x) below*'.

5. **Autodynamics—dynamical autonomy:** This pertains to the fact that the dynamical (vacuum) Einstein equations in ADG-gravity is expressed solely in terms of the (curvature of the) connection field $\mathcal{D}$ 'in-itself' (*sdb* 12, 23), without reference to a background spacetume (whether 'discrete' or 'continuous').

6. **Background independence:** The currently fashionable in QG research notion of 'background independence' acquires a new, deeper meaning in ADG-gravity. Unlike in various perturbative and non-perturbative approaches to QG where this term means 'background *metric* independence' (*ie*, no background geometry), while a background manifold is still employed in order to be able to do CDG (CDG-conservatism and monopoly; *sdb* 8), in ADG-gravity not even a smooth base (spacetime) manifold, let alone a smooth metric, is used in the theory. In this sense, ADG-gravity is genuinely and completely background independent.

7. **Categoricity of (gravitational) dynamics:** The (vacuum) gravitational dynamics is expressed via sheaf morphisms. In particular, via the curvature of the connection, which is an **A**-morphism—a fact which bears on the functoriality of the dynamics in ADG-gravity (*sdb* 20).

8. **CDG and manifold conservatism and monopoly in physics:** By this we mean that all physical theories that have been formulated (so far) in the language of differential geometry (basically, the dynamics that define them as physical theories proper are modelled after differential equations) employ in one way or another a base differential manifold—a (locally) Euclidean space, thus effectively they use the conceptual and technical means of Classical Differential Geometry (CDG). Even in the case of singularities, which are arguably 'internal blemishes' or 'faults' of the background manifold employed, we have so far used persistently CDG (Analysis) to deal with them (*sdb* 8).

9. **CDG-vicious circle vis-à-vis singularities:** This refers to the fact that since spacetime singularities are 'innate' anomalies and pathologies of the background spacetime manifold, the employment of the manifold based CDG (Analysis) to deal with (*eg*, resolve) them is essentially an 'oxymoron'. In other words, there are genuine singularities—*ie*, singularities that CDG cannot cope with—in the manifold based GR.

10. **Conflict between the PGC and singularities:** Since singularities are 'inherent' in the manifold, the Principle of General Covariance (PGC) of GR, which is modelled in the usual theory via the diffeomorphism group $\mathrm{Diff}(M)$ of the underlying differential spacetime manifold M, appears to come in conflict with them and moreover it makes a precise definition of them, within the base manifold confines of CDG, an impossible task.

11. **Connection field:** This is the 'gauge field proper' part $\mathcal{D}$ of the ADG-field $(\mathcal{E}, \mathcal{D})$. Technically, it is a $\mathbf{C} \equiv \mathbf{K}$-morphism, not an $\mathbf{A}$-morphism. This property qualifies $\mathcal{D}$ as a purely algebraic entity, and not as a 'geometrical object' proper (*sdb* 25). From a gauge-theoretic point of view, the connection may be identified with the gauge (potential) field, although by the latter term theoretical physicists usually understand only the local part $\mathcal{A}$ of $\mathcal{D} := \partial + \mathcal{A}$.

12. **Coordinate (virtual) singularities:** In the context of the manifold and *in extenso* CDG-based GR, the notion 'coordinate singularities' pertains to certain *loci* of the spacetime continuum that appear to host singularities for some of the $\mathcal{C}^{\infty}(M)$-components of the smooth spacetime metric-solution of Einstein's equations, simply because those *loci* have been charted by (referred to) an 'inappropriate' system of coordinates (frame). It is therefore understood that by a suitable change of system of (smooth) coordinates (*eg*, by an appropriate smooth extension), coordinate singularities disappear, as for instance when one changes coordinates from the Cartesian-Schwarzschild frame to the Eddington-Finkelstein one in order to 'resolve' the exterior Schwarzschild singularity.

13. **Curvature (of a connection):** In ADG, the curvature of a connection, $R(\mathcal{D})$, is also a sheaf morphism; albeit, an $\mathbf{A}$-morphism. This property qualifies R as a 'geometrical object' proper (*sdb* 25). From a gauge-theoretic vantage, R may be identified with the gauge field strength.

14. **DGSs, SFSs and VESs:** So far, the Analytic (CDG-based) taxonomy of spacetime singularities splits into three general classes: (i) differential geometric singularities (DGSs)—points at a suitably defined (topological) boundary ∂M of the spacetime manifold M for which there is no C^k-differential extension of (the solution-metric on) M so as to incorporate them with the other regular points in M's interior; (ii) solution field singularities (SFSs)—again, boundary points of M for which there is no (analytic) extension of the (metric on the) latter that removes them and at the same time it is a distributional solution of the Einstein field equations; and (iii), various energy singularities (VESs)—again, boundary points for which there is no (analytic) extension of (the metric on) M that removes them satisfying at the same time various energy conditions.

15. **Differentiability, differential (geometric) mechanism:** By differentiability, in a certain theoretical framework (for doing differential geometry), we mean the possibility of defining a differential operator $d \equiv \partial$—*ie*, the ability to differentiate mathematical entities within that framework. Since differentiation is usually perceived as a (local) topologico-algebraic (analytic) notion, in the usual theory (CDG) it is the (locally) Euclidean character of the base

space (manifold) that secures the definition of d and, *in extenso*, differentiability. In CDG it is the background space (manifold) that provides one with 'differentiability' (differential structure). In contradistinction, in ADG, which is base manifold free, the topological ('spatial background') aspect of differentiability becomes atrophic and what is pronounced is d's algebraic character. Indeed, the basic recognition of ADG is that d is a particular instance of the general notion of connection $\mathcal{D}$ ('generalized derivative') and to secure the (definition of the) latter, all that one needs is a structure algebra sheaf $\mathbf{A}$ of generalized arithmetics—abstract 'differentiable functions' ('coordinates')—on an in principle arbitrary topological space X. In turn, the classical theory (CDG) is obtained when one assumes $\mathbf{A} \equiv \mathcal{C}^\infty_X$ for structure sheaf of generalized coordinates (or what amounts to the same by Gel'fand duality, a base differential manifold $X \equiv M$). *In toto*, we maintain that in ADG the essentially algebraic differential geometric mechanism derives from the stalk of the sheaves involved and not from the base space, which is merely a topological space. ADG is an algebraization of Analysis emphasizing the latter's algebraic (relational) qualities and at the same time deemphasizing the latter's topological (spatial-geometrical) 'dependencies'.

16. **Differential geometric monads (with 'windows'):** Differential triads are the '*ur*'-elements, the basic building units of ADG. On them one can erect the entire differential geometric edifice (*eg*, modules of higher-order differential forms, connection, curvature *etc.*), while the base space employed plays absolutely no role in the purely algebraic (:sheaf-theoretic) differential geometric mechanism 'encoded' in (or 'carried' by) those units. In particular, for the most important ADG-notion of connection (field) $\mathcal{D}$, this background independence has made us qualify the ADG-fields $(\mathcal{E}, \mathcal{D})$ as (dynamically) autonomous, 'unitary' entities—entities in no need of a background space(time) for their (differential geometric) subsistence. Due to their (differential geometric) autonomy and self-sufficiency, triads may be coined, in honor of Leibniz (and of the purely relational character of Calculus that he had envisaged), 'differential geometric monads'. Granted that an external space does not influence at all the purely algebraic differential mechanism inherent in the triads, it does not mean that the latter, like the Leibnizian monads, are 'windowless': on the contrary, they form a category $\mathfrak{DT}$ and the arrows in it simply show that triads 'communicate' with each other (*ie*, they have windows). For the connections (fields) in particular, which are sheaf morphisms, the fact that the laws of physics (as differential equations) are represented by equations between sheaf morphisms, shows that fields actually 'communicate' (*eg*, they interact).

17. **Dynamics as algebra ('cause'):** This pertains to the fact that in ADG-gravity the algebraic $\mathbf{A}$-connection field $\mathcal{D}$ defines the gravitational dynamics—the actual law of motion

of the gravitational field—without an *ab initio* commitment to an *a priori* fixed, external (ambient), kinematical (possibility) space(time). Antonio Machado's verses come to mind: "*Traveller there are no paths; paths are made by walking*" [250], and it is the precisely the fields that 'do the walking' by their dynamics, in the manifest absence of an ambient/external, pre-existing, geometrical space-time.) This is consistent with the aforesaid 'dynamical autonomy' of the ADG-fields and it suggests that in ADG-gravity the notion of the causal nexus (between events) in the external, geometrical space(time) is replaced by the autonomous, 'bootstrapping' actions of the spacetimeless ADG-fields (and their inherent particle-quanta). Moreover, the geometrical physical space(time) is in a sense 'inherent' in those algebraic (relational) fields—it is the 'result' ('solution space') of the dynamics that the fields define.

18. **Field—generalized causality:** This, as also briefly alluded to above, refers to the observation that in ADG-gravity, causality and the 'geometrical pattern' of causal ties between spacetime events is replaced by the (gravitational) field and the dynamics that this defines—nothing more, nothing less.

19. **Field-particle duality:** This is closely related to the notion of third quantization (*sdb*) and it refers to the self-dual character of the autonomous, unitary ADG-fields $\mathfrak{F} := (\mathcal{E}, \mathcal{D})$: on the one hand we have the 'proper' field aspect $\mathcal{D}$ of $\mathfrak{F}$, and on the other, from a geometric prequantization vantage, the local sections of $\mathcal{E}$ represent the local quantum-particle states of the field—$\mathfrak{F}$'s particle aspect.

20. **Functional sheaves:** This refers to sheaves of functions that can be used as structure sheaves in ADG, although ADG in principle admits also non-functional sheaves, as long as the latter can accommodate (provide one with) a differential operator d (with which one can actually do differential geometry).

21. **Functoriality (kinematical and dynamical):** In general terms, functoriality of a construction (in a categorical setting) pertains to the quality of the construction that all entities involved in it respect the relevant categories concerned. In the standard kinematical sense, functoriality means that in the process of the construction certain key kinematical (structural) features are preserved—*eg*, quantization perceived as a structural-functorial procedure: geometric prequantization being functorial, first quantization being non-functorial, and second quantization being functorial. By contrast, in ADG-gravity functoriality is not of a kinematical, but of a dynamical kind. This pertains to the fact the dynamical equations of Einstein are functorial with respect to the structure sheaf $\mathbf{A}$ of generalized coordinates, as the curvature (of the gravitational connection field) involved in them is an $\mathbf{A}$-morphism,

a $\otimes_{\mathbf{A}}$-tensor, with the latter being the homological tensor product functor. On this functoriality hinges the ADG-theoretic generalization of the PGC of the manifold based GR (namely, the principle of Synvariance) as well as ADG-gravity's evasion of $\mathcal{C}^\infty$-singularities by first absorbing (or integrating) them into $\mathbf{A}$, and then by making the dynamics manifestly $\mathbf{A}$-independent (*ie*, $\mathbf{A}$-functorial).

22. **Gauge theory of the third kind:** In contradistinction to the so-called gauge theories of the first (global gauge symmetries) and second (local gauge symmetries) kind, ADG-gravity—in fact, the ADG-theoresis also of Maxwellian electrodynamics (abelian gauge symmetries) and Yang-Mills theories (non-abelian gauge symmetries)—is formulated solely in terms of the (gravitational) connection field $\mathcal{D}$ (half-order formalism); moreover and more importantly, no external, background spacetime (manifold) is involved in the theory. Due to these two features, the ADG-theoresis of gravity (and of the other fundamental gauge forces) is coined gauge theory of the third kind.

23. **Generalized arithmetics (coordinates)—structure sheaf:** In ADG, where the principal motto is that '*differentiability is independent of smoothness*' [271], one can use a structure sheaf $\mathbf{A}$ of (algebras of) 'differentiable coordinates' different from the usual (classical) one $\mathcal{C}^\infty_M$ of smooth ones on a differential manifold M. Such $\mathbf{A}$s are called generalized arithmetics and they may be far from smooth (*eg*, 'discrete', or distributional and 'ultra-singular'). From an ADG-theoretic viewpoint, all differential geometry boils down to $\mathbf{A}$, since, for one thing, it is the 'domain' of (or 'source space' for) the basic differential ∂ and effectively of the connection field $\mathcal{D}$ (since by definition $\mathcal{E}$ is locally a finite power of $\mathbf{A}$). From a physical point of view, the elements of $\mathbf{A}$ model our acts of coordinatization, representation, measurement and concomitant 'geometrization' of the fields involved.

24. **'Geometrical' objects, 'algebraic' objects:** By 'geometrical' objects we mean fields that our generalized arithmetics or measurements in $\mathbf{A}$ respect. Geometrical objects are mathematically modelled in ADG by $\mathbf{A}$-sheaf morphisms, or perhaps better, by $\otimes_{\mathbf{A}}$-tensors. The canonical example of a geometrical object is the curvature (field) of the connection (field), $R(\mathcal{D})$. By contrast, 'algebraic' objects are those fields not respected by our generalized coordinates—fields that elude our attempts (in $\mathbf{A}$) at localizing, representing and 'geometrizing' (measuring) them. Algebraic objects are not $\otimes_{\mathbf{A}}$-tensors and they evade our acts at sharply localizing (measuring) them by using $\mathbf{A}$. The canonical example of an algebraic object is the connection field $\mathcal{D}$ itself, which is not an $\mathbf{A}$-sheaf morphism, but only a $\mathbf{C} \equiv \mathbf{K}$-morphism. That $\mathcal{D}$ cannot be sharply determined by $\mathbf{A}$ (*ie*, the field cannot be localized and be quantum particle-represented by $\mathcal{E}$, which is locally isomorphic to $\mathbf{A}^n$) is reflected by the fact that it acts (as a 'derivation') on the local particle (generalized position)

states (*ie*, the local sections) of $\mathcal{E}$ and changes them as a momentum-like operator. This of course bears on the fact that the unitary ADG-field $(\mathcal{E}, \mathcal{D})$ is third- or self-quantum.

25. **Half-order formalism:** The formulation of ADG-gravity solely in terms of the connection field $\mathcal{D}$, as opposed to the second order formalism—the original formulation of GR in terms of the metric (Einstein)—as well as to the so-called first order formalism whose basic variables are the (smooth) connection and the (smooth) vierbein/tetrad frame (Palatini, Ashtekar). Furthermore, in contrast to both the first and the second order formalism, in the third order ADG-formalism for gravity no $\mathcal{C}^{\infty}$-smooth base spacetime manifold is involved at all.

26. **Internal, 'auto-symmetries':** This refers to the 'symmetries' of the ADG-gravitational field $(\mathcal{E}, \mathcal{D})$, and concomitantly, to the 'invariances' of the dynamical law (Einstein equations) that the field defines. Because the field is independent of an external, background spacetime (manifold) and it is of a purely gauge (third gauge) character, these symmetries may be coined 'internal', or 'esoteric'—they are the invariances of the field (law) 'in-itself', what one could call 'auto' or 'self-symmetries'. Mathematically, they are organized into the principal group sheaf $\mathcal{A}ut\mathcal{E}$ of (dynamical) 'auto-transmutations' of the field ($\mathcal{D}$) and its particle-quanta ($\mathcal{E}$).

27. **Kinematics as geometry ('effect'):** This pertains to the fact that in ADG-gravity the role of kinematics and dynamics is reversed, in the sense that one does not think of a geometrical kinematical (possibility) space as being *a priori* fixed—*ie*, prescribed before the 'algebraic' dynamics is given—by the theoretician. Rather, kinematics (geometry) is the result of dynamics (algebra), and in this respect one may think of the traditional term 'physical spacetime geometry' as the 'solution space(time)'—the 'place' where the dynamical field law of gravity holds. In terms of the algebraic and geometrical objects distinction made above and of our perception of $\mathcal{D}$ as an abstract causal nexus (causality/causal connection), one may formally write this 'geometry is the effect/result of algebra' (or equivalently, that 'algebra is the cause of geometry') by $\mathcal{D} \Rightarrow R(\mathcal{D})$.

28. **Kleinian field-particle geometry:** This pertains to identifying *à la* Felix Klein the internal, 'esoteric' geometry of the particle-field pair $(\mathcal{E}, \mathcal{D})$ with its group (sheaf) $\mathcal{A}ut\mathcal{E}$ of dynamical self-transmutations—the field's automorphisms.

29. **Local (open) gauges, local frames, local measurements:** The commonly used term in local gauge theory (of the second kind) 'local gauge', in the context of ADG-gravity refers simply to an open subset U of the base topological (localization) space X (of the sheaves) involved. U may be thought of as an abstract 'local laboratory', a 'local reference frame', or even better, a 'local measuring device' (*ie*, a local gauge!), to which our local measurements

(field-coordinatizations) in $\mathbf{A}|_U \equiv \mathbf{A}(U)$ belong. Accordingly, given a 'coordinatizing open cover' $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of X, e^U defines a local frame or a local choice of basis (or gauge!) for the vector sheaf $\mathcal{E}$. The e_is in e^U are local sections of $\mathcal{E}$ (*ie*, they are elements of $\mathcal{E}|_U \equiv \mathcal{E}(U) \equiv \Gamma(U, \mathcal{E})$) constituting a basis of $\mathcal{E}(U)$—*ie*, they span uniquely, with linear coefficients in $\mathbf{A}(U)$, every local section of $\mathcal{E}$ living in $\mathcal{E}(U)$.

30. **Localizing-sheafifying-gauging-curving-relativizing-dynamicalizing:** From an ADG-theoretic viewpoint, these six terms are essentially synonyms. Localization pertains to soldering the various algebraic structures involved on the base topological space X. Technically, and in the sheaf-theoretic context of ADG, this is accomplished by sheafification. From a physical viewpoint localization of the various (algebraic) structures involved and their symmetries is tautosemous to 'gauging'—*ie*, endowing the (locally independent) structures with a non-trivial connection—whose geometrical expression is the curvature field. Finally, the assignment of a $\mathcal{D}$ means essentially that the 'relativized' structures involved (*ie*, structures referred to and expressed in a particular local gauge) are variable—in fact, that they are dynamically variable, and their dynamical variation is expressed differential geometrically by the differential equations that the said connection field defines (here, the Einstein equations, which are geometrically expressed not directly in terms of the connection which anyway is not a geometrical, but an algebraic, object, but indirectly via the curvature of the connection, which is the geometrical object *par excellence*).

31. **Natural transformation theory:** This is ADG's categorical version of the general notion of 'transformation theory' (*eg*, in the sense of Dirac) underlying any process of relativization. To explain this, in GR, relativization—*ie*, the Principle of Relativity (PR)—can be expressed by saying that the description of the gravitational dynamics is 'independent' of reference to any particular system of (smooth) coordinates, which culminates in the standard $\mathrm{Diff}(M)$-representation of the PGC of the differential manifold based CDG underlying GR. In the background manifoldless ADG-gravity however, the PGC is generalized to the notion of Synvariance, which is in turn equivalent to the dynamical $\mathbf{A}$-functoriality of the ADG-formulated Einstein equations (with $\mathbf{A}$ not necessarily restricted to be $\mathcal{C}^\infty_M$ as in the manifold based CDG). This functoriality then amounts to a relativization of $\mathbf{A}$ in ADG-gravity, which is expressed via the Principle of Algebraic Relativity of Differentiability (PARD). Finally, the latter is mathematically represented by a pair of 'second order' functors (natural transformations) mapping the $\mathbf{A}_1$-functorial ADG-gravitational dynamics expressed in a certain (chosen) structure sheaf $\mathbf{A}_1$ of generalized coordinates, to the $\mathbf{A}_2$-functorial ADG-Einstein equations expressed in another (different) structure sheaf $\mathbf{A}_2$. The physical upshot of PARD is the Principle of Field Realism (PFR) in ADG-gravity.

32. **Newtonian spark—background space(time) forgetfulness:** This pertains to the general 'phenomenon' in ADG whereby once one has 'extracted' the inherently algebraic differential geometric mechanism from the stalk of the algebra and vector sheaves involved, or even possibly from a locally Euclidean base space (manifold) as in the classical theory (CDG), we develop all our differential geometric concepts and constructions independently of that surrogate background (sheaf-theoretic) localization space(time). As it were, we totally forget about the base space(time) and work exclusively in the (algebra inhabited) sheaf space occupied by the really (physically) significant objects—the fields and their algebraic interrelations.

33. **Physical (spacetime) geometry:** By 'physical (spacetime) geometry' we understand the 'solution space' of the dynamical law, or more generally, the 'space' where the dynamical law holds. In ADG-gravity, this is the representation (carrier) sheaf (space) $\mathcal{E}$ for (of) the gravitational field $\mathcal{D}$—the 'space(time)' (defined and occupied by the quantum particle states of the field) where the (vacuum) Einstein equations of ADG-gravity hold.

34. **Principal sheaf:** In ADG-gravity, this is the group sheaf $\mathcal{A}ut\mathcal{E}$ of (dynamical) 'auto-symmetries' of the (dynamically) autonomous ADG-gravitational field $(\mathcal{E}, \mathcal{D})$. In turn, it is understood that $\mathcal{A}ut\mathcal{E}$ has $\mathcal{E}$ as its associated (representation) vector sheaf.

35. **Principle of Algebraic Relativity of Differentiability:** Since one of the basic morals of ADG is that all differential geometry boils down to $\mathbf{A}$, the idea is that by changing $\mathbf{A}$ (with these changes dictated by actual physical problems—*eg*, when one needs to change structure sheaf of differentiable functions in order to include in the new structure sheaf an apparently singular, from the point of view of the old structure sheaf, function; *alias*, when one wishes to incorporate or 'absorb' the singular function into the structure sheaf of generalized arithmetics),

36. **Principle of Field Realism:** The dynamical laws of physics which are (mathematically) defined (as differential equations) by the ADG-fields $(\mathcal{E}, \mathcal{D})$ are independent of our 'generalized measurements'. That is, they are independent of the structure sheaves $\mathbf{A}$ that we—the external (to the fields) 'observers', 'measurers' ('geometers'), or 'coordinators'—employ to represent ('geometrize', or 'coordinatize on some geometrical spacetime') those fields. This is reflected in the aforesaid functoriality of the ADG-field dynamics, and *in extenso* in the PARD, with its natural transformations' representation.

37. **Real (genuine) singularities:** Let it be stressed up-front that from the perspective of ADG-gravity, this notion is both conceptually and technically a chimera. In the usual CDG-based GR, real (true or genuine) singularities are *loci* in the spacetime continuum that 'resist'

analytic extension of that manifold so as to include them with the other regular points of M (in other words, one cannot further enlarge one's algebra of differentiable coordinate functions, always staying within the realm of $\mathcal{C}^\infty_M$, so as to include the singular ones with the other regular, smooth ones). As a result, true singularities are thought of as being situated at the boundary of a maximally (analytically) extended, albeit incomplete, spacetime manifold. They are 'defined' by exclusion (they are sites where all Analysis fails!), so that there is no precise and straightforward definition of real singularities in the manifold and CDG-based GR. In contradistinction, in the base smooth manifoldless ADG-gravity where all singularities are integrated or 'absorbed' into (a suitably chosen) **A** while the Einstein law that the ADG-gravitational field defines is **A**-functorial (synvariant) hence not at all impeded by any singularity, there is no real (genuine) singularity (at least in the DGS and SFS sense above—*ie*, as sites where the differential equation represented field law breaks down, or where even a generalized, distributional solution fails to be one). From an ADG-theoretic vantage, in a deep sense all singularities are coordinate (virtual) ones as they are embodied into (a suitably chosen) **A** and the **A**-functorial gravitational dynamics is not at all impeded by their presence (Einstein's equations hold in their very presence).

38. **Singularities as differential geometric solution-metric anomalies:** In view of the fundamental conflict between the PGC of GR and the existence of singularities, the latter may be thought of as 'differential geometric solution-metric anomalies' in the sense that while the field law (Einstein equations) are Diff(M)-invariant, its solution-metric may possess singularities where 'the smoothness of the law' appears to be broken.

39. **Singularity absorption and evasion:** This refers to the basic singularity-evasion 'strategy' of ADG: upon encountering a singularity, one should look for a structure sheaf **A** that contains it while at the same time it provides one with the (essentially algebraic) differential geometric mechanism with which one can write the Einstein equations (as differential equations proper) in the very presence of the said singularity.

40. **Synvariance:** Due to the manifest absence of an external (to the gravitational field itself) background spacetime manifold in ADG-gravity, the Diff(M)-modelled PGC of the manifold based GR is replaced by the group (sheaf) $\mathcal{A}ut\mathcal{E}$ of dynamical self-transmutations of the ADG-gravitational field $(\mathcal{E}, \mathcal{D})$ 'in-itself' (to be precise, $\mathcal{A}ut\mathcal{E}$ is the symmetry group sheaf of the local quantum-particle states of the field, which, at least from a geometric prequantization vantage, are modelled after the local sections of $\mathcal{E}$). There is no external spacetime structure dynamically varying with ('covarying' with) the gravitational field. All there is 'out there' is the gravitational field $\mathcal{D}$ alone and the dynamics (the vacuum Einstein equations) that it defines via the action of its Ricci curvature on its own local particle states (*ie*, the sections

of) $\mathcal{E}$: $\mathcal{R}(\mathcal{E}) = 0$. At the same time, thanks to the **A**-functoriality of the ADG-gravitational dynamics, no matter what **A** we employ to coordinatize or geometrize (*ie*, localize and particle-represent) the gravitational connection field $\mathcal{D}$, we do not 'perturb' it (and the dynamics that it defines via its curvature) at all, no matter what singularities that (the function-like objects in) **A** may be assumed to carry.

41. **Third Quantization:** In contradistinction to both first (non-relativistic quantum particle mechanics) and second (relativistic quantum field mechanics) quantization, for the formulation of which an external space-time manifold is invariably involved in one way or another, third quantization (better, third- or self-quantum field theory) pertains to the ADG-scheme of doing field theory solely in terms of the fields $(\mathcal{E}, \mathcal{D})$ 'in-themselves' without a base space-time manifold; and moreover, with quantum traits built into these autonomous fields from the very start (*ie*, virtually 'from their very definition').

42. **Unitary ('monadic') field theory:** In the context of ADG, the epithet 'unitary' to 'field theory' pertains less to Einstein's original vision of 'one single field for all forces' and more to (also Einstein's, less popular however, vision of) a 'total field' obeying (defining) total dynamical (differential) equations that are 'free' from (*ie*, in no way impeded by, let alone breaking down in the presence of) singularities, it incorporates its particle-quanta as singularities in the law (differential equation) that it defines, and (as a bonus to Einstein's vision—and something that Einstein could not have possibly envisioned in the CDG-based field theory that he had in mind and was advocating) the background spacetime continuum plays no role whatsoever in the field's autonomous dynamics ('autodynamics'). *In summa*, a genuinely unitary field theory in our ADG-theoretic sense is concerned only with the field, the whole field and nothing but the field.

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