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Smooth Static Solutions of the Einstein/Yang-Mills Equations

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Abstract. We consider the Einstein/Yang-Mills equations in $3+1$ space time dimensions with $SU(2)$ gauge group and prove rigorously the existence of a globally defined smooth static solution. We show that the associated Einstein metric is asymptotically flat and the total mass is finite. Thus, for non-abelian gauge fields the Yang-Mills repulsive force can balance the gravitational attractive force and prevent the formation of singularities in spacetime.

1. Introduction

It is well-known that there are no non-singular symmetric static solutions of the vacuum Einstein equations, $R_{ij}-\frac{1}{2}Rg_{ij}=0$; indeed, the unique static solution is the celebrated Schwarzschild metric which is singular at $r=0$ [1]. Similarly, the pure Yang-Mills equations $d^*F=0$ have no static regular solutions [3, 4] and if one couples Einstein's equations to Maxwell's equations,

$$R_{ij}-\tfrac{1}{2}Rg_{ij}=\sigma T_{ij}, \quad d^*F_{ij}=0$$

(where T_{ij} is the stress energy tensor associated to the electromagnetic field $F_{ij}dx^i \wedge dx^j$), then the only static solution is the Reissner-Nordström metric which is again singular at the origin [1]. Finally, in [5] it is shown that for any gauge group, in $2+1$ spacetime dimensions, the Einstein/Yang-Mills (EYM) equations likewise have no regular static solutions. Deser has asked the question as to whether there exist non-singular static solutions in $3+1$ spacetime dimensions. In this paper we prove that in $3+1$ spacetime dimensions with $SU(2)$ gauge group, the EYM equations admit non-singular static solutions, whose metric is asymptoti-

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cally flat Minkowskian. Thus for non-abelian gauge fields, the Yang-Mills repulsive force can balance gravitational attraction and prevent the formation of singularities in spacetime. Viewed differently, from a mathematical perspective, it is the nonlinearity of the corresponding Yang-Mills equations which precludes singularities.

The problem of finding static non-singular solutions of the EYM equations, with $SU(2)$ gauge group reduces to the study of a coupled system of ordinary differential equations of the form

$$r^2Aw''+\Phi(w,A,r)w'+w(1-w^2)=0,$$

$$rA'+(1+2w'^2)A=1-\frac{(1-w^2)^2}{r^2}$$

in $r\geqq 0$, where $\Phi(w,A,r)=r(1-A)-(1-w^2)^2/r$, and the unknowns are $A=A(r)$, and $w=w(r)$. Together with these equations, we are also given a one-parameter family ($\lambda\geqq 0$) of initial conditions which are chosen precisely so as to avoid a singularity at $r=0$. The problem is then to find a particular parameter value λ for which the corresponding solution $(w_\lambda(r), w'_\lambda(r), A_\lambda(r))$ has a finite limit as $r\to\infty$. (We note that such solutions were previously observed numerically, by Bartnik and McKinnon, [2]; they also derived the above system of equations. This important paper was the starting point of our investigations.)

In the appendix (Sect. 7) we show that given any $\lambda>0$, the above system has a unique solution, defined on an interval $0<r<R(\lambda)$, satisfying the initial conditions $w(0)=1$, $w'(0)=0$, $w''(0)=-\lambda$. This gives us a one-parameter family of local solutions which are non-singular at $r=0$. However, for large values of λ, the solution develops a singularity at some finite $\bar{r}_\lambda\sim\frac{1}{\lambda}$, with $w(\bar{r}_\lambda)>0$. That is, w' is unbounded near $\bar{r}_\lambda$, and $\lim_{r\to\bar{r}_\lambda} A(r)=0$[1]. It follows that the solution cannot be continued beyond $\bar{r}_\lambda$. Furthermore, at $\bar{r}_\lambda$, the associated Einstein metric becomes singular, and the solution is no longer of physical interest.

We now briefly describe the contents of the paper. Section 2 is devoted to a quick derivation of the equations, whereby we also put them into a form suitable for our purposes. In the next section, we find a range of parameter values for which A stays positive and w' remains bounded. We also analyze the solutions for small λ. In Sect. 4 we show that for $\lambda>2$ the solution develops a singularity in the region $w>0$. In Sect. 5 we give a rigorous proof of the existence of some $\bar{\lambda}<2$ for which the equations have a bounded solution. In the final section, we show that for our solution, the corresponding Einstein metric is asymptotically Minkowskian, and the total mass is finite.

Added in proof. We have learned through a preprint of G. W. Gibbons (Self-gravitating magnetic monopoles, global monopoles, and black holes), that H. P. Künzle and A. K. M. Masood-ul-Alam [J. Math. Phys. **31**, 928 (1990)] have also considered these equations and have done both theoretical and numerical studies; they conclude that for $\lambda>7.7$, w' becomes infinite before w reaches -1. However, they do not provide a theoretical proof of the existence of a bounded solution as indicated numerically in [2].

Next, Straumann and Zhou [Phys. Lett. B **237**, 353 (1990)] have numerically investigated the linearized stability of our solution. They have indicated that there is a small positive eigenvalue.

[1] This, too, was first seen numerically in [2]. We shall give a rigorous proof of this fact (for $\lambda>2$) in Sect. 4 (Theorem 4.1)

2. The Equations

The EYM equations with $SU(2)$ gauge group have been derived in [2]; for completeness, we shall give a short description of the derivation.

Let τ_1, τ_2, τ_3 denote the standard basis for $su(2)$, the Lie algebra of $SU(2)$; i.e.

$$\tau_1 = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Using gauge invariance, and seeking solutions of a particular form, the $su(2)$ connection takes the form

$$\mathfrak{A} = w\tau_1 d\Theta + [\cos\theta\tau_2 + w\sin\theta\tau_2]d\phi,$$

where $w = w(r)$. The Yang-Mills curvature F, associated to this connection, is obtained from the usual formula $F = d\mathfrak{A} + \mathfrak{A} \wedge \mathfrak{A}$, and can be written as

$$F = w'\tau_1 dr \wedge d\theta + w'\tau_2 dr \wedge (\sin\theta d\phi) - (1 - w^2)\tau_3 d\theta \wedge (\sin\theta d\phi), \tag{2.1}$$

where "prime" denotes differentiation with respect to r.

The static, spherically symmetric metric can be written as

$$ds^2 = -T^{-2}dt^2 + R^2 dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.2}$$

where T and R are both functions of r. If $|F|^2 = g^{ik}g^{jl}F_{ij}F_{kl}$, then an easy calculation gives

$$|F|^2 = \frac{2w'^2}{r^2} + \frac{(1-w^2)^2}{r^4}. \tag{2.3}$$

The Yang-Mills equations, $d^*F = 0$, in this set-up reduce to the single equation

$$\left(\frac{w'}{RT}\right)' + \frac{R}{r^2 T}(1 - w^2) = 0. \tag{2.4}$$

The EYM equations are derived from the action

$$\int (-\mathcal{R} + |F|^2)\sqrt{|g|}dx,$$

where $\mathcal{R}$ is the scalar curvature associated to the metric (2.2). These equations,

$$R_{ij} = 2F_{ik}F_j^k - \tfrac{1}{2}|F|^2 g_{ij},$$

become

$$\left.\begin{aligned} (RT)\left(\frac{T'}{RT^2}\right)' + \frac{2R'}{rR} &= \frac{2w'^2}{r^2} - \frac{R^2(1-w^2)^2}{r^4} \\ -(RT)\left(\frac{T'}{RT^2}\right)' - \frac{2T'}{rT} &= \frac{2w'^2}{r^2} = \frac{R^2(1-w^2)^2}{r^4} \\ r\left(\frac{R'}{R} + \frac{T'}{T}\right) + R^2 - 1 &= \frac{R^2}{r^2}(1-w^2)^2. \end{aligned}\right\} \tag{2.5}$$

If we define

$$A = R^{-2},$$

then (2.5) reduces to the two equations

$$rA' + (2w'^2+1)A = 1 - \frac{(1-w^2)^2}{r^2}, \tag{2.6}$$

$$2rA\frac{T'}{T} = \frac{(1-w^2)^2}{r^2} + (1-2w'^2)A - 1. \tag{2.7}$$

In this notation, (2.4) becomes

$$r^2Aw'' + \left[r(1-A) - \frac{(1-w^2)^2}{r}\right]w' + w(1-w^2) = 0. \tag{2.8}$$

Equations (2.6)–(2.8) constitute the entire system. Observe that remarkably, (2.6) and (2.8) do not involve T, so we can first solve these for A and w, and use (2.7) to obtain T.

Next if

$$\lim_{r\to\infty} A(r) = 1, \tag{2.9}$$

then $R(r)\to 1$ as $r\to\infty$ so the metric (2.2) will be asymptotically flat if in addition $T(r)\to 1$ as $r\to\infty$. Also, if

$$\lim_{r\to\infty} (w(r), w'(r)) \quad \text{is finite}, \tag{2.10}$$

then we show in Corollary 6.4 that the total mass is finite. Thus we seek solutions of (2.6) and (2.8) having asymptotic behavior (2.9), (2.10).

There are two explicit solutions of our system (2.6)–(2.8); namely, if $w\equiv 1$ we recover the Schwarzschild solution, while if $w\equiv 0$, we obtain the Reissner-Nordström metric with $u(1)$-valued YM curvature,

$$F = \left[\frac{w}{r^2}\, dr \wedge d\theta + d\theta \wedge (\sin\theta d\phi)\right]\tau_3 .$$

The conditions at $r=0$ needed to ensure that we formally have *non-singular* solutions at $r=0$ are easily obtained:

$$w(0)=1, \qquad w'(0)=0, \qquad A(0)=1. \tag{2.11}$$

But Eqs. (2.6) and (2.8) are nonlinear and singular at $r=0$. However, it follows from the local existence theorem outlined in the appendix that the non-singular solutions are parametrized by $w''(0)$; we make this explicit by writing

$$w''(0) = -\lambda, \qquad \lambda > 0. \tag{2.12}$$

That is, in the appendix we outline a proof that given any $\lambda>0$, there is an interval I_λ: $0\leqq r<R_\lambda$, for which the corresponding solution of Eqs. (2.6), (2.8), together with the initial conditions (2.11), (2.12) has a $C^{2+\alpha}$ solution on I_λ which is analytic on the interior and which depends continuously on λ.

Thus our problem is to prove that there is a $\bar\lambda>0$ for which the system (2.6) and (2.8), subject to the initial conditions (2.11) has a solution which satisfies both (2.9) and (2.10). In fact, we shall prove that Eqs. (2.6) and (2.8) admit a "connecting orbit"; i.e., a solution $(w(r,\bar\lambda), w'(r,\bar\lambda))$ satisfying both (2.9) and

$$\lim_{r\to\infty} (w(r,\bar\lambda), w'(r,\bar\lambda)) = (-1, 0).$$

We will then show that $T(r)\to 1$ as $r\to\infty$. Furthermore, if we define the quantity μ by

$$\mu(r)=r(1-R(r)^{-2})=r(1-A(r)), \tag{2.13}$$

then we will prove

$$\lim_{r\to\infty}\mu(r)=m<\infty;$$

i.e., that the "total mass" is finite (cf. [2.6]). We note that [cf. (2.6)], μ satisfies the equation

$$\mu'=2Aw'^2+\frac{(1-w^2)^2}{r^2}. \tag{2.14}$$

3. The Case $\lambda\leqq 1$

We rewrite Eqs. (2.6), (2.8):

$$rA'+(2w'^2+1)A=1-\frac{(1-w^2)^2}{r^2}, \tag{3.1}$$

$$r^2Aw''+\left[r(1-A)-\frac{(1-w^2)^2}{r}\right]w'+w(1-w^2)=0, \tag{3.2}$$

subject to the initial conditions

$$w(0)=1, \quad w'(0)=0, \quad w''(0)=-\lambda<0, \quad A(0)=1. \tag{3.3}$$

Our main objective in this section is to prove that for $\lambda\leqq 1$, the solution develops no singularities in the region $w^2\leqq 1$; that is, w' stays bounded, and $A>0$. Furthermore, we shall also show that for λ near 0, all orbits (w,w') exit this region through the line $w=-1$.

We shall often have occasion to use the following "self-adjoint" form of the above equations:

$$(rAe^Q)'=\left(1-\frac{(1-w^2)^2}{r^2}\right)e^Q, \tag{3.4}$$

$$(e^Pw')'+e^P\frac{w(1-w^2)}{r^2A}=0, \tag{3.5}$$

where

$$Q'=\frac{2w'^2}{r}, \quad \text{and} \quad P'=\frac{1}{r^2A}\left[r(1-A)-\frac{(1-w^2)^2}{r}\right].$$

It is also desirable to define the important function Φ by

$$\Phi(r)=r(1-A)-\frac{(1-w^2)^2}{r}. \tag{3.6}$$

Then (3.1) and (3.2) can be rewritten as

$$rA'+2w'^2A=\Phi/r \tag{3.1$'$}$$

$$r^2Aw''+\Phi w'+w(1-w^2)=0. \tag{3.2$'$}$$

As we have remarked above, in the appendix, we show that given λ, $2 \geqq \lambda \geqq 0$, there is an $R>0$ such that the problem (3.3)–(3.5) has a (unique) smooth-solution (A_λ, w_λ) defined on $0<r<R$, and $A_\lambda(r)>0$ on this range.

Our first goal in this section is to show that $|w'|$ is bounded and $\Phi>0$ if $\lambda \leqq 1$. We will then show (in Theorem 3.1) that $A>0$ in the region

$$\Gamma=\{(w,w'):\ w^2 \leqq 1,\ w' \leqq 0\}. \tag{3.7}$$

Theorem 3.1. *Fix $\lambda \leqq 1$; then in Γ, $A(r,\lambda)>0$, and $|w'(r,\lambda)|$ is bounded.*

This theorem will follow from a series of lemmas. First, we define g by

$$g(r)=cr^2+w^2-1; \tag{3.8}$$

we then have

Lemma 3.2. *If $c=1$, $\lambda \leqq 1$, and $A(r)>0$ on $0<r<R_1$, it follows that $g(r)>0$ and $g'(r) \geqq 0$ on this interval.*

(In Sects. 4 and 5, we will use this function with $c=2$ and $c=\frac{1}{3}$, respectively.)

Proof. We have

$$g'(r)=2cr+2ww', \tag{3.9}$$

so

$$g(0)=0=g'(0). \tag{3.10}$$

Also

$$g''(r)=2c+2w'^2+2ww'', \tag{3.11}$$

so that

$$g''(0)=2(c-\lambda). \tag{3.12}$$

Next, a calculation yields the following differential equation for g:

$$r^2Ag''+\Phi g'=2[g+r^2Aw'^2+(1-c)(1-w^2)^2]. \tag{3.13}$$

Now if $\lambda<1$ and $c=1$, then from (3.12), $g''(0)>0$ so g and g' are positive for r near 0; then (3.13) shows that were there a smallest $\bar{r}$, $0<\bar{r}<R_1$, for which $g'(\bar{r})=0$, then $g''(\bar{r})>0$. It follows that $g'>0$ and $0<r<R_1$. Thus our result holds if $\lambda<1$. If $\lambda=1$, then $g' \geqq 0$ follows by continuity. □

Corollary 3.3. *Let $0<\lambda \leqq 1$, and assume $A(r)>0$ and $w(r) \geqq \varepsilon>0$ on $0 \leqq r<R_1$. Then on this range, $w'(r) \geqq -r/\varepsilon$*

Lemma 3.4. *Let $0<\lambda \leqq 1$, and define*

$$h(r,\lambda)=A(r,\lambda)-w^2(r,\lambda). \tag{3.14}$$

Then $h>0$ as long as $A>0$ in the region $w>0$.

Proof. We have $h(0)=0=h'(0)$, and that $h''(0)=2\lambda(1-\lambda)>0$ if $\lambda<1$. Moreover, h satisfies the equation

$$h'=\frac{(1+2w'^2)}{r}h-\frac{ww'}{r}g'+\frac{(1-w^2)^2}{r^3}g; \tag{3.15}$$

[cf. (3.8) with $c=1$]. Then if $h(r)=0$, (3.15) shows that $h'(r)>0$; hence $h(r)>0$. Thus our lemma holds if $\lambda<1$. If $\lambda=1$, then $h\geqq 0$ follows by continuity. If $h(\bar{r})=0$, then (3.15) shows $h'(\bar{r})>0$, since $g'\geqq 0$ and $0<w<1$, and this would be a contradiction. □

Corollary 3.5. *If $\lambda\leqq 1$, then $A(r)\geqq\varepsilon^2>0$ in the region $w(r)\geqq\varepsilon>0$.*

We now obtain some properties of the function Φ; cf. (3.6).

Lemma 3.6. *Let $\lambda\leqq 1$ and assume $A(r)>0$ on $0<r<R_1$. Then $\Phi(r)>0$ and $\Phi'(r)\geqq 0$ on this range.*

Proof. An easy computation shows

$$\frac{r^2}{2}\Phi'(r)=(1-w^2)^2+2w(1-w^2)(rw')+A(rw')^2. \tag{3.16}$$

Notice that $\Phi'>0$ if $w\leqq 0$. If $w>0$, then we consider the right-hand-side of (3.16) as a quadratic form in $u=1-w^2$ and $v=rw'$; i.e.,

$$\frac{r^2}{2}\Phi'(r)=u^2+(2w)uv+Av^2.$$

$r^2\Phi'/2$ is clearly positive if $u=0$ and $v>0$, and using Lemma 3.4 its discriminant is $4(w^2-A)<0$. Hence the lemma is proved if $\lambda<1$, and the case $\lambda=1$ again follows by continuity. □

Note that Corollaries 3.3 and 3.5 imply that if $\lambda\leqq 1$, w' is bounded from below in the region $w\geqq\varepsilon>0$. The next lemma shows that w' is always finite in Γ. For this, we introduce the following notation: Let $r_0=r_0(\lambda)$ be defined by

$$w(r_0(\lambda),\lambda)=0.$$

(Note that r_0 is well-defined since $w'<0$.) We can now complete the

Proof of Theorem 3.1. By the local existence theorem in the appendix, there is an $R>0$ such that w, w', and A are all continuous on $[0,R]$, $A>0$ and w' is bounded from below on this interval; say $w'(r)>-K$. We shall show that w' is bounded below up to the first zero of A, and then that A is never zero in the region $w^2\leqq 1$, $w'\leqq 0$. Thus if w' were unbounded in the region where $A>0$, we could find a sequence of points $\{r_n\}$ such that $A(r_n)>0$, $w'(r_n)=-n$, and $w''(r_n)\leqq 0$ [if $w''(r)>0$ whenever $w'(r)=-n$, then w' would be bounded below by $-n$]. If n is large, say $n>M$, then $r_n>R$ and thus by Lemma 3.7, $\Phi(r_n)>\Phi(R)>0$, so $\Phi(r_n)w'(r_n)<-n\Phi(R)$. Then using (3.2)′, we have

$$r_n^2A(r_n)w''(r_n)+\Phi(r_n)w'(r_n)+w(r_n)(1-w^2(r_n))=0. \tag{3.17}$$

But this cannot hold for large n since the left-hand side of (3.17) tends to $-\infty$ as $n\to\infty$. Thus no such sequence $\{r_n\}$ can exist, and w' is bounded below up to the first zero of A. On the other hand, if $\lim_{r\nearrow\bar{r}}A(r)=0$, we set $A(\bar{r})=0$ and A is continuous on $[0,\bar{r}]$; we thus conclude that $\lim_{r\nearrow\bar{r}}Aw'^2=0$. Then for small $\varepsilon>0$,

$$-A(\bar{r}-\varepsilon)=A(\bar{r})-A(\bar{r}-\varepsilon)=A'(\xi)\varepsilon>0$$

from (3.1)′, where ξ is an intermediate point. This contradiction shows that $A>0$ in Γ. □

Thus there is no blowup for $\lambda \leqq 1$ in the region $w^2 \leqq 1$, $w' \leqq 0$, and $A > 0$ here. But for $\lambda > 2$ things are quite different, as we shall see in Sect. 4.

We shall next show that for λ near zero, the corresponding orbit exits the region Γ through the line $w = -1$. To this end, we make the following change of variables in Eqs. (3.1, (3.2)

$$\tilde{\mu}(r) = \frac{r}{\lambda}(1 - A(r/\sqrt{\lambda})), \qquad w(r) = \tilde{w}(r/\sqrt{\lambda}), \tag{3.18}$$

and find that $\tilde{\mu}$ and $\tilde{w}$ satisfy

$$\tilde{\mu}' = 2\left(1 - \frac{\lambda\tilde{\mu}}{r}\right)\tilde{w}'^2 + \frac{(1 - \tilde{w}^2)^2}{r^2}, \tag{3.19}$$

$$r^2\left(1 - \frac{\lambda\tilde{\mu}}{r}\right)\tilde{w}'' + \lambda\left[\tilde{\mu} - \frac{(1 - \tilde{w}^2)^2}{r}\right]\tilde{w}' + \tilde{w}(1 - \tilde{w}^2) = 0, \tag{3.20}$$

together with the initial conditions

$$\tilde{\mu}(0) = 0, \quad \tilde{w}(0) = 1, \quad \tilde{w}'(0) = 0, \quad \tilde{w}''(0) = -1. \tag{3.21}$$

We have thus transformed our equations to a new system in which the parameter λ appears explicitly in the equations but not in the initial data. Now as we have shown in the appendix, the solutions depend continuously on λ, so here, too, solutions of these equations depend continuously on λ, provided that $\lambda \leqq 1$ (so that the coefficient of $\tilde{w}''$ doesn't vanish for $r > 0$ in the relevant region $\tilde{w}^2 \leqq 1$, $\tilde{w}' \leqq 0$). We can consider Eqs. (3.19), (3.20) in their own right, for a moment, without reference to our original system. Thus consider (3.19), (3.20) for $\lambda = 0$, together with the initial conditions (3.21); then these become

$$r^2\tilde{w}'' + \tilde{w}(1 - \tilde{w}^2) = 0, \quad \tilde{w}(0) = 1, \quad \tilde{w}'(0) = 0, \quad \tilde{w}''(0) = -1, \tag{3.22}$$

$$\tilde{\mu}' = 2\tilde{w}'^2 + \frac{(1 - \tilde{w}^2)^2}{r^2}, \quad \tilde{\mu}(0) = 0. \tag{3.23}$$

If we make the change of variables $t = \ln r$, $\tilde{w}(r) = \tilde{w}(e^t) = z(t)$, then (3.22) goes over into

$$z'' - z' + z(1 - z^2) = 0, \quad z(-\infty) = 1, \quad z'(-\infty) = 0 = z''(-\infty).$$

If we write the equations as a first-order system $z' = y$, $y' = y - z(1 - z^2)$, and define $H(z, y) = \frac{1}{2}\left(z^2 - \frac{z^4}{2} + y^2\right)$, then $H' = y^2$, so H increases on orbits. The orbit (z, y) satisfying the boundary conditions $z = 1$, $y = 0$ at $t = -\infty$ cannot ever return to $y = 0$ since the graph of $H(z, 0)$ has the form as depicted in Fig. 1 below; this orbit must then exit the region $z^2 \leqq 1$, $z' \leqq 0$ through $z = -1$, $z' < 0$, at some finite time $\bar{t} > 0$; see Fig. 1.

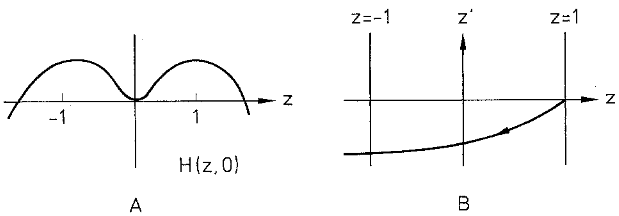

Fig. 1 A and B

Thus for $\lambda=0$ the solution to (3.19)–(3.21) exits the region Γ through the line $w=-1$; moreover, the orbit crosses this line transversally at some finite $\bar{t}$ because $w'<0$. Hence for small $\lambda>0$, the solution to (3.19)–(3.21) must also exit Γ through $w=-1$ at time t near $\bar{t}$ because $A>0$. Transforming back to (3.1)–(3.3), we see that for $0<\lambda\leqq\varepsilon$, the orbit exits Γ through the line $w=-1$. Thus we have shown

Lemma 3.7. *The set of all* $\lambda\leqq 1$ *for which there is a solution* $(w(r,\lambda), A(r,\lambda))$ *of* (3.1)–(3.3), *for which the orbit* $(w(r,\lambda), w'(r,\lambda))$ *exits* Γ *through* $w=-1$, $w'<0$, *is a non-void open set.*

Numerical approximations to the solution for $\lambda=1$ indicate that w' goes positive in the region $-1<w<0$. If this could be established rigorously, it would follow that the set of λ for which the corresponding orbits exits Γ through $w'=0$ is again an open non-void set. Thus there would exist a $\bar{\lambda}$, $0<\bar{\lambda}<1$ for which the corresponding orbit stays in Γ for all $r\geqq 0$. This would solve our problem as originally stated [cf. (2.10)]. In fact, we shall *prove* in Sect. 6 that for some $\bar{\lambda}<2$,

$$\lim_{r\to\infty}(w(r,\bar{\lambda}), w'(r,\bar{\lambda}), A(r,\bar{\lambda}))=(-1,0,1);$$

cf. Theorem 6.1.

4. The Case $\lambda>2$

We shall show that for $\lambda>2$, the solution must become singular in the region $w>0$. In fact, we have the following theorem.

Theorem 4.1. *If* $\lambda>2$, *then the solution of* (3.3)–(3.5) *cannot exist up to* $w=0$.

Proof. Thus suppose that $\lambda>2$, and the solution of (3.1)–(3.3) exists up to $r_0=r_0(\lambda)$. That is, for $0\leqq r\leqq r_0$, w and A are defined, $A>0$, and Eqs. (3.3)–(3.5) are satisfied. [Recall $w(r_0(\lambda),\lambda)=0$.]

We recall the functions g and h [cf. (3.8) and (3.14)], which we now consider on $0\leqq r\leqq r_0$:

$$h(r)=A(r)-w^2(r) \tag{4.1}$$

and

$$g(r)=2r^2-(1-w^2(r))\,. \tag{4.2}$$

Thus, from the proof of Lemma 3.5, $h(0)=0=h'(0)$, and $h''(0)=2\lambda(1-\lambda)$; hence

$$h(r)<0 \quad \text{for } r \text{ near } 0, \text{ if } \quad \lambda>1\,. \tag{4.3}$$

Moreover, $g(0)=0=g'(0)$, and $g''(0)=(2-\lambda)$; thus

$$g(r)<0 \quad \text{for } r \text{ near } 0, \text{ if } \quad \lambda>2\,. \tag{4.4}$$

Concerning these functions, we have two lemmas, the first of which is

Lemma 4.2. *Let* $\lambda>1$; *then* $h(r)<0$ *as long as* $g(r)<0$.

Proof. Suppose $g(r)<0$, $0<r\leqq\tilde{r}\leqq r_0$, and $h(\tilde{r})=0$, $\tilde{r}$ being minimal with respect to this property, $0<\tilde{r}\leqq r_0$. Then using (3.3),

$$h'=-2ww'+\frac{2w'^2A}{r}+\frac{\Phi}{r^2},$$

so

$$h'(\tilde{r}) = -2ww' - \frac{2}{\tilde{r}}(ww')^2 + \frac{\Phi}{\tilde{r}^2}. \tag{4.5}$$

Now consider the right-hand side of (4.5) as a formal quadratic form in $s=(ww')$. We will show that the discriminant is negative; the form is clearly negative for large s, and hence the form is everywhere negative.

The discriminant Δ of the right-hand side of (4.5) is

$$\begin{aligned}\Delta &= 4\left(1+\frac{2\Phi}{\tilde{r}^3}\right)_{r=\tilde{r}} = 4\left[1+\frac{2(1-A)}{r^2}-\frac{2(1-w^2)^2}{r^4}\right]_{r=\tilde{r}} \\ &= 4\left[1+\frac{2(1-w^2)}{r^2}-\frac{2(1-w^2)^2}{r^4}\right]_{r=\tilde{r}} \\ &= 4\left[1+\frac{2(1-w^2)}{r^2}\left(1-\frac{1-w^2}{r^2}\right)\right]_{r=\tilde{r}} \\ &< 4\left[1-\frac{2(1-w^2)}{r^2}\right]_{r=\tilde{r}} < 4(1-4) < 0,\end{aligned}$$

again because $g(\tilde{r}) \leqq 0$. Thus $h'(\tilde{r})<0$, and this is impossible. It follows that no such $\tilde{r}$ can exist. □

Lemma 4.3. *Let $\lambda>2$; then $g'(r)<0$ as long as $h(r)<0$.*

Proof. Suppose $h(r)<0$, $0<r\leqq\tilde{r}\leqq r_0$ and $g'(\tilde{r})=0$, $\tilde{r}$ being minimal with respect to this property, $0<\tilde{r}\leqq r_0$. We will show that $g''(\tilde{r})<0$, so that by the mean-value theorem, no such $\tilde{r}$ can exist.

We recall that g satisfies the differential equation

$$g''(r) = \frac{-\Phi}{r^2A}g' + \frac{2}{r^2A}\left[r^2Aw'^2-(1-w^2)^2+g\right].$$

Now at $r=\tilde{r}$, we have $w^2>A$, since $h(\tilde{r})<0$, and $-ww'=2\tilde{r}$, since $g'(\tilde{r})=0$. Thus

$$\begin{aligned} g''(\tilde{r}) &= \frac{2}{\tilde{r}^2A}\left[r^2Aw'^2-(1-w^2)^2+g\right]_{r=\tilde{r}} \\ &\leqq \frac{2}{\tilde{r}^2A}\left[r^2w^2w'^2-(1-w^2)^2+g\right]_{r=\tilde{r}} \\ &= \frac{2}{\tilde{r}^2A}\left[4\tilde{r}^4-(1-w^2)^2+g\right]_{r=\tilde{r}} \\ &= \frac{2}{\tilde{r}^2A}\left[g(1+2\tilde{r}^2+(1-w^2))\right]_{r=\tilde{r}} < 0,\end{aligned}$$

since $g(\tilde{r})<0$. Thus no such $\tilde{r}$ can exist. □

We now can complete the proof of Theorem 4.1. First, note that $h(r)<0$ for r near zero, and $h(r_0)=A(r_0)>0$. Hence $h(r)=0$ for some $r<r_0$; let r_1 be the first zero of h. Similarly, $g'(r)<0$ for r near 0, and $g'(r_0)=4r_0>0$, so let r_2 be the first zero of g'; thus $g(r)<0$, $0<r\leqq r_2$.

Now since $g(r_2)<0$, Lemma 4.2 implies $r_1>r_2$. On the other hand, Lemma 4.3 shows that $r_2\geqq r_1$. It follows that for $\lambda>2$, $w(r,\lambda)$ cannot reach the line $w=0$. □

5. Existence of a Connecting Orbit

In this section we shall prove the existence of some $\lambda \leqq 2$ for which the corresponding solution of (3.3)–(3.5) satisfies

$$\lim_{r\to\infty} (w(r,\bar{\lambda}), w'(r,\bar{\lambda}), A(r,\bar{\lambda})) = (-1,0,1).$$

The basic idea of our approach is easy to describe. Thus, as we have shown in Lemma 4.1, for λ near 0, there is an $r_\lambda > 0$ such that the following statement holds:

$$w(r_\lambda,\lambda) = -1, \quad \text{and for} \quad 0 < r \leqq r_\lambda, \quad w'(r,\lambda) < 0 \quad \text{and} \quad A(r,\lambda) > 0. \quad (5.1)$$

Next, from Theorem 4.1, it follows that for $\lambda > 2$, the solution $(w(r,\lambda), A(r,\lambda))$ *cannot* satisfy (5.1). Define the set Λ by

$$\Lambda = \{\lambda \geqq 0\colon \text{there exists an } r_\lambda > 0 \text{ for which (5.1) holds}\},$$

and set

$$\bar{\lambda} = \sup \Lambda .$$

We will prove that the $\bar{\lambda}$-orbit is the desired solution. We first list the a-priori possibilities for the behavior of this orbit; then we shall eliminate all but the desired case.

Now for the solution $(A(r,\bar{\lambda})), w(r,\bar{\lambda}))$, one of the following six possibilities must hold:

1) $\bar{\lambda} \in \Lambda$.
2) There exists an $\bar{r} > 0$ such that $w'(\bar{r},\bar{\lambda}) = 0$, $w(\bar{r},\bar{\lambda}) \geqq -1$, and $w'(r,\bar{\lambda}) < 0$, $A(r,\bar{\lambda}) > 0$ for $0 < r < \bar{r}$.
3) There exists an $\bar{r} > 0$ such that $\lim_{r\nearrow\bar{r}} A(r,\bar{\lambda}) = 0$, $w'(r,\bar{\lambda})$ is bounded from below on $0 \leqq r < \bar{r}$, and $w(r,\bar{\lambda}) \geqq -1$, $w'(r,\bar{\lambda}) < 0$, $A(r,\bar{\lambda}) > 0$ on $0 < r < \bar{r}$.
4) There exists an $\bar{r} > 0$ such that $\lim_{r\nearrow\bar{r}} A(r,\bar{\lambda}) = 0$, $w'(r,\bar{\lambda})$ is not bounded from below on $0 \leqq r < \bar{r}$, and $w(r,\bar{\lambda}) \geqq -1$, $w'(r,\bar{\lambda}) < 0$, $A(r,\bar{\lambda}) > 0$ on $0 < r < \bar{r}$.
5) There exists an $\bar{r} > 0$ such that the solution $(w(r,\bar{\lambda}), A(r,\bar{\lambda}))$ is singular at $\bar{r}$; i.e., $w'(r,\bar{\lambda})$ is unbounded on $0 \leqq r < \bar{r}$.
6) For all $r > 0$, $w(r,\bar{\lambda}) > -1$, $w'(r,\bar{\lambda}) < 0$, and $A(r,\bar{\lambda}) > 0$.

To see that these are in fact all the cases, we note that if A is always positive, the solution can be continued unless $|w'|$ becomes unbounded for some finite r. (We shall show below, in Proposition 5.2, that $\lim_{r\nearrow\bar{r}} A(r,\bar{\lambda})$ exists.)

Before discussing these cases, we shall need two propositions (which will repeatedly be used throughout this section), the first of which is

Proposition 5.1. *There is a constant* $L > 0$ *such that if* $0 \leqq \lambda \leqq 2$, *and* $w(r,\lambda)^2 \leqq 1$, *then* $(Aw'^2)(r,\lambda) \leqq L$.

Proof. Let $f = Aw'^2$; then f satisfies the equation

$$r^2 f' + (2rf + \Phi)w'^2 + 2ww'(1-w^2) = 0 .$$

By the local existence theorem in the appendix, there exists an $R > 0$ such that the solution to (3.1)–(3.3) is non-singular on $0 \leqq r \leqq R$, and $0 \leqq \lambda \leqq 2$. Thus f is continuous, and hence bounded on this compact set; say $f \leqq L_1$. If $n > L_1$, then either $f(r,\lambda)\leqq n$ for all r and $\lambda\leqq 2$, or else there exist $r_n>R$, $\lambda_n\leqq 2$ such that $f(r_n,\lambda_n)=n$, and $f'(r_n,\lambda_n)\geqq 0$. Since

$$\Phi(r,\lambda)=r(1-A)-\frac{(1-w^2)^2}{r}>-\frac{1}{r}>-\frac{1}{R},\quad \text{for}\quad r>R,$$

We have $r_n^2 f'(r_n,\lambda_n)\geqq 0$, and $2r_n f(r_n,\lambda_n)+\Phi(r_n,\lambda_n)\geqq 2Rn-\frac{1}{R}>2$, for n large. Since $w(1-w^2)\geqq -1$, and $w'^2(r_n,\lambda_n)>|w'(r_n,\lambda_n)|$, we would have

$$r_n^2 f'(r_n,\lambda_n)+(2r_n f(r_n,\lambda_n)+\Phi(r_n,\lambda_n))w'^2(r_n,\lambda_n)+2ww'(1-w^2)(r_n,\lambda_n)>0.$$

This contradiction establishes the result. □

Proposition 5.2. *$A(r,\lambda)$ is continuous for $0\leqq\lambda\leqq\bar{\lambda}$ in the region $w^2\leqq 1$.*

Proof. It suffices to show that A is continuous at $(\bar{r},\bar{\lambda})$. If $\lim_{r\nearrow\bar{r}} A(r,\bar{\lambda})\neq 0$, then by the last result, $w'(r,\bar{\lambda})$ is bounded on $0\leqq r\leqq\bar{r}$, so the solution (A,w) of (3.1)–(3.3) continues beyond $\bar{r}$, and depends continuously on the parameter λ. Thus it only remains to consider the case where $\lim_{r\nearrow\bar{r}} A(r,\bar{\lambda})=0$. If we set $A(\bar{r},\bar{\lambda})=0$, then we see that A is a continuous function of r.

We now claim that A is continuous in λ at $\lambda=\bar{\lambda}$. For this, it suffices to show that $\mu(\bar{r},\lambda)$ is continuous at $\bar{\lambda}$ [cf. (2.13)]. Thus, let $\lambda<\bar{\lambda}$, and choose $r<\bar{r}$; then

$$\begin{aligned}0<\bar{r}-\mu(\bar{r},\lambda)&=\mu(\bar{r},\bar{\lambda})-\mu(\bar{r},\lambda)\\&=\mu(\bar{r},\bar{\lambda})-\mu(r,\bar{\lambda})+\mu(r,\bar{\lambda})-\mu(r,\lambda)+\mu(r,\lambda)-\mu(\bar{r},\lambda)\\&\leqq\mu(\bar{r},\bar{\lambda})-\mu(r,\bar{\lambda})+\mu(r,\bar{\lambda})-\mu(r,\lambda),\end{aligned}$$

because $\mu'>0$ [cf. (2.14)]. Now given $\varepsilon>0$, choose $r<\bar{r}$ so close to r that $|\mu(\bar{r},\bar{\lambda})-\mu(r,\bar{\lambda})|<\varepsilon/2$. For this fixed r, choose $\delta>0$ such that $0<\bar{\lambda}-\lambda<\delta$ implies $|\mu(r,\bar{\lambda})-\mu(r,\lambda)|<\varepsilon/2$; then for these λ, $|\mu(\bar{r},\bar{\lambda})-\mu(\bar{r},\lambda)|<\varepsilon$, and this proves our claim.

Next, we show that $A'(r,\lambda)$ is bounded for (r,λ) near $(\bar{r},\bar{\lambda})$. Thus, from (3.2)′,

$$A'=\frac{-2w'^2A}{r}+\Phi/r^2.$$

Since Aw'^2 is bounded (Lemma 5.1), and $r\geqq\Phi\geqq-\frac{1}{r}$, we see that Φ/r^2 is bounded for r near $\bar{r}$; thus A' is bounded.

We can now complete the proof of Proposition 5.2; namely, let $\varepsilon>0$ be given, and let k be a bound for $|A'|$. Let $\delta=\varepsilon/6k$, and choose $|r-\bar{r}|<\delta$. Choose r_1, $\bar{r}-\delta<r_1<\bar{r}$ so that $|A(\bar{r},\bar{\lambda})-A(r_1,\bar{\lambda})|<\varepsilon/3$. Then choose τ such that $0<\bar{\lambda}-\lambda<\tau$ implies $|A(r_1,\lambda)-A(r_1,\bar{\lambda})|<\varepsilon/3$. Then

$$\begin{aligned}|A(r,\lambda)-A(\bar{r},\bar{\lambda})|&\leqq|A(r,\lambda)-A(r_1,\lambda)|+|A(r_1,\lambda)-A(r_1,\bar{\lambda})|+|A(r_1,\bar{\lambda})-A(r,\bar{\lambda})|\\&<K|r-r_1|+\varepsilon/3+\varepsilon/3\\&<K\left(\frac{2\varepsilon}{6K}\right)+\frac{2\varepsilon}{3}=\varepsilon,\end{aligned}$$

if $|r-\bar{r}|<\delta$, and $0<\bar{\lambda}-\lambda<\tau$. This completes the proof of Proposition 5.2. □

Corollary 5.3. *Let $v(r,\lambda)=(Aw')(r,\lambda)$, and define $v(\bar{r},\bar{\lambda})=0$. Then v is continuous in the region $w^2\leqq 1$, and if $0\leqq\lambda\leqq\bar{\lambda}$, v is bounded in this region.*

Proof. We need only investigate the behavior of v near $(\bar{r}, \bar{\lambda})$. Since $v^2=(Aw'^2)A$, Aw'^2 is bounded, and $A(r,\lambda)\to 0$ as $(r,\lambda)\to(\bar{r},\bar{\lambda})$, the results follow. □

We are now in a position to discuss the various cases 1)–6) above. First, if $\bar{\lambda}\in\Lambda$, then by transversality, there are $\lambda>\bar{\lambda}$ such that $\lambda\in\Lambda$, thereby violating the definition of $\bar{\lambda}$; thus case 1) cannot occur. Next, in cases 2)–5), we may assume that $\bar{r}$ is the smallest such r – since $w'(r,\lambda)<0$ for r near 0, and $A(0,\bar{\lambda})=1$, such an $\bar{r}$ exists. Now consider case 2). If $A(\bar{r},\bar{\lambda})=0$, we are then in case 3); thus we can assume $A(\bar{r},\bar{\lambda})>0$. If $w(\bar{r},\bar{\lambda})=-1$, it follows from standard o.d.e. theorems that $w(r,\bar{\lambda})\equiv 1$, and this violates $w(0,\bar{\lambda})=1$. If $w'(\bar{r},\bar{\lambda})=0$, $A(\bar{r},\bar{\lambda})>0$ and $w(\bar{r},\bar{\lambda})>-1$, then if $w''(\bar{r},\bar{\lambda})\neq 0$, an easy transversality argument shows that for some $\lambda<\bar{\lambda}$, and some r near $\bar{r}$, that $w'(\bar{r},\bar{\lambda})=0$, thereby violating the definition of $\bar{\lambda}$, while if $w''(\bar{r},\bar{\lambda})=0$, we find from (3.2) that $w(r,\bar{\lambda})\equiv 0$, or $w(r,\bar{\lambda})\equiv 1$; both of these violate $w''(0,\bar{\lambda})=-\bar{\lambda}<0$. Thus case 2) is subsumed by case 3). Next, consider case 5). At the singular point $\bar{r}$, if $A(\bar{r},\bar{\lambda})\neq 0$, this would violate Proposition 5.1. Thus $A(\bar{r},\bar{\lambda})=0$, so case 5) is subsumed by cases 3) or 4).

Thus, we may assume that only the following two possibilities occur:

(A) $A(\bar{r},\bar{\lambda})=0$, and for $0\leqq r<\bar{r}$, $w(r,\bar{\lambda})>-1$, $w'(r,\bar{\lambda})<0$, $A(r,\bar{\lambda})>0$,

or

(B) for all $r>0$, $w(r,\bar{\lambda})>-1$, $w'(r,\bar{\lambda})<0$, and $A(r,\bar{\lambda})>0$.

We shall prove that only Case (B) can occur, by ruling out Case (A), taking into account the two possibilities; namely, that $w'(r,\bar{\lambda})$ is bounded or unbounded near $\bar{r}$. *We now assume in what follows in this section that Case (A) holds*, and we shall arrive at a contradiction. This will be accomplished by eliminating all the alternative cases; namely,

Case 1. $\bar{w}\geqq 0$, $w'(r,\bar{\lambda})$ bounded for r near $\bar{r}$.

Case 2. $\bar{w}<0$.

Case 3. $\bar{w}>0$, $w'(r,\bar{\lambda})$ unbounded for r near $\bar{r}$.

Case 4. $\bar{w}=0$, $w'(r,\bar{\lambda})$ unbounded for r near $\bar{r}$.

We will find it convenient to define $r_a(\lambda)$ by

$$w(r_a(\lambda),\lambda)=a\,,\quad \text{for}\quad -1<a<1\,. \tag{5.4}$$

Note that if $\lambda\leqq 1$, then $A(r,\lambda)>0$ in the region $\Gamma=\{w^2\leqq 1,\ w'\leqq 0\}$, cf. Theorem 3.1. Since we are in Case (A), we may assume $1<\bar{\lambda}\leqq 2$. We now have the following result.

Lemma 5.4. *Let* $1\leqq\lambda\leqq 2$; *then there exists* $\sigma>0$, *and* w_1, $-1<w_1<0$ *such that* $\Phi(r,\lambda)\geqq\sigma$ *whenever* $-1<w(r,\lambda)<w_1$.

Proof. By the local existence theorem in the appendix there exists an $R>0$ such that the solution is defined on $[0,R]$ for $0\leqq\lambda\leqq 2$. If $1\leqq\lambda\leqq 2$ and $\mu(r,\lambda)$ is defined by $\mu(r,\lambda)=r(1-A(r,\lambda))$ [cf. (2.13)], then $\mu(0,\lambda)=0$, and (2.14) implies that $\mu'(r,\lambda)>0$; thus by compactness, we can find $\sigma>0$ such that $\mu(R,\lambda)\geqq 2\sigma>0$, for $1\leqq\lambda\leqq 2$. Now as

$$\Phi(r,\lambda)=\mu(r,\lambda)-\frac{(1-w^2(r,\lambda))^2}{r},$$

we have, for $r>R$,

$$\Phi(r,\lambda)\geqq 2\sigma-\frac{(1-w^2(r,\lambda))^2}{R}.$$

If $\sigma\geqq\frac{1}{R}$, then $\Phi(r,\lambda)\geqq\sigma$, so w_1 can be chosen to be any number in $(-1,0)$. If $\sigma<\frac{1}{R}$, then there is a unique w_1, $-1<w_1<0$ satisfying $2\sigma-\frac{(1-w^2)^2}{R}=\sigma$. Then if $-1<w(r,\lambda)<w_1$,

$$\Phi(r,\lambda)>2\sigma-\frac{(1-w^2)^2}{r}>2\sigma-\frac{(1-w_1^2)^2}{R}=\sigma;$$

this completes the proof. □

Lemma 5.5. *Fix $\lambda\leqq 2$, and suppose that there is a $\sigma>0$ such that $\Phi(r,\lambda)\geqq\sigma$ for $r_1\leqq r\leqq r_2$. Then if $-1\leqq w(r_2,\lambda)<w(r_1,\lambda)<0$, we have*

$$w'(r_2,\lambda)\geqq -kr_2^2, \tag{5.5}$$

where $k=\frac{L}{\sigma}(w(r_1,\lambda)-w(r_2,\lambda))^{-1}$.

Proof. Since λ is fixed, we suppress it. From (3.2)′ for $r_1\leqq r\leqq r_2$, we get

$$w''=\frac{-\Phi w'}{r^2A}-\frac{w(1-w^2)}{r^2A}\geqq\frac{-\Phi w'}{r^2A},$$

so

$$\frac{w''}{w'^2}\geqq\frac{\Phi(-w')}{r^2Aw'^2}\geqq\frac{\sigma}{Lr^2}(-w').$$

Integrating from r_1 to r_2 gives

$$\frac{-1}{w'(r_2)}+\frac{1}{w'(r_1)}=\int_{r_1}^{r_2}\frac{w''}{w'^2}dr\geqq\frac{\sigma}{Lr_2^2}\int_{r_1}^{r_2}(-w')dr$$

$$=\frac{\sigma}{Lr_2^2}(w(r_1)-w(r_2)).$$

Hence

$$\frac{-1}{w'(r_2)}\geqq\frac{\sigma}{Lr_2^2}(w(r_1)-w(r_2)),$$

so

$$\frac{Lr_2^2}{\sigma}(w(r_1)-w(r_2))^{-1}\geqq -w'(r_2), \tag{5.6}$$

and this is (5.5). □

In order to obtain the desired contradiction, we shall often prove that the function v defined by

$$v(r,\lambda)=(Aw')(r,\lambda) \tag{5.7}$$

is zero for some $\lambda<\bar{\lambda}$. We have shown in Corollary 5.3 that v is continuous at $(\bar{r},\bar{\lambda})$. An easy calculation shows that v satisfies the equation

$$v'+\frac{2w'^2}{r}v+\frac{w(1-w^2)}{r^2}=0. \tag{5.8}$$

We now choose numbers w_0, w_2, w_3 satisfying

$$-1<w_3<w_2<w_1<w_0<0$$

(where w_1 is obtained from Lemma 5.4), and let

$$c^2=\min\{-w(1-w^2):\ w_3\leqq w\leqq w_0\}. \tag{5.9}$$

We recall that there exists $L>0$ satisfying (cf. Proposition 5.1)

$$(Aw'^2)(r,\lambda)\leqq L, \quad \text{if} \quad 1\leqq\lambda\leqq 2, \tag{5.10}$$

for all orbits under consideration. We shall now prove some lemmas.

Lemma 5.6. *Let $\lambda_n \nearrow \bar{\lambda}$, and suppose that there are positive constants B, τ, and a sequence $\{r_n\}$ of positive numbers satisfying the following conditions:*

$$r_n\leqq B, \tag{5.11}$$

$$v(r_n,\lambda_n)\to 0 \quad as \quad n\to\infty, \tag{5.12}$$

$$w_2<w(r_n,\lambda_n)<w_0 \quad for\ large\ n, \tag{5.13}$$

$$r_{w_3}(\lambda_n)-r_n\geqq\tau>0 \quad (cf.\ (5.4)). \tag{5.14}$$

Then there exists an integer N such that if $n>N$, $v(r,\lambda_n)=0$, and $w_3<w(r,\lambda_n)<w_0$, for some $r=r(n)$.

Proof. From (5.7), for $r_{w_3}(\lambda_n)\geqq r\geqq r_n$, we have

$$\begin{aligned} v' &= \frac{-2w'^2}{r}v-\frac{w(1-w^2)}{r^2} \\ &\geqq \frac{-w(1-w^2)}{r^2}\geqq\frac{c^2}{r^2}. \end{aligned}$$

Thus, integrating gives

$$\begin{aligned} v(r_{w_3}(\lambda_n),\lambda_n) &\geqq v(r_n,\lambda_n)+c^2\int_{r_n}^{r_{w_3}(\lambda_n)}\frac{dr}{r^2} \\ &= v(r_n,\lambda_n)+c^2\frac{(r_{w_3}-r_n)}{r_n r_{w_3}(\lambda_n)}. \end{aligned}$$

Now if for some $k>0$, $r_{w_3}(\lambda_n)\leqq k$ for infinitely many n, then

$$v(r_{w_3}(\lambda_n),\lambda_n)\geqq v(r_n,\lambda_n)+\frac{c^2\tau}{kB}>0,$$

for sufficiently large n. This inequality means that we may assume $r_{w_3}(\lambda_n)\to\infty$. Thus for large n, $r_{w_3}(\lambda_n)\geqq 2B$. Then

$$\begin{aligned} v(r_{w_3}(\lambda_n),\lambda_n) &\geqq v(r_n,\lambda_n)+\frac{1}{r_n}-\frac{1}{r_{w_3}(\lambda_n)} \\ &\geqq v(r_n,\lambda_n)+\frac{1}{2B}>0 \end{aligned}$$

for sufficiently large n. This completes the proof. □

We can actually strengthen the last result by eliminating hypothesis (5.14).

Corollary 5.7. *Let $\lambda_n \nearrow \bar{\lambda}$ and suppose that there is a positive constant B and a sequence $\{r_n\}$ of positive numbers satisfying the following conditions:*

$$r_n \leqq B, \tag{5.11}$$

$$v(r_n, \lambda_n) \to 0 \quad as \quad n \to \infty, \tag{5.12}$$

$$w_2 < w(r_n, \lambda_n) < w_0 \quad for\ large\ n. \tag{5.13}$$

Then there exists an integer N such that if $n > N$, $v(r, \lambda_n) = 0$, and $w_0 > w(r, \lambda_n) > w_3$ for some $r = r(n)$.

Proof. We need only show that (5.14) holds. If $r_{w_3(\lambda_n)} - r_n \geqq 1$ for infinitely-many n, then Lemma 5.6 applies; we may thus assume (without loss of generality) that for all n

$$r_{w_3}(\lambda_n) - r_n < 1. \tag{5.15}$$

Now from Lemma 5.4, $\Phi(r, \lambda_n) \geqq \sigma$ whenever $-1 < w(r, \lambda_n) < w_1$. From Lemma 5.5, with $r_1 = r_{w_1}(\lambda_n)$, $r_2 = r_{w_2}(\lambda_n)$, we have $w'(r_{w_2}(\lambda_n), \lambda_n) \geqq -k r_{w_2}(\lambda_n)^2$, where $k = \frac{L}{\sigma}(w_1 - w_2)^{-1}$. Since $w_1 < 0$, $w''(r, \lambda_n) > 0$ if $r > r_{w_2}(\lambda_n)$; it follows that $w'(r, \lambda_n) \geqq -k r_{w_2}(\lambda_n)^2$. Thus

$$r_{w_3}(\lambda_n) - r_{w_2}(\lambda_n) = \frac{w_3 - w_2}{w'(\xi, \lambda_n)},$$

for some $\xi = \xi(\lambda_n)$, $r_{w_3}(\lambda_n) > \xi > r_{w_2}(\lambda_n)$. Hence

$$r_{w_3}(\lambda_n) - r_n \geqq r_{w_3}(\lambda_n) - r_{w_2}(\lambda_n) \geqq \frac{w_2 - w_3}{k r_{w_2}(\lambda_n)^2}.$$

But from (5.15) and (5.11)

$$r_{w_2}(\lambda_n) < r_{w_3}(\lambda_n) < 1 + B,$$

so that

$$r_{w_3}(\lambda_n) - r_n \geqq \frac{w_2 - w_3}{k(1+B)^2} \equiv \tau.$$

This completes the proof. □

We can now show that Cases 1–4 above, are impossible. We begin with the easy case.

Proposition 5.8. *Case 1 ($\bar{w} \geqq 0$, $w'(r, \bar{\lambda})$ bounded for r near $\bar{r}$) is impossible.*

Proof. For $\lambda = \bar{\lambda}$, set $Q'(r) = \frac{2w'^2(r)}{r}$, $Q(0) = 0$. Then $Q(\bar{r}) < \infty$, and from (5.7), for $r < \bar{r}$,

$$(e^Q v)' = \frac{-w(1-w^2)}{r^2} e^Q < 0.$$

On the other hand, $e^{Q(r)} v(r) = 0$ for $r = 0$, and $r = \bar{r}$ (Corollary 5.3), so $(e^Q v)'$ must vanish for some r between 0 and $\bar{r}$. This contradiction establishes the result. □

We next eliminate the case $\bar{w} < 0$.

Proposition 5.9. *Case 2* ($\bar{w}<0$) *is impossible.*

Proof. First, suppose $\bar{w}=-1$. Since Lemma 5.4 implies that $\Phi(r)>0$ if $-1\leqq w(r)<w_1$, we see that $r^2Aw''=-\Phi w'-w(1-w^2)>0$ for $-1\leqq w(r)<w_1$. Hence $w'(r,\bar{\lambda})$ is bounded for r near $\bar{r}$, and

$$\lim_{r\nearrow\bar{r}} A'(r)=A'(\bar{r})=\Phi(\bar{r})/\bar{r}^2>0,$$

which is impossible. We may thus assume $\bar{w}>-1$.

We shall use Corollary 5.7 to show that $v(\bar{r},\bar{\lambda})=0$ (cf. Corollary 5.3) implies that $v(r,\lambda)=0$ for some $\lambda<\bar{\lambda}$, where $w(r,\lambda)>-1$.

By Lemma 5.4, we choose $w_1<\bar{w}$ such that $\Phi\geqq\sigma$ on $-1\leqq w\leqq w_1$. Now choose w_0, w_2, w_3 such that $-1<w_3<w_2<w_1<\bar{w}<w_0<0$. We shall now verify the hypotheses of Corollary 5.7. To this end we set $r_n=\bar{r}$ for all n, and let $\lambda_n\nearrow\bar{\lambda}$. Then (5.11) holds, and from Corollary 5.3, (5.12) also holds. Furthermore, as $w(\bar{r},\bar{\lambda})=\bar{w}$, we see that (5.13) holds. Thus all the hypotheses of Corollary 5.7 hold, so that for n large, $v(r,\lambda_n)=0$, and $w(r,\lambda_n)>-1$. This completes the proof. □

In order to rule out the remaining Cases 3 and 4, we need the following result. (Recall that we continue to assume that we are in Case A.)

Lemma 5.10. *Assume that* $w'(r,\lambda)$ *is unbounded for* r *near* $\bar{r}$; *then* $\Phi(\bar{r},\bar{\lambda})\leqq 0$.

Proof. If $\Phi(\bar{r},\bar{\lambda})=\theta>0$, we choose $r_n\nearrow\bar{r}$ such that $w'(r_n,\bar{\lambda})=-n$, $w''(r_n,\bar{\lambda})\leqq 0$ [if $w''(r,\bar{\lambda})>0$ whenever $w'(r,\bar{\lambda})=-n$, then $-n$ would be a lower bound for $w'(r,\bar{\lambda})$]. Since $\Phi(r_n,\bar{\lambda})w'(r_n,\bar{\lambda})\to-\infty$, this would violate (3.2)′; hence $\Phi(\bar{r},\bar{\lambda})\leqq 0$. □

Our next goal is to strengthen this last result.

Proposition 5.11. *Assume that* $\bar{w}\geqq 0$, *and that* $w'(r,\bar{\lambda})$ *is unbounded for* r *near* $\bar{r}$; *then* $\Phi(\bar{r},\bar{\lambda})=0$.

Before giving the proof, we will need the following lemma. [This lemma would be trivial if Φ were a continuous function of w and λ. But we must work harder because w' is unbounded near $(\bar{r},\bar{\lambda})$.]

Lemma 5.12. *Assume that* $\bar{w}\geqq 0$, $w'(r,\bar{\lambda})$ *is unbounded for* r *near* $\bar{r}$, *and* $\Phi(\bar{r},\bar{\lambda})=-2\theta<0$. *Then there exists* $\varepsilon>0$, *and* $\delta>0$ *such that* $0<\bar{\lambda}-\lambda<\delta$, *and* $|w(r,\lambda)-\bar{w}|<\varepsilon$, *imply* $\Phi(r,\lambda)<-\theta$.

Proof. Consider first the case $\bar{w}>0$. Let $\lambda<\bar{\lambda}$, and consider (3.16) in the region $w\geqq 0$:

$$\begin{aligned}\phi'(r)&=\frac{2(1-w^2)^2}{r^2}+2Aw'^2+\frac{4w(1-w^2)}{r}w'\\&<2Aw'^2+\frac{2}{r^2}\\&<2L+\frac{2}{r^2},\end{aligned}\tag{5.16}$$

where we have used (5.9). Choose r_1, $\bar{r}>r_1>\frac{\bar{r}}{2}$ such that both of the following hold:

$$\Phi(r_1,\bar{\lambda})<\frac{-7}{4}\theta,\quad\text{and}\quad -w'(r_1,\bar{\lambda})>\frac{4k}{\theta}.\tag{5.17}$$

Choose $\delta>0$ such that $0<\bar{\lambda}-\lambda\leqq\delta$ implies that the following hold:

$$\Phi(r_1,\lambda)<-\tfrac{3}{2}\theta, \tag{5.18}$$

$$-w'(r_1,\lambda)>\frac{2k}{\theta}, \tag{5.19}$$

$$w(r_1,\lambda)>\bar{w}. \tag{5.20}$$

For $0<\lambda-\lambda\leqq\delta$, define $r_0(\lambda)$ by [cf. (5.4)]

$$w(r_0(\lambda),\lambda)=0. \tag{5.21}$$

[Note that this implies that (5.16) holds for $r\leqq r_0(\lambda)$.]

Now suppose that for some r, $r_1\leqq r\leqq r_0(\lambda)$, $0<\bar{\lambda}-\lambda\leqq\delta$, we have $\Phi(r,\lambda)=-\theta$; let $r_2(\lambda)$ be the first such r-value. Then for $r_1\leqq r\leqq r_2(\lambda)$, and $0<\bar{\lambda}-\lambda\leqq\delta$, (5.16) and (5.18) give

$$\begin{aligned}\Phi(r,\lambda)&=\Phi(r_1,\lambda)+\int_{r_1}^{r}\Phi'(r,\lambda)dr\\&\leqq-\tfrac{3}{2}\theta+k(r_2(\lambda)-r_1),\end{aligned} \tag{5.22}$$

with $k=2L+8/\bar{r}^2$, and where we have used the fact that $r\geqq r_1>\dfrac{\bar{r}}{2}$. Now on this λ-range,

$$-1<w(r_2(\lambda),\lambda)-w(r,\lambda)=w'(\xi,\lambda)(r_2(\lambda)-r_1)$$

for some $\xi=\xi(\lambda)$, $r_1<\xi<r_2(\lambda)$. Then on this interval, $\Phi(r,\lambda)<0$, and so (3.2)′ shows that $w''(r,\lambda)<0$, so $w'(r_1,\lambda)>w'(\xi(\lambda),\lambda)$. Thus from (5.22), for $0<\bar{\lambda}-\lambda\leqq\delta$,

$$\begin{aligned}\Phi(r,\lambda)&\leqq-\frac{3}{2}\theta-\frac{k}{w'(\xi,\lambda)}\\&\leqq-\frac{3}{2}\theta-\frac{k}{w'(r_1,k\lambda)}\\&\leqq-\frac{3}{2}\theta-\frac{k\theta}{4k}\\&<-\theta,\end{aligned}$$

where we have used (5.19). This is a contradiction. Hence for $0<\bar{\lambda}-\lambda\leqq\delta$,

$$\Phi(r,\lambda)<-\theta \quad \text{if} \quad r_1\leqq r\leqq r_0(\lambda), \qquad 0<\bar{\lambda}-\lambda\leqq\delta. \tag{5.23}$$

On the other hand, from (5.20), we see that for $0<\lambda-\bar{\lambda}\leqq\delta$, $w(r_1,\lambda)>\bar{w}$, and $w(r_1,\bar{\lambda})>\bar{w}$ because $r_1<\bar{r}$. If we set

$$w_1=\inf\{w(r_1,\lambda):\ 0\leqq\bar{\lambda}-\lambda\leqq\delta\},$$

then $w_1>\bar{w}$, and thus from (5.23), if $0<\bar{\lambda}-\lambda\leqq\delta$, $\Phi(r,\lambda)<-\theta$ if $0\leqq w(r,\lambda)<w_1$. This completes the proof of the lemma in the case $\bar{w}>0$.

If $\bar{w}=0$, we make the following modifications in the proof. We choose r_1 as before, satisfying (5.17), and in addition, so as to also satisfy $w(r_1,\bar{\lambda})<\frac{1}{8}$. Then we can find a $\delta>0$ such that if $0<\bar{\lambda}-\lambda\leqq\delta$, (5.18)–(5.20) hold, and in addition

$$w(r_1,\lambda)<\tfrac{1}{4}. \tag{5.24}$$

Now choose $\varepsilon>0$ so that both of the following hold:

$$\varepsilon<\frac{1}{4} \quad\text{and}\quad \frac{8\varepsilon^2}{\bar{r}}<\theta/4.$$

Then from (3.16), for $w(r,\lambda)\geqq-\varepsilon$, and $r\geqq r_1$,

$$\begin{aligned}\Phi(r,\lambda)&=\Phi(r_1,\lambda)+\int_{r_1}^{r}\left(2w'^2A+\frac{2(1-w^2)^2}{r^2}\right)dr\\&\quad+\int_{r_1}^{r_0(\lambda)}\frac{4w(1-w^2)}{r}w'dr+\int_{r_0(\lambda)}^{r}\frac{4w(1-w^2)}{r}w'dr\\&\leqq-\frac{3}{2}\theta+k(r-r_1)+\int_{r_0(\lambda)}^{r}\frac{4w(1-w^2)}{r}w'dr.\end{aligned}\tag{5.25}$$

Suppose that for some $r>r_1$ with $w(r_1,\lambda)\geqq\varepsilon$, we have $\phi(r,\lambda)=-\theta$; let $r_2(\lambda)$ be the first such r-value. Then for $r_1\leqq r\leqq r_2(\lambda)$, as before

$$(r-r_1)\leqq\frac{w(r_1,\lambda)-w(r,\lambda)}{-w'(r_1,\lambda)}<\frac{1}{2}\frac{\theta}{2k}=\frac{\theta}{4k}.\tag{5.26}$$

Thus from (5.25), with $r=r_2(\lambda)$,

$$\begin{aligned}\Phi(r_2(\lambda),\lambda)&\leqq-\frac{3}{2}\theta+\frac{\theta}{4}+\int_{r_0(\lambda)}^{r_2(\lambda)}\frac{4w(1-w^2)}{r}w'dr\\&=-\frac{5\theta}{4}+\int_{0}^{w(r_2(\lambda),\lambda)}\frac{4w(1-w^2)}{\bar{r}/2}dw\\&\leqq-\frac{5\theta}{4}+\frac{8}{\bar{r}}\varepsilon^2\\&<-\frac{5\theta}{4}+\frac{\theta}{4}=-\theta,\end{aligned}$$

where we have used (5.24). This contradiction shows that if $\bar{w}=0$, then $\Phi(r,\lambda)<-\theta$ if $-\varepsilon\leqq w(r,\lambda)\leqq w_1$, for $0<\bar{\lambda}-\lambda\leqq\delta$, where w_1 is defined as before (in the $\bar{w}>0$ case). This completes the proof of Lemma 5.12. □

Notice that under the hypotheses of Lemma 5.12, we have found an $\varepsilon>0$ such that if $\bar{w}\geqq0$, then

$$\Phi(r,\lambda)\leqq-\theta,\quad\text{whenever}\quad \bar{w}-\varepsilon\leqq w(r,\lambda)\leqq\bar{w}.\tag{5.27}$$

To complete the proof of Proposition 5.11, we need one more lemma.

Lemma 5.13. *Suppose that for* $0<\bar{\lambda}-\lambda\leqq\delta$, *there are constants* $a>b$, $k>0$, *and* $R_1>0$ *such that the following hypotheses hold:*

$$\Phi(r,\lambda)<-k \quad \textit{if} \quad b\leqq w(r,\lambda)\leqq a,\tag{5.28}$$

$$r_a(\lambda)\leqq R_1,\tag{5.29}$$

$$\overline{\lim_{\lambda\nearrow\bar{\lambda}}}\,-w'(r_a(\lambda),\lambda)=+\infty.\tag{5.30}$$

Then there is a $\lambda<\bar{\lambda}$ *and an* $r_3=r_3(\lambda)$ *such that* $b<w(r_3,\lambda)<a$ *and* $\lim_{r\nearrow r_3}-w'(r,\lambda)=+\infty$.

Proof. The idea is to use the fact that given any $\varepsilon_1 > 0$, solutions of the inequality $y' \geqq c^2 y^2$ blow up in time $\leqq (\varepsilon_1 + t_0)$, provided that $y(t_0)$ is sufficiently large.

We can find a convergent sequence $r_a(\lambda_n) \to \varrho \leqq R_1$, for which $\lim_{n\to\infty} -w'(r_a(\lambda_n), \lambda_n) = +\infty$. Now choose N so large that $n > N$ implies that

$$-w'(r_a(\lambda_n), \lambda_n) > \max\left(\frac{2}{k}, a-b\right).$$

Then at $r = r_a(\lambda_n)$, (3.2)′ gives

$$r_a(\lambda_n)^2 A w'' = -\Phi w' - w(1-w^2) < kw' + 1 < 0;$$

hence $w''(r_a(\lambda_n), \lambda_n) < 0$ so $w''(r, \lambda_n) < 0$ for $b \leqq w(r, \lambda_n) \leqq a$, and $-w'(r, \lambda_n)$ satisfies

$$-w'(r, \lambda_n) > \max\left(\frac{2}{k}, a-b\right), \tag{5.31}$$

for $r_b(\lambda_n) \geqq r \geqq r_a(\lambda_n)$. Then for some ξ, $r_b(\lambda_n) > \xi > r_a(\lambda_n)$,

$$r_b(\lambda_n) - r_a(\lambda_n) = \frac{b-a}{w'(\xi, \lambda_n)} < 1,$$

and thus (5.29) implies

$$r_b(\lambda_n) < R_1 + 1, \tag{5.32}$$

for $n > N$. Using (3.2)′ again, for $n > N$, and $r_b(\lambda_n) \geqq r \geqq r_a(\lambda_n)$,

$$\begin{aligned} \frac{dw'}{dw} &= \frac{-\Phi - \dfrac{w(1-w^2)}{w'}}{r^2 A} \geqq \frac{-\Phi/2}{r^2 A} \\ &\geqq \frac{k}{2(R_1+1)^2} \frac{w'^2}{Aw'^2} \\ &\geqq \frac{k}{2(R_1+1)^2 L} w'^2 \equiv c^2 w'^2, \end{aligned}$$

where we have used (5.9) and (5.31). Note that c^2 is independent of λ. Thus if we take n sufficiently large [so as to make $-w'(r_a(\lambda_n), \lambda_n)$ as large as we please], we see that $-w'(r, \lambda_n)$ will blow up for some r with $r_b(\lambda_n) > r > r_a(\lambda_n)$. This completes the proof of Lemma 5.13. □

We can now complete the proof of Proposition 5.11. Assume $\Phi(\bar{r}, \bar{\lambda}) = -2\theta < 0$. The idea is to use Lemma 5.13. For this, we set $a = \bar{w}$, and $b = \bar{w} - \varepsilon$, see (5.27). Then (5.28) holds (with $k = \theta$, from Lemma 5.12) and (5.30) clearly holds. To show (5.29), we choose $N > 0$ such that $-N\theta + 1 < 0$. Then choose $\varepsilon > 0$ and $\tilde{r}$, $\bar{r} > \tilde{r} > \bar{r}/2$ so that $w'(\tilde{r}, \bar{\lambda}) < -2N$ and $w < w(\tilde{r}, \bar{\lambda}) < \bar{w} + 2\varepsilon$. Then choose δ so that $0 < \bar{\lambda} - \lambda < \delta$ implies that $w'(\tilde{r}, \lambda) < -N$ and $-1 < w(\tilde{r}, \lambda) < \bar{w} + \varepsilon$. From (3.2)′, we have at $r = \tilde{r}$, for these λ, $\tilde{r}^2 A w'' = \Phi w' - w(1-w^2) < -kN + 1 < 0$. Thus $w''(\tilde{r}, \lambda) < 0$, and so $w'(r, \lambda) < -N$ provided that $b \leqq w(r, \lambda) \leqq \bar{w}$. Thus for $0 < \bar{\lambda} - \lambda < \delta$,

$$r_a(\lambda) - \tilde{r} = \frac{w(\tilde{r}, \lambda) - w(r_a(\lambda), \lambda)}{w'(\xi, \lambda)} < \frac{2}{N},$$

where $\bar{r}<\xi<r_a(\lambda)$. It follows that for $0<\bar{\lambda}-\lambda<\delta$, $r_a(\lambda)<\bar{r}+\frac{2}{N}$, and this proves (5.29). We have thus verified the hypotheses of Lemma 5.13. The conclusion of this lemma gives us the desired contradiction. This completes the proof of Proposition 5.11. □

We can now rule out the case $\bar{w}>0$ with $w'(r,\bar{\lambda})$ unbounded for r near $\bar{r}$. We will apply Lemma 5.13 to show that $-w'(r,\lambda)$ blows up for some $\lambda<\bar{\lambda}$.

Proposition 5.14. *Case 3 ($\bar{w}>0$, $w'(r,\bar{\lambda})$ unbounded for r near $\bar{r}$) is impossible.*

Proof. From Proposition 5.11, we may assume that $\Phi(\bar{r},\bar{\lambda})=0$. Choose η, so that $0<\eta<\bar{w}$. We now prove some lemmas which we shall use to verify conditions (5.28)–(5.30) in Lemma 5.13.

Lemma 5.15. *If $\bar{w}>0$, and $w'(r,\bar{\lambda})$ is unbounded for r near $\bar{r}$, then given $N>0$, there exist $\varepsilon>0$, and $\delta>0$ such that if $0<\bar{\lambda}-\lambda<\delta$, and $\eta\leqq w(r,\lambda)\leqq\bar{w}+\varepsilon$, then $w'(r,\lambda)<-N$ and $w''(r,\lambda)<0$.*

Proof. Choose N_1 so large that both of the following hold [cf. (5.9)]:

$$\frac{2}{N_1}<\bar{r} \quad \text{and} \quad 2L+\frac{8}{\bar{r}^2}-\frac{2\eta N_1}{\bar{r}}<-2. \tag{5.33}$$

We claim that for $\bar{r}/2\leqq r\leqq 2\bar{r}$, and $\lambda<\bar{\lambda}$, that if $w'(r,\lambda)<-N_1$, we have $\Phi'(r,\lambda)<-2$ provided that $w(r,\lambda)\geqq\eta$. (To see this, note that from (3.16)

$$\Phi'(r,\lambda)=2w'^2A+\frac{2(1-w^2)^2}{r^2}+\frac{4w(1-w^2)w'}{r}<-2$$

in view of (5.33).) Next we claim that for $\lambda<\bar{\lambda}$, if for some r_1, $\frac{\bar{r}}{2}<r_1<\bar{r}$, $w'(r_1,\lambda)<-N_1$, $w''(r_1,\lambda)<0$, and $w(r_1,\lambda)>\bar{w}$, then for $r>r_1$, $w'(r,\lambda)$ decreases as long as $w(r,\lambda)\geqq\eta$. To see this, suppose that there were a first point $r_2>r_1$ for which $w''(r_2,\lambda)=0$. Then

$$r_2-r_1\leqq\frac{w(r_1,\lambda)-w(r_2,\lambda)}{N_1}\leqq\frac{2}{N_1},$$

so that $\bar{r}/2<r_2<2\bar{r}$, in view of (5.33). Then differentiating (3.2)′ gives (suppressing λ)

$$r_2^2A(r_2)w'''(r_2)+[\Phi'(r_2)+(1-3w^2(r_2))]w'(r_2)=0.$$

Since $w'(r_2,\lambda)<-N_1$, our earlier claim gives $\Phi'(r_2,\lambda)<-2$ so that $[\Phi'(r_2)+1-3w^2(r_2)]<-1$ and thus $w'''(r_2,\lambda)<0$. This contradiction shows that no such r_2 exists, and proves our second claim.

Now given $N>N_1$, we can find $\varepsilon>0$ such that $w'(r_{\bar{w}+\varepsilon}(\bar{\lambda}),\bar{\lambda})<-2N$ and $w''(r_{\bar{w}+\varepsilon}(\bar{\lambda}),\bar{\lambda})<0$. Then for λ near $\bar{\lambda}$, say $0<\bar{\lambda}-\lambda<\delta$, $w'(r_{\bar{w}+\varepsilon}(\lambda),\lambda)<-N$, and $w''(r_{\bar{w}+\varepsilon}(\lambda),\lambda)<0$. Applying our last claim (with N_1 replaced by N) shows that for $0<\bar{\lambda}-\lambda<\delta$, $w'(r,\lambda)<-N$ if $r_\eta(\lambda)\geqq r\geqq r_{\bar{w}+\varepsilon}(\lambda)$. □

Lemma 5.16. *If $\bar{w}>0$, and $w'(r,\bar{\lambda})$ is unbounded for r near $\bar{r}$, then there exists a $k>0$ such that if $0<\bar{\lambda}-\lambda<\delta$ then $\Phi(r,\lambda)<-k$ if $0\leqq w(r,\lambda)\leqq\eta$. (Here δ is as in the last lemma.)*

Proof. Since $A(\bar{r}, \bar{\lambda})=0$, we have [cf. (3.6)],

$$0=\Phi(\bar{r}, \bar{\lambda})=\bar{r}-\frac{(1-\bar{w}^2)^2}{\bar{r}}>\bar{r}-\frac{(1-\eta^2)^2}{\bar{r}}\equiv -2k.$$

By continuity, there exists $\varepsilon_1>0$ such that if $|r-\bar{r}|<\varepsilon_1$, then $r-(1-\eta^2)^2/r<-k$. Thus for $0\leqq w\leqq\eta$,

$$\Phi(r,\lambda)\leqq r-\frac{(1-w^2(r,\lambda))^2}{r}<r-\frac{(1-\eta^2)^2}{r}<-k,$$

provided that $|r-\bar{r}|<\varepsilon_1$. Choose N, such that $\frac{1}{N}<\frac{1}{2}\varepsilon_1$, and $-Nk+1<0$. Now from Lemma 5.15, for $0<\bar{\lambda}-\lambda<\delta$,

$$w'(r_\eta(\lambda),\lambda)<-N \quad \text{and} \quad w''(r_\eta(\lambda),\lambda)<0.$$

Thus $w'(r,\lambda)<-N$ for r slightly larger than $r_\eta(\lambda)$; say $r_\eta(\lambda)\leqq r<r_1$. Then as

$$0<r_1-r_\eta(\lambda)\leqq\frac{2}{N}<\varepsilon_1,$$

we have $\Phi(r,\lambda)<-k$ for $r_\eta(\lambda)\leqq r\leqq r_1$ and thus for $0<\bar{\lambda}-\lambda<\delta$, and $r_\eta(\lambda)\leqq r\leqq r_1$ (again suppressing λ)

$$\begin{aligned} r^2A(r)w''(r)&=-\Phi(r)w'(r)-w(r)(1-w^2(r))\\ &<-Nk+1<0; \end{aligned}$$

hence $w''(r,\lambda)<0$, so $w'(r,\lambda)<-N$, and we may repeat the argument to conclude that $\Phi(r,\lambda)<-k$ for $0\leqq w(r,\lambda)\leqq\eta$, provided that $0<\bar{\lambda}-\lambda<\delta$. □

We can now complete the proof of Proposition 5.14. In order to apply Lemma 5.13, we set $b=0$, and $a=\eta$. From Lemma 5.16, for $0<\lambda-\lambda<\delta$, we see that (5.28) holds, and Lemma 5.15 implies (5.30). Thus the proof will be complete provided that we show $r_\eta(\lambda)$ is bounded above, on this range of λ's. But this is easy; namely, since $r_{\bar{w}+\varepsilon}(\lambda)$ is bounded for $1\leqq\lambda\leqq\bar{\lambda}$, and

$$r_\eta-r_{\bar{w}+\varepsilon}=\frac{\bar{w}+\varepsilon-\eta}{-w'(\xi,\lambda)}\leqq\frac{1}{N}$$

[where $\xi=\xi(\lambda)$ is an intermediate point], we see that (5.30) holds. Thus using Lemma 5.12, we obtain the contradiction $-w'(r,\lambda)$ blows up in the region $0\leqq w\leqq\eta$. The proof of Proposition 5.15 is complete. □

Finally, we shall rule out the case $\bar{w}=0$ and $w'(r,\bar{\lambda})$ is unbounded for r near $\bar{r}$.

Proposition 5.17. *Case 4 ($\bar{w}=0$, $w'(r,\bar{\lambda})$ unbounded for r near $\bar{r}$) is impossible.*

Proof. From Proposition 5.11, we may assume that $\Phi(\bar{r},\bar{\lambda})=0$. Notice that $A(\bar{r},\bar{\lambda})=0=\bar{w}$ implies that $\bar{r}=1$. Under these hypotheses, we have the following lemma.

Lemma 5.18. *If there is an $\eta>0$ such that $w(r_n,\lambda_n)=-\eta$ for some sequence (r_n,λ_n), where $\lambda_n\nearrow\bar{\lambda}$, and the r_n's are bounded, then for sufficiently large n, there exists an $\tilde{r}_n$ such that $(Aw')(\tilde{r}_n,\lambda_n)=0$, and $-1<w(\tilde{r}_n,\lambda_n)<0$; i.e., Proposition 5.17 holds.*

Proof. We shall show that the hypotheses of Corollary 5.7 hold. Thus, by hypothesis, there is a $B>0$ such that $r_n\leqq B$; hence (5.11) holds. Now choose w_0

satisfying $-\eta<w_0<0$, and $w_3<w_2<w_1<-\eta$, where w_1 is obtained from Lemma 5.4. Then clearly (5.13) holds. It remains to prove (5.12). Now if $w(1,\lambda_n)\leqq 0$ for infinitely-many n, then since (5.6) implies that $v'(r,\lambda_n)>0$ if $r>r_0(\lambda_n)$, we have, for large n, $0>v(r,\lambda_n)>v(1,\lambda_n)$. Since v is continuous (Corollary 5.3), $v(1,\lambda_n)\to 0$ so (5.12) holds. We may thus assume that $w(1,\lambda_n)>0$ for all n. Now let $\varepsilon>0$ be given. Choose $\delta>0$ such that $w(1-\delta,\bar{\lambda})<\varepsilon/2$; then for n large, $w(1-\delta,\lambda_n)<\varepsilon$. Hence

$$
\begin{aligned}
0 &\geqq v(r_n,\lambda_n)>v(r_0(\lambda_n),\lambda_n)=v(1,\lambda_n)+\int_1^{r_0(\lambda_n)} v'dr\\
&>v(1,\lambda_n)+\int_1^{r_0(\lambda_n)} \frac{-w(1-w^2)}{r^2}dr\\
&>v(1,\lambda_n)+\int_1^{r_0(\lambda_n)} -w\,dr\\
&>v(1,\lambda_n)-Bw(1,\lambda_n)\\
&>v(1,\lambda_n)-Bw(1-\delta,\lambda_n)\\
&>v(1,\lambda_n)-B\varepsilon .
\end{aligned}
$$

Since $v(1,\lambda_n)\to 0$, and ε was arbitrary, we see that $v(r_n,\lambda_n)\to 0$. Thus Corollary 5.7 can be applied to show that for n large, there is an $\tilde{r}_n$ such that $v(\tilde{r}_n,\lambda_n)=0$, with $-1<w(\tilde{r}_n,\lambda_n)<0$. This contradiction completes the proof. □

We now return to the proof of Proposition 5.16. In view of our last lemma, we may assume that if r_n is a bounded sequence and $\lambda_n\nearrow\bar{\lambda}$, then $\underline{\lim}\, w(r_n,\lambda_n)\geqq 0$. This can be used to give the following statement:

$$
\begin{cases}\text{If } r_n \text{ is a bounded sequence satisfying } \underline{\lim}\, r_n>1,\\ \text{and } \lambda_n\nearrow\bar{\lambda}, \text{ then } \lim\limits_{n\to\infty} w(r_n,\lambda_n)=0.\end{cases} \tag{5.35}
$$

[For, given $\varepsilon>0$, there exists a $\delta>0$ such that $w(1-\delta,\bar{\lambda})<\varepsilon/2$. If n is large, $w(1-\delta,\lambda_n)<\varepsilon$, and $r_n>1$ so that $w(r_n,\lambda_n)<w(1,\lambda_n)<w(1-\delta,\lambda_n)<\varepsilon$. On the other hand, our earlier statement shows that $\underline{\lim}\, w(r_n,\lambda_n)\geqq 0>-\varepsilon$.] We now have.

Lemma 5.19. *Given any sequence $\lambda_n\nearrow\bar{\lambda}$, and any $B>1$, there is a subsequence λ_{n_k}, and a sequence $r_k\to B$ such that $(w(r_k,\lambda_{n_k}), w'(r_k,\lambda_{n_k}))\to(0,0)$ as $k\to\infty$.*

Proof. We have, for any $k\in\mathbb{Z}_+$,

$$
w(B,\lambda_n)-w\left(B-\frac{1}{k},\lambda_n\right)=\frac{1}{k}w'(\xi_n^k,\lambda_n),
$$

for some ξ_n^k, $B-\frac{1}{k}<\xi_n^k<B$. For large k, $B-\frac{1}{k}>1$, so that (5.35) implies that $w'(\xi_n^k,\lambda_n)\to 0$ as $n\to\infty$. Thus there is a subsequence $\{\xi_{n_k}\}\subset\{\xi_n^k\}$ such that $|w'(\xi_{n_k},\lambda_{n_k})|<\frac{1}{k}$. Thus $w'(\xi_{n_k},\lambda_{n_k})=0$ and since $\lim\limits_{k\to\infty}\xi_n^k=B$, we have $\lim\limits_{k\to\infty}\xi_{n_k}^k=B$; thus (5.35) implies that $\lim\limits_{k\to\infty} w(\xi_n^k,\lambda_{n_k})=0$. This proves the lemma if we set $r_k=\xi_{n_k}$. □

Using this last lemma, and passing twice to subsequences, we produce sequences λ_n, r_n, s_n satisfying $r_n\to 2$, $s_n\to 3/2$, $\lambda_n\nearrow\bar{\lambda}$, and such that both of the following hold:

$$(w(s_n, \lambda_n), w'(s_n, \lambda_n)) \to (0,0), \tag{5.36}$$

$$(w(r_n, \lambda_n), w'(r_n, \lambda_n)) \to (0,0). \tag{5.37}$$

We thus have (by passing to a further subsequence, if necessary) a sequence (r_n, λ_n) such that

$$(w(r_n, \lambda_n), w'(r_n, \lambda_n)), A(r_n, \lambda_n), r_n) \to (0, 0, \bar{A}, 2).$$

Consider first the case $\bar{A} > 0$; i.e.,

$$\lim_{n\to\infty} A(r_n, \lambda_n) = \bar{A} > 0. \tag{5.38}$$

In this case we have an explicit solution to Eqs. (3.1)′, (3.2)′; namely, $w_e(r) \equiv 0$, $w_e'(r) \equiv 0$, $A_e(r) = 1 + \frac{1}{r^2} - \frac{c}{r}$, where $c = \frac{5}{2} - 2\bar{A}$. This solution is even defined for all complex $r \neq 0$. We shall obtain a contradiction by showing that on the one hand

$$\lim_{\lambda_n \to \bar{\lambda}} w\left(\frac{1}{10}, \lambda_n\right) = w\left(\frac{1}{10}, \bar{\lambda}\right) \equiv w_0 > 0$$

(this is merely the continuity of w as a function of λ), and on the other hand,

$$\lim_{n\to\infty} w\left(\frac{1}{10}, \lambda_n\right) = w_e\left(\frac{1}{10}\right) \equiv 0.$$

From standard results, the solution (A, w) of (3.1), (3.2) is an analytic function of both r and λ, if $A > 0$, see e.g. [6, p. 73], and the appendix.

We proceed now with the details. The orbit of the explicit solution through the point $p_e = (0, 0, \bar{A}, 2)$ in (w, w', A, r)-space is $\left(0, 0, 1 + \frac{1}{r^2} - \frac{c}{r}, r\right)$, where $c = \frac{5}{2} - 2\bar{A}$; $\frac{5}{2} \geqq c \geqq \frac{1}{2}$. If we consider r as a complex variable, then the two-point set $\left\{r \in \mathbb{C}: 1 + \frac{1}{r^2} - \frac{c}{r} = 0\right\}$ does not separate the point $r = 2$ from the point $r = \frac{1}{10}$. Hence we may choose a compact contour γ joining $r = 2$ to $r = \frac{1}{10}$ and such that $A_e(r) = 1 + \frac{1}{r^2} - \frac{c}{r} \neq 0$ for $r \in \gamma$. If p is any point in $\mathbb{R}^4$ sufficiently close to p_e, then the orbit through p, $(w(r), w'(r), A(r), r)$, will be close to the explicit solution for $r \in \gamma$ by continuous dependence on initial conditions [6, p. 73]. In particular, the orbit through p will have $A(r) > 0$, and $\left|w\left(\frac{1}{10}\right) - w_e\left(\frac{1}{10}\right)\right|$ small. Taking λ_n close to λ yields $p_n = (w(2, \lambda_n), w'(2, \lambda_n), A(2, \lambda_n), 2)$ arbitrarily close to p_e, and hence $w\left(\frac{1}{10}, \lambda_n\right)$ arbitrarily close to $w_e\left(\frac{1}{10}\right) = 0$. But $w\left(\frac{1}{10}, \lambda_n\right)$ is also close to $w\left(\frac{1}{10}, \bar{\lambda}\right) > 0$. This is a contradiction. This completes the proof in the case $\bar{A} > 0$. Now consider the case $\bar{A} = 0$; i.e.,

$$\lim_{n\to\infty} A(r_n, \lambda_n) = 0 \quad (r_n \to 2). \tag{5.39}$$

Now if $\lim_{n\to\infty} A(s_n, \lambda_n)>0$, then just as above, we can arrive at a contradiction; hence we may assume that we also have

$$\lim_{n\to\infty} A(s_n, \lambda_n)=0 \qquad (s_n\to 3/2). \tag{5.40}$$

We will show that it is inconsistent to have both (5.39) and (5.40), by using (3.4),

$$(re^{Q}A)'=\left(1-\frac{(1-w^2)^2}{r^2}\right)e^{Q}\geqq 1-\frac{4}{9}=\frac{5}{9}$$

and showing that $-w'(r, \lambda_n)\leqq$ const on the interval $3/2\leqq r\leqq 2$, for large n. For this we need the following lemma $\left(\text{cf. (3.8) with } c=\frac{1}{3}\right)$.

Lemma 5.20. *Define* $g(r,\lambda)=\frac{1}{3}r^2-(1-w^2(r,\lambda))$; *then for large n, the following statements hold:*

(i) $g\left(\frac{3}{2}, \lambda_n\right)>g(1.1, \lambda_n)$,

(ii) $\left(g+\frac{2}{3}(1-w^2)^2+r^2Aw'^2\right)(r,\lambda_n)\geqq k>0$, *if* $1.1\leqq r$, *where k is a constant independent of n and r.*

Proof. We have

$$g\left(\frac{3}{2}, \lambda_n\right)-g(1.1, \lambda_n)=\frac{9}{12}-\left(1-w^2\left(\frac{3}{2}, \lambda_n\right)\right)-\left[\frac{1.21}{3}-(1-w^2(1.1, \lambda_n))\right]$$
$$>\frac{1}{3}+w^2\left(\frac{3}{2}, \lambda_n\right)-w^2(1.1, \lambda_n).$$

But from Lemma 5.18 passing to a subsequence if necessary, we may assume that

$$\lim_{n\to\infty} w^2\left(\frac{3}{2}, \lambda_n\right)=0=\lim_{n\to\infty} w^2(1.1, \lambda_n),$$

and thus (i) holds. For (ii), writing $u=1-w^2$ gives at (r, λ_n),

$$g+\frac{2}{3}u^2+r^2Aw'^2\geqq \frac{r^2}{3}-u+\frac{2}{3}u^2$$
$$\geqq \frac{r^2}{3}-\frac{3}{8}$$
$$\geqq \frac{9.68-9}{24}\geqq \frac{0.68}{24}\equiv k>0,$$

if $r\geqq 1.1$ because $\frac{2}{3}u^2-u\geqq -\frac{3}{8}$ if $0\leqq u\leqq 1$. This proves (ii) and completes the proof of the lemma.

As a consequence of part (i) of the last result, we have the following corollary.

Corollary 5.21. $g'(r, \lambda_n)>0$ *for* $\frac{3}{2}\leqq r\leqq 2$ *if n sufficiently is large.*

Proof. From part (i) of the last lemma, we see that for sufficiently large n, there is a ξ_n, $1.1<\xi_n<\frac{3}{2}$, for which $g'(\xi_n, \lambda_n)>0$. Also, from (3.13), with $c=\frac{1}{3}$, we have that g satisfies the equation

$$r^2Ag''+\Phi g'=2\left[g+\frac{2}{3}(1-w^2)+r^2Aw'^2\right],$$

so that if $P'=\Phi/r^2A$, we may write

$$(e^Pg')'=\frac{2}{r^2A}e^P\left[g+\frac{2}{3}(1-w^2)+r^2Aw'^2\right]. \tag{5.41}$$

Using part (ii) of the last lemma, we see that $(e^Pg')'>0$ if $r>1.1$. Thus for large n, say $n>N$, $g'(r,\lambda_n)>0$ if $\frac{3}{2}\leqq r\leqq 2$; i.e., for $n>N$,

$$r>-3(ww')(r,\lambda_n),\quad \text{if}\quad \frac{3}{2}\leqq r\leqq 2.\quad \square \tag{5.42}$$

Now consider the function Φ. Using (3.16) we have, for $\lambda=\lambda_n$, and $\frac{3}{2}\leqq r\leqq 2$,

$$\begin{aligned}\Phi'&=\frac{2(1-w^2)^2}{r^2}+\frac{4w(1-w^2)w'}{r}+2w'^2A\\ &\geqq\frac{2(1-w^2)^2}{r^2}+\frac{4w(1-w^2)w'}{r}\\ &\geqq\frac{2(1-w^2)^2}{r^2}-\frac{4}{3}(1-w^2)>\frac{4}{3},\end{aligned}$$

where we have used (5.42); thus

$$\Phi'(r,\lambda_n)>-\frac{4}{3},\quad \text{if}\quad \frac{3}{2}\leqq r\leqq 2. \tag{5.43}$$

Now as $\Phi(r)=r(1-A)-(1-w^2)/r\geqq r(1-A)-\frac{1}{r}$, we have

$$\begin{aligned}\Phi\left(\frac{3}{2},\lambda_n\right)&=\frac{3}{2}-\frac{3}{2}A\left(\frac{3}{2},\lambda_n\right)-\frac{2}{3}\left(1-w^2\left(\frac{3}{2},\lambda_n\right)\right)^2\\ &\geqq\frac{5}{6}-\frac{3}{2}A\left(\frac{3}{2},\lambda_n\right).\end{aligned}$$

Thus for $\frac{3}{2}\leqq r\leqq 2$, if n is large, we have for some σ_n, $\frac{3}{2}<\sigma_n<2$,

$$\begin{aligned}\Phi(r,\lambda_n)&\geqq\frac{5}{6}-\frac{3}{2}A\left(\frac{3}{2},\lambda_n\right)+\Phi'(\sigma_n,\lambda_n)\left(r-\frac{3}{2}\right)\\ &>\frac{5}{6}-\frac{3}{2}A\left(\frac{3}{2},\lambda_n\right)-\frac{4}{3}\left(r-\frac{3}{2}\right)\\ &\geqq\frac{5}{6}-\frac{3}{2}A\left(\frac{3}{2},\lambda_n\right)-\frac{4}{3}\left(\frac{1}{2}\right)\\ &>\delta>0,\end{aligned}$$

where we have used (5.43) and (5.40); here δ is independent of n. Now for large n, say $n>N_1>N$, $-w'\left(\frac{3}{2},\lambda_n\right)<\frac{1}{\delta}$, so if $\frac{3}{2}<r\leqq 2$, and $\lambda=\lambda_n$, n large, we have from (3.2)',

$$r^2Aw''=-\Phi w'-w(1-w^2)>\delta(-w')-1\,.$$

Thus if $w'(r,\lambda_n)=\frac{1}{\delta}$ for some r, $\frac{3}{2}<r\leqq 2$, then $w''(r,\lambda_n)>0$ and this is impossible. It follows that $-w'(r,\lambda_n)<\frac{1}{\delta}$ if $\frac{3}{2}\leqq r\leqq 2$, and $n>N_1$. Then if

$$Q(r,\lambda)=\int_{3/2}^{r}\frac{2w'^2}{s}\,ds\,,$$

we have, from (3.4) for $\lambda=\lambda_n$, and $\frac{3}{2}\leqq r\leqq 2$,

$$(rAe^Q)'=\left(1-\frac{(1-w^2)^2}{r^2}\right)e^Q\geqq\frac{5}{9}\,.$$

Thus for $\frac{3}{2}<r\leqq 2$, and $\lambda=\lambda_n$, n large,

$$rA(r)e^{Q(r)}\geqq\frac{3}{2}A\left(\frac{3}{2}\right)e^{Q(3/2)}+\frac{5}{9}\left(r-\frac{3}{2}\right)$$
$$>\frac{5}{9}\left(r-\frac{3}{2}\right),$$

so that for large n

$$2A(2,\lambda_n)\geqq\frac{5}{9}e^{-Q(2,\lambda_n)}\frac{1}{2}\,.$$

But as $-w'(2,\lambda_n)<\frac{1}{\delta}$, we see that Q is bounded near $r=2$, and hence

$$\overline{\lim_{n\to\infty}}\,A(2,\lambda_n)>0\,,$$

thereby violating (5.39). This completes the proof of Proposition 5.16.

We have thus proved that Case (A) cannot occur. Thus Case (B) holds so that the orbit $(w(r,\bar\lambda),w'(r,\bar\lambda))$ stays in the region $\Gamma=\{w^2\leqq 1,\ w'\leqq 0\}$ for all $r>0$, and $A(r,\bar\lambda)>0$ for all $r\geqq 0$. In the next section we shall prove that

$$\lim_{r\to\infty}(w(r,\bar\lambda),w'(r,\bar\lambda),A(r,\bar\lambda))=(-1,0,1)\,.$$

6. Concluding Remarks

In this final section, we shall show that when $\lambda=\bar\lambda$, the corresponding Einstein metric is asymptotically Minkowskian, and the total mass is finite.

Proposition 6.1. *If* $(w(r,\bar\lambda),w'(r,\bar\lambda))$ *is a bounded non-singular orbit of* (3.3)–(3.5), *which stays in* $w^2\leqq 1$, $w'\leqq 0$ *for* $r>0$, *then*

$$\lim_{r\to\infty}(w(r,\bar\lambda),w'(r,\bar\lambda),A(r,\bar\lambda))=(-1,0,1)\,.$$

Proof. Since $\bar{\lambda}$ is fixed, we shall suppress the $\bar{\lambda}$-dependence in what follows.

Since $w'(r)<0$ for all r and $w^2(r)\leqq 1$, we see that $\lim_{r\to\infty} w(r)$ exists, and $\lim_{r\to\infty} w(r) \geqq -1$. Also as $\Phi(r)=\mu(r)-\frac{(1-w^2)^2}{r}$ [cf. (3.6)], and $\mu'>0$ [cf. (2.14)], it follows that $\Phi(r)\geqq\sigma>0$ for some $\sigma>0$, for all sufficiently large r. From (2.14), μ satisfies the equation

$$\mu'=2Aw'^2+\frac{(1-w^2)^2}{r^2}.$$

Thus since $w'(r)\geqq -c^2$ for some $c>0$ and all $r\geqq 0$,

$$\mu'(r)\leqq 2c^2(-w')+\frac{1}{r^2},$$

$$\mu(r)\leqq\mu(r_0)+2c^2(w(r_0)-w(r))+\frac{1}{r_0}$$

$$\leqq\mu(r_0)+4c^2+\frac{1}{r_0},$$

so that the total mass is finite; cf. Sect. 2. Thus

$$\lim_{r\to\infty} A(r)=\lim_{r\to\infty}\left(1-\frac{\mu(r)}{r}\right)=1.$$

Now suppose that $\lim w(r)\geqq 0$; we shall show that this is impossible. Thus if not, note that from (3.5), $(e^P w')'<0$, and also $e^P w'<0$, so $e^P w'$ has a negative limit as $r\to\infty$. Since μ is bounded, it follows easily that Φ is bounded [cf. (3.6)], and as $A\to 1$, $e^{P(r)}$ is bounded. Hence $e^P w'$ has a strictly negative finite limit, and thus w' tends to a (finite) negative limit as $r\to\infty$; say $w'(r)\to -L^2$. But then as

$$w(r)-w(r_0)=\int_{r_0}^{r} w'(s)ds<-L^2(r-r_0)$$

we see that w cannot stay positive. This contradiction shows that the orbit (w,w') cannot stay in $w\geqq 0$ for all r. It follows that

$$\lim_{r\to\infty} w(r)<0.$$

Now since $w''>0$ for large r (since $\Phi>0$), we see that $\lim_{r\to\infty} w'(r)$ exists, and this limit is $\leqq 0$. But by what we have just seen,

$$\lim_{r\to\infty} w'(r)=0.$$

Also, $\lim_{r\to\infty} w(r)>-1$ cannot hold for if it were true, then as follows from (3.4) for large r, we would have

$$r^2Aw''=-\Phi w'-w(1-w^2)\geqq -w(1-w^2)\geqq \text{const}>0.$$

Thus as A is bounded from below, $w''(r)\geqq\frac{c^2}{r^2}$ for large r, where $c\neq 0$. Then for large r,

$$-w'(r)=w'(\infty)-w'(r)=\int_r^{\infty} w''(s)ds\geqq\int_r^{\infty}\frac{c^2}{s^2}ds=\frac{c^2}{r},$$

but this is impossible since it implies that w' is non-integrable. Hence

$$\lim_{r\to\infty} w(r) = -1,$$

and the proof is complete. □

Corollary 6.2. *If* $\lambda = \bar{\lambda}$, *then* $\lim_{r\to\infty} R(r) = 1$ *(cf. Eq.* (2.2)*).*

Proof. We have seen that $\mu(r)$ is bounded, and as $R(r) = \left(1 - \frac{\mu(r)}{r}\right)^{-1/2}$, the result follows.

As a consequence of this last result, we see that the metric (2.2) will be asymptotically flat provided that for $\lambda = \bar{\lambda}$, the solution T of Eq. (2.7) satisfies

$$\lim_{r\to\infty} T(r) = 1.$$

This will be demonstrated in the next theorem.

Theorem 6.3. *Let* (w, A) *be the solution of* (3.3)–(3.5) *satisfying* (4.1). *Then the corresponding metric* (2.2) *is asymptotically Minkowskian; i.e.,* $T(0)$ *can be chosen such that*

$$\lim_{r\to\infty} T(r) = 1 = \lim_{r\to\infty} R(r).$$

Proof. To see that $T(r)\to 1$, we first recall that T satisfies (2.7). If we write (as before) $A = 1 - \mu/r$, then the equation for T is

$$\begin{aligned} 2rAT' &= \left[\frac{(1-w^2)^2}{r^2} - 2Aw'^2 + (A-1)\right]T \\ &= [-2Aw'^2 - \Phi/r]T. \end{aligned} \tag{6.1}$$

An easy calculation shows that $T'(0) = 0$, but $T(0)$ is free. If

$$\psi(r) = \frac{1}{2A}\left[\frac{-2Aw'^2}{r} - \Phi/r^2\right],$$

then it is not difficult to show that $\psi \in L^1(0, \infty)$, and (6.1) implies

$$T(r) = T(0) \exp \int_0^r \psi(s)ds.$$

Thus choosing $T(0) = \exp\left[-\int_0^\infty \psi(s)ds\right]$, implies that $T(r)\to 1$ as $r\to\infty$. □

Corollary 6.4. *For the solution* $(w(r, \bar{\lambda}), A(r, \bar{\lambda}))$ *of* (3.1), (3.2), *the corresponding total mass is finite.*

Proof. This is merely the statement that $\mu(r, \bar{\lambda})$ has a finite limit as $r\to\infty$.

7. Appendix

We outline here a proof of the local existence of solutions of (2.6), (2.8), (2.11), (2.12), for each $\lambda \in \mathbb{R}$. (This is not a trivial exercise since these nonlinear equations are singular at $r=0$.) We shall also show that the solution depends continuously on λ, and is analytic in $r>0$.

We set

$$w(r)=1-\frac{\lambda}{2}r^2+v(r), \qquad z=v',$$

and then write our differential equations in the form

$$\begin{aligned} v' &= z, \\ z' &= \lambda+\frac{\left[\frac{(1-w^2)^2}{r}-\mu\right](z-\lambda r)+w(1-w^2)}{r^2\left(1-\frac{\mu}{r}\right)}, \\ \mu' &= 2\left(1-\frac{\mu}{r}\right)(z-\lambda r)^2+\frac{(1-w^2)^2}{r^2}. \end{aligned} \tag{7.1}$$

In what follows, *we assume $\lambda>0$ is fixed.* We assume that $v\in C^{2+\alpha}_{000}[0,R]$, $z,\mu\in C^{1+\alpha}_{00}[0,R]$; here the zero subscripts denote $v(0)=v'(0)=v''(0)=0$, and so on. We rewrite (7.1) as integral equations, and seek a local solution via iteration:

$$\begin{aligned} \tilde{v}(r) &= \int_0^r z(s)ds, \\ \tilde{z}(r) &= \int_0^r \left\{\lambda+\frac{w(w^2-1)+(z-\lambda s)\left[\frac{(1-w^2)^2}{2}-\mu\right]}{s^2\left(1-\frac{\mu(s)}{s}\right)}\right\}ds, \\ \tilde{\mu}(r) &= \int_0^r \left[2\left(1-\frac{\mu(s)}{s}\right)(z-\lambda s)^2+\frac{(1-w^2)}{s^2}\right]ds. \end{aligned} \tag{7.2}$$

We abbreviate (7.2) as $(\tilde{v},\tilde{z},\tilde{\mu})\equiv T(v,z,w)$. Let $X=(C^{2+\alpha}_{000}\times C^{1+\alpha}_{00}\times C^{1+\alpha}_{00})[0,R]$, and define norms on X as follows: Fix α, b in $(0,1)$, then set

$$|v|_{2+\alpha}=b\sup_{r_1\neq r_2}\left|\frac{v''(r_2)-v''(r_1)}{(r_2-r_1)^\alpha}\right|,$$

$$|z|_{1+\alpha}=\sup_{r_1\neq r_2}\left|\frac{z'(r_2)-z'(r_1)}{(r_2-r_1)^\alpha}\right|,$$

$$|\mu|_{1+\alpha}=\sup_{r_1\neq r_2}\left|\frac{\mu'(r_2)-\mu'(r_1)}{(r_2-r_1)^\alpha}\right|,$$

and

$$\|(v,z,\mu)\|\equiv\|(v,z,\mu)\|_X=\max(|v|_{2+\alpha},|z|_{1+\alpha},|\mu|_{1+\alpha}).$$

Fix a real number $\varrho>0$, and assume $\|(v,z,\mu)\|<\varrho$; i.e., $(v,z,\mu)\in B_\varrho(X)$. We shall indicate that for small R (R independent of λ on compact λ-intervals) the following hold:

a) $T(B_\varrho)\subset B_\varrho$,
b) T is a contraction;

these will imply local existence in the space X.

To show a), we first note that it is straightforward to verify that $(\tilde{v}, \tilde{z}, \tilde{\mu}) \in X$. To show a) and b), we observe that by L'Hospital's rule, $\left(1-\frac{\mu(r)}{r}\right) \to 1$ as $r \to 0$. Next we need the following lemma.

Lemma 7.1. a) *If* $v \in C^{2+\alpha}_{00}[0,R]$, *then* $\left|\frac{v}{r^2}\right|_\alpha \leqq \frac{|v|_{2+\alpha}}{b(1+\alpha)(2+\alpha)}$.

b) *If* $v \in C^{1+\alpha}_0[0,R]$, *then* $\left|\frac{v}{r}\right|_\alpha \leqq \frac{|v|_{1+\alpha}}{(1+\alpha)}$.

c) *If* $v \in C^{1+\alpha}_{00}[0,R]$, *then* $|v|_\alpha \leqq \frac{R^{1+\alpha}}{1+\alpha}|v|_{1+\alpha}$.

d) *If* $g \in C^\alpha_0[0,R]$, *then* $|g|_\infty \leqq R^\alpha |g|_\alpha$.

e) *If* $f \in C^\beta[0,R]$ *and* $\alpha < \beta$, *then* $|f|_\alpha \leqq R^{\beta-\alpha}|f|_\beta$.

f) *If* $f, g \in C^\alpha[0,R]$, *then* $|fg|_\alpha \leqq |f|_\alpha |g|_\infty + |f|_\infty |g|_\alpha$.

Proof. For a), we have

$$\frac{v(r)}{r} = \int_0^1 v'(\mathrm{tr})dt = \int_0^1 \left(\int_0^1 \frac{dv'(str)}{ds}ds\right)dt$$
$$= \int_0^1 \int_0^1 \mathrm{tr}\, v''(str)ds dt,$$

so

$$\frac{v(r_2)}{r_2^2} - \frac{v(r_2)}{r_1^2} = \int_0^1 \int_0^1 t[v''(str_2) - v''(str_1)]ds dt$$
$$= (r_2 - r_1)^\alpha \int_0^1 \int_0^1 t\left[\frac{v''(str_2) - v''(str_1)}{(st)^\alpha (r_2 - r_1)^\alpha}\right](st)^\alpha ds dt.$$

so that

$$\left|\frac{\frac{v(r_2)}{r_2^2} - \frac{v(r_1)}{r_1^2}}{(r_2 - r_1)^\alpha}\right| \leqq \frac{|v|_{2+\alpha}}{b}\int_0^1 \int_0^1 s^\alpha t^{1+\alpha} ds dt$$
$$= \frac{|v|_{2+\alpha}}{b}\frac{1}{(\alpha+1)}\frac{1}{(\alpha+2)}.$$

The proofs of the other statements are similar. □

Using this lemma, it is straightforward to show $T(B_\varrho) \subset B_\varrho$ if R is small (R independent of λ on compact λ-intervals); we omit the details. To show that T is a contraction for small r, we consider the differential dT evaluated at a point $(v, z, \mu) \in X$, and show that $\|dT\| \leqq c < 1$, if R is small (again R is independent of λ on compact λ-intervals); here $\|dT\|$ is defined by

$$\|dT\| = \sup_{\|(\bar{\alpha}, \bar{\beta}, \bar{\gamma})\| = 1} \|d_{(v,z,\mu)}T(\alpha, \beta, \gamma)\|$$
$$= \max_{i=1,2,3} \sup_{\|(\alpha, \beta, \gamma)\| = 1} \|d_{(v,z,\mu)}(\pi_i \circ T)(\bar{\alpha}, \bar{\beta}, \bar{\gamma})\|,$$

where $\tilde{v} = \pi_1 \circ T(v, z, \mu)$, $\tilde{z} = \pi_2 \circ T(v, z, \mu)$, $\tilde{\mu} = \pi_3 \circ T(v, z, \mu)$, and $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \in X$. To illustrate, we have $\pi_1 \circ T = \tilde{v}$ so

$$d(\pi_1 \circ T)(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) = \frac{\partial \tilde{v}}{\partial v}\bar{\alpha} + \frac{\partial \tilde{v}}{\partial z}\bar{\beta} + \frac{\partial \tilde{v}}{\partial \mu}\bar{\gamma}$$
$$= \int_0^r \bar{\beta}(s)ds,$$

so

$$|d(\pi_1 \circ T)(\bar{\alpha}, \bar{\beta}, \bar{\gamma})|_{2+\alpha} = \left| \int_0^r \bar{\beta}(s)ds \right|_{2+\alpha} = b|\bar{\beta}|_{1+\alpha} \leqq b\|\bar{\alpha}, \bar{\beta}, \bar{\gamma}\| < 1 .$$

Also $\pi_2 \circ T = \tilde{\mu}$, so

$$|d(\pi_3 \circ T)(\bar{\alpha}, \bar{\beta}, \bar{\gamma})|_{1+\alpha} = \left| \frac{\partial \tilde{\mu}}{\partial v} \bar{\alpha} + \frac{\partial \tilde{\mu}}{\partial z} \bar{\beta} + \frac{\partial \tilde{\mu}}{\partial \mu} \bar{\gamma} \right|_{1+\alpha} .$$

We may write

$$\sigma_1 = d\tilde{\mu}_{(v,z,\mu)}(0,0,\bar{\gamma}) = \frac{d}{dt} \tilde{\mu}(v,z,\mu + t\bar{\gamma}) \Big|_{t=0} = \frac{\partial \tilde{\mu}}{\partial \mu} \bar{\gamma} = \int_0^r \frac{-2\bar{\gamma}}{s} (z - \lambda s)^2 ds ,$$

$$\sigma_2 = d\tilde{\mu}_{(v,z,\mu)}(0, \bar{\beta}, 0) = \int_0^r 4 \left(1 - \frac{\mu}{s}\right)(z - \lambda s)\bar{\beta}(s) ds ,$$

$$\sigma_3 = d\tilde{\mu}_{(v,z,\mu)}(\bar{\alpha}, 0, 0) = \int_0^r \frac{2}{s^2}(1 - w^2)(-2w)\bar{\alpha} ds ,$$

and

$$\begin{aligned} &|d_{(v,z,\mu)}(\pi_3 \circ T)(\bar{\alpha}, \bar{\beta}, \bar{\gamma})|_{1+\alpha} = |\sigma_1 + \sigma_2 + \sigma_3|_{1+\alpha} \\ &= \left| \frac{-2\bar{\gamma}}{s}(z - \lambda s)^2 + 4\left(1 - \frac{\mu}{s}\right)(z\text{-}\lambda s)\bar{\beta} - \frac{4w(1-w^2)}{s^2}\bar{\alpha} \right|_{\alpha} \\ &\leqq \left| \frac{2(z-\lambda s)^2}{\sqrt{s}} \frac{\bar{\gamma}}{\sqrt{s}} \right|_{\alpha} + 4\left| \left(1 - \frac{\mu}{s}\right)(z - \lambda s)\bar{\beta} \right|_{\alpha} + 4\left| \frac{(1-w^2)}{r} \frac{\bar{\alpha}w}{r} \right|_{\alpha} . \end{aligned} \tag{7.3}$$

We can estimate each of these terms separately. (In what follows, C's will denote constants depending only on ϱ, and λ.) For example,

$$\left| \frac{(1-w^2)}{r} \frac{\bar{\alpha}w}{r} \right|_{\alpha} \leqq \left| \frac{(1-w^2)w}{r} \right|_{\infty} \left| \frac{\bar{\alpha}}{r} \right|_{\alpha} + \left| \frac{w(1-w^2)}{r} \right|_{\alpha} \left| \frac{\bar{\alpha}}{r} \right|_{\infty} . \tag{7.4}$$

Now as $w = 1 - \frac{\lambda}{2} r^2 + v$, $(1-w^2)/r = \frac{\lambda^2}{4} r^2 + \frac{v^2}{r^2} - \lambda r + \frac{2v}{r} - \lambda r v$, and since $(w^2-1)/r \in C^{1+\alpha}$, we see $|w(1-w^2)/r|_{\infty} \leqq C$. Then using Lemma 7.1c and e, $\left|\frac{\bar{\alpha}}{r}\right|_{\alpha} \leqq \frac{1}{\alpha+1} |\bar{\alpha}|_{1+\alpha} \leqq \frac{R}{\alpha+1} |\bar{\alpha}|_{2+\alpha}$; thus the first term in (7.4) is bounded by CR. Similarly, as $\left| \frac{(1-w^2)}{r} w \right|_{\alpha} \leqq R \left| \frac{(1-w^2)}{r} w \right|_{1+\alpha} \leqq CR$, and $\left|\frac{\bar{\alpha}}{r}\right|_{\infty} \leqq \left|\frac{\bar{\alpha}}{r}\right| R^{\alpha} \leqq \frac{R^{\alpha}}{\alpha+1} |\bar{\alpha}|_{1+\alpha}$ (where we have used Lemma 7.1d and c), we see that the second term in (7.4) is bounded by CR. Thus the third term in (7.3) is bounded by CR. Similar estimates can be made on the first two terms in (7.3). Next, as

$$|d(\pi_2 \circ T(\bar{\alpha}, \bar{\beta}, \bar{\gamma})|_{1+\alpha} = |d\tilde{z}(\bar{\alpha}, 0, 0) + d\tilde{z}(0, \bar{\beta}, 0) + d\tilde{z}(0, 0, \bar{\gamma})|_{1+\alpha} ,$$

we can use similar techniques to show that this term is bounded by CR^{α}; we leave these straightforward estimates to the reader. Thus since $|T(v_1, z_1, \mu_1) - T(v_2, z_2, \mu_2)| \leqq \|dT\| \cdot \|v_2 - v_1\|_X$, it follows that T is a contraction, and hence that given any $\lambda \in \mathbb{R}$, Eqs. (2.6), (2.8), (2.11), and (2.12) have a unique solution in X on $0 \leqq r \leqq R$, where R is independent of λ on compact λ-intervals.

Next we note that $\lambda = 0$ implies $w \equiv 1$ and $\mu \equiv 0$; this holds since $w \equiv 1$ and $\mu \equiv 0$ solves our equations and initial conditions, and our solutions are unique.

We now show that the solution depends continuously on λ. To see this let $\lambda_0 \in \mathbb{R}$, and define the fiber bundle E over $[\lambda_0-\delta, \lambda_0+\delta]$ by

$$[\lambda_0-\delta, \lambda_0+\delta] \times C^{2+\alpha}_{000}[0,R] \times C^{1+\alpha}_{00}[0,R] \times C^{1+\alpha}_{00}[0,R] \xrightarrow{\pi} [\lambda_0-\delta, \lambda_0+\delta].$$

We define a mapping $S: E \to E$ by $S(\lambda, v, z, \mu) = (\lambda, T(v, z, \mu)) = (\lambda, \tilde{v}, \tilde{z}, \tilde{\mu})$; cf. (7.2). Now S is continuous, preserves fibers, and has a *fixed* section $s(\lambda)$ (i.e., a section which is fixed under S), because we have a unique fixed point of T for each λ. To show that s is continuous, we let $\lambda_n \to \lambda_0$, and we shall show $s(\lambda_n) \to s(\lambda_0)$. Thus let $p(\lambda)$ be any continuous (local) section of $[\lambda_0-\delta, \lambda_0+\delta]$, with $p(\lambda_0)=s(\lambda_0)$. Let d be the maximum distance from $p(\lambda)$ to $T(p(\lambda))$.

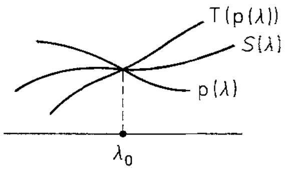

Fig. 2

Let $\varepsilon > 0$ be given, and let $\|dT\| \leqq C < 1$. For any integer N

$$\operatorname{dist}(T^N(p(\lambda)), s(\lambda)) \leqq \frac{C^N d}{1-C}, \qquad \lambda \in [\lambda_0-\delta, \lambda_0+\delta_0].$$

Now choose N so large that

$$\frac{C^N d}{1-C} < \frac{\varepsilon}{2} \quad \text{and} \quad \operatorname{dist}(T^N(p(\lambda_n)), p(\lambda_0)) < \varepsilon/2;$$

then

$$\operatorname{dist}(s(\lambda_n), s(\lambda_0)) = \operatorname{dist}(s(\lambda_n), p(\lambda_0)) < \varepsilon,$$

so $s(\lambda_n) \to s(\lambda_0)$, and the solution depends continuously on λ.

Finally, we note that, given any λ, as follows from the Cauchy-Kowaleski theorem the solution is an analytic function of r on its domain of definition, $0 < r < R(\lambda)$, provided that $A(r, \lambda) \neq 0$.

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