

SMOOTH SOLUTIONS OF SYSTEMS OF QUASILINEAR PARABOLIC EQUATIONS

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Abstract. We consider in this article diagonal parabolic systems arising in the context of stochastic differential games. We address the issue of finding smooth solutions of the system. Such a regularity result is extremely important to derive an optimal feedback proving the existence of a Nash point of a certain class of stochastic differential games. Unlike in the case of scalar equation, smoothness of solutions is not achieved in general. A special structure of the nonlinear Hamiltonian seems to be the adequate one to achieve the regularity property. A key step in the theory is to prove the existence of Hölder solution.

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1. INTRODUCTION

In this paper, we consider diagonal parabolic systems arising in the context of stochastic differential games. These systems are of the form

$$\frac{\partial}{\partial t} u_\nu - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(t, x) \frac{\partial}{\partial x_j} u_\nu \right) = H_\nu(x, u, \nabla u), \quad \nu = 1, \dots, N, \quad (1.1)$$

where the “Hamiltonians” H_ν have quadratic growth in ∇u . In application to game theory, N corresponds to the number of players. The aim is to prove global existence and regularity results under the assumptions of uniform ellipticity for a_{ij} and special structure conditions for H_ν , say

$$|H_\nu(x, u, p)| \leq K |p_\nu| |p| + K \sum_{\mu < \nu} |p_\mu|^2 + K, \quad (1.2)$$

provided some L^∞ -estimate is available, for which in turn, simple conditions are presented.

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The theory of scalar parabolic equations (1.1) for the right-hand side having general quadratic growth is well understood since C^α -estimates are available without smallness conditions for the growth-factor a ,

$$|H(x, u, \nabla u)| \leq a|\nabla u|^2 + K$$

cf. the book [15]. For examples of scalar parabolic equations arising from stochastic optimal control, *cf.* Bensoussan and Lions [8].

For parabolic systems with “large” growth factor, C^α -estimates can not be achieved in general. Struwe presented an example of the same type of system as (1.1), where a discontinuous weak solution develops from the zero solution!

C^α -regularity and C^α -bounds follow from a smallness condition for the growth-factor a , in fact $|a| < \|u\|_\infty \lambda$, λ ellipticity constant *cf.*, Struwe [16]. However, for applications in the theory of stochastic games such a condition for the growth factor a is not acceptable since the ellipticity constant may be very small compared to the growth-factor. So, in order to avoid smallness conditions the authors [4] assumed the “specific structure condition” (1.2) in the case of two players and obtained C^α -estimates and hence $L^q(W^{2,q})$ -solutions, $q < \infty$, for (1.1). In fact [4] contains also cases with N players, $N \geq 3$, but not with the “simple” assumption (1.2). In [3, 5–7] and [2] examples of games mainly with two players, where (1.2) holds, are presented and can be extended to the parabolic case.

Note, that a similar discussion concerning the growth factor holds for the elliptic analogue; references for C^α -estimates and -regularity under rather optimal smallness condition are [13, 17].

Note further, that, without C^α -estimates, in general it is false that a sequence of (possibly approximate) solutions to (1.2), which is bounded in $L^\infty(L^2) \cap L^2(H^1)$, has a subsequence which converges strongly in $L^2(H^1)$. This type of convergence allows to pass to the limit in the terms $\int H_\nu \varphi dx$. For a counter example in the elliptic case, see the example [11]. Therefore, for systems (1.1) and its elliptic analogue the well known theory of Leray and Lions [14] can not be applied. This explains the difficulty of the analysis.

The analysis for obtaining C^α -estimates (and hence $W^{2,p}$ -estimates and existence theorems) for the above system with $N > 2$ and the specific structure condition (1.2) is the main contribution of this paper.

In the elliptic analogue, the corresponding case of three and four players has been worked out in [12]. In fact, the methods [3, 4, 7] and [2] are extended to the general case of N players and the parabolic situation. Note that the main results of this paper apply to the elliptic case (assume $u(t, x)$ is constant in time) and that, also in the elliptic case, the results are new.

2. SETTING OF THE PROBLEM AND STATEMENT OF RESULTS

2.1. Preliminary setting and assumptions

Let Ω be a smooth bounded domain of \mathbb{R}^n and $Q = \Omega \times (0, T)$. We write

$$\Sigma = \partial\Omega \times (0, T).$$

Let $a_{ij}(x, t)$, $i, j = 1, \dots, n$ be given functions satisfying

$$a_{ij} \in C^1(\bar{Q}), \tag{2.1}$$

$$a_{ij}\xi_j\xi_i \geq \alpha|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \quad \alpha > 0. \tag{2.2}$$

We define the family of 2nd order elliptic operators in divergence form

$$A \equiv A(t) = -\frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j} \quad (2.3)$$

with Dirichlet boundary conditions.

We consider a vector function $H(x, t; \lambda, p)$ from $\mathbb{R}^n \times [0, T] \times \mathbb{R}^N \times \mathbb{R}^{Nn}$ into \mathbb{R}^N , such that

$$H \text{ is measurable, and continuous on } \lambda, p. \quad (2.4)$$

We denote by H_ν , $\nu = 1, \dots, N$, the components of the vector H . In fact, we shall make use of linear manipulations on the components of H as follows.

Let Γ be an $N \times N$ -matrix which is invertible, then to Γ we associate the transform of H , denoted H^Γ and defined by

$$H^\Gamma(x, t; \lambda, p) = \Gamma H(x, t; \Gamma^{-1}\lambda, \Gamma^{-1}p). \quad (2.5)$$

We shall say that a matrix Γ satisfies the maximum principle, whenever

$$\text{if } \Gamma\lambda \geq 0, \quad \text{then } \lambda \geq 0. \quad (2.6)$$

We shall make use of three alternative sets of assumptions. These conditions will us give L^∞ -estimates.

$$|H_\nu(x, t, \lambda, p)|_{p_\nu=0} \leq K, \quad (2.7)$$

or, alternatively, the following two conditions (2.8) and (2.9):

$$\sum_\nu H_\nu(x, t; \lambda, p) \geq -\gamma \text{ for those } p \text{ such that } \sum_{\nu=1}^N p_\nu = 0 \quad (2.8)$$

and

$$H_\nu^\Gamma(x, t; \lambda, p)|_{p_\nu=0} \leq \gamma_\nu \text{ for some matrix } \Gamma \text{ satisfying the maximum principle} \quad (2.9)$$

or, alternatively, the following conditions (2.10) and (2.11):

$$\sum_\nu H_\nu(x, t; \lambda, p) \leq \gamma \text{ for those } p \text{ such that } \sum_{\nu=1}^N p_\nu = 0 \quad (2.10)$$

and

$$H_\nu^\Gamma(x, t; \lambda, p)|_{p_\nu=0} \geq -\gamma_\nu \text{ for some matrix } \Gamma \text{ which satisfies the maximum principle.} \quad (2.11)$$

The last two pairs of conditions increases the applicability of the theory to stochastic differential games considerably, *cf.* [2, 5] and [7].

The next assumption specifies what we shall call the specific structure, namely there exists a matrix Γ such that

$$|H_\nu^\Gamma(x, t; \lambda, p)| \leq K_\nu |p| |p_\nu| + \sum_{\mu=1}^{\nu} K_\mu^\nu |p_\mu|^2 + k_\nu(x, t), \quad \nu = 1, \dots, N-1 \quad (2.12)$$

where K_ν, K_μ^ν are positive constants and k_ν positive functions $\in L^q(Q)$, with $q > \frac{n}{2} + 1$. Moreover

$$|H_N^\Gamma(x, t; \lambda, p)| \leq K_N |p|^2 + k_N(x, t) \tag{2.13}$$

with a positive constant K_N , a positive function $k_N \in L^q(Q)$, with $q > \frac{n}{2} + 1$.

Without loss of generality, the special structure (2.12, 2.13) can be formulated as follows, dropping Γ to simplify the notation

$$H_\nu(x, t; \lambda, p) = Q_\nu(x, t; \lambda, p)p_\nu + H_\nu^0(x, t; \lambda, p), \quad \nu = 1, \dots, N \tag{2.14}$$

with

$$\begin{aligned} &Q_\nu \text{ measurable, continuous in } \lambda, p, \text{ for } p_\nu \neq 0, \\ &|Q_\nu(x, t; \lambda, p)| \leq K_\nu |p| \quad \nu = 1, \dots, N-1, \\ &Q_N = Q_{N-1}, \end{aligned} \tag{2.15}$$

$$\begin{aligned} &H_\nu^0 \text{ measurable, continuous in } \lambda, p, \nu = 1, \dots, N-1, \\ &|H_\nu^0(x, t; \lambda, p)| \leq \sum_{\mu=1}^{\nu} K_\mu^\nu |p_\mu|^2 + k_\nu(x, t), \nu = 1, \dots, N, \end{aligned} \tag{2.16}$$

where we have defined

$$K_\mu^N = K_N + \frac{K_N - 1}{2}, \mu = 1, \dots, N-1; \quad K_N^N = K_N + K_{N-1}.$$

Indeed, we set, for $\nu = 1, \dots, N-1$

$$\sigma_\nu(x, t; \lambda, p) = \frac{H_\nu(x, t; \lambda, p)}{K_\nu |p| |p_\nu| + \sum_{\mu=1}^{\nu} K_\mu^\nu |p_\mu|^2 + k_\nu(x, t)} \tag{2.17}$$

and successively

$$\begin{aligned} Q_\nu(x, t; \lambda, p) &= K_\nu \sigma_\nu(x, t; \lambda, p) |p| \frac{p_\nu}{|p_\nu|}, \quad \nu = 1, \dots, N-1, \\ Q_N &= Q_{N-1}. \end{aligned} \tag{2.18}$$

$$H_\nu^0(x, t; \lambda, p) = H_\nu(x, t; \lambda, p) - Q_\nu(x, t; \lambda, p)p_\nu, \quad \nu = 1, \dots, N \tag{2.19}$$

and (2.14–2.16) are satisfied.

2.2. The problem

Let be given

$$u^0 \in (W_0^{1,\infty}(\Omega))^N, \tag{2.20}$$

we look for a function u such that

$$u \in \left(L^q(0, T; W_0^{2,q}(\Omega)) \right)^N, \quad \frac{\partial u}{\partial t} \in (L^q(Q))^N, \quad (2.21)$$

$$\frac{\partial u}{\partial t} + A(t)u = H(x, t, u, Du), \quad \forall x, t, \quad (2.22)$$

$$u(0) = u^0, \quad u|_{\Sigma} = 0. \quad (2.23)$$

Thanks to the transformation Γ we can imbed the problem (2.21–2.23) into a family of problems as follows:

If u is a solution of (2.21–2.23) then set

$$z = \Gamma u \quad (2.24)$$

and clearly z is a solution of the same problem, with H replaced by H^Γ , and u^0 by Γu^0 .

We state the following:

Theorem 2.1. *Let the smoothness conditions (2.1, 2.4, 2.20) on the data and the ellipticity condition (2.2) be satisfied. Assume the specific growth conditions (2.12, 2.13) on the Hamiltonians and finally the structure conditions (2.8, 2.9) or, alternatively (2.10, 2.11) instead of (2.8, 2.9). Then there exists a global solution of the parabolic system (2.21–2.23).*

By “global” we mean that no smallness assumption for T is assumed.

Remark 2.1. The structure conditions (2.8, 2.9) resp. (2.10, 2.11) are “responsible” for the L^∞ -estimate, the structure conditions (2.12, 2.13) for the step from L^∞ to $C^\alpha \cap H^1$.

3. FIRST A PRIORI ESTIMATES

3.1. A fundamental inequality

We consider a solution of (2.21–2.23). We shall write

$$\rho = \|u\|_{(L^\infty(Q))^N}. \quad (3.1)$$

To the solution u we associate any constant vector c such that

$$\|c\| \leq \rho \quad (3.2)$$

and we write

$$\tilde{u} = u - c. \quad (3.3)$$

Let also

$$\begin{aligned} \psi &\geq 0, & \psi &\in L^2(0, T, H^1(\Omega)), & \psi &\in L^\infty(Q), \\ \psi(x, t)|_{\partial\Omega} &= 0, & & \forall t \in (0, T) \text{ if } c \neq 0. \end{aligned} \quad (3.4)$$

We introduce the notation

$$\beta(x) = e^x - x - 1 \quad (3.5)$$

and the map $X(s) : \mathbb{R}^N \rightarrow \mathbb{R}^N$, defined backwards by the formulas

$$\begin{aligned} X_N(s) &= \exp[\beta(\gamma_N s_N) + \beta(-\gamma_N s_N)], \\ X_\nu(s) &= \exp[\beta(\gamma_\nu s_\nu) + \beta(-\gamma_\nu s_\nu) + X_{\nu+1}], \quad \nu = 1, \dots, N-1, \end{aligned} \quad (3.6)$$

where γ_ν are positive constants and $s = (s_1, \dots, s_N)$.

We note that

$$\frac{\partial X_\nu}{\partial s_\mu} = \begin{cases} 0 & \text{if } \mu < \nu \\ \gamma_\mu X_\nu \dots X_\mu (\beta'(\gamma_\mu s_\mu) - \beta'(-\gamma_\mu s_\mu)) & \text{if } \mu \geq \nu. \end{cases} \quad (3.7)$$

We call

$$X(x, t) = X(\tilde{u}(x, t)) \quad (3.8)$$

hence, clearly

$$DX_\nu = \sum_{\mu=\nu}^N \gamma_\mu X_\nu \dots X_\mu (\beta'(\gamma_\mu \tilde{u}_\mu) - \beta'(-\gamma_\mu \tilde{u}_\mu)) Du_\mu, \quad (3.9)$$

$$\frac{\partial X_\nu}{\partial t} = \sum_{\mu=\nu}^N \gamma_\mu X_\nu \dots X_\mu (\beta'(\gamma_\mu \tilde{u}_\mu) - \beta'(-\gamma_\mu \tilde{u}_\mu)) \frac{\partial u_\mu}{\partial t}, \quad (3.10)$$

from which we have the estimates

$$|DX| \leq c(\rho) |\tilde{u}| |Du|, \quad (3.11)$$

$$\left| \frac{\partial X}{\partial t} \right| \leq c(\rho) |\tilde{u}| \left| \frac{\partial u}{\partial t} \right|, \quad (3.12)$$

where in the sequel $c(\rho)$ will denote a constant (not precised explicitly) depending only on ρ (this assumes that the constants γ_ν are defined only as functions of ρ , which will be the case as shown below).

Note also, that thanks to formulas (3.7)

$$|X - X(0)| \leq c(\rho) |\tilde{u}|^2, \quad (3.13)$$

where $X(0)$ is the value of $X(s)$, for $s = 0$.

Moreover,

$$X \geq X(0) \geq 1. \quad (3.14)$$

We state the following:

Proposition 3.1. *If we assume (2.1, 2.2, 2.12, 2.13, 2.20), and if we have a solution u of (2.21–2.23), then there exist $\gamma_\nu(\rho)$, $c(\rho)$, such that, for any constant vector c subject to (3.3), and any ψ such that (3.4) holds,*

one has

$$\int_Q \psi \frac{\partial X_1}{\partial t} dx dt + \int_Q a_{ij} \frac{\partial X_1}{\partial x_j} \frac{\partial \psi}{\partial x_i} dx dt + \alpha \int_Q \psi |Du|^2 dx dt \leq c(\rho) \int_Q \psi \sum_{\nu=1}^N k_\nu dx dt. \quad (3.15)$$

Proof of Proposition 3.1. We take as a test function in (2.22)

$$v_\nu = \psi \gamma_\nu (\beta'(\gamma_\nu \tilde{u}_\nu) - \beta'(-\gamma_\nu \tilde{u}_\nu)) \prod_{\mu=1}^{\nu} X_\mu,$$

then from (3.10) it is easy to check that

$$\sum_{\nu=1}^N \frac{\partial u_\nu}{\partial t} v_\nu = \psi \frac{\partial X_1}{\partial t}.$$

Similarly,

$$\begin{aligned} \int_Q A(t) u_\nu v_\nu dx dt &= \int_Q a_{ij} \frac{\partial u_\nu}{\partial x_j} \frac{\partial \psi}{\partial x_i} \gamma_\nu (\beta'(\gamma_\nu \tilde{u}_\nu) - \beta'(-\gamma_\nu \tilde{u}_\nu)) \prod_{\mu=1}^{\nu} X_\mu dx dt \\ &\quad + \int_Q a_{ij} \frac{\partial u_\nu}{\partial x_j} \psi \gamma_\nu^2 (\beta''(\gamma_\nu \tilde{u}_\nu) + \beta''(-\gamma_\nu \tilde{u}_\nu)) \frac{\partial u_\nu}{\partial x_i} \prod_{\mu=1}^{\nu} X_\mu dx dt \\ &\quad + \int_Q a_{ij} \frac{\partial u_\nu}{\partial x_j} \psi \gamma_\nu (\beta'(\gamma_\nu \tilde{u}_\nu) - \beta'(-\gamma_\nu \tilde{u}_\nu)) \frac{\partial}{\partial x_i} \prod_{\mu=1}^{\nu} X_\mu dx dt \\ &= I + II + III, \end{aligned}$$

and as easily seen

$$\begin{aligned} I &= \int_Q a_{ij} \frac{\partial \psi}{\partial x_i} \frac{\partial X_1}{\partial x_j} dx dt, \\ II &\geq \alpha \int_Q \psi \gamma_\nu^2 |Du|^2 (e^{\gamma_\nu \tilde{u}_\nu} + e^{-\gamma_\nu \tilde{u}_\nu}) \prod_{\mu=1}^{\nu} X_\mu dx dt, \\ III &= \int_Q a_{ij} \frac{\partial F_\nu}{\partial x_j} \frac{\partial F_\nu}{\partial x_i} \psi \prod_{\mu=1}^{\nu} X_\mu dx dt, \end{aligned}$$

where $F_\nu = \log X_\nu$.

Now

$$\begin{aligned} \sum_{\nu=1}^{N-1} \int_Q Q_\nu Du_\nu v_\nu dx dt &= \sum_{\nu=1}^{N-1} \int_Q \psi Q_\nu (DF_\nu - DX_{\nu+1}) \prod_{\mu=1}^{\nu} X_\mu dx dt \\ &= \sum_{\nu=1}^{N-1} \int_Q \psi (Q_\nu - Q_{\nu-1}) DF_\nu \prod_{\mu=1}^{\nu} X_\mu dx dt - \int_Q \psi Q_{N-1} DF_N \prod_{\mu=1}^N X_\mu dx dt, \end{aligned}$$

where we have set $Q_0 = 0$. Since $Q_N = Q_{N-1}$, it follows that

$$\sum_{\nu=1}^{N-1} \int_Q Q_\nu Du_\nu v_\nu \, dx \, dt = \sum_{\nu=1}^{N-1} \int_Q \psi \tilde{Q}_\nu DF_\nu \prod_{\mu=1}^{\nu} X_\mu \, dx \, dt$$

with

$$\tilde{Q}_\nu = Q_\nu - Q_{\nu-1}, \quad \nu = 1, \dots, N-1.$$

Collecting results and performing additional majorations yields

$$\begin{aligned} \int_Q \psi \frac{\partial X_1}{\partial t} \, dx \, dt &+ a_{ij} \frac{\partial \psi}{\partial x_i} \frac{\partial X_1}{\partial x_j} \, dx \, dt + \alpha \sum_{\nu=1}^N \int_Q \psi \gamma_\nu^2 |Du_\nu|^2 \left(e^{\gamma u_\nu} + e^{-\gamma u_\nu} \right) \prod_{\mu=1}^{\nu} X_\mu \, dx \, dt \leq \sum_{\nu=1}^{N-1} \int_Q \psi \frac{a^{-1} \tilde{Q}_\nu \cdot \tilde{Q}_\nu}{4} \\ &+ \sum_{\nu=1}^N \int_Q \psi H_\nu^0 \gamma_\nu \left(e^{\gamma_\nu \tilde{u}_\nu} - e^{-\gamma_\nu \tilde{u}_\nu} \right) \prod_{\mu=1}^{\nu} X_\mu \, dx \, dt \end{aligned}$$

and thanks to (2.16),

$$\begin{aligned} &\leq \sum_{\nu=1}^N \int_Q \psi |Du_\nu|^2 \left[\frac{\|a^{-1}\|}{4} \sum_{\sigma=1}^{N-1} (K_\sigma + K_{\sigma-1})^2 \prod_{\mu=1}^{\sigma} X_\mu \right. \\ &\quad \left. + \sum_{\sigma=\nu}^N \gamma_\sigma K_\nu^\sigma |e^{\gamma_\sigma \tilde{u}_\sigma} - e^{-\gamma_\sigma \tilde{u}_\sigma}| \prod_{\mu=1}^{\sigma} X_\mu \right] \, dx \, dt \\ &\quad + \sum_{\nu=1}^N \int_Q \gamma_\nu \psi k_\nu |e^{\gamma_\nu \tilde{u}_\nu} - e^{-\gamma_\nu \tilde{u}_\nu}| \prod_{\mu=1}^{\nu} X_\mu \, dx \, dt. \end{aligned}$$

Suppose $\gamma_\nu(\rho)$ are chosen so that

$$\begin{aligned} \alpha \gamma_\nu^2 - 2\gamma_\nu K_\nu^\nu &\geq \frac{\|a^{-1}\|}{4} \left[\sum_{\sigma=1}^{\nu} (K_\sigma + K_{\sigma-1})^2 \right. \\ &\quad \left. + \sum_{\sigma=\nu+1}^{N-1} (K_\sigma + K_{\sigma-1})^2 \sum_{\mu=\nu+1}^{\sigma} X_\mu \right] + \sum_{\sigma=\nu+1}^N \gamma_\sigma K_\nu^\sigma |e^{\gamma_\sigma \tilde{u}_\sigma} - e^{-\gamma_\sigma \tilde{u}_\sigma}| \prod_{\mu=\nu+1}^{\sigma} X_\mu \end{aligned} \quad (3.16)$$

and $c(\rho)$ is such that

$$\gamma_\nu |e^{\gamma_\nu \tilde{u}_\nu} - e^{-\gamma_\nu \tilde{u}_\nu}| \prod_{\mu=1}^{\nu} X_\mu \leq c(\rho). \quad (3.17)$$

Then one obtains (3.15). The constants γ_ν depending only on ρ , can be defined from the inequalities (3.16) backwards, observing that X_μ can be majorized by a number depending only on $\rho, \gamma_\mu, \dots, \gamma_N$. The proof has been completed. \square

³We implicitly assume the matrix $a = a_{ij}$ symmetric. If not replace a for $\frac{a+a^*}{2}$ in the formula.

3.2. $L^2(0, T; H^1)$ estimate

If we pick $c = 0$ and $\psi = 1$ in (3.15), we obtain

$$\alpha \int_Q |Du|^2 \, dx \, dt \leq \int_\Omega X(u^0) \, dx + c(\rho) \int_Q \left(\sum_\nu k_\nu \right) \, dx \, dt \leq c(\rho), \quad (3.18)$$

and thus an estimate of the norm in $(L^2(0, T; H_0^1(\Omega)))^N$ follows from the L^∞ norm of u , denoted by ρ .

3.3. L^∞ estimate

We state the

Proposition 3.2. *We assume (2.1, 2.2, 2.7) (or (2.8, 2.9)) (or (2.10, 2.11)). If we have a solution u of (2.21–2.23), then there exists a number ρ depending only on the constants γ, γ_ν , the matrix Γ and $\|u^0\|_{L^\infty}$, so that*

$$\|u\|_{(L^\infty(Q))^N} \leq \rho. \quad (3.19)$$

Proof. We treat only the second alternative. The proof is typical for the other cases.

Since $u_t \in L^q$, $\nabla^2 u \in L^q$, q large, we have $\nabla u \in C^\alpha$. Suppose that the boundary ∂Q contains no minimum point of the function $\varphi = e^{-t} \sum_\nu u_\nu$. Let M be the set of points where φ attains its minimum m . M is closed and $\nabla \varphi = 0$ and $\nabla \sum_\nu u_\nu = 0$. We have $\sum_\nu H_\nu \geq -\gamma$ on M (assume (2.8, 2.9), to fix ideas). Let $U \subset\subset Q$ be an open neighborhood of M such that $\varphi = m + \varepsilon$ on ∂U , $\varepsilon > 0$ small. In addition, we may choose $\text{dist}(\partial U, M)$ so small, that $\sum_\nu H_\nu \geq -\gamma - \delta$ on U . (For simplicity we assume here that H_ν is also continuous in x . The general case is treated by approximating H_ν .)

$$\left(\frac{\partial}{\partial t} + A(t) \right) \sum_\nu u_\nu \geq -\gamma - \delta \text{ on } U$$

and

$$\left(\frac{\partial}{\partial t} + A(t) \right) \varphi + \varphi \geq -e^{-t}(\gamma + \delta) \text{ on } U.$$

Now, suppose that $m + \varepsilon \leq -e^{-t}(\gamma + \delta)$, $0 \leq t \leq T$. Then

$$\left(\frac{\partial}{\partial t} + A(t) \right) \varphi + (\varphi - (m + \varepsilon)) \geq 0 \text{ on } U,$$

and we obtain a contradiction if we test by $\min(\varphi - (m + \varepsilon), 0)$. (Take into account that $\varphi - (m + \varepsilon) = 0$ on ∂U and $\nabla \varphi$ cannot be zero a.e. on U .) This means that either $m + \varepsilon \geq -e^{-t}(\gamma + \delta)$ or the minimum is attained at ∂Q . In both cases, we arrive at a bound for $\sum_\nu u_\nu$ from below:

$$\sum_\nu u_\nu \geq -c. \quad (3.20)$$

With similar arguments, we achieve a bound for u_ν , $\nu = 1, \dots, N$ from above. Let $z = \Gamma u$. Then z is a solution of

$$\begin{aligned} \left(\frac{\partial}{\partial t} + A(t) \right) z &= H^\Gamma(z, Dz), \\ z|_\Sigma &= 0, \\ z(x, 0) &= \Gamma u^0(x) = z_0(x). \end{aligned}$$

Suppose that $\varphi_\nu = e^{-t} z_\nu$ does not attain its maximum at the boundary and let M_ν be the set of its maximum points. Then $\nabla \varphi_\nu = \nabla z_\nu = 0$ on M and

$$\left(\frac{\partial}{\partial t} + A(t) \right) \varphi_\nu + \varphi_\nu \leq e^{-t} \gamma_\nu \text{ on } M_\nu.$$

Introducing an adequate neighbourhood U_ν of M_ν as above we obtain

$$\left(\frac{\partial}{\partial t} + A(t) \right) \varphi_\nu + \varphi_\nu \leq e^{-t} (\gamma_\nu + \delta_\nu) \text{ on } U_\nu,$$

and if we suppose that $m_\nu = \max \varphi_\nu$ and $m_\nu - \varepsilon \geq e^{-\gamma} (\gamma_\nu + \delta_\nu)$, we obtain a contradiction by testing with $\max(0, \varphi_\nu - (m_\nu - \varepsilon))$. This gives a bound for z_ν from above:

$$z_\nu \leq c.$$

Since Γ satisfies the maximum principle, we conclude a bound for the u_ν from above. This concludes the proof of the proposition. \square

4. HÖLDER *A PRIORI* ESTIMATES

4.1. Notation

The Hölder estimates represent the core of the *a priori* estimates. We recall that for $\delta > 0$

$$C^{\delta, \frac{\delta}{2}}(Q) = \{ \varphi \mid |\varphi(x_1, t_1) - \varphi(x_2, t_2)| \leq c(|x_1 - x_2|^\delta + |t_1 - t_2|^{\delta/2}), \forall x_1, t_1; x_2, t_2 \in Q \}. \quad (4.1)$$

We shall use the characterization of $C^{\delta, \frac{\delta}{2}}(Q)$ as a Campanato space. We need some notation. If $z_0 = (x_0, t_0) \in Q$, we set

$$\begin{aligned} B_R(x_0) &= \{x \mid |x - x_0| \leq R\}, \\ \tilde{B}_R(x_0) &= B_R(x_0) \cap \Omega, \\ Q_R(z_0) &= B_R(x_0) \cap [(t_0 - R^2)^+, t_0], \\ \tilde{Q}(z_0) &= \tilde{B}_R(x_0) \cap [(t_0 - R^2)^+, t_0]. \end{aligned}$$

We shall write also B_R, Q_R etc. when there is no ambiguity on the center.

$$\varphi_{R, z_0} = \frac{\int_{\tilde{Q}_R(z_0)} \varphi \, dx \, dt}{|\tilde{Q}_R(z_0)|},$$

then the characterization of $C^{\delta, \frac{\delta}{2}}(Q)$ is given by (*cf.* Campanato [9], Da Prato [10])

$$\int_{\tilde{Q}_R(z_0)} |\varphi - \varphi_{R, z_0}|^2 dx dt \leq cR^{n+2+2\delta}, \quad \forall z_0 \in Q, R < R_0. \quad (4.2)$$

4.2. Green function

We shall make use of the Green functions associated to a point $z_0 \in Q$ and a number $\theta > 0$, denoted by $G_{x_0, t_0+\theta}(x, t)$, abbreviated as G_θ , defined for $x \in R^n$, $t < t_0 + \theta$, solution of

$$\begin{aligned} \frac{\partial G_\theta}{\partial t} + \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial G_\theta}{\partial x_i} \right) &= 0, \\ G_\theta(x_0, t_0 + \theta) &= \delta(x - x_0). \end{aligned} \quad (4.3)$$

This writing is formal, but G_θ is well defined for $t < t_0 + \theta$, and satisfies the estimates (Aronson [1]):

$$k_1(t_0 + \theta - t)^{-\frac{n}{2}} \exp\left(-\delta_1 \frac{|x - x_0|^2}{t_0 + \theta - t}\right) \leq G_\theta(x, t) \leq k_2(t_0 + \theta - t)^{-\frac{n}{2}} \exp\left(-\delta_2 \frac{|x - x_0|^2}{t_0 + \theta - t}\right), \quad (4.4)$$

where $k_1, \delta_1, k_2, \delta_2$ are positive and depend only on α and the L^∞ norm of the a_{ij} .

The function $s^{-\frac{n}{2}} \exp -\frac{\beta}{s}$ attains its maximum for $s > 0$ at $\hat{s} = \frac{2\beta}{n}$, and

$$s^{-\frac{n}{2}} \exp -\frac{\beta}{s} \leq (\hat{s})^{-\frac{n}{2}},$$

therefore we have, applying with $\beta = \delta_2|x - x_0|^2$,

$$G_\theta(x, t) \leq c|x - x_0|^{-n}. \quad (4.5)$$

A sharper estimate can be obtained if $t_0 + \theta - t$ is small compared to $|x - x_0|^2$. Indeed, if

$$0 < t_0 + \theta - t < \varepsilon^2|x - x_0|^2 \quad \text{with } \varepsilon^2 < \frac{2\delta_2}{n}, \quad (4.6)$$

then the above function is on its increasing side, and thus can be majorized by its value at $\varepsilon^2|x - x_0|^2$, hence

$$G_\theta(x, t) \leq \delta(\varepsilon)|x - x_0|^{-n} \quad (4.7)$$

with $\delta(\varepsilon) = k_2\varepsilon^{-n} \exp -\frac{\delta_2}{\varepsilon^2}$ tending to 0 as $\varepsilon \rightarrow 0$.

Furthermore, if

$$\varepsilon^2|x - x_0|^2 \leq t_0 + \theta - t \leq m^2|x - x_0|^2, \quad (4.8)$$

we can write

$$G_\theta(x, t) \geq \delta_0(\varepsilon)|x - x_0|^{-n} \quad (4.9)$$

with $\delta_0(\varepsilon) = k_1m^{-n} \exp -\frac{\delta_1}{\varepsilon}$, tending to 0 as $\varepsilon \rightarrow 0$.

4.3. Basic inequalities

We consider two cut off functions as follows:

$$\tau(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 2, \end{cases}$$

$$\tau \in C_0^\infty(\mathbb{R}^m), \quad 0 \leq \tau \leq 1$$

$$\beta(t) = \begin{cases} 1 & \text{if } t \geq -1, \\ 0 & \text{if } t \leq -4, \end{cases}$$

$\beta \in C^\infty(\mathbb{R})$, $0 \leq \beta \leq 1$, and we set

$$\begin{aligned} \tau_{R,x_0} &= \tau_R = \tau\left(\frac{x-x_0}{R}\right), \\ \beta_{R,t_0} &= \beta_R = \beta\left(\frac{t-t_0}{R^2}\right), \\ \eta_{R,z_0}(x,t) &= \eta_R = \tau_{R,x_0}\beta_{R,t_0}. \end{aligned}$$

Note that $0 \leq \eta_R \leq 1$, $\eta_R \in C^\infty(\mathbb{R}^{n+1})$, and

$$\eta_R = \begin{cases} 1 & \text{if } (x,t) \in Q_R, \\ 0 & \text{if } x \notin B_{2R} \text{ or } t \leq t_0 - 4R^2. \end{cases}$$

We apply (3.15) as follows: we pick the constant c in (3.2) such that

$$c = c^R = 0, \quad \text{if } B_{2R}(x_0) \cap (R^n - \Omega) \neq \emptyset, \quad (4.10)$$

and pick

$$\psi = \begin{cases} \eta_R^2 G_\theta & \text{if } t < t_0, \\ 0 & \text{if } t > t_0. \end{cases} \quad (4.11)$$

Thanks to the choice of the constant (4.10) we can assert that

$$(X_1 - X_1(0))\eta_R^2 = 0 \text{ on } \Sigma. \quad (4.12)$$

We write

$$\begin{aligned} & \int_Q \left(\psi \frac{\partial X_1}{\partial t} + a_{ij} \frac{\partial X_1}{\partial x_j} \frac{\partial \psi}{\partial x_i} \right) dx dt \\ &= \int_0^{t_0} \int_\Omega \left(\eta_R^2 G_\theta \frac{\partial}{\partial t} (X_1 - X_1(0)) + a_{ij} \frac{\partial}{\partial x_j} (X_1 - X_1(0)) \left(\frac{\partial G_\theta}{\partial x_i} \eta_R^2 + G_\theta \frac{\partial \eta_R^2}{\partial x_i} \right) \right) dx dt \end{aligned}$$

and using (4.3), we obtain

$$\begin{aligned}
&= \int_{\Omega} \eta_R^2 G_{\theta}(X_1 - X_1(0))(x, t_0) \, dx - \int_{\Omega} \eta_R^2 G_{\theta}(X_1 - X_1(0))(x, 0) \, dx \\
&\quad + \int_0^{t_0} \int_{\Omega} G_{\theta} \left(-(X_1 - X_1(0)) \frac{\partial \eta_R^2}{\partial t} + a_{ij} \frac{\partial X_1}{\partial x_j} \frac{\partial}{\partial x_i} \eta_R^2 \right) \, dx \, dt \\
&\quad - \int_0^{t_0} \int_{\Omega} \frac{\partial G_{\theta}}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j} \eta_R^2 (X_1 - X_1(0)) \, dx \, dt.
\end{aligned}$$

Thus from (3.15) we deduce

$$\begin{aligned}
\alpha \int_{\tilde{Q}_R(z_0)} |Du|^2 G_{\theta} \, dx \, dt &\leq \int_{\tilde{B}_{2R}(x_0)} G_{\theta}(X_1 - X_1(0))(x, 0) \, dx 1_{t_0 < 4R^2} \\
&\quad + \int_0^{t_0} \int_{\Omega} G_{\theta} \left((X_1 - X_1(0)) \frac{\partial \eta_R^2}{\partial t} - a_{ij} \frac{\partial X_1}{\partial x_j} \frac{\partial}{\partial x_i} \eta_R^2 \right) \, dx \, dt \\
&\quad + \int_0^{t_0} \int_{\Omega} \frac{\partial G_{\theta}}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j} \eta_R^2 (X_1 - X_1(0)) \, dx \, dt \\
&\quad + c(\rho) \int_{\tilde{Q}_{2R}} \left(G_{\theta} \sum_{y=1}^N k_{\nu} \right) \, dx \, dt. \tag{4.13}
\end{aligned}$$

Noting that, from the assumption on k_{ν}

$$\int_{\tilde{Q}_{2R}} \left(G_{\theta} \sum_{\nu=1}^N k_{\nu} \right) \, dx \, dt \leq c \left(\int_{\tilde{Q}_{2R}} G_{\theta}^q \, dx \, dt \right)^{\frac{1}{q}} \leq cR^{2 - \frac{n+2}{q}}.$$

Setting $\beta_0 = 2 - \frac{n+2}{q} > 0$, and making use of (3.11, 3.13) and the definition of the cut off functions, we obtain the basic inequality

Lemma 4.1. *The following inequality holds:*

$$\begin{aligned}
\alpha \int_{\tilde{Q}_R(z_0)} |Du|^2 G_{\theta} \, dx \, dt &\leq c(\rho) \int_{\tilde{B}_{2R}(x_0)} |u^0 - c^R|^2 G_{\theta}(x, 0) 1_{t_0 < 4R^2} \\
&\quad + c(\rho) \int_{\tilde{Q}_{2R}(z_0) - \tilde{Q}_R(z_0)} \left(\frac{|u - c^R|^2}{R^2} + |Du|^2 \right) G_{\theta} \, dx \, dt \\
&\quad + c(\rho) \int_{(t_0 - 4R^2)^+}^{t_0} \int_{\tilde{B}_{2R} - \tilde{B}_R} \eta_R^2 |u - c^R|^2 |DG_{\theta}|^2 G_{\theta}^{-1} \, dx \, dt + c(\rho) R^{\beta_0}. \tag{4.14}
\end{aligned}$$

4.4. Auxiliary result

To take care of the last integral in the right hand side of (4.14) one estimates the quantity

$$Z = \int_{(t_0-4R^2)^+}^{t_0} \int_{\tilde{B}_{2R}-\tilde{B}_R} \eta_R^2 |u - c^R|^2 |DG_\theta|^2 G_\theta^{-\frac{3}{2}} dx dt. \tag{4.15}$$

Let ρ be a new cut off function such that

$$\begin{aligned} \rho &\in C_0^\infty(\mathbb{R}^n), \quad \rho = 0 \text{ for } |x| \leq \frac{1}{2}, \\ 0 \leq \rho &\leq \tau, \quad \rho = \tau \text{ for } |x| \geq 1, \end{aligned} \tag{4.16}$$

and set $\rho_R(x) = \rho\left(\frac{x-x_0}{R}\right)$ and

$$\varphi_R(x, t) = \rho_R(x)\beta_R(t) \tag{4.17}$$

so that

$$\varphi_R = \eta_R \text{ on } (\tilde{B}_{2R} - \tilde{B}_R) \times [(t_0 - 4R^2)^+, t_0].$$

We test the Green function equation (4.3) with $G_\theta^{-\frac{1}{2}}|u - c^R|^2\varphi_R^2$ and (2.22) with $(u - c^R)\varphi_R^2 G_\theta^{\frac{1}{2}}$, integrating on $[(t_0 - 4R^2)^+, t_0]$ and Ω , and noting that these functions vanish on Σ . Combining we obtain

$$\begin{aligned} &\int_{(t_0-4R^2)^+}^{t_0} \int_{\tilde{B}_{2R}-\tilde{B}_{R/2}} a_{ij} \frac{\partial G_\theta}{\partial x_j} \frac{\partial G_\theta}{\partial x_i} G_\theta^{-\frac{3}{2}} |u - c^R|^2 \varphi_R^2 dx dt \leq 4 \int_{\tilde{B}_{2R}-\tilde{B}_{R/2}} \rho_R^2 |u^0 - c^R|^2 G_\theta^{\frac{1}{2}}(x, 0) dx 1_{t_0 < 4R^2} \\ &+ 8 \int_{(t_0-4R^2)^+}^{t_0} \int_{\tilde{B}_{2R}-\tilde{B}_{R/2}} H(u - c^R) \varphi_R^2 G_\theta^{\frac{1}{2}} dx dt + 4 \int_{(t_0-4R^2)^+}^{t_0} \int_{\tilde{B}_{2R}-\tilde{B}_{R/2}} a_{ij} \frac{\partial}{\partial x_i} G_\theta^{\frac{1}{2}} \frac{\partial}{\partial x_j} \varphi_R^2 |u - c^R|^2 dx dt \\ &+ 4 \int_{(t_0-4R^2)^+}^{t_0} \int_{\tilde{B}_{2R}-\tilde{B}_{R/2}} G_\theta^{\frac{1}{2}} \left[|u - c^R|^2 \frac{\partial \varphi_R^2}{\partial t} - a_{ij} \frac{\partial}{\partial x_j} |u - c^R|^2 \frac{\partial}{\partial x_i} \varphi_R^2 \right] dx dt. \end{aligned}$$

Majorizing the third integral on the right hand side by Young's inequality and combining with the left hand side, noting that Z is smaller than the left hand side leads to

$$Z \leq c \int_{\tilde{B}_{2R}-\tilde{B}_{R/2}} |u^0 - c^R|^2 G_\theta^{\frac{1}{2}}(x, 0) dx 1_{t_0 < 4R^2} + c \int_{(t_0-4R^2)^+}^{t_0} \int_{\tilde{B}_{2R}-\tilde{B}_{R/2}} \left(\frac{|u - c^R|^2}{R^2} + |Du|^2 \right) G_\theta^{\frac{1}{2}} dx dt + c(\rho)R^{\frac{n}{2} + \beta_0}. \tag{4.18}$$

4.5. Transformation of the basic inequality

We want to make use of (4.18) to estimate the last integral in the right hand side of (4.14). We write it as the sum of two terms (ε small)

$$I_\varepsilon = \int_{\tilde{B}_{2R}-\tilde{B}_R} \int_{(t_0-4R^2)^+}^{t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2)^+} \eta_R^2 |u - c^R|^2 |DG_\theta|^2 G_\theta^{-1} dx dt,$$

$$II_\varepsilon = \int_{\tilde{B}_{2R}-\tilde{B}_R} \int_{t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2)^+}^{t_0} \eta_R^2 |u - c^R|^2 |DG_\theta|^2 G_\theta^{-1} dx dt.$$

Consider II_ε . For $t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2)^+ < t < t_0$, we have

$$t_0 + \theta - t < \varepsilon^2 R^2 < \varepsilon^2 |x - x_0|^2$$

since we integrate on $\tilde{B}_{2R} - \tilde{B}_R$, hence $R < |x - x_0| < 2R$. Therefore from (4.7)

$$G_\theta(x, t) \leq \delta(\varepsilon) |x - x_0|^{-n} \leq \delta(\varepsilon) R^{-n},$$

hence, clearly

$$II_\varepsilon \leq \delta^{1/2}(\varepsilon) R^{-\frac{n}{2}} Z,$$

which implies from (4.18)

$$II_\varepsilon \leq c \int_{\tilde{B}_{2R}-\tilde{B}_{R/2}} |u^0 - c^R|^2 (G_\theta(x, 0) + \delta(\varepsilon) R^{-n}) dx 1_{t_0 < 4R^2}$$

$$+ c \int_{(t_0-4R^2)^+}^{t_0} \int_{\tilde{B}_{2R}-\tilde{B}_{R/2}} \left(|Du|^2 + \frac{|u - c^R|^2}{R^2} \right) (G_\theta + \delta(\varepsilon) R^{-n}) dx dt + c(\rho) R^{\beta_0}. \quad (4.19)$$

In I_ε , we use (4.5) and since

$$R < |x - x_0| < 2R$$

we have $G_\theta \leq cR^{-n}$, hence

$$I_\varepsilon \leq cR^{-\frac{n}{2}} \int_{\tilde{B}_{2R}-\tilde{B}_R} \int_{(t_0-4R^2)^+}^{t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2)^+} \eta_R^2 |u - c^R|^2 |DG_\theta|^2 G_\theta^{-3/2} dx dt.$$

This integral is similar to Z up to the upper level of integration of t . This leads to

$$\begin{aligned} I_\varepsilon &\leq cR^{-\frac{n}{2}} \int_{\tilde{B}_{2R} - \tilde{B}_{R/2}} |u^0 - c^R|^2 G_\theta^{\frac{1}{2}}(x, 0) dx 1_{t_0 < 4R^2} \\ &\quad + cR^{-\frac{n}{2}} \int_{(t_0 - 4R^2)^+}^{t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2)^+} \int_{\tilde{B}_{2R} - \tilde{B}_{R/2}} \left(\frac{|u - c^R|^2}{R^2} + |Du|^2 \right) G_\theta^{\frac{1}{2}} dx dt + c(\rho)R^{\beta_0}. \end{aligned}$$

For

$$(t_0 - 4R^2)^+ < t < t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2)^+$$

we have, since $x \in \tilde{B}_{2R} - \tilde{B}_{R/2}$

$$\frac{\varepsilon^2}{4} |x - x_0|^2 < \varepsilon^2 R^2 < t_0 + \theta - t < 4R^2 + \theta.$$

Suppose we restrict θ so that

$$\theta \leq q^2 R^2, \quad q > 1 \tag{4.20}$$

then

$$\frac{\varepsilon^2}{4} |x - x_0|^2 < t_0 + \theta - t < (4 + q^2)R^2 < 4(4 + q^2)|x - x_0|^2$$

and thus (4.8) applies with ε changed into $\frac{\varepsilon}{2}$ and $m^2 = 4(4 + q^2)$.

It follows that

$$G_\theta(x, t) \geq \delta \left(\frac{\varepsilon}{2} \right) |x - x_0|^{-n} \geq \delta_0 \left(\frac{\varepsilon}{2} \right) 2^{-n} R^{-n}.$$

Using

$$G_\theta(x, 0) \leq cR^{-n}$$

we finally obtain

$$\begin{aligned} I_\varepsilon &\leq cR^{-n} \int_{\tilde{B}_{2R}} |u^0 - c^R|^2 dx 1_{t_0 < 4R^2} + cK(\varepsilon) \int_{(t_0 - 4R^2)^+}^{t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2)^+} \int_{\tilde{B}_{2R} - \tilde{B}_{R/2}} \left(\frac{|u - c^R|^2}{R^2} + |Du|^2 \right) G_\theta dx dt + c(\rho)R^{\beta_0} \end{aligned} \tag{4.21}$$

with $K(\varepsilon) = 2^{\frac{n}{2}} \delta_0^{-\frac{1}{2}} \left(\frac{\varepsilon}{2} \right) \rightarrow +\infty$ as $\varepsilon \rightarrow 0$.

Collecting results we transform the basic inequality (4.14) as follows:

Lemma 4.2. *The following inequality holds (provided (4.20) holds)*

$$\begin{aligned}
\alpha \int_{\tilde{Q}_R(z_0)} |Du|^2 G_\theta \, dx \, dt &\leq c(\rho) \int_{B_{2R}(x_0)} |u^0 - c^R|^2 (G_\theta(x, 0) + R^{-n}) \mathbf{1}_{t_0 < 4R^2} \\
&+ c(\rho) \int_{\tilde{Q}_{2R}(z_0) - \tilde{Q}_{R/2}(z_0)} |Du|^2 (K(\varepsilon)G_\theta + \delta(\varepsilon)R^{-n}) \, dx \, dt \\
&+ c(\rho) \int_{\tilde{Q}_{2R}(z_0) - \tilde{Q}_{R/2}(z_0)} \frac{|u - c^R|^2}{R^2} (G_\theta + \delta(\varepsilon)R^{-n}) \, dx \, dt \\
&+ c(\rho)K(\varepsilon) \int_{(t_0 - 4R^2)^+}^{t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2)^+} \int_{\tilde{B}_{2R} - \tilde{B}_{R/2}} \frac{|u - c^R|^2}{R^2} G_\theta \, dx \, dt + c(\rho)R^{\beta_0} \tag{4.22}
\end{aligned}$$

with $\delta(\varepsilon) \rightarrow 0$, $K(\varepsilon) \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

4.6. Choice of the constant c^R

In the case (4.10) we have $c^R = 0$. Since u^0 vanishes on the boundary, we have

$$|u^0(x)| \leq cR, \quad \text{for } x \in \tilde{B}_{2R}(x_0)$$

and thus

$$\int_{\tilde{B}_{2R}(x_0)} |u^0|^2 (G_\theta(x, 0) + R^{-n}) \, dx \leq cR^2. \tag{4.23}$$

Next, considering the integral

$$\int_{(t_0 - 4R^2)^+}^{t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2)^+} \int_{\tilde{B}_{2R} - \tilde{B}_{R/2}} \frac{|u|^2}{R^2} G_\theta \, dx \, dt,$$

we may use inside the integral

$$G_\theta \leq cR^{-n} \leq K(\varepsilon)G_\theta$$

together with Poincaré's inequality to assert that

$$I_1 \leq cK(\varepsilon) \int_{\tilde{Q}_{4R} - \tilde{Q}_{R/2}} |Du|^2 G_\theta \, dx \, dt.$$

Next

$$I_2 = \int_{\tilde{Q}_{2R} - \tilde{Q}_{R/2}} \frac{|u|^2}{R^2} G_\theta \, dx \, dt = \int_{\tilde{B}_{2R} - \tilde{B}_{R/2}} \int_{(t_0 - 4R^2)^+}^{t_0} \frac{|u|^2}{R^2} G_\theta \, dx \, dt + \int_{\tilde{B}_{R/2}} \int_{(t_0 - 4R^2)^+}^{(t_0 - \frac{R^2}{4})^+} \frac{|u|^2}{R^2} G_\theta \, dx \, dt.$$

In the second integral, we may use inside

$$cR^{-n} \leq G_\theta \leq cR^{-n}$$

together with Poincaré’s inequality. In the first integral we decompose the interval of time into $[(t_0 - 4R^2)^+, t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2)^+]$ and $[t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2)^+, t_0]$. In the first interval we recover I_1 . In the second interval we can majorize G_θ by $\delta(\varepsilon)R^{-n}$, and use Poincaré again.

Collecting results we can assert that

$$\alpha \int_{\tilde{Q}_R(z_0)} |Du|^2 G_\theta \, dx \, dt \leq \int_{\tilde{Q}_{4R}(z_0) - \tilde{Q}_{R/2}(z_0)} |Du|^2 (G_\theta K_1(\varepsilon) + \delta_1(\varepsilon)R^{-n}) \, dx \, dt + c(\rho)R^{\beta_0}, \tag{4.24}$$

where again

$$\delta_1(\varepsilon) \rightarrow 0, \quad K_1(\varepsilon) \rightarrow \infty \quad \text{as } \varepsilon \rightarrow 0.$$

We now assume that

$$B_{2R}(x_0) \subset \Omega. \tag{4.25}$$

Note that $\tilde{B}_{2R} = B_{2R}$, $\tilde{Q}_{2R} = Q_{2R}$.

To proceed, we introduce the notation

$$u_{R,x_0,t}^\rho = \frac{\int_{B_{2R}} u(x,t) \rho_R \, dx}{\int_{B_{2R}} \rho_R \, dx},$$

where ρ_R has been defined in (4.16). We shall need the following result:

Lemma 4.3. *Let $(t_0 - 4R^2)^+ \leq s \leq t \leq t_0$, then one has*

$$|u_{R,x_0,t}^\rho - u_{R,x_0,s}^\rho|^2 \leq cR^{-n} \int_s^t \int_{B_{2R} - B_{R/2}} |Du|^2 + cR^{2\beta_0}. \tag{4.26}$$

Proof. It is obtained by testing (2.22) with ρ_R and integrating over x and the interval s, t . □

Consider the case

$$t_0 + \theta - \varepsilon^2 R^2 \leq 0, \tag{4.27}$$

then we take

$$c^R = (u^0)^\rho = u_{R,x_0,0}^\rho. \tag{4.28}$$

Note that in $B_{2R}(x_0)$, we have $|u(x) - (u^0)^\rho| \leq cR$, hence the first term in the right hand side of (4.22) is bounded by $c(\rho)R^2$. Next applying (4.26) we have

$$|u_{R,x_0,t}^\rho - c^R|^2 \leq cR^{-n} \int_0^{t_0} \int_{B_{2R}-B_{R/2}} |Du|^2 + cR^{2\beta_0}. \quad (4.29)$$

Moreover, in (4.22) the last integral vanishes. Consider

$$\begin{aligned} \int_{Q_{2R}-Q_{R/2}} \frac{|u - c^R|^2}{R^2} (G_\theta + \delta(\varepsilon)R^{-n}) \, dx \, dt &= \int_0^{t_0} \int_{B_{2R}-B_{R/2}} \frac{|u - c^R|^2}{R^2} (G_\theta + \delta(\varepsilon)R^{-n}) \, dx \, dt \\ &\leq \delta'(\varepsilon)R^{-n} \int_0^{t_0} \int_{B_{2R}-B_{R/2}} \frac{|u - c^R|^2}{R^2} \, dx \, dt \end{aligned}$$

since in the integral $G_\theta(x, t) \leq 2^{-n}\delta(2\varepsilon)R^{-n}$, by (4.7).

Using

$$\int_0^{t_0} \int_{B_{2R}-B_{R/2}} \frac{|u - c^R|^2}{R^2} \, dx \, dt \leq 2 \int_0^{t_0} \int_{B_{2R}-B_{R/2}} \frac{|u - u_{R,x_0,t}^\rho|^2}{R^2} \, dx \, dt + 2R^{n-2} \int_0^{t_0} |u_{R,x_0,t}^\rho - c^R|^2 \, dt$$

and Poincaré's inequality, we obtain again that the inequality (4.24) holds.

It remains to consider the case

$$t_0 + \theta - \varepsilon^2 R^2 > 0. \quad (4.30)$$

We take this time

$$c^R = \frac{\int_{(t_0-4R^2)^+}^{t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2)} u_{R,x_0,t}^\rho \, dt}{\int_{(t_0-4R^2)^+}^{t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2)} dt}. \quad (4.31)$$

To evaluate the first term on the right hand side of (4.22), we restrict ourselves to $t_0 < 4R^2$. We remark that

$$|c^R - (u^0)^\rho|^2 \leq \frac{\int_0^{t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2)} |u_{R,x_0,t}^\rho - (u^0)^\rho|^2 \, dt}{\int_0^{t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2)} dt},$$

and from Lemma 4.3

$$|c^R - (u^0)^\rho|^2 \leq cR^{-n} \int_0^{t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2)} \int_{B_{2R}-B_{R/2}} |Du|^2 \, dx \, dt + cR^{2\beta_0}.$$

Using inside the integral $G_\theta(x, t) \geq \delta_0(\frac{\varepsilon}{2})2^{-n}R^{-n}$, we obtain

$$|c^R - (u^0)^\rho|^2 \leq K'(\varepsilon) \int_{Q_{2R}(z_0) - Q_{R/2}(z_0)} |Du|^2 G_\theta(x, t) \, dx \, dt + cR^{2\beta_0},$$

which is similar to the right hand side of (4.24).

Consider the term

$$J = \int_{Q_{2R}(z_0) - Q_{R/2}(z_0)} \frac{|u - c^R|^2}{R^2} (G_\theta + \delta(\varepsilon)R^{-n}) \, dx \, dt,$$

then we can reduce it to the study of

$$J_1 = \int_{Q_{2R}(z_0) - Q_{R/2}(z_0)} \frac{|u - u_{R,x_0,t}^\rho|^2}{R^2} (G_\theta + \delta(\varepsilon)R^{-n}) \, dx \, dt$$

and

$$J_2 = \int_{Q_{2R}(z_0) - Q_{R/2}(z_0)} \frac{|u_{R,x_0,t}^\rho - c^R|^2}{R^2} (G_\theta + \delta(\varepsilon)R^{-n}) \, dx \, dt.$$

We treat J_1 as in the case when $c^R = 0$ (see above). Note that

$$J_2 \leq c \int_{(t_0 - 4R^2)^+}^{t_0} \frac{|u_{R,x_0,t}^\rho - c^R|^2}{R^2} \, dt$$

and the interval of integration in time is split into

$$[(t_0 - 4R^2)^+, t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2)] \quad \text{and} \quad [t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2), t_0].$$

In the second interval, we use the fact that

$$\int_{t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2)}^{t_0} \frac{|u_{R,x_0,t}^\rho - u_{R,x_0,t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2)}^\rho|^2}{R^2} \, dt \leq c\varepsilon^2 \left[R^{-n} \int_{(t_0 - 4R^2)^+}^{t_0} \int_{B_{2R} - B_{R/2}} |Du|^2 \, dx \, dt + R^{2\beta_0} \right],$$

which is a term like in the right hand side of (4.24).

For $|u_{R,x_0,t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2)}^\rho - c^R|^2$ and the first interval of time we need to use (4.26) with $s \leq t \leq t_0 \wedge (t_0 + \theta - \varepsilon^2 R^2)$, in which case we rely on the fact that $R^{-n} \leq K(\varepsilon)G_\theta(x, t)$ to obtain contributions of the type of the right hand side of (4.24).

Collecting results, thanks to the appropriate choice of c^R , we can assert that:

Lemma 4.4. *Provided (4.20) holds, the inequality (4.24) applies, with $\delta_1(\varepsilon) \rightarrow 0$, $K_1(\varepsilon) \rightarrow \infty$ as $\varepsilon \rightarrow 0$.*

4.7. Statement of the main property

We can now state the

Proposition 4.1. *Under the conditions of Proposition 3.1 one has the property*

$$\int_{\tilde{Q}_{R(z_0)}} |Du|^2 dx dt \leq c(\rho)R^{n+2\beta} \quad (4.32)$$

for some $\beta > 0$, $2\beta < \beta_0$.

Proof. Using

$$cR^{-n} \leq G_{R^2}(x, t) \quad \text{for } t < t_0, x \in B_{4R}(x_0)$$

we assert from (4.24), that for $\theta < q^2 R^2$,

$$\int_{\tilde{Q}_{\frac{R}{2}(z_0)}} |Du|^2 G_\theta dx dt \leq K(\varepsilon) \int_{\tilde{Q}_{4R(z_0)} - \tilde{Q}_{\frac{R}{2}(z_0)}} |Du|^2 G_\theta dx dt + \delta(\varepsilon) \int_{\tilde{Q}_{4R(z_0)} - \tilde{Q}_{\frac{R}{2}(z_0)}} G_{R^2} dx dt + c(\rho)R^{\beta_0}.$$

Then for $2\beta < \beta_0$ and $8^{2\beta} < 1$, setting

$$\varphi(R) = R^{-2\beta} \sup_{\theta < q^2 R^2} \int_{\tilde{Q}_{R/2}} |Du|^2 G_\theta dx dt,$$

we deduce, by the hole filling trick that

$$\varphi(R) \leq \nu(\varepsilon)\varphi(8R) + c(\rho),$$

where $\nu(\varepsilon) < 1$ for a convenient choice of ε sufficiently small. This implies

$$\sup_{0 < R < R_1} R^{-2\beta} \sup_{\theta < 4q^2 R^2} \int_{\tilde{Q}_R} |Du|^2 G_\theta dx dt \leq c(\rho),$$

and in particular

$$R^{-2\beta} \int_{\tilde{Q}_R} |Du|^2 G_{R^2} dx dt \leq c(\rho)$$

hence also (4.32), and the proof has been completed. \square

To obtain the Hölder property from (4.32), it is sufficient to observe that

$$\int_{\tilde{Q}_R(z_0)} |u - u_{R,z_0}|^2 dx dt \leq cR^2 \int_{\tilde{Q}_{4R(z_0)}} |Du|^2 dx dt + cR^{n+2+2\beta_0}. \quad (4.33)$$

When $B_{2R}(x_0) \cap (R^n - \Omega) \neq \emptyset$, this follows from Poincaré's inequality. When $B_{2R}(x_0) \subset \Omega$, it is obtained by using the analogue of Lemma 4.2 with τ_R as a test function.

From (4.32) and (4.33), thanks to the characterization (4.2), we obtain the Hölder estimate

$$\int_{\tilde{Q}_R(z_0)} |u - u_{R,z_0}|^2 dx dt = c(\rho) R^{n+2+2\beta}. \quad (4.34)$$

Remark 4.1. Note that β does not depend on ρ , but only on $\beta_0 = 2 - \frac{n+2}{q}$.

5. A PRIORI ESTIMATES IN $L^p(0, T; W^{2,p}(\Omega))$

5.1. Statement of the result

We want to show now an estimate of u in $L^p(0, T; W^{2,p})$ and $\frac{\partial u}{\partial t} \in L^p(Q)$, with $p \leq q$.

We shall rely on the method used in the elliptic theory (see Frehse [11]). We need to know that u is Hölder and the following property of the Hamiltonian summarizes what is needed

$$|H(x, t, u, Du)| \leq K|Du|^2 + f, \quad f \in L^q > \frac{n}{2} + 1, \quad (5.1)$$

which of course is satisfied in the conditions of Proposition 3.1. Without loss of generality we can take

$$u^0 = 0. \quad (5.2)$$

Indeed defining z by

$$\begin{aligned} \frac{\partial z}{\partial t} + A(t)z &= 0, \\ z|_{\Sigma} &= 0, \quad z(x, 0) = u^0, \end{aligned}$$

then from the linear theory and the assumption (2.20), z has full regularity. By a simple translation, we obtain a problem where the property (5.1) is satisfied as well as (5.2).

We state the

Proposition 5.1. *Under the conditions of Proposition 3.1 one has the property*

$$\left\| \frac{\partial u}{\partial t} \right\|_{L^p(Q)} + \|u\|_{L^p(0, T; W^{2,p}(\Omega))} \leq c(\rho) \quad \forall p \leq q. \quad (5.3)$$

5.2. Proof of Proposition 5.1

Let $Q_R(x_0)$ be the cube of center x_0 and side length R . We note

$$\tilde{Q}_R(x_0) = \Omega \cap Q_R(x_0).$$

Let τ be a Lipschitz continuous function such that

$$\tau = \begin{cases} 1 & \text{on } Q_1(0), \\ 0 & \text{outside } Q_2(0), \end{cases}$$

and denote

$$\tau_R(x) = \tau\left(\frac{x - x_0}{R}\right) = \begin{cases} 1 & \text{on } Q_R(x_0), \\ 0 & \text{outside } Q_{2R}(x_0). \end{cases}$$

We shall denote by K constants depending only on the data, $K(\rho)$ those depending also on the L^∞ norm of u , which has been denoted by ρ , and K_R constants depending on R . Calculating

$$\frac{\partial}{\partial t}(u^\nu \tau_R) + A(t)(u^\nu \tau_R),$$

and applying the linear theory of parabolic equations, we can assert, making use of (5.1):

$$\left(\int_0^T \int_{\tilde{Q}_R} \left| \frac{\partial u}{\partial t} \right|^p dx dt \right)^{\frac{1}{p}} + \left(\int_0^T \int_{\tilde{Q}_R} |D^2 u|^p dx dt \right)^{\frac{1}{p}} \leq K \left(\int_0^T \int_{\tilde{Q}_{2R}} |Du|^{2p} dx dt \right)^{\frac{1}{p}} + K_R, \quad \forall p \leq q. \quad (5.4)$$

We notice that

$$\int_0^T \int_{\tilde{Q}_{2R}} |Du|^{2p} dx dt \leq \int_0^T \int_{\tilde{Q}_{4R}} |\tau_{2R} Du|^{2p} dx dt. \quad (5.5)$$

Let $c^R(t)$ be a vector, depending on t but not on x , such that

$$\begin{aligned} c^R &= 0 \quad \text{if } Q_{4R}(x_0) \cap (\mathbb{R}^n - \Omega) \neq \emptyset \\ c^R(t) &= \frac{1}{2} \left(\min_{x \in Q_{4R}} u(x, t) + \max_{x \in Q_{4R}} u(x, t) \right) \quad \text{if } Q_{4R}(x_0) \subset \Omega. \end{aligned} \quad (5.6)$$

Therefore $\tau_{2R}(u - c^R) = 0$ on Σ .

Performing an integration by parts we have

$$\begin{aligned} \int_0^T \int_{\tilde{Q}_{4R}} |\tau_{2R} Du|^{2p} dx dt &= - \int_0^T \int_{\tilde{Q}_{4R}} \tau_{2R}^{2p} |Du|^{2p-2} \Delta u_\nu (u_\nu - c_\nu^R) dx dt \\ &\quad - 2p \int_0^T \int_{\tilde{Q}_{4R}} \tau_{2R}^{2p-1} \frac{\partial \tau_{2R}}{\partial x_i} |Du|^{2p-2} \frac{\partial u_\nu}{\partial x_i} (u_\nu - c_\nu^R) dx dt \\ &\quad - 2(p-1) \int_0^T \int_{\tilde{Q}_{4R}} \tau_{2R}^{2p} |Du|^{2p-4} \frac{\partial u_\mu^2}{\partial x_i \partial x_j} \frac{\partial u_\mu}{\partial x_j} \frac{\partial u_\nu}{\partial x_i} (u_\nu - c_\nu^R) dx dt \\ &\leq c_p \int_0^T \int_{\tilde{Q}_{4R}} \tau_{2R}^{2p} |Du|^{2p-2} |D^2 u| |u - c^R| dx dt \\ &\quad + c_p \int_0^T \int_{\tilde{Q}_{4R}} \tau_{2R}^{2p-1} |D\tau_{2R}| |Du|^{2p-1} |u - c^R| dx dt \end{aligned}$$

hence

$$\leq c_p \int_0^T \int_{\tilde{Q}_{4R}} \tau_{2R}^{2p} |D^2 u|^p |u - c^R| \, dx \, dt + c_p \int_0^T \int_{\tilde{Q}_{4R}} \tau_{2R}^{2p} |Du|^{2p} |u - c^R| \, dx \, dt + c_p \int_0^T \int_{\tilde{Q}_{4R}} |D\tau_{2R}|^{2p} |u - c^R| \, dx \, dt,$$

and from the Hölderianity of u

$$\leq K_p(\rho) R^\beta \int_0^T \int_{\tilde{Q}_{4R}} \tau_{2R}^{2p} |Du|^{2p} \, dx \, dt + K_p(\rho) R^\beta \int_0^T \int_{\tilde{Q}_{4R}} \tau_{2R}^{2p} |D^2 u|^p \, dx \, dt + K_{p,R}(\rho).$$

Combining with (5.4, 5.5), we arrive at

$$\int_0^T \int_{\tilde{Q}_R} |D^2 u|^p \, dx \, dt \leq \frac{K'_p(\rho) R^\beta}{1 - K_p(\rho) R^\beta} \int_0^T \int_{\tilde{Q}_{4R}} |D^2 u|^p \, dx \, dt + K'_{p,R}(\rho), \quad (5.7)$$

provided of course $1 - K_p(\rho) R^\beta > 0$. So take $R < R_1$ with $1 - K_p(\rho) R_1^\beta > 0$ and define

$$\xi = \sup_{\substack{x_0 \in \Omega \\ R < R_1}} \int_0^T \int_{\tilde{Q}_R} |D^2 u|^p \, dx \, dt,$$

then notice that

$$\sup_{\substack{x_0 \in \Omega \\ R < R_1}} \int_0^T \int_{\tilde{Q}_{4R}} |D^2 u|^p \, dx \, dt \leq K \xi,$$

and thus

$$\xi \leq \frac{K''_p(\rho) R^\beta}{1 - K_p(\rho) R^\beta} \xi + K'_{p,R}(\rho).$$

We may assume R_1 such that

$$\frac{K''_p(\rho) R_1^\beta}{1 - K_p(\rho) R_1^\beta} < 1,$$

which leads to an estimate on ξ . By a finite covering of Ω by cubes Q_R , we easily conclude the estimate (5.3). The proof has been completed.

6. PROOF OF THEOREM 2.1

If we consider the approximated Hamiltonians

$$H^\varepsilon(x, t, \lambda, p) = \frac{H(x, t, \lambda p)}{1 + \varepsilon |H(x, t, \lambda, p)|},$$

then it is clear that it satisfies all the assumptions of H , with the same constants, independently of ε . Now if we consider the approximate problem

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} + A(t)u^\varepsilon &= H^\varepsilon(x, t, u^\varepsilon, Du^\varepsilon), \\ u^\varepsilon(0) &= u^0, \quad u^\varepsilon|_\varepsilon = 0, \end{aligned}$$

whose solution in $L^2(0, T; W^{2,p}(\Omega))$, $\frac{\partial u^\varepsilon}{\partial t} \in L^p(Q)$, exists for any p , since H^ε is bounded by $\frac{1}{\varepsilon}$, then the a priori estimates obtained in (3.18, 3.19, 4.34) and (5.3) are valid for u^ε . By extracting a subsequence which converges in the appropriate spaces, it is fairly easy to see that the limit is a solution of (2.21–2.23).

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