# Smoothable del Pezzo surfaces with quotient singularities 

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#### Abstract

We classify del Pezzo surfaces with quotient singularities and Picard rank 1 which admit a $\mathbb{Q}$-Gorenstein smoothing. These surfaces arise as singular fibres of del Pezzo fibrations in the 3 -fold minimal model program and also in moduli problems.


## 1. Introduction

We give a complete classification of del Pezzo surfaces with quotient singularities and Picard rank 1 which admit a $\mathbb{Q}$-Gorenstein smoothing. This solves a problem posed by Kollár, cf. [Kol08, §4].

One of the possible end products of the 3 -fold minimal model program is a del Pezzo fibration. A del Pezzo fibration is a morphism $f: Y \rightarrow S$ with connected fibres such that $Y$ is a 3 -fold with terminal singularities, $S$ is a smooth curve, $-K_{Y}$ is relatively ample, and the relative Picard number $\rho(Y / S)$ equals 1. In particular, a general fibre of $f$ is a smooth del Pezzo surface. Typically, a singular fibre $X=Y_{s}$ of a del Pezzo fibration $Y / S$ is a normal del Pezzo surface with quotient singularities. Moreover, if we work locally analytically at $s \in S$, we can run a relative minimal model program over $S$ to reduce to the case $\rho(X)=1$. This is a key motivation for our work.

Let $X$ be a normal surface with quotient singularities. We say $X$ admits a $\mathbb{Q}$-Gorenstein smoothing if there exists a deformation $\mathcal{X} /(0 \in T)$ of $X$ over a smooth curve germ such that the general fibre is smooth and $K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier. (The requirement that $K_{\mathcal{X}}$ be $\mathbb{Q}$-Cartier is natural from the point of view of the minimal model program and is important in moduli problems, cf. [KS88, 5.4]. It is automatically satisfied if $X$ is Gorenstein.)

Theorem 1.1. Let $X$ be a projective surface with quotient singularities such that $-K_{X}$ is ample, $\rho(X)=1$, and $X$ admits a $\mathbb{Q}$-Gorenstein smoothing. Then $X$ is one of the following:
(1) A toric surface as in Thm. 4.1.
(2) A deformation of a toric surface from (1), determined by specifying the subset of singularities to be partially smoothed as in Cor. 2.7.
(3) A sporadic surface as in Ex. 8.3.

There are 14 infinite families of toric examples, see Thm. 4.1. The surfaces in each family correspond to solutions of a Markov-type equation. The solutions of the (original) Markov equation

$$
a^{2}+b^{2}+c^{2}=3 a b c
$$

correspond to the vertices of an infinite tree such that each vertex has degree 3 . Here two vertices are joined by an edge if they are related by a so called mutation of the form

$$
(a, b, c) \mapsto(a, b, 3 a b-c) .
$$

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The solutions of the other equations are described similarly.
Given one of the toric surfaces $Y$, the $\mathbb{Q}$-Gorenstein deformations of $Y$ which preserve the Picard number are as follows. First, there are no locally trivial deformations and no local-to-global obstructions to deformations. Second, for each singularity $Q \in Y$, the deformation is either locally trivial or a deformation of a singularity of index $>1$ to a Du Val singularity of type $A$, see Cor. 2.7. Moreover, in the second case, the deformation is essentially unique (it is obtained from a fixed one parameter deformation by base change).

There are 20 isolated sporadic surfaces and one family of sporadic surfaces parametrised by $\mathbb{A}^{1}$, see Ex. 8.3. Every sporadic surface has index $\leqslant 2$. In particular, they occur in the list of Alexeev and Nikulin [AN06].

Our methods produce many examples of smoothable del Pezzo surfaces $X$ with quotient singularities of Picard rank $\rho(X)>1$. Indeed, let $Z$ be one of the toric surfaces enumerated in Thm. 4.1 and let $X$ be any partial $\mathbb{Q}$-Gorenstein smoothing of $Z$. Then the Picard number $\rho(X)$ can be computed by the formula in Prop. 2.6.

In the case $K_{X}^{2}=9$ we obtain the following stronger result. This completely solves the problem studied by Manetti in [Man91].

Corollary 1.2. Let $X$ be a projective surface with quotient singularities which admits a smoothing to the plane. Then $X$ is a $\mathbb{Q}$-Gorenstein deformation of a weighted projective plane $\mathbb{P}\left(a^{2}, b^{2}, c^{2}\right)$, where ( $a, b, c$ ) is a solution of the Markov equation.

Proof. If $X$ is a surface with quotient singularities which admits a smoothing to the plane, then $\rho(X)=1,-K_{X}$ is ample, and the smoothing is $\mathbb{Q}$-Gorenstein by [Man91, $\left.\S 1\right]$.

We note that a partial classification of the surfaces with $K_{X}^{2} \geqslant 5$ was obtained by Manetti [Man91],[Man93].

As a consequence of our techniques we verify a particular case of Reid's general elephant conjecture (see, e.g., [Ale94]).

Theorem 1.3. Let $f: V \rightarrow(0 \in T)$ be a del Pezzo fibration over the germ of a smooth curve. That is, $V$ is a 3 -fold with terminal singularities, $f$ has connected fibres, $-K_{V}$ is ample over $T$, and $\rho(V / T)=1$. Assume in addition that the special fibre is reduced and normal, and has only quotient singularities. Then a general member $S \in\left|-K_{V}\right|$ is a normal surface with Du Val singularities.

Notation. Throughout this paper, we work over the field $k=\mathbb{C}$ of complex numbers. The symbol $\boldsymbol{\mu}_{n}$ denotes the group of $n$th roots of unity.

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## 2. $T$-singularities

$T$-singularities are by definition the quotient singularities of dimension 2 which admit a $\mathbb{Q}$-Gorenstein smoothing. We recall the classification of $T$-singularities from $[\mathrm{KS} 88, \mathrm{Sec} .3]$ and establish some basic results.

## 2.1 $\mathbb{Q}$-Gorenstein deformations

Let $X$ be a normal surface such that $K_{X}$ is $\mathbb{Q}$-Cartier. A deformation $\mathcal{X} /(0 \in S)$ of $X$ over a germ $(0 \in S)$ is $\mathbb{Q}$-Gorenstein if locally analytically at each singular point $P \in X$ it is induced by an
equivariant deformation of the canonical covering of $P \in X$. This definition was originally proposed by Kollár [Kol91] and the general theory is worked out in [Hac04, Sec. 3]. We only use the explicit version in Sec. 2.2. We note that, if $X$ has quotient singularities and $S$ is a smooth curve, then a deformation $\mathcal{X} /(0 \in S)$ is $\mathbb{Q}$-Gorenstein iff $K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier.

### 2.2 Definition and classification of $T$-singularities

Definition 2.1. [KS88, Def. 3.7] Let $P \in X$ be a quotient singularity of dimension 2. We say $P \in X$ is a $T$-singularity if it admits a $\mathbb{Q}$-Gorenstein smoothing. That is, there exists a $\mathbb{Q}$-Gorenstein deformation of $P \in X$ over a smooth curve germ such that the general fibre is smooth.

For $n, a \in \mathbb{N}$ with $(a, n)=1$, let $\frac{1}{n}(1, a)$ denote the cyclic quotient singularity $\left(0 \in \mathbb{A}_{u, v}^{2} / \boldsymbol{\mu}_{n}\right)$ given by

$$
\boldsymbol{\mu}_{n} \ni \zeta:(u, v) \mapsto\left(\zeta u, \zeta^{a} v\right) .
$$

The following result is due to J. Wahl [Wah81, 5.9.1], [LW86, Props. 5.7,5.9]. It was proved by a different method in [KS88, Prop. 3.10].

Proposition 2.2. A T-singularity is either a Du Val singularity or a cyclic quotient singularity of the form $\frac{1}{d n^{2}}(1, d n a-1)$ for some $d, n, a \in \mathbb{N}$ with $(a, n)=1$.

The singularity $\frac{1}{d n^{2}}(1, d n a-1)$ has index $n$ and canonical covering $\frac{1}{d n}(1,-1)$, the Du Val singularity of type $A_{d n-1}$. We have an identification

$$
\frac{1}{d n}(1,-1)=\left(x y=z^{d n}\right) \subset \mathbb{A}_{x, y, z}^{3},
$$

where $x=u^{d n}, y=v^{d n}$, and $z=u v$. Taking the quotient by $\boldsymbol{\mu}_{n}$ we obtain

$$
\frac{1}{d n^{2}}(1, d n a-1)=\left(x y=z^{d n}\right) \subset \frac{1}{n}(1,-1, a) .
$$

Hence a $\mathbb{Q}$-Gorenstein smoothing is given by

$$
\left(x y=z^{d n}+t\right) \subset \frac{1}{n}(1,-1, a) \times \mathbb{A}_{t}^{1} .
$$

More generally, a versal $\mathbb{Q}$-Gorenstein deformation of $\frac{1}{d n^{2}}(1, d n a-1)$ is given by

$$
\begin{equation*}
\left(x y=z^{d n}+t_{d-1} z^{(d-1) n}+\cdots+t_{0}\right) \subset \frac{1}{n}(1,-1, a) \times \mathbb{A}_{t_{0}, \ldots, t_{d-1}}^{1} . \tag{1}
\end{equation*}
$$

We call a $T$-singularity of the form $\frac{1}{d n^{2}}(1, d n a-1)$ a $T_{d^{-}}$-singularity.
Proposition 2.3. Let $(P \in \mathcal{X}) /(0 \in S)$ be a $\mathbb{Q}$-Gorenstein deformation of $\frac{1}{d n^{2}}(1, d n a-1)$. Then the possible singularities of a fibre of $\mathcal{X} / S$ are as follows: either $A_{e_{1}-1}, \ldots, A_{e_{s}-1}$ or $\frac{1}{e_{1} n^{2}}\left(1, e_{1} n a-1\right)$, $A_{e_{2}-1}, \ldots, A_{e_{s}-1}$, where $e_{1}, \ldots, e_{s}$ is a partition of $d$.
Proof. The family $\mathcal{X} / S$ is pulled back from the versal $\mathbb{Q}$-Gorenstein deformation (1). Hence each fibre of $\mathcal{X} / S$ has the form

$$
\left(x y=z^{d n}+a_{d-1} z^{(d-1) n}+\cdots+a_{0}\right) \subset \frac{1}{n}(1,-1, a)
$$

for some $a_{0}, \ldots, a_{d-1} \in k$. Write

$$
z^{d n}+a_{d-1} z^{(d-1) n}+\cdots+a_{0}=\prod\left(z^{n}-\gamma_{i}\right)^{e_{i}}
$$

where the $\gamma_{i}$ are distinct. Then the fibre has singularities as described in the statement (the second case occurs if $\gamma_{i}=0$ for some $i$ ).

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### 2.3 Noether's formula

For $P \in X$ a $T$-singularity, let $M$ be the Milnor fibre of a $\mathbb{Q}$-Gorenstein smoothing. Thus ( $M, \partial M$ ) is a smooth 4 -manifold with boundary, and is uniquely determined by $P \in X$ since the $\mathbb{Q}$-Gorenstein deformation space of $P \in X$ is smooth. Let $\mu_{P}=b_{2}(M)$, the Milnor number.

Lemma 2.4. [Man91, Sec. 3] If $P \in X$ is a Du Val singularity of type $A_{r}, D_{r}$, or $E_{r}$, then $\mu_{P}=r$. If $P \in X$ is of type $\frac{1}{d n^{2}}(1, d n a-1)$ then $\mu_{P}=d-1$.

Remark 2.5. If $M$ is the Milnor fibre of a smoothing of a normal surface singularity $P \in X$ then $M$ has the homotopy type of a CW complex of real dimension 2 by Morse theory and $b_{1}(M)=0$ [GS83]. In particular the Euler number $e(M)=1+\mu_{P}$.

Proposition 2.6. Let $X$ be a projective surface with $T$-singularities. Then

$$
K_{X}^{2}+e(X)+\sum_{P \in \operatorname{Sing} X} \mu_{P}=12 \chi\left(\mathcal{O}_{X}\right)
$$

and

$$
\chi\left(\mathcal{O}_{X}\left(m K_{X}\right)\right)=\chi\left(\mathcal{O}_{X}\right)+\frac{1}{2} m(m-1) K_{X}^{2}
$$

for $m \in \mathbb{Z}$.
In particular, if $X$ is rational then

$$
K_{X}^{2}+\rho(X)+\sum_{P \in \operatorname{Sing} X} \mu_{P}=10
$$

and if $-K_{X}$ is big and nef then

$$
h^{0}\left(\mathcal{O}_{X}\left(-n K_{X}\right)\right)=1+\frac{1}{2} n(n+1) K_{X}^{2}
$$

for $n \in \mathbb{Z}_{\geqslant 0}$.
Proof. For $X$ a projective normal surface with quotient singularities there is a singular Noether formula

$$
K_{X}^{2}+e(X)+\sum_{P} c_{P}=12 \chi\left(\mathcal{O}_{X}\right)
$$

where the sum is over the singular points $P \in X$, and the correction term $c_{P}$ depends only on the local analytic isomorphism type of the singularity $P \in X$. (Indeed, let $\pi: \tilde{X} \rightarrow X$ be the minimal resolution of $X$ and $E_{1}, \ldots, E_{n}$ the exceptional curves. Noether's formula on $\tilde{X}$ gives $K_{\tilde{\tilde{X}}}^{2}+e(\tilde{X})=12 \chi\left(\mathcal{O}_{\tilde{X}}\right)$. Write $K_{\tilde{X}}=\pi^{*} K_{X}+\sum a_{i} E_{i}=\pi^{*} K_{X}+A$. Then $K_{\tilde{X}}^{2}=K_{X}^{2}+A^{2}$, $e(\tilde{X})=e(X)+n$ (by the Mayer-Vietoris sequence), and $\chi\left(\mathcal{O}_{\tilde{X}}\right)=\chi\left(\mathcal{O}_{X}\right)$ (because $X$ has rational singularities). Hence $K_{X}^{2}+e(X)+\left(A^{2}+n\right)=12 \chi\left(\mathcal{O}_{X}\right)$.) Similarly, if $D$ is a Weil divisor on $X$ we have a singular Riemann-Roch formula

$$
\chi\left(\mathcal{O}_{X}(D)\right)=\chi\left(\mathcal{O}_{X}\right)+\frac{1}{2} D\left(D-K_{X}\right)+\sum c_{P}(D)
$$

where the sum is over points $P \in X$ where the divisor $D$ is not Cartier and the correction term $c_{P}(D)$ depends only on the local analytic isomorphism type of the singularity $P \in X$ and the local analytic divisor class of $D$ at $P \in X$ [Bla95, 1.2].

For each $T$-singularity $P \in X$, there exists a projective surface $Y$ with a unique singularity isomorphic to $P \in X$ and a $\mathbb{Q}$-Gorenstein smoothing $\mathcal{Y} /(0 \in T)$ by Looijenga's globalisation theorem [Loo85, App.]. We use $\mathcal{Y} / T$ to compute the correction terms $c_{P}$ and $c_{P}\left(m K_{X}\right)$. Let $Y^{\prime}$ denote the general fibre. We have $K_{Y^{\prime}}^{2}=K_{Y}^{2}, \chi\left(\mathcal{O}_{Y^{\prime}}\right)=\chi\left(\mathcal{O}_{Y}\right)$, and $e\left(Y^{\prime}\right)=e(Y)+\mu_{P}$ (because the Milnor fibre of the smoothing has Euler number $\left.1+\mu_{P}\right)$. Hence $K_{Y}^{2}+e(Y)+\mu_{P}=12 \chi\left(\mathcal{O}_{Y}\right)$, so
$c_{P}=\mu_{P}$. The Riemann-Roch formula for the line bundle $\mathcal{O}_{Y^{\prime}}\left(m K_{Y^{\prime}}\right)$ on $Y^{\prime}$ gives $\chi\left(\mathcal{O}_{Y^{\prime}}\left(m K_{Y^{\prime}}\right)\right)=$ $\chi\left(\mathcal{O}_{Y^{\prime}}\right)+\frac{1}{2} m(m-1) K_{Y^{\prime}}^{2}$. We have $\chi\left(\mathcal{O}_{Y^{\prime}}\left(m K_{Y^{\prime}}\right)\right)=\chi\left(\mathcal{O}_{Y}\left(m K_{Y}\right)\right)$ (because $\omega_{\mathcal{Y} / T}^{[m]}$ is flat over $T$ and commutes with base change since $\mathcal{Y} / T$ is $\mathbb{Q}$-Gorenstein). So $\chi\left(\mathcal{O}_{Y}\left(m K_{Y}\right)\right)=\chi\left(\mathcal{O}_{Y}\right)+\frac{1}{2} m(m-1) K_{Y}^{2}$ and $c_{P}\left(m K_{X}\right)=0$.

Finally, if $-K_{X}$ is nef and big then $H^{i}\left(\mathcal{O}_{X}\left(-n K_{X}\right)\right)=0$ for $i>0$ and $n \geqslant 0$ by KawamataViehweg vanishing, so $h^{0}\left(\mathcal{O}_{X}\left(-n K_{X}\right)\right)=1+\frac{1}{2} n(n+1) K_{X}^{2}$, as required.

Corollary 2.7. Let $X$ be a projective surface with $T$-singularities and $X^{\prime}$ a fibre of a $\mathbb{Q}$-Gorenstein deformation $\mathcal{X} /(0 \in T)$ of $X$ over a smooth curve germ. Then $e(X)=e\left(X^{\prime}\right)$ iff at each singular point $P \in X$, the deformation is either locally trivial or a deformation of a $T_{d}$-singularity to an $A_{d-1}$ singularity.

Proof. This follows immediately from Props. 2.3 and 2.6.

### 2.4 Minimal resolutions of $T$-singularities

Given a cyclic quotient singularity $\frac{1}{n}(1, a)$, let $\left[b_{1}, \ldots, b_{r}\right]$ be the expansion of $n / a$ as a HirzebruchJung continued fraction [Ful93, p. 46]. Then the exceptional locus of the minimal resolution of $\frac{1}{n}(1, a)$ is a chain of smooth rational curves with self-intersection numbers $-b_{1}, \ldots,-b_{r}$. The strict transforms of the coordinate lines $(u=0)$ and $(v=0)$ intersect the right and left end components of the chain respectively.

Remark 2.8. Note that $\left[b_{r}, \ldots, b_{1}\right]$ corresponds to the same singularity as $\left[b_{1}, \ldots, b_{r}\right]$ with the roles of the coordinates $u$ and $v$ interchanged. Thus, if $\left[b_{1}, \ldots, b_{r}\right]=n / a$ then $\left[b_{r}, \ldots, b_{1}\right]=n / a^{\prime}$ where $a^{\prime}$ is the inverse of $a$ modulo $n$.

We recall the description of the minimal resolution of the cyclic quotient singularities of class $T$ due to J. Wahl. Let a $T_{d}$-string be a string $\left[b_{1}, \ldots, b_{r}\right]$ which corresponds to a $T_{d}$-singularity.

Proposition 2.9. [KS88, Prop. 3.11], [Man91, Thm. 17]
(1) [4] is a $T_{1}$-string and, for $d \geqslant 2,[3,2, \ldots, 2,3]$ (where there are $(d-2) 2$ 's) is a $T_{d}$-string.
(2) If $\left[b_{1}, \ldots, b_{r}\right]$ is a $T_{d}$-string, then so are $\left[b_{1}+1, b_{2}, \ldots, b_{r}, 2\right]$ and $\left[2, b_{1}, \ldots, b_{r}+1\right]$.
(3) For each $d$, all $T_{d}$-strings are obtained from the example in (1) by iterating the steps in (2).

## 3. Unobstructedness of deformations

Proposition 3.1. Let $X$ be a projective surface with $\log$ canonical singularities such that $-K_{X}$ is big. Then there are no local-to-global obstructions to deformations of $X$. In particular, if $X$ has $T$-singularities then $X$ admits a $\mathbb{Q}$-Gorenstein smoothing.

Proof. The local-to-global obstructions to deformations of $X$ lie in $H^{2}\left(T_{X}\right)$, where $T_{X}=\mathcal{H o m}\left(\Omega_{X}, \mathcal{O}_{X}\right)$ is the tangent sheaf of $X$. This follows from either a direct cocycle computation (cf. [Wah81, Prop. 6.4]) or the theory of the cotangent complex [IIl71, 2.1.2.3]. Since $H^{2}\left(T_{X}\right)=\operatorname{Hom}\left(T_{X}, \mathcal{O}_{X}\left(K_{X}\right)\right)^{*}$ by Serre duality, it suffices to show $\operatorname{Hom}\left(T_{X}, \mathcal{O}_{X}\left(K_{X}\right)\right)=0$, or, equivalently, $\operatorname{Hom}\left(\mathcal{O}_{X}\left(-K_{X}\right), \Omega_{X}^{\vee \vee}\right)=$ 0 . If $L \subset \Omega_{X}^{\vee \vee}$ is a rank one reflexive subsheaf, then the Kodaira-Iitaka dimension $\kappa(X, L) \leqslant 1$ by Bogomolov-Sommese vanishing for log canonical varieties [GGK08, Thm. 1.4]. So $-K_{X}$ big implies that $\operatorname{Hom}\left(\mathcal{O}_{X}\left(-K_{X}\right), \Omega_{X}^{\vee \vee}\right)=0$ as required.

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## 4. Toric surfaces

Theorem 4.1. The projective toric surfaces with $T$-singularities and Picard rank 1 are as follows. There are 14 infinite families (1), .., (8.4) which we list in the tables below. In cases (1), .., (4), the surface $X$ is a weighted projective plane $\mathbb{P}\left(w_{0}, w_{1}, w_{2}\right)$, and the weights $w_{0}, w_{1}, w_{2}$ are determined by a solution $(a, b, c)$ of a Markov-type equation. In the remaining cases, the surface $X$ is a quotient of one of the above weighted projective planes $Y$ by $\boldsymbol{\mu}_{e}$ acting freely in codimension 1. The action is diagonal with weights $\left(m_{0}, m_{1}, m_{2}\right)$, i.e.,

$$
\boldsymbol{\mu}_{e} \ni \zeta:\left(X_{0}, X_{1}, X_{2}\right) \mapsto\left(\zeta^{m_{0}} X_{0}, \zeta^{m_{1}} X_{1}, \zeta^{m_{2}} X_{2}\right)
$$

where $X_{0}, X_{1}, X_{2}$ are homogeneous coordinates on $Y$. We also record $K_{X}^{2}$ and the values of $d=\mu+1$ for the singularities of $X$.

| $X$ | $w_{0}, w_{1}, w_{2}$ | Markov-type equation | $K_{X}^{2}$ | $d$ |
| :--- | :--- | :--- | :--- | :--- |
| $(1)$ | $a^{2}, b^{2}, c^{2}$ | $a^{2}+b^{2}+c^{2}=3 a b c$ | 9 | $1,1,1$ |
| $(2)$ | $a^{2}, b^{2}, 2 c^{2}$ | $a^{2}+b^{2}+2 c^{2}=4 a b c$ | 8 | $1,1,2$ |
| $(3)$ | $a^{2}, 2 b^{2}, 3 c^{2}$ | $a^{2}+2 b^{2}+3 c^{2}=6 a b c$ | 6 | $1,2,3$ |
| $(4)$ | $a^{2}, b^{2}, 5 c^{2}$ | $a^{2}+b^{2}+5 c^{2}=5 a b c$ | 5 | $1,1,5$ |


| $X$ | $Y$ | $e$ | $m_{0}, m_{1}, m_{2}$ | $K_{X}^{2}$ | $d$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $(5)$ | $(2)$ | 2 | $0,1,-1$ | 4 | $2,2,4$ |
| $(6.1)$ | $(1)$ | 3 | $0,1,-1$ | 3 | $3,3,3$ |
| $(6.2)$ | $(3)$ | 2 | $0,1,-1$ | 3 | $1,2,6$ |
| $(7.1)$ | $(2)$ | 4 | $0,1,1$ | 2 | $1,1,8$ |
| $(7.2)$ | $(2)$ | 4 | $0,1,-1$ | 2 | $2,4,4$ |
| $(7.3)$ | $(3)$ | 3 | $0,1,-1$ | 2 | $1,3,6$ |
| $(8.1)$ | $(1)$ | 9 | $0,1,-1$ | 1 | $1,1,9$ |
| $(8.2)$ | $(2)$ | 8 | $0,1,-1$ | 1 | $1,2,8$ |
| $(8.3)$ | $(3)$ | 6 | $0,1,-1$ | 1 | $2,3,6$ |
| $(8.4)$ | $(4)$ | 5 | $0,1,-1$ | 1 | $1,5,5$ |

Remark 4.2. With notation as above, let $X^{0} \subset X$ be the smooth locus and $p^{0}: Y^{0} \rightarrow X^{0}$ the restriction of the cover $Y \rightarrow X$. Then $p^{0}$ is the universal cover of $X^{0}$. In particular $\pi_{1}\left(X^{0}\right)$ is cyclic of order $e$.

The solutions of the Markov-type equations in Thm. 4.1 may be described as follows [KN98, 3.7]. We say a solution ( $a, b, c$ ) is minimal if $a+b+c$ is minimal. The equations (1),(2),(3) have a unique minimal solution ( $1,1,1$ ), and (4) has minimal solutions $(1,2,1)$ and $(2,1,1)$. Given one solution, we obtain another by regarding the equation as a quadratic in one of the variables, $c$ (say), and replacing $c$ by the other root. Explicitly, if the equation is $\alpha a^{2}+\beta b^{2}+\gamma c^{2}=\lambda a b c$, then

$$
\begin{equation*}
(a, b, c) \mapsto\left(a, b, \frac{\lambda}{\gamma} a b-c\right) \tag{2}
\end{equation*}
$$

This process is called a mutation. Every solution is obtained from a minimal solution by a sequence of mutations.

For each equation, we define an infinite graph $\Gamma$ such that the vertices are labelled by the solutions and two vertices are joined by an edge if they are related by a mutation. For equation (1), $\Gamma$ is an infinite tree such that each vertex has degree 3, and there is an action of $S_{3}$ on $\Gamma$ given by permuting the variables $a, b, c$. The other cases are similar, see [KN98, 3.8] for details.

Proof of Theorem 4.1. Let $X$ be a projective toric surface such that $X$ has only $T$-singularites and $\rho(X)=1$. The surface $X$ is given by a complete fan $\Sigma$ in $N_{\mathbb{R}} \simeq \mathbb{R}^{2}$, where $N \simeq \mathbb{Z}^{2}$ is the group of 1 -parameter subgroups of the torus. The fan $\Sigma$ has 3 rays because $\rho(X)=1$. Let $v_{0}, v_{1}, v_{2} \in N$ be the minimal generators of the rays. There is a unique relation

$$
w_{0} v_{0}+w_{1} v_{1}+w_{2} v_{2}=0
$$

where $w_{0}, w_{1}, w_{2} \in \mathbb{N}$ are pairwise coprime. Let $N_{Y} \subseteq N$ denote the subgroup generated by $v_{0}, v_{1}, v_{2}$. Let $p: Y \rightarrow X$ be the finite toric morphism corresponding to the inclusion $N_{Y} \subseteq N$. Then $Y$ is isomorphic to the weighted projective plane $\mathbb{P}\left(w_{0}, w_{1}, w_{2}\right)$ and $p$ is a cyclic cover of degree $e=$ $\left|N / N_{Y}\right|$ which is étale over the smooth locus $X^{0} \subset X$. The surface $Y$ has only $T$-singularities because a cover of a $T$-singularity which is étale in codimension 1 is again a $T$-singularity (this follows easily from the classification of $T$-singularities).

The surface $X$ has 3 cyclic quotient singularities of class $T$. Let the singularities of $X$ be $\frac{1}{d_{i} n_{i}^{2}}\left(1, d_{i} n_{i} a_{i}-1\right)$ for $i=0,1,2$. Then

$$
\begin{equation*}
d_{0}+d_{1}+d_{2}+K_{X}^{2}=12 \tag{3}
\end{equation*}
$$

by Prop. 2.6. The singularities of $X$ are quotients of the singularities $\frac{1}{w_{0}}\left(w_{1}, w_{2}\right), \frac{1}{w_{1}}\left(w_{0}, w_{2}\right)$, $\frac{1}{w_{2}}\left(w_{0}, w_{1}\right)$ of $Y$ by $\boldsymbol{\mu}_{e}$. Hence $d_{i} n_{i}^{2}=e w_{i}$. Also $K_{Y}^{2}=e K_{X}^{2}$ because $p: Y \rightarrow X$ has degree $e$ and is étale in codimension 1. Let $H$ be the ample generator of the class group of $Y$. Then $K_{Y} \sim-\left(w_{0}+w_{1}+w_{2}\right) H$, and $H^{2}=\frac{1}{w_{0} w_{1} w_{2}}$. We deduce that

$$
\begin{equation*}
d_{0} n_{0}^{2}+d_{1} n_{1}^{2}+d_{2} n_{2}^{2}=\sqrt{K_{X}^{2} d_{0} d_{1} d_{2}} \cdot n_{0} n_{1} n_{2} \tag{4}
\end{equation*}
$$

In particular

$$
\sqrt{K_{X}^{2} d_{0} d_{1} d_{2}}=\sqrt{\left(12-\sum d_{i}\right) d_{0} d_{1} d_{2}} \in \mathbb{Z}
$$

We compute all triples $d=\left(d_{0}, d_{1}, d_{2}\right)$ satisfying this condition. They are as listed in the last column of the tables above.

We first treat the cases $d=(1,1,1),(1,1,2),(1,2,3)$, and $(1,1,5)$. These are the cases for which $K_{X}^{2} \geqslant 5$. Since $K_{Y}^{2}=e K_{X}^{2} \leqslant 9$ by Prop. 2.6 we deduce that $e=1$. Thus $X$ is isomorphic to a weighted projective plane. The weights $d_{i} n_{i}^{2}$ are determined by the solution ( $n_{0}, n_{1}, n_{2}$ ) of (4), which is the Markov-type equation given in the statement. Conversely, we check that for any solution of (4) the weighted projective plane $X=\mathbb{P}\left(d_{0} n_{0}^{2}, d_{1} n_{1}^{2}, d_{2} n_{2}^{2}\right)$ has $T$-singularities and the expected value of $d$. We use the description of the solutions of (4) given above. We write $\lambda=\sqrt{K_{X}^{2} d_{0} d_{1} d_{2}}$, and note that $d_{0} d_{1} d_{2}$ divides $\lambda$ in each case. By induction using (2) we find that $n_{0}, n_{1}, n_{2}$ are pairwise coprime and $\operatorname{gcd}\left(n_{i}, \frac{\lambda}{d_{i}}\right)=1$ for each $i$. In particular, the $d_{i} n_{i}^{2}$ are pairwise coprime. Now consider the singularity $\frac{1}{d_{0} n_{0}^{2}}\left(d_{1} n_{1}^{2}, d_{2} n_{2}^{2}\right)$. We have

$$
d_{1} n_{1}^{2}+d_{2} n_{2}^{2}=\lambda n_{0} n_{1} n_{2} \quad \bmod d_{0} n_{0}^{2}
$$

by (4), and so $\operatorname{gcd}\left(d_{1} n_{1}^{2}+d_{2} n_{2}^{2}, d_{0} n_{0}^{2}\right)=d_{0} n_{0}$ because $\operatorname{gcd}\left(\frac{\lambda}{d_{0}} n_{1} n_{2}, n_{0}\right)=1$. Thus this singularity is of type $T_{d_{0}}$.

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For the remaining values of $d$, we determine the degree $e$ of the cover $p: Y \rightarrow X$ as follows. We have $e=\operatorname{gcd}\left(d_{0} n_{0}^{2}, d_{1} n_{1}^{2}, d_{2} n_{2}^{2}\right)$. By inspecting the equation (4) we find a factor of $e$, and, together with the inequality $e K_{X}^{2}=K_{Y}^{2} \leqslant 9$, this is sufficient to determine $e$ in each case. For example, let $d=(1,2,8)$. Then we find that $n_{0}$ is divisible by 4 and $n_{1}$ is even, so $e$ is divisible by 8 , hence equal to 8 . In each case we have $K_{Y}^{2} \geqslant 5$, so $Y$ is one of the surfaces classified above.

We now classify the possible actions of $\boldsymbol{\mu}_{e}$ on the covering surface $Y$. We have $Y=\mathbb{P}\left(d_{0} n_{0}^{2}, d_{1} n_{1}^{2}, d_{2} n_{2}^{2}\right)$ where $d=d_{Y}=(1,1,1),(1,1,2),(1,2,3)$, or $(1,1,5)$, and $\left(n_{0}, n_{1}, n_{2}\right)$ is a solution of (4). The action is given by

$$
\boldsymbol{\mu}_{e} \ni \zeta:\left(X_{0}, X_{1}, X_{2}\right) \mapsto\left(\zeta^{m_{0}} X_{0}, \zeta^{m_{1}} X_{1}, \zeta^{m_{2}} X_{2}\right)
$$

where $X_{0}, X_{1}, X_{2}$ are the homogeneous coordinates on the weighted projective plane $Y$. In each case $d_{0} n_{0}^{2}=n_{0}^{2}$ is coprime to $e$. So we may assume that $m_{0}=0$. We may also assume that $m_{1}=1$ (because the action is free in codimension 1 ). Consider the singularity $P_{0} \in X$ below $(1: 0: 0) \in Y$. This singularity admits a covering by $\frac{1}{e}\left(1, m_{2}\right)$ (which is étale in codimension 1 ). Hence $\frac{1}{e}\left(1, m_{2}\right)$ is a $T$-singularity. If $e$ is square-free, it follows that $m_{2}=-1$. If $e=4$, then $m_{2}= \pm 1$. If $e=8$ then $d_{Y}=(1,1,2)$ and $d_{X}=(1,2,8)$, so we may assume that $P_{0} \in X$ is a $T_{8}$-singularity (note that a $\boldsymbol{\mu}_{8}$-quotient of a $T_{2}$-singularity cannot be a $T_{8}$-singularity). Thus $P_{0} \in X$ is covered by $\frac{1}{8}(1,-1)$ and so $m_{2}=-1$. Similarly if $e=9$ then $d_{Y}=(1,1,1)$ and $d_{X}=(1,1,9)$, so we may assume that $P_{0} \in X$ is a $T_{9}$-singularity, and $m_{2}=-1$. This gives the list of group actions above. Finally, it remains to check that for each such quotient $X=Y / \boldsymbol{\mu}_{e}$, the surface $X$ has $T$-singularities with the expected values of $d$. This is a straightforward toric calculation, so we omit it.

## 5. Surfaces with a $D$ or $E$ singularity

A log del Pezzo surface is a normal projective surface $X$ such that $X$ has only quotient singularities and $-K_{X}$ is ample.

Theorem 5.1. Let $X$ be a $\log$ del Pezzo surface such that $\rho(X)=1$, and assume that $\operatorname{dim}\left|-K_{X}\right| \geqslant$ 1.
(1) If $X$ has a $D u$ Val singularity of type $E$ then $K_{X}$ is Cartier.
(2) If $X$ has a $D u$ Val singularity of type $D$ then either $K_{X}$ is Cartier or there is a unique non Du Val singularity of type $\frac{1}{m}(1,1)$ for some $m \geqslant 3$.
Moreover, in both cases, a general member of $\left|-K_{X}\right|$ is irreducible and does not pass through the Du Val singularities.
Proof. Assume that $X$ has a $D$ or $E$ singularity $P \in X$ and $K_{X}$ is not Cartier. Let $\nu: \hat{X} \rightarrow X$ be the minimal resolution of the non Du Val singularities of $X$ and write $\hat{P}=\nu^{-1}(P)$. So $\hat{X}$ has only Du Val singularities and $\hat{P} \in \hat{X}$ is a $D$ or $E$ singularity. Let $\left\{E_{i}\right\}$ be the exceptional curves of $\nu$ and write $E=\sum E_{i}$.

Write $\left|-K_{\hat{X}}\right|=|M|+F$ where $F$ is the fixed part and $M$ is general in $|M|$. We have an equality

$$
K_{\hat{X}}=\nu^{*} K_{X}+\sum a_{i} E_{i}
$$

where $a_{i}<0$ for all $i$ because $\nu$ is minimal and we only resolve the non Du Val singularities [KM98, Lem. 3.41]. Hence $\operatorname{dim}\left|-K_{\hat{X}}\right|=\operatorname{dim}\left|-K_{X}\right|$ and $F \geqslant E$.

We run the minimal model program on $\hat{X}$. We obtain a birational morphism $\phi: \hat{X} \rightarrow \bar{X}$ such that $\bar{X}$ has Du Val singularities and exactly one of the following holds.
(1) $K_{\bar{X}}$ is nef.
(2) $\rho$

$$
\rho(\bar{X})=2 \text { and there is a fibration } \psi: \bar{X} \rightarrow \mathbb{P}^{1} \text { with } K_{\bar{X}} \cdot f<0 \text { for } f \text { a fibre. }
$$

(3) $\rho(\bar{X})=1$ and $-K_{\bar{X}}$ is ample.

Clearly $K_{\bar{X}}$ is not nef because $\operatorname{dim}\left|-K_{\bar{X}}\right| \geqslant \operatorname{dim}\left|-K_{\hat{X}}\right| \geqslant 1$.
In the minimal model program for surfaces with Du Val singularities, the birational extremal contractions are weighted blowups $f: X \rightarrow Y$ of a smooth point $P \in Y$ with weights $(1, n)$ for some $n \in \mathbb{N}$. In particular the exceptional divisor $E \subset X$ is a smooth rational curve and passes through a unique singularity of $X$ which is of type $\frac{1}{n}(1,-1)=A_{n-1}$. See [KM99, Lem. 3.3].

Therefore, the birational morphism $\phi$ is an isomorphism near the $D$ or $E$ singularity $\hat{P} \in \hat{X}$ and $\bar{E}:=\phi_{*} E$ is contained in the smooth locus of $\bar{X}$. Note also that $\bar{E} \neq 0$ because $\rho(X)=1$ and $X$ has a non Du Val singularity.

Suppose first we are in case (3). We have $-K_{\bar{X}} \sim \bar{M}+\bar{F}$ where $\bar{M}:=\phi_{*} M$ is mobile and $\bar{F}:=\phi_{*} F \geqslant \bar{E}$. In particular, $\operatorname{Pic}(\bar{X})$ is not generated by $-K_{\bar{X}}$ because $\bar{M}+\bar{F}>\bar{E}$ and $\bar{E}$ is Cartier. Hence $\bar{X}$ is isomorphic to $\mathbb{P}^{2}$ or $\mathbb{P}(1,1,2)$ by the classification of Gorenstein log del Pezzo surfaces [Dem80]. (Indeed, if $Y$ is a Gorenstein del Pezzo surface, let $f: \tilde{Y} \rightarrow Y$ be the minimal resolution. Then either $Y$ is isomorphic to $\mathbb{P}^{2}$ or $\mathbb{P}(1,1,2)$, or $\tilde{Y}$ is obtained from $\mathbb{P}^{2}$ by a sequence of blowups. In the last case, let $C \subset \tilde{Y}$ be a $(-1)$-curve. Then

$$
K_{Y} \cdot f_{*} C=f^{*} K_{Y} \cdot C=K_{\tilde{Y}} \cdot C=-1 .
$$

It follows that $-K_{Y}$ is a generator of $\operatorname{Pic} Y$ if $\rho(Y)=1$.) So $\bar{X}$ does not have a $D$ or $E$ singularity, a contradiction.

So we are in case (2). Write $p=\psi \circ \phi: \hat{X} \rightarrow \mathbb{P}^{1}$. The divisor $E$ has a $p$-horizontal component, say $E_{1}$ (because $\rho(X)=1$ so there does not exist a morphism $X \rightarrow \mathbb{P}^{1}$ ). If $f$ is a general fibre of $p$ then

$$
2=-K_{\hat{X}} \cdot f \geqslant E_{1} \cdot f \geqslant 1
$$

If $E_{1} \cdot f=1$ then all fibres of $\psi$ are reduced (because $\bar{E}_{1}$ is contained in the smooth locus of $\bar{X}$ ), so $\bar{X}$ is smooth [KM99, Lem. 11.5.2], a contradiction. So $E_{1} \cdot f=2$. Then $\left(M+\left(F-E_{1}\right)\right) \cdot f=0$, so $M$ and $F-E_{1}$ are $p$-vertical. In particular $M$ is basepoint free and $E_{1}$ has coefficient 1 in $F$. Since

$$
2 \geqslant 2-2 p_{a}\left(E_{1}\right)=-\left(K_{\hat{X}}+E_{1}\right) \cdot E_{1}=\left(M+\left(F-E_{1}\right)\right) \cdot E_{1} \geqslant M \cdot E_{1} \geqslant 2,
$$

we find $M \cdot E_{1}=2$ and $\left(F-E_{1}\right) \cdot E_{1}=0$. Thus $M$ is a fibre of $\psi$ and the divisors $M+E_{1}$ and $F-E_{1}$ have disjoint support. But $M+F \sim-K_{\hat{X}}$ is connected because

$$
H^{1}\left(\mathcal{O}_{\hat{X}}(-M-F)\right)=H^{1}\left(K_{\hat{X}}\right)=H^{1}\left(\mathcal{O}_{\hat{X}}\right)^{*}=0
$$

Hence $F=E=E_{1}$. In particular, $X$ has a unique non Du Val singularity of type $\frac{1}{m}(1,1)$ (where $E_{1}^{2}=-m$ ). Also, a general member of $\left|-K_{X}\right|$ is irreducible and does not pass through any Du Val singularities. Finally $\bar{X}$ does not have a singularity of type $E$ by the classification of fibres of $\mathbb{P}^{1}$ fibrations with Du Val singularities [KM99, Lem. 11.5.12]. So $X$ does not have an $E$ singularity.

If $K_{X}$ is Cartier then a general member of $\left|-K_{X}\right|$ is smooth and misses the singular points by [Dem80].

## 6. Surfaces of index $\leqslant 2$

Alexeev and Nikulin classified $\log$ del Pezzo surfaces $X$ of index $\leqslant 2$ [AN06]. They prove that $X$ is a $\mathbb{Z} / 2 \mathbb{Z}$ quotient of a K3 surface and use the Torelli theorem for K3 surfaces to obtain the classification. In this section, we deduce the index $\leqslant 2$ case of our main theorem from their result.

We note that the quotient singularities of index $\leqslant 2$ are the Du Val singularities and the cyclic quotient singularities of type $\frac{1}{4 d}(1,2 d-1)$, see [AN06]. In particular, they are $T$-singularities.

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Proposition 6.1. Let $X$ be a log del Pezzo surface of index $\leqslant 2$ such that $\rho(X)=1$. Then exactly one of the following holds.
(1) $X$ is a $\mathbb{Q}$-Gorenstein deformation of a toric surface.
(2) $X$ has either a $D$ singularity, an $E$ singularity, or $\geqslant 4 D u$ Val singularities.

Proof. We first observe that the two conditions cannot both hold. If $X$ is a $\mathbb{Q}$-Gorenstein deformation of a toric surface $Y$, then necessarily $\rho(Y)=1$ and $Y$ has only $T$-singularities. In particular, $Y$ has at most 3 singularities. Moreover, since the deformation preserves the Picard number, the only possible non-trivial deformation of a singularity of $Y$ is a deformation of a $T_{d}$ singularity to a $A_{d-1}$ singularity by Cor. 2.7. Finally, note that $Y$ does not have a $D$ or $E$ singularity because $Y$ is toric. Hence $X$ has at most 3 singularities and does not have a $D$ or $E$ singularity.

We now use the classification of log del Pezzo surfaces of index $\leqslant 2$ and Picard rank 1 [AN06, Thms. 4.2,4.3]. We check that each such surface $X$ which does not satisfy condition (2) is a deformation of a toric surface $Y$. By [AN06], $X$ is determined up to isomorphism by its singularities. So it suffices to exhibit a toric surface $Y$ such that $\rho(Y)=1$ and the singularities of $X$ are obtained from the singularities of $Y$ by a $\mathbb{Q}$-Gorenstein deformation which preserves the Picard number. We list the surfaces $Y$ in the tables below.

In the following tables, for each log del Pezzo surface $X$ of Picard rank 1 and index $\leqslant 2$ such that $X$ does not satisfy condition (2) of Prop. 6.1, we exhibit a toric surface $Y$ such that $X$ is a $\mathbb{Q}$-Gorenstein deformation of $Y$. We give the number of the surface $X$ in the list of Alexeev and Nikulin [AN06, p. 93-100]. We use the description of the toric surfaces $Y$ given in Thm. 4.1. We give the number of the infinite family to which $Y$ belongs and the solution $(a, b, c)$ of the Markov-type equation corresponding to $Y$. We record the value of $d=\mu+1$ for each singularity in the last column of the table.

| $X$ | $\operatorname{Sing} X$ | $Y$ | $\operatorname{Sing} Y$ | $d$ |
| :--- | :--- | :--- | :--- | :--- |
| 1 |  | $(1),(1,1,1)$ |  | $1,1,1$ |
| 2 | $A_{1}$ | $(2),(1,1,1)$ | $A_{1}$ | $1,1,2$ |
| 5 | $A_{1}, A_{2}$ | $(3),(1,1,1)$ | $A_{1}, A_{2}$ | $1,2,3$ |
| 6 | $A_{4}$ | $(4),(1,2,1)$ | $\frac{1}{4}(1,1), A_{4}$ | $1,1,5$ |
| 7 b | $2 A_{1}, A_{3}$ | $(5),(1,1,1)$ | $2 A_{1}, A_{3}$ | $2,2,4$ |
| 8 b | $A_{1}, A_{5}$ | $(6.2),(1,1,1)$ | $\frac{1}{4}(1,1), A_{1}, A_{5}$ | $1,2,6$ |
| 8 c | $3 A_{2}$ | $(6.1),(1,1,1)$ | $3 A_{2}$ | $3,3,3$ |
| 9 b | $A_{7}$ | $(7.1),(1,1,1)$ | $2 \frac{1}{4}(1,1), A_{7}$ | $1,1,8$ |
| 9 c | $A_{2}, A_{5}$ | $(7.3),(1,1,1)$ | $\frac{1}{9}(1,2), A_{2}, A_{5}$ | $1,3,6$ |
| 9 d | $A_{1}, 2 A_{3}$ | $(7.2),(1,1,1)$ | $\frac{1}{8}(1,3), 2 A_{3}$ | $2,4,4$ |
| 10 b | $A_{8}$ | $(8.1),(1,1,1)$ | $2 \frac{1}{9}(1,2), A_{8}$ | $1,1,9$ |
| 10 c | $A_{1}, A_{7}$ | $(8.2),(1,1,1)$ | $\frac{1}{16}(1,3), \frac{1}{8}(1,3), A_{7}$ | $1,2,8$ |
| 10 d | $A_{1}, A_{2}, A_{5}$ | $(8.3),(1,1,1)$ | $\frac{1}{18}(1,5), \frac{1}{12}(1,5), A_{5}$ | $2,3,6$ |
| 10 e | $A_{4}, A_{4}$ | $(8.4),(1,2,1)$ | $\frac{1}{25}(1,9), \frac{1}{20}(1,9), A_{4}$ | $1,5,5$ |
| 11 | $\frac{1}{4}(1,1)$ | $1,(1,1,2)$ | $\frac{1}{4}(1,1)$ | $1,1,1$ |
| 15 | $\frac{1}{4}(1,1), A_{4}$ | $4,(1,2,1)$ | $\frac{1}{4}(1,1), A_{4}$ | $1,1,5$ |
| 10 |  |  |  |  |

Smoothable del Pezzo surfaces with quotient singularities

| $X$ | Sing $X$ | $Y$ | $\operatorname{Sing} Y$ | $d$ |
| :--- | :--- | :--- | :--- | :--- |
| 18 | $\frac{1}{4}(1,1), A_{1}, A_{5}$ | $6.2,(1,1,1)$ | $\frac{1}{4}(1,1), A_{1}, A_{5}$ | $1,2,6$ |
| 19 | $\frac{1}{4}(1,1), A_{7}$ | $7.1,(1,1,1)$ | $2 \frac{1}{4}(1,1), A_{7}$ | $1,1,8$ |
| 21 | $\frac{1}{8}(1,3), A_{2}$ | $3,(1,2,1)$ | $\frac{1}{8}(1,3), A_{2}$ | $1,2,3$ |
| 25 | $2 \frac{1}{4}(1,1), A_{7}$ | $7.1,(1,1,1)$ | $2 \frac{1}{4}(1,1), A_{7}$ | $1,1,8$ |
| 26 | $\frac{1}{8}(1,3), 2 A_{3}$, | $7.2,(1,1,1)$ | $\frac{1}{8}(1,3), 2 A_{3}$ | $2,4,4$ |
| 27 | $\frac{1}{8}(1,3), A_{7}$ | $8.2,(1,1,1)$ | $\frac{1}{16}(1,3), \frac{1}{8}(1,3), A_{7}$ | $1,2,8$ |
| 30 | $\frac{1}{12}(1,5), 2 A_{2}$ | $6.1,(1,1,2)$ | $\frac{1}{12}(1,5), 2 A_{2}$ | $3,3,3$ |
| 33 | $A_{1}, \frac{1}{12}(1,5), A_{5}$ | $8.3,(1,1,1)$ | $\frac{1}{18}(1,5), \frac{1}{12}(1,5), A_{5}$ | $2,3,6$ |
| 40 | $\frac{1}{20}(1,9)$ | $4,(1,3,2)$ | $\frac{1}{9}(1,2), \frac{1}{20}(1,9)$ | $1,1,5$ |
| 44 | $\frac{1}{20}(1,9), A_{4}$ | $8.4,(1,2,1)$ | $\frac{1}{25}(1,9), \frac{1}{20}(1,9), A_{4}$ | $1,5,5$ |
| 46 | $A_{2}, \frac{1}{24}(1,11)$ | $7.3,(1,2,1)$ | $\frac{1}{9}(1,2), A_{2}, \frac{1}{24}(1,11)$, | $1,3,6$ |
| 50 | $\frac{1}{36}(1,17)$ | $8.1,(2,1,1)$ | $2 \frac{1}{9}(1,2), \frac{1}{36}(1,17)$ | $1,1,9$ |

## 7. Existence of special fibrations

Let $X$ be a $\log$ del Pezzo surface such that $\rho(X)=1$ and let $\pi: \tilde{X} \rightarrow X$ be its minimal resolution. We show that, under certain hypotheses, $\tilde{X}$ admits a morphism $p: \tilde{X} \rightarrow \mathbb{P}^{1}$ with general fibre a smooth rational curve such that the exceptional locus of $\pi$ has a particularly simple form with respect to the ruling $p$. When $X$ has only $T$-singularities (and satisfies the hypotheses), we use this structure to construct a toric surface $Y$ such that $X$ is a $\mathbb{Q}$-Gorenstein deformation of $Y$, see Sec. 8.

We first establish the existence of a so called 1-complement of $K_{X}$. We recall the definition and basic properties. For more details and motivation, see [Kol92, Sec. 19], [Pro01]. Let $X$ be a projective surface with quotient singularities. A 1-complement of $K_{X}$ is a divisor $D \in\left|-K_{X}\right|$ such that the pair $(X, D)$ is $\log$ canonical. In particular, by the classification of $\log$ canonical singularities of pairs [KM98, Thm. 4.15], $D$ is a nodal curve, and, at each singularity $P \in X$, either $D=0$ and $P \in X$ is a Du Val singularity, or the pair $(P \in X, D)$ is locally analytically isomorphic to the pair $\left(\frac{1}{n}(1, a),(u v=0)\right)$ for some $n$ and $a$. Moreover $D$ has arithmetic genus 1 because $2 p_{a}(D)-2=\left(K_{X}+D\right) \cdot D=0$ (note that the adjunction formula holds because $K_{X}+D$ is Cartier [Kol92, 16.4.3]). Thus $D$ is either a smooth elliptic curve or a cycle of smooth rational curves.
Theorem 7.1. Let $X$ be a log del Pezzo surface such that $\rho(X)=1$. Assume that dim $\left|-K_{X}\right| \geqslant 1$ and every singularity of $X$ is either a cyclic quotient singularity or a Du Val singularity. Then there exists a 1-complement of $K_{X}$, i.e., a divisor $D \in\left|-K_{X}\right|$ such that the pair $(X, D)$ is $\log$ canonical.
Proof. Write $-K_{X} \sim M+F$ where $M$ is an irreducible divisor such that $\operatorname{dim}|M|>0$ and $F$ is effective (we do not assume that $F$ is the fixed part of $\left|-K_{X}\right|$ ). Let $M$ be general in $|M|$.

Suppose first that $(X, M)$ is purely log terminal (plt). Then $M$ is a smooth curve. We may assume that $F \neq 0$ (otherwise $M$ is a 1 -complement). Then $-\left(K_{X}+M\right) \sim F$ is ample (because $\rho(X)=1$ ). Recall that for $X$ a normal variety and $S \subset X$ an irreducible divisor the different Diff $S_{S}(0)$ is the effective $\mathbb{Q}$-divisor on $S$ defined by the equation

$$
\left.\left(K_{X}+S\right)\right|_{S}=K_{S}+\operatorname{Diff}_{S}(0)
$$

That is, $\operatorname{Diff}_{S}(0)$ is the correction to the adjunction formula for $S \subset X$ due to the singularities of $X$ at $S$. See [Kol92, Sec. 16]. If $S$ is a normal variety and $B$ is an effective $\mathbb{Q}$-divisor on $S$ with

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coefficients less than 1, a 1-complement of $K_{S}+B$ is a divisor $D \in\left|-K_{S}\right|$ such that $(S, D)$ is $\log$ canonical and $D \geqslant\lfloor 2 B\rfloor$. By [Pro01, Prop. 4.4.1] it's enough to show that $K_{M}+\operatorname{Diff}_{M}(0)$ has a 1-complement.

The curve $M$ is smooth and rational and $\operatorname{deg}\left(K_{M}+\operatorname{Diff}_{M}(0)\right)<0$ because

$$
2 p_{a}(M)-2 \leqslant \operatorname{deg}\left(K_{M}+\operatorname{Diff}_{M}(0)\right)=\left(K_{X}+M\right) \cdot M=-F \cdot M<0
$$

Moreover, at each singular point $P_{i}$ of $X$ on $M$, the pair $(X, M)$ is of the form $\left(\frac{1}{m_{i}}\left(1, a_{i}\right),(x=0)\right)$, and

$$
\operatorname{Diff}_{M}(0)=\sum_{i}\left(1-\frac{1}{m_{i}}\right) P_{i}
$$

by [Kol92, 16.6.3]. So, if $K_{M}+\operatorname{Diff}_{M}(0)$ does not have a 1-complement, then, by [Kol92, Cor. 19.5] or direct calculation, there are exactly 3 singular points of $X$ on $M$, and ( $m_{1}, m_{2}, m_{3}$ ) is a Platonic triple $(2,2, m)$ (for some $m \geqslant 2),(2,3,3),(2,3,4)$, or $(2,3,5)$. The divisor $F$ passes through each singular point $P_{i}$ because $F \sim-\left(K_{X}+M\right)$ is not Cartier there. So $F \cdot M \geqslant \sum \frac{1}{m_{i}}$, and

$$
0=\left(K_{X}+M+F\right) \cdot M=\operatorname{deg}\left(K_{M}+\operatorname{Diff}_{M}(0)\right)+F \cdot M \geqslant 1,
$$

a contradiction.
Now suppose that the pair $(X, M)$ is not plt, and let $c$ be its $\log$ canonical threshold, i.e.,

$$
c=\sup \{t \in \mathbb{Q} \geqslant 0 \mid(X, t M) \text { is } \log \text { canonical }\} .
$$

Then there exists a projective birational morphism $f: Y \rightarrow X$ with exceptional locus an irreducible divisor $E$ such that the discrepancy $a(E, X, c M)=-1$ and $(Y, E)$ is plt. See [Pro01, Prop. 3.1.4]. So

$$
K_{Y}+c M^{\prime}+E=f^{*}\left(K_{X}+c M\right)
$$

where $M^{\prime}$ is the strict transform of $M$. Now

$$
-\left(K_{Y}+E\right)=c M^{\prime}-f^{*}\left(K_{X}+c M\right)
$$

is nef (note $M^{\prime}$ is nef because it moves). Moreover $-\left(K_{Y}+E\right)$ is big unless $M^{\prime 2}=0$ and $K_{X}+c M \sim_{\mathbb{Q}}$ 0 , in which case $c=1, F=0$, and $M$ is a 1 -complement. So we may assume $-\left(K_{Y}+E\right)$ is nef and big. Thus, by [Pro01, Prop. 4.4.1] again, it's enough to show that $K_{E}+\operatorname{Diff}_{E}(0)$ has a 1-complement. Suppose not. Then $E$ passes through 3 cyclic quotient singularities on $Y$ as above. Let $\tilde{Y} \rightarrow Y$ be the minimal resolution of $Y, E^{\prime}$ the strict transform of $E$, and consider the composition $g: \tilde{Y} \rightarrow X$. Let $P \in X$ be the point $f(E)$. Then $g^{-1}(P)$ is the union of $E^{\prime}$ and 3 chains of smooth rational curves (the exceptional loci of the minimal resolutions of the cyclic quotient singularities), and $E^{\prime}$ meets each chain in one of the end components. Let $-b_{i}$ be the self-intersection number of the end component $F_{i}$ of the $i$ th chain that meets $E^{\prime}$. Then $b_{i} \leqslant m_{i}$ where $m_{i}$ is the order of the cyclic group for the $i$ th quotient singularity. If we contract the $F_{i}$ and let $\bar{E}^{\prime}$ denote the image of $E^{\prime}$, then

$$
0>{\bar{E}^{\prime 2}}^{\prime 2}=E^{\prime 2}+\sum \frac{1}{b_{i}} \geqslant E^{\prime 2}+\sum \frac{1}{m_{i}}>E^{\prime 2}+1 .
$$

Hence $E^{2} \leqslant-2$ and $g$ is the minimal resolution of $X$. So $P \in X$ is a $D$ or $E$ singularity by our assumption. But $P \in X$ is a basepoint of $\left|-K_{X}\right|$, so this contradicts Thm. 5.1.

We describe the types of degenerate fibres which occur in the ruling we construct. We first introduce some notation.

Definition 7.2. Let $a, n \in \mathbb{N}$ with $a<n$ and $(a, n)=1$. We say the fractions $n / a$ and $n /(n-a)$ are conjugate.

Lemma 7.3. If $\left[b_{1}, \ldots, b_{r}\right]$ and $\left[c_{1}, \ldots, c_{s}\right]$ are conjugate, then so are $\left[b_{1}+1, b_{2}, \ldots, b_{r}\right]$ and $\left[2, c_{1}, \ldots, c_{s}\right]$. Conversely, every conjugate pair can be constructed from [2],[2] by a sequence of such steps. Also, if $\left[b_{1}, \ldots, b_{r}\right]$ and $\left[c_{1}, \ldots, c_{s}\right]$ are conjugate then so are $\left[b_{r}, \ldots, b_{1}\right]$ and $\left[c_{s}, \ldots, c_{1}\right]$.

Proof. If $\left[b_{1}, \ldots, b_{r}\right]=n / a$ and $\left[c_{1}, \ldots, c_{s}\right]=n /(n-a)$ then $\left[b_{1}+1, b_{2}, \ldots, b_{r}\right]=(n+a) / a$ and $\left[2, c_{1}, \ldots, c_{s}\right]=(n+a) / n$. The last statement follows immediately from Rem. 2.8.

Proposition 7.4. Let $S$ be a smooth surface, $T$ a smooth curve, and $p: S \rightarrow T$ a morphism with general fibre a smooth rational curve. Let $f$ be a degenerate fibre of $p$. Suppose that $f$ contains a unique ( -1 -curve and the union of the remaining irreducible components of $f$ is a disjoint union of chains of smooth rational curves. Then the dual graph of $f$ has one of the following forms.


Here the black vertex denotes the ( -1 -curve and a white vertex with label $a \geqslant 2$ denotes a smooth rational curve with self-intersection number $-a$. In both types the strings $\left[a_{1}, \ldots, a_{r}\right]$ and $\left[b_{1}, \ldots, b_{s}\right]$ are conjugate. In type (II) there are $t(-2)$-curves in the branch containing the ( -1 )-curve.

Conversely, any configuration of curves of this form is a degenerate fibre of a fibration $p: S \rightarrow T$ as above.

Proof. The morphism $p: S \rightarrow T$ is obtained from a $\mathbb{P}^{1}$-bundle $F \rightarrow T$ by a sequence of blowups. The statements follow by induction on the number of blowups.

We refer to the fibres above as fibres of types $(I)$ and $(I I)$. We also call a fibre of the form

a fibre of type $(O)$.
Remark 7.5. The curves of multiplicity one in the fibre are the ends of the chain in types $(O)$ and $(I)$ and the ends of the branches not containing the $(-1)$-curve in type ( $I I$ ). In particular, a section of the fibration meets the fibre in one of these curves.

Theorem 7.6. Let $X$ be a $\log$ del Pezzo surface such that $\rho(X)=1$. Assume that dim $\left|-K_{X}\right| \geqslant 1$ and every singularity of $X$ is either a cyclic quotient singularity or a Du Val singularity. Let $\pi$ : $\tilde{X} \rightarrow$ $X$ be the minimal resolution of $X$. Then one of the following holds.
(1) There exists a morphism $p: \tilde{X} \rightarrow \mathbb{P}^{1}$ with general fibre a smooth rational curve satisfying one of the following.
(a) Exactly one component $\tilde{E}_{1}$ of the exceptional locus of $\pi$ is $p$-horizontal. The curve $\tilde{E}_{1}$ is a section of $p$. The fibration $p$ has at most two degenerate fibres and each is of type (I) or (II).
(b) Exactly two components $\tilde{E}_{1}, \tilde{E}_{2}$ of the exceptional locus of $\pi$ are p-horizontal. The curves $\tilde{E}_{1}, \tilde{E}_{2}$ are sections of $p$. Either $\tilde{E}_{1}$ and $\tilde{E}_{2}$ are disjoint and $p$ has two degenerate fibres of types $(O)$ and either $(I)$ or $(I I)$, or $\tilde{E}_{1} \cdot \tilde{E}_{2}=1$ and $p$ has a single degenerate fibre of type $(O)$. The sections $\tilde{E}_{1}$ and $\tilde{E}_{2}$ meet distinct components of the degenerate fibres.
(2) The surface $X$ has at most 2 non $D u$ Val singularities and each is of the form $\frac{1}{m}(1,1)$ for some $m \geqslant 3$.

Proof. Assume that $K_{X}$ is not Cartier. As in the proof of Thm. 5.1, let $\nu: \hat{X} \rightarrow X$ be the minimal resolution of the non Du Val singularities, $\left\{E_{i}\right\}$ the exceptional divisors, and $E=\sum E_{i}$. Write $\left|-K_{\hat{X}}\right|=|M|+F$ where $F$ is the fixed part and $M \in|M|$ is general. Then $F \geqslant E$ and $\operatorname{dim}|M|=$ $\operatorname{dim}\left|-K_{X}\right| \geqslant 1$.

We run the MMP on $\hat{X}$. We obtain a birational morphism $\phi: \hat{X} \rightarrow \bar{X}$ such that $\bar{X}$ has Du Val singularities and either $\rho(\bar{X})=2$ and there is a fibration $\psi: \bar{X} \rightarrow \mathbb{P}^{1}$ such that $-K_{\bar{X}}$ is $\psi$-ample or $\rho(\bar{X})=1$ and $-K_{\bar{X}}$ is ample. Moreover, $\phi$ is a composition

$$
\hat{X}=X_{1} \xrightarrow{\phi_{1}} X_{2} \xrightarrow{\phi_{2}} \ldots \xrightarrow{\phi_{n}} X_{n+1}=\bar{X}
$$

where $\phi_{i}$ is a weighted blowup of a smooth point of $X_{i+1}$ with weights $\left(1, n_{i}\right)$ (by the classification of birational extremal contractions in the MMP for surfaces with Du Val singularities).
Claim 7.7. Given $\phi: \hat{X} \rightarrow \bar{X}$, we can direct the MMP so that the components of $E$ contracted by $\phi$ are contracted last. That is, for some $1 \leqslant m \leqslant n$, the exceptional divisor of $\phi_{i}$ is (the image of) a component of $E$ iff $i>m$.

Proof. We have $K_{\hat{X}}=\nu^{*} K_{X}+\sum a_{i} E_{i}$ where $-1<a_{i}<0$ for each $i$. Write $\Delta=\sum\left(-a_{i}\right) E_{i}$. So $\nu^{*} K_{X}=K_{\hat{X}}+\Delta$ and $\Delta$ is an effective divisor such that $\lfloor\Delta\rfloor=0$ and $\operatorname{Supp} \Delta=E$. Hence $-\left(K_{\hat{X}}+\Delta\right)$ is nef and big and $(\hat{X}, \Delta)$ is Kawamata log terminal (klt). These properties are preserved under the $K_{\hat{X}}$-MMP.

Let $R=\sum R_{i}$ be the sum of the $\phi$-exceptional curves that are not contained in $E$ and $R^{\prime} \subset R$ a connected component. Then $R^{\prime} \cdot E>0$ (otherwise $\nu$ is an isomorphism near $R^{\prime}$ which contradicts $\rho(X)=1$ ). Let $R_{i}$ be a component of $R^{\prime}$ such that $R_{i} \cdot E>0$. Then $\left(K_{\hat{X}}+\Delta\right) \cdot R_{i} \leqslant 0$ and $R_{i} \cdot \Delta>0$. So $K_{\hat{X}} \cdot R_{i}<0$, and we can contract $R_{i}$ first in the $K_{\hat{X}}$-MMP. Repeating this procedure, we contract all of $R$, obtaining a birational morphism $\hat{X} \rightarrow \hat{X}^{\prime}$. Finally we run the MMP on $\hat{X}^{\prime}$ over $\bar{X}$ to contract the remaining curves.

Claim 7.8. We may assume $\rho(\bar{X})=2$.
Proof. Suppose $\rho(\bar{X})=1$. Write $\bar{M}=\phi_{*} M$, etc. Then $-K_{\bar{X}} \sim \bar{M}+\bar{F}, \bar{F} \geqslant \bar{E}>0$, and $\bar{E}$ is contained in the smooth locus of $\bar{X}$. Thus, as in the proof of Thm. 5.1, $-K_{\bar{X}}$ is not a generator of Pic $\bar{X}$, so $\bar{X} \simeq \mathbb{P}^{2}$ or $\bar{X} \simeq \mathbb{P}(1,1,2)$ by the classification of log del Pezzo surfaces with Du Val singularities. In particular, it follows that $\bar{E}$ has at most 2 components.

Suppose first that $\phi$ does not contract any component of $E$. Then $E$ has at most 2 components. So, either we are in case (2), or $E=E_{1}+E_{2}, E_{1} \cap E_{2} \neq \emptyset, \bar{X} \simeq \mathbb{P}^{2}$, and $\bar{M}, \bar{E}_{1}, \bar{E}_{2} \sim l$, where $l$ is the class of a line. In this case $\rho(\hat{X})=\rho(X)+2=3$, so $\phi: \hat{X} \rightarrow \bar{X}$ is a composition of two weighted blowups of weights $\left(1, n_{1}\right),\left(1, n_{2}\right)$. These must have centres two distinct points $P_{1} \in \bar{E}_{1}, P_{2} \in \bar{E}_{2}$, and in each case the local equation of $\bar{E}_{i}$ is a coordinate with weight $n_{i}$ (because $E_{i}$ is contained in the smooth locus of $\hat{X}$ ). Let $l_{12}$ be the line through $P_{1}$ and $P_{2}$. Then these blowups are toric with respect to the torus $\bar{X} \backslash l_{12}+\bar{E}_{1}+\bar{E}_{2}$. We find that the minimal resolution $\tilde{X}$ is a toric surface with boundary divisor a cycle of smooth rational curves with self-intersection numbers

$$
-2, \ldots,-2,-1,-\left(n_{1}-1\right),-\left(n_{2}-1\right),-1,-2, \ldots,-2,-1
$$

where $\tilde{E}_{1}$ and $\tilde{E}_{2}$ are the curves with self-intersection numbers $-\left(n_{1}-1\right),-\left(n_{2}-1\right)$, the first two $(-1)$-curves are the strict transforms of the exceptional curves of the blowups of $P_{1}$ and $P_{2}$, the last $(-1)$-curve is the strict transform of $l_{12}$, and the chains of $(-2)$-curves are the exceptional loci of the resolutions of the singularities of $\hat{X}$ and have lengths $\left(n_{1}-1\right)$ and $\left(n_{2}-1\right)$. In particular, there
is a fibration $p: \tilde{X} \rightarrow \mathbb{P}^{1}$ with two degenerate fibres of types $-1,-2, \ldots,-2,-1$ (where there are $\left(n_{2}-1\right)(-2)$-curves) and $-2, \ldots,-2,-1,-\left(n_{1}-1\right)$ (where there are $\left(n_{1}-2\right)(-2)$-curves), and two $\pi$-exceptional sections with self-intersection numbers $-\left(n_{2}-1\right)$ and -2 . So we are in case (1b).

Now suppose $\phi$ contracts some component of $E$. Then $\phi_{n}: X_{n} \rightarrow X_{n+1}=\bar{X}$ is an (ordinary) blowup of a smooth point $Q \in \bar{X}$. If $\bar{X} \simeq \mathbb{P}^{2}$ then $X_{n} \simeq \mathbb{F}_{1}$ and there is a fibration $\psi: X_{n} \rightarrow \mathbb{P}^{1}$. So we may assume $\rho(\bar{X})=2$. If $\bar{X} \simeq \mathbb{P}(1,1,2)$, the quadric cone, let $L$ be the ruling of the cone through $Q$. Then the strict transform $L^{\prime}$ of $L$ on $X_{n}$ satisfies $K_{X_{n}} \cdot L^{\prime}<0$ and $L^{\prime 2}<0$. Contracting $L^{\prime}$ we obtain a morphism $\phi_{n}^{\prime}: X_{n} \rightarrow \bar{X}^{\prime} \simeq \mathbb{P}^{2}$. So, replacing $\phi_{n}$ by $\phi_{n}^{\prime}$, we may assume $\bar{X} \simeq \mathbb{P}^{2}$.

We now assume $\rho(\bar{X})=2$. We have a diagram

where $\pi: \tilde{X} \rightarrow X$ is the minimal resolution. Let $l$ be a general fibre of $p$ and $\tilde{E}$ the strict transform of $E$ on $\tilde{X}$. Note that, by construction, the components of the exceptional locus of $\pi$ over Du Val singularities are contained in fibres of $p$. Write $\left|-K_{\tilde{X}}\right|=|\tilde{M}|+\tilde{F}$ where $\tilde{F}$ is the fixed part and $\tilde{M} \in|\tilde{M}|$ is general. Then $\tilde{F} \geqslant \tilde{E}$.

There is a 1-complement of $K_{X}$ by Thm. 7.1. This can be lifted to $\tilde{X}$. (Indeed, if $D$ is a 1complement of $K_{X}$, define $\tilde{D}$ by $K_{\tilde{X}}+\tilde{D}=\pi^{*}\left(K_{X}+D\right)$ and $\pi_{*} \tilde{D}=D$. Note that $\tilde{D}$ is an effective $\mathbb{Z}$-divisor because $K_{\tilde{X}}$ is $\pi$-nef and $K_{X}+D$ is Cartier. Then $\tilde{D}$ is a 1-complement of $K_{\tilde{X}}$.) Hence $(\tilde{X}, \tilde{M}+\tilde{F})$ is $\log$ canonical. In particular, $\tilde{F}$ is reduced and $\tilde{M}+\tilde{F}$ is a cycle of smooth rational curves.

There exists a $p$-horizontal component $\tilde{E}_{1}$ of $\tilde{E}$ (because $\rho(X)=1$ ). Then

$$
1 \leqslant \tilde{E}_{1} \cdot l \leqslant(\tilde{F}+\tilde{M}) \cdot l=-K_{\tilde{X}} \cdot l=2 .
$$

Suppose first that $\tilde{E}_{1} \cdot l=2$. Then $\tilde{M}$ and $\tilde{F}-\tilde{E}_{1}$ are $p$-vertical. Hence $\tilde{M} \sim l$ and $\tilde{F}=\tilde{E}_{1}$, so $\tilde{E}=\tilde{E}_{1}$ and we are in case (2).

Suppose now that $\tilde{E}_{1} \cdot l=1$. Since $\mu\left(\tilde{E}_{1}\right)$ is contained in the smooth locus of $\bar{X}$, the fibres of $\psi$ have multiplicity 1 , so $\psi$ is smooth by [KM99, Lem. 11.5.2]. Thus $\bar{X} \simeq \mathbb{F}_{n}$ for some $n \geqslant 0$.

If $\tilde{E}_{1}$ is the only $p$-horizontal component of $\tilde{E}$ we are in case (1a). Suppose there is another $p$-horizontal component $\tilde{E}_{2}$. Then, since $-K_{\tilde{X}} \cdot l=2$, we have $\tilde{E}_{2} \cdot l=1$ and $\tilde{M}$ and $\tilde{F}-\tilde{E}_{1}-\tilde{E}_{2}$ are contained in fibres of $p$. If $M \sim 2 l$ then $\tilde{F}=\tilde{E}=\tilde{E}_{1}+\tilde{E}_{2}$ and $\tilde{E}_{1} \cap \tilde{E}_{2}=\emptyset$ so we are in case (1b). So we may assume $\tilde{M} \sim l$. Then the components of $\tilde{F}$ form a chain, with ends $\tilde{E}_{1}$ and $\tilde{E}_{2}$.

We note that a component $\Gamma$ of a degenerate fibre of $p$ that is not contracted by $\pi$ is necessarily a ( -1 )-curve, because $K_{\tilde{X}}=\pi^{*} K_{X}-\tilde{\Delta}$ where $\tilde{\Delta}$ is effective and $\pi$-exceptional, so

$$
K_{\tilde{X}} \cdot \Gamma \leqslant \pi^{*} K_{X} \cdot \Gamma=K_{X} \cdot \pi_{*} \Gamma<0
$$

Hence, since $\rho(X)=1$, there exists a unique degenerate fibre of $p$ containing exactly two ( -1 )curves, and any other degenerate fibres contain exactly one $(-1)$-curve. Let $\tilde{G}$ denote the reduction of the fibre containing two $(-1)$-curves.

If $\tilde{F}=\tilde{E}_{1}+\tilde{E}_{2}$ then $\tilde{E}_{1} \cdot \tilde{E}_{2}=1$ and any degenerate fibre of $p$ consists of $(-1)$-curves and $(-2)$-curves. It follows that $\tilde{G}$ is of type $(O)$ and there are no other degenerate fibres, so we are in case (1b). So assume $\tilde{F}>\tilde{E}_{1}+\tilde{E}_{2}$. Then $\tilde{E}_{1} \cap \tilde{E}_{2}=\emptyset$.

Suppose first that $\tilde{G}$ is the only degenerate fibre. Then $\tilde{F} \leqslant \tilde{G}+\tilde{E}_{1}+\tilde{E}_{2}$. Write $\tilde{G}=\tilde{G}^{\prime}+\tilde{G}^{\prime \prime}$ where $\tilde{G}^{\prime}=\tilde{F}-\tilde{E}_{1}-\tilde{E}_{2}$. So $\tilde{G}^{\prime}$ is a chain of smooth rational curves. It follows that each connected component of $\tilde{G}^{\prime \prime}$ is a chain of smooth rational rational curves such that one end component is a $(-1)$-curve adjacent to $\tilde{G}^{\prime}$ and the remaining curves are ( -2 -curves. We construct an alternative ruling $p^{\prime}: \tilde{X} \rightarrow \mathbb{P}^{1}$ with only one horizontal $\pi$-exceptional curve by inductively contracting ( -1 )curves as follows. First contract the components of $\tilde{G}^{\prime \prime}$. Second, contract ( -1 )-curves in $\tilde{G}^{\prime}$ until the image of $\tilde{E}_{1}$ or $\tilde{E}_{2}$ is a ( -1 )-curve. Now contract this curve, and continue contracting ( -1 )-curves until we obtain a ruled surface $\bar{X}^{\prime} \simeq \mathbb{F}_{m}$. Then $\tilde{M} \sim l$ is horizontal for the induced ruling $p^{\prime}$. Moreover, if $C$ is a $p^{\prime}$-horizontal $\pi$-exceptional curve then $C \not \subset \tilde{G}^{\prime \prime}$ by construction. Hence $C \subset \tilde{F}$. Thus there exists a unique such $C$, and $C$ is a section of $p^{\prime}$. So we are in case (1a).

Finally, suppose there is another degenerate fibre of $p$, and let $\tilde{V}$ denote its reduction. Then $\tilde{V}$ contains a unique ( -1 )-curve $C$. The surface $X$ has only cyclic quotient singularities by assumption. Therefore $\tilde{V}-C$ is a union of chains of smooth rational curves. It follows that $\tilde{V}$ is a fibre of type $(I)$ or $(I I)$. Now $\tilde{E}_{1} \cdot C=\tilde{E}_{2} \cdot C=0$ because $C$ has multiplicity greater than 1 in the fibre. So $\tilde{V}$ contains a component of $\tilde{F}$ (because $1=-K_{\tilde{X}} \cdot C=(\tilde{M}+\tilde{F}) \cdot C$ ). Hence $\tilde{F}-\tilde{E}_{1}-\tilde{E}_{2} \leqslant \tilde{V}$ (because $\tilde{M}+\tilde{F}$ is a cycle of rational curves and $\tilde{M} \sim l$ ). In particular, $\tilde{G}$ consists of two ( -1 )-curves and some ( -2 )-curves. Hence $\tilde{G}$ is of type ( $O$ ) and we are in case (1b). This completes the proof.

## 8. Proof of Main Theorem

Theorem 8.1. Let $X$ be a log del Pezzo surface such that $\rho(X)=1$ and $X$ has only $T$-singularities. Then exactly one of the following holds
(1) $X$ is a $\mathbb{Q}$-Gorenstein deformation of a toric surface $Y$, or
(2) $X$ is one of the sporadic surfaces listed in Example 8.3.

Remark 8.2. Note that the surface $Y$ in Thm. 8.1(1) necessarily has only $T$-singularities and $\rho(Y)=$ 1. Thus $Y$ is one of the surfaces listed in Thm. 4.1.

Example 8.3. We list the log del Pezzo surfaces $X$ such that $X$ has only $T$-singularities and $\rho(X)=1$, but $X$ is not a $\mathbb{Q}$-Gorenstein deformation of a toric surface. In each case $X$ has index $\leqslant 2$. If $X$ is Gorenstein, the possible configurations of singularities are

$$
\begin{gathered}
D_{5}, E_{6}, E_{7}, A_{1} D_{6}, 3 A_{1} D_{4}, E_{8}, D_{8}, A_{1} E_{7}, \\
A_{2} E_{6}, 2 A_{1} D_{6}, A_{3} D_{5}, 2 D_{4}, 2 A_{1} 2 A_{3}, 4 A_{2} .
\end{gathered}
$$

The configuration determines the surface uniquely with the following exceptions: there are two surfaces for $E_{8}, A_{1} E_{7}, A_{2} E_{6}$, and an $\mathbb{A}^{1}$ of surfaces for $2 D_{4}$. See [AN06, Thm 4.3]. If $X$ has index 2 , the possible configurations of singularities are

$$
\frac{1}{4}(1,1) D_{8}, \frac{1}{4}(1,1) 2 A_{1} D_{6}, \frac{1}{4}(1,1) A_{3} D_{5}, \frac{1}{4}(1,1) 2 D_{4}
$$

and the configuration determines the surface uniquely. See [AN06, Thm 4.2].
Remark 8.4. Note that the case $K_{X}^{2}=7$ does not occur. This may be explained as follows. If $X$ is a del Pezzo surface with $T$-singularities such that $\rho(X)=1$, then there exists a $\mathbb{Q}$-Gorenstein smoothing $\mathcal{X} / T$ of $X$ over $T:=\operatorname{Spec} k[[t]]$ such that the generic fibre $\mathcal{X}_{K}$ is a smooth del Pezzo surface over $K=k((t))$ with $\rho\left(\mathcal{X}_{K}\right)=1$. (Indeed, if $\mathcal{X} / T$ is a smoothing of $X$ over $T$, the restriction map $\mathrm{Cl}(\mathcal{X}) \rightarrow \mathrm{Cl}\left(\mathcal{X}_{K}\right)=\operatorname{Pic}\left(\mathcal{X}_{K}\right)$ is an isomorphism because the closed fibre $X$ is irreducible and the restriction map $\operatorname{Pic}(\mathcal{X}) \rightarrow \operatorname{Pic}(X)$ is an isomorphism because $H^{1}\left(\mathcal{O}_{X}\right)=H^{2}\left(\mathcal{O}_{X}\right)=0$. Thus $\rho\left(\mathcal{X}_{K}\right) \geqslant \rho(X)=1$ with equality iff the total space $\mathcal{X}$ of the deformation is $\mathbb{Q}$-factorial. Since there are no local-to-global obstructions for deformations of $X$, there exists a $\mathbb{Q}$-Gorenstein smoothing
$\mathcal{X} / T$ such that $P \in \mathcal{X}$ is smooth for $P \in X$ a Du Val singularity and $P \in \mathcal{X}$ is of type $\frac{1}{n}(1,-1, a)$ for $P \in X$ a singularity of type $\frac{1}{d n^{2}}(1, d n a-1)$ (see Sec. 2.2). In particular, $\mathcal{X}$ is $\mathbb{Q}$-factorial.) Note that $K_{\mathcal{X}_{K}}^{2}=K_{X}^{2}$. If $Y$ is a smooth del Pezzo surface with $K_{Y}^{2}=7$ over a field (not necessarily algebraically closed) then $\rho(Y)>1$, see, e.g., [Man86]. Hence there is no $X$ with $K_{X}^{2}=7$.
Proof of Thm. 8.1. First assume that $X$ does not have a $D$ or $E$ singularity. Note that $\operatorname{dim}\left|-K_{X}\right|=$ $K_{X}^{2} \geqslant 1$ by Prop. 2.6, so we may apply Thm. 7.6. We use the notation of that theorem.

Suppose first that we are in case $(1 a)$. We construct a toric surface $Y$ and prove that $X$ is a $\mathbb{Q}$-Gorenstein deformation of $Y$. We first describe the surface $Y$. Let $\tilde{E}_{1}^{2}=-d$. There is a uniquely determined toric blowup $\mu_{Y}: \tilde{Y} \rightarrow \mathbb{F}_{d}$ such that $\mu_{Y}$ is an isomorphism over the negative section $B \subset \mathbb{F}_{d}$, and the degenerate fibres of the ruling $p_{Y}: \tilde{Y} \rightarrow \mathbb{P}^{1}$ are fibres of type $(I)$ associated to the degenerate fibres of $p: \tilde{X} \rightarrow \mathbb{P}^{1}$ as follows. Let $f$ be a degenerate fibre of $p$ of type $(I)$ or (II) as in Prop. 7.4, and assume that $\tilde{E}_{1}$ intersects the left end component. If $f$ is of type $(I)$ then the associated fibre $f_{Y}$ of $p_{Y}$ has the same form. If $f$ is of type $(I I)$ then $f_{Y}$ is a fibre of type ( $I$ ) with self-intersection numbers

$$
-a_{r}, \ldots,-a_{1},-t-2,-b_{1}, \ldots,-b_{s},-1,-d_{1}, \ldots,-d_{u}
$$

Note that the sequence $d_{1}, \ldots, d_{u}$ is uniquely determined (see Prop. 7.4). In each case the strict transform $B^{\prime}$ of $B$ again intersects the left end component of $f_{Y}$.

Let $Y$ be the toric surface obtained from $\tilde{Y}$ by contracting the strict transform of the negative section of $\mathbb{F}_{d}$ and the components of the degenerate fibres of the ruling with self-intersection number at most -2 . For each fibre $f$ of $p$ of type $(I I)$ as above, the chain of rational curves with self-intersections $-d_{1}, \ldots,-d_{u}$ in the associated fibre $f_{Y}$ of $p_{Y}$ contracts to a $T_{t+1}$ singularity by Lem. 8.5(1). This singularity replaces the $A_{t}$ singularity on $X$ obtained by contracting the chain of $t(-2)$-curves in $f$. In particular, the surface $Y$ has $T$-singularities. Moreover $\rho(Y)=1$, and $K_{Y}^{2}=K_{X}^{2}$ by Prop. 2.6. A $T_{d}$-singularity admits a $\mathbb{Q}$-Gorenstein deformation to an $A_{d-1}$ singularity (see Prop. 2.3). Hence the singularities of $X$ are a $\mathbb{Q}$-Gorenstein deformation of the singularities of $Y$. There are no local-to-global obstructions for deformations of $Y$ by Prop. 3.1. Hence there is a $\mathbb{Q}$-Gorenstein deformation $X^{\prime}$ of $Y$ with the same singularities as $X$. We prove below that $X \simeq X^{\prime}$.

Let $f$ be a degenerate fibre of $p$ of type $(I I)$ as above and $f_{Y}$ the associated fibre of $p_{Y}$. Let $P \in Y$ be the $T$-singularity obtained by contracting the chain of rational curves in $f_{Y}$ with selfintersections $-d_{1}, \cdots,-d_{u}$. Let $X^{\prime}$ be the general fibre of a $\mathbb{Q}$-Gorenstein deformation of $Y$ over the germ of a curve which deforms $P \in Y$ to an $A_{t}$ singularity and is locally trivial elsewhere. Let $\hat{Y} \rightarrow Y$ and $\hat{X}^{\prime} \rightarrow X^{\prime}$ be the minimal resolutions of the remaining singularities (where the deformation is locally trivial). Thus $\hat{Y}$ has a single $T$-singularity and $\hat{X}^{\prime}$ a single $A_{t}$ singularity. The ruling $p_{Y}: \tilde{Y} \rightarrow \mathbb{P}^{1}$ descends to a ruling $\hat{Y} \rightarrow \mathbb{P}^{1}$; let $A$ be a general fibre of this ruling. Then $A$ deforms to a 0-curve $A^{\prime}$ in $\hat{X}^{\prime}$ (because $H^{1}\left(\mathcal{N}_{A / \hat{Y}}\right)=H^{1}\left(\mathcal{O}_{A}\right)=0$ ) which defines a ruling $\hat{X}^{\prime} \rightarrow \mathbb{P}^{1}$. Let $\tilde{X}^{\prime} \rightarrow \hat{X}^{\prime}$ be the minimal resolution of $\hat{X}^{\prime}$ and consider the induced ruling $p_{X^{\prime}}: \tilde{X}^{\prime} \rightarrow \mathbb{P}^{1}$. Note that the exceptional locus of $\hat{Y} \rightarrow Y$ deforms without change by construction. Moreover, the ( -1 )curve in the remaining degenerate fibre (if any) of $p_{Y}$ also deforms. There is a unique horizontal curve in the exceptional locus of $\pi_{X^{\prime}}: \tilde{X}^{\prime} \rightarrow X^{\prime}$, and $\rho\left(X^{\prime}\right)=1$ by Prop. 2.6. Hence each degenerate fibre of $p_{X^{\prime}}$ contains a unique $(-1)$-curve, and the remaining components of the fibre are in the exceptional locus of $\pi_{X^{\prime}}$. We can now describe the degenerate fibres of $p_{X^{\prime}}$. If $p_{Y}$ has a degenerate fibre besides $f_{Y}$, then $p_{X^{\prime}}$ has a degenerate fibre of the same form. We claim that there is exactly one additional degenerate fibre of $p_{X^{\prime}}$, which is of type $(I I)$ and has the same form as the fibre $f$ of $p$. Indeed, the union of the remaining degenerate fibres consists of the chain of rational curves with self-intersections $-a_{r}, \ldots,-a_{1},-t-2, b_{1}, \ldots, b_{s}$ (the deformation of the chain of the same form in $f_{Y}$, the chain of $(-2)$-curves which contracts to the $A_{t}$ singularity, and some $(-1)$-curves. The claim follows by the description of degenerate fibres in Prop. 7.4. If there is a second degenerate

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fibre of $p$ of type (II) we repeat this process. We obtain a $\mathbb{Q}$-Gorenstein deformation $X^{\prime}$ of $Y$ with minimal resolution $\pi_{X^{\prime}}: \tilde{X}^{\prime} \rightarrow X^{\prime}$, and a ruling $p_{X^{\prime}}: \tilde{X}^{\prime} \rightarrow \mathbb{P}^{1}$ such that the exceptional locus of $\pi_{X^{\prime}}$ has the same form with respect to the ruling $p_{X^{\prime}}$ as that of $\pi$ with respect to $p$.

We claim that $X \simeq X^{\prime}$. Indeed, there is a smooth toric surface $Z$ and, for each fibre $f_{i}$ of $p$ of type (II), an irreducible toric boundary divisor $\Delta_{i} \subset Z$ and points $P_{i}, P_{i}^{\prime}$ in the torus orbit $O_{i} \subset \Delta_{i}$, such that $\tilde{X}$ (respectively $\tilde{X}^{\prime}$ ) is obtained from $Z$ by successively blowing up the points $P_{i}$ (respectively $\left.P_{i}^{\prime}\right) t_{i}+1$ times, where $t_{i}$ is the length of the chain of $(-2)$-curves in $f_{i}$. It remains to prove that we may assume $P_{i}=P_{i}^{\prime}$ for each $i$. Let $T$ be the torus acting on $Z$ and $N$ its lattice of 1-parameter subgroups. Let $\Sigma \subset N_{\mathbb{R}}$ be the fan corresponding to $X$ and $v_{i} \in N$ the minimal generator of the ray in $\Sigma$ corresponding to $\Delta_{i}$. Then $T_{i}=\left(N /\left\langle v_{i}\right\rangle\right) \otimes \mathbb{G}_{m}$ is the quotient torus of $T$ which acts faithfully on $\Delta_{i}$. Thus, there is an element $t \in T$ taking $P_{i}$ to $P_{i}^{\prime}$ for each $i$ except in the following case: there are two fibres of $p$ of type (II), and $v_{1}+v_{2}=0$. In this case, there is a toric ruling $q: Z \rightarrow \mathbb{P}^{1}$ given by the projection $N \rightarrow N /\left\langle v_{1}\right\rangle$. The toric boundary of $Z$ decomposes into two sections (given by $\Delta_{1}, \Delta_{2}$ ) and two fibres of $q$. But one of these fibres (the one containing the image of $\tilde{E}_{1} \subset \tilde{X}$ ) is a chain of rational curves of self-intersections at most -2 , a contradiction.

Next assume that we are in case (1b). There is a ruling $p: \tilde{X} \rightarrow \mathbb{P}^{1}$ with two $\pi$-exceptional sections $\tilde{E}_{1}$ and $\tilde{E}_{2}$. Suppose first that $\tilde{E}_{1} \cap \tilde{E}_{2}=\emptyset$. Then there are two degenerate fibres of types $(O)$ and either $(I)$ or $(I I)$. We use the notation of Prop. 7.4. The exceptional locus of $\pi$ consists of the components of the degenerate fibres of self-intersection $\leqslant-2$ and the two disjoint sections $\tilde{E}_{1}$ and $\tilde{E}_{2}$ of $p$ which meet the first fibre in the two ( -1 )-curves and the second fibre in the components labelled $-a_{r}$ and $-b_{s}$ respectively. If the degenerate fibres are of types $(O)$ and $(I)$ then $X$ is toric. So we may assume the degenerate fibres are of types $(O)$ and $(I I)$. Set $\tilde{E}_{1}^{2}=-a_{r+1}$ and $\tilde{E}_{2}^{2}=-b_{s+1}$. Let $m$ be the number of $(-2)$-curves in the fibre of type $(O)$. Then $X$ has singularities $A_{m}, A_{t}$, and the cyclic quotient singularity whose minimal resolution has exceptional locus the chain of rational curves with self-intersections $-a_{r+1}, \ldots,-a_{1},-(t+2),-b_{1}, \ldots,-b_{s+1}$.

The ruling $p: \tilde{X} \rightarrow \mathbb{P}^{1}$ is obtained from a $\mathbb{P}^{1}$-bundle by a sequence of blowups. It follows that $m=a_{r+1}+b_{s+1}-2$.

We construct a toric surface $Y$ and prove that $X$ is a $\mathbb{Q}$-Gorenstein deformation of $Y$. The minimal resolution of $\tilde{Y}$ is the toric surface which fibres over $\mathbb{P}^{1}$ with two degenerate fibres, one of type $(O)$ (where there are $m(-2)$-curves as above) and one of type ( $I$ ) with self-intersection numbers

$$
-a_{r}, \ldots,-a_{1},-(t+2),-b_{1}, \ldots,-b_{s+1},-1,-d_{1}, \ldots,-d_{u},
$$

and two disjoint torus-invariant sections with self-intersection numbers $-a_{r+1}$ and $-b_{s+1}$ which intersect the first fibre in the two $(-1)$-curves and the second in the end components labelled $-a_{r}$ and $-d_{u}$ respectively. Note that the sequence $d_{1}, \ldots, d_{u}$ is uniquely determined. Note also that, as above, the equality $m=a_{r+1}+b_{s+1}-2$ ensures that this does define a toric surface (it is obtained as a toric blowup of a $\mathbb{P}^{1}$-bundle). The surface $Y$ has singularities an $A_{m}$ singularity and the cyclic quotient singularities obtained by contracting the chains of smooth rational curves with self-intersection numbers $-a_{r+1}, \ldots,-a_{1},-(t+2),-b_{1}, \ldots,-b_{s+1}$ and $-d_{1}, \ldots,-d_{u},-b_{s+1}$. This last singularity is of type $T_{t+1}$ by Lem. 8.5(2). Hence the singularities of $X$ are $\mathbb{Q}$-Gorenstein deformations of the singularities of $Y$ - the first two singularities are not deformed, and the $T_{t+1^{-}}$ singularity is deformed to an $A_{t}$ singularity. Moreover, this deformation does not change the Picard number. Let $X^{\prime}$ be the general fibre of a 1-parameter deformation of $X$ inducing this deformation of the singularities. We show that $X^{\prime} \simeq X$.

Let $\hat{Y} \rightarrow Y$ and $\hat{X}^{\prime} \rightarrow X^{\prime}$ be the minimal resolutions of the singularities we do not deform. Thus $\hat{Y}$ has a single $T_{t+1}$ singularity given by contracting the chain of smooth rational curves with self-intersection numbers $-d_{1}, \ldots,-d_{u},-b_{s+1}$ on $\tilde{Y}$. Let $C_{1}$ and $C_{2}$ be the images of the ( -1 )-curves
on $\tilde{Y}$ incident to the ends of this chain. Then $C_{1}$ and $C_{2}$ are smooth rational curves meeting in a node at the singular point. We claim that $C=C_{1}+C_{2}$ deforms to a smooth ( -1 )-curve on $\hat{X}^{\prime}$ (not passing through the singular point). First, by Lem. 8.6 we have $C^{2}=-1$. Second, we prove that $C$ deforms. We work on the canonical covering stack $q: \hat{\mathcal{Y}} \rightarrow \hat{Y}$ of $\hat{Y}$. (Here, for a normal $\mathbb{Q}$-Gorenstein surface $Z$, the canonical covering stack is the Deligne-Mumford stack $\mathcal{Z} \rightarrow Z$ with coarse moduli space $Z$ defined by the local canonical coverings of $Z$. That is, if $P \in Z$ is a point of index $n$, and $V \rightarrow U$ is a canonical covering of a neighbourhood $U$ of $P$ with group $G \simeq \mathbb{Z} / n \mathbb{Z}$, then $\left.\mathcal{Z}\right|_{U}$ is isomorphic to $[V / G]$ over $\left.U\right)$. Note that the deformation of $\hat{Y}$ lifts to a deformation of $\hat{\mathcal{Y}}$ (because it is a $\mathbb{Q}$-Gorenstein deformation). Let $\mathcal{C} \rightarrow C$ be the restriction of the covering $\hat{\mathcal{Y}} \rightarrow \hat{Y}$. The closed substack $\mathcal{C} \subset \hat{\mathcal{Y}}$ is a Cartier divisor. Hence the obstruction to deforming $\mathcal{C} \subset \hat{\mathcal{Y}}$ lies in $H^{1}\left(\mathcal{N}_{\mathcal{C}} \hat{\mathcal{Y}}\right)$, where $\mathcal{N}_{\mathcal{C} / \hat{\mathcal{Y}}}$ is the normal bundle $\mathcal{O}_{\hat{\mathcal{Y}}}(\mathcal{C}) \mid \mathcal{C}$. We compute that this obstruction group is zero. Consider the exact sequence

$$
0 \rightarrow \mathcal{N}_{\mathcal{C} / \hat{\mathcal{Y}}} \rightarrow \oplus \mathcal{N}_{\mathcal{C} / \hat{\mathcal{V}}} \mid \mathcal{C}_{i} \rightarrow \mathcal{N}_{\mathcal{C} / \hat{\mathcal{Y}}} \otimes k(Q) \rightarrow 0
$$

where $\mathcal{C}_{i} \rightarrow C_{i}$ are the restrictions of $q$ and $Q \in \hat{\mathcal{Y}}$ is the point over the singular point $P \in \hat{Y}$. Now push forward to the coarse moduli space $\hat{Y}$. (Recall that if $\mathcal{X}$ is a Deligne-Mumford stack and $q: \mathcal{X} \rightarrow X$ is the map to its coarse moduli space, then locally over $X$ the map $q$ is of the form $[U / G] \rightarrow U / G$ where $U$ is a scheme and $G$ is a finite group acting on $U$. A sheaf $\mathcal{F}$ over $[U / G]$ corresponds to a $G$-equivariant sheaf $\mathcal{F}_{U}$ over $U$, and $q_{*} \mathcal{F}=\left(\pi_{*} \mathcal{F}_{U}\right)^{G}$ where $\pi: U \rightarrow U / G$ is the quotient map.) Let $n$ be the index of the singularity $P \in Y$. Then $n>1$ and the $\boldsymbol{\mu}_{n}$ action on $\mathcal{N}_{\mathcal{C} / \hat{\mathcal{Y}}} \otimes k(Q)$ is non-trivial. So $q_{*}\left(\mathcal{N}_{\mathcal{C} / \hat{\mathcal{Y}}} \otimes k(Q)\right)=0$ and $q_{*} \mathcal{N}_{\mathcal{C} / \hat{\mathcal{Y}}}=\oplus q_{*} \mathcal{N}_{\mathcal{C} / \hat{\mathcal{Y}}} \mid \mathcal{C}_{i}$ by the exact sequence above. The sheaf $q_{*} \mathcal{N}_{\mathcal{C} / \hat{\mathcal{Y}}} \mid \mathcal{C}_{i}$ is a line bundle on $C_{i} \simeq \mathbb{P}^{1}$ of degree $\left\lfloor C \cdot C_{i}\right\rfloor$. Let $\alpha: \tilde{Y} \rightarrow \hat{Y}$ denote the minimal resolution of $\hat{Y}$ and $C_{i}^{\prime}$ the strict transform of $C_{i}$ for each $i$. Then

$$
C \cdot C_{i}=\alpha^{*} C \cdot C_{i}^{\prime}>C_{i}^{\prime 2}=-1
$$

Hence $H^{1}\left(q_{*} \mathcal{N}_{\mathcal{C} / \hat{\mathcal{Y}}} \mid \mathcal{C}_{i}\right)=0$. We deduce that $H^{1}\left(\mathcal{N}_{\mathcal{C} / \hat{\mathcal{Y}}}\right)=0$ as required.
We now compute locally that $C$ deforms to a smooth curve that does not pass through the singular point of $\hat{X}^{\prime}$. Locally at the singular point of $\hat{Y}$, the deformation of $\hat{Y}$ is of the form

$$
\left(x y=\left(z^{n}-w\right)^{d}\right) \subset \frac{1}{n}(1,-1, a) \times \mathbb{C}_{w}^{1}
$$

where $d=t+1$. The deformation of $C$ is given by an equation $(z+w \cdot h=0)$, where $h \in k[[x, y, w]]$ has $\boldsymbol{\mu}_{n}$-weight $a$. So, eliminating $z$, the abstract deformation of $C$ is given by $\left(x y=u \cdot w^{d}\right) \subset$ $\frac{1}{n}(1,-1) \times \mathbb{C}_{w}^{1}$, where $u$ is a unit. In particular the general fibre is smooth and misses the singular point of the ambient surface $\hat{X}^{\prime}$.

We deduce that, on $\hat{X}^{\prime}$, we have a cycle of smooth rational curves of self-intersections

$$
-a_{r+1}, \ldots,-a_{1},-(t+2),-b_{1}, \ldots,-b_{s+1},-1,-2, \ldots,-2,-1
$$

(where the chain of $(-2)$-curves has length $m$ ). Indeed the chains $-a_{r+1}, \ldots,-b_{s+1}$ and $-2, \ldots,-2$ are the exceptional loci of the minimal resolutions of two of the singular points of $X^{\prime}$, the first $(-1)$-curve is the deformation of $C$ described above, and the last $(-1)$-curve is the deformation of the $(-1)$-curve on $\hat{Y}$. Moreover $\hat{X}^{\prime}$ has a unique singular point of type $A_{t}$ which does not lie on this cycle. Let $\tilde{X}^{\prime} \rightarrow \hat{X}^{\prime}$ be the minimal resolution. Observe that the chain $-1,-2, \ldots,-2,-1$ defines a ruling of $\tilde{X}^{\prime}$. If $f$ is another degenerate fibre, then $f$ contains a unique ( -1 )-curve and its remaining components are exceptional over $X^{\prime}$ (because $\rho\left(X^{\prime}\right)=1$ ). We deduce that there is exactly one additional degenerate fibre, which is the union of the chain $-a_{r}, \ldots,-b_{s}$, the chain $-2, \ldots,-2$ of length $t$ (the exceptional locus of the minimal resolution of the $A_{t}$ singularity) and a ( -1 )-curve. This determines the fibre uniquely. We conclude that $X^{\prime} \simeq X$.

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A similar argument works when $\tilde{E}_{1} \cdot \tilde{E}_{2}=1$. In this case the ruling $p: \tilde{X} \rightarrow \mathbb{P}^{1}$ has a unique degenerate fibre of type $(O)$ and the two sections $\tilde{E}_{1}$ and $\tilde{E}_{2}$ meet this fibre in the two ( -1 )-curves. Set $\tilde{E}_{1}^{2}=-a$ and $\tilde{E}_{2}^{2}=-b$ and let $m$ be the number of $(-2)$-curves in degenerate fibre. Then $X$ has singularities $A_{m}$ and the cyclic quotient singularity whose minimal resolution has exceptional locus $\tilde{E}_{1}+\tilde{E}_{2}$. (In particular, $(a, b)=(2,2),(3,3)$, or $(2,5)$ because $X$ has $T$-singularities, but we give a uniform treatment of these cases.) We compute that $m=a+b+1$ by expressing $p$ as a blowup of a $\mathbb{P}^{1}$-bundle.

We construct a toric surface $Y$ and prove that $X$ is a $\mathbb{Q}$-Gorenstein deformation of $Y$. The minimal resolution of $\tilde{Y}$ is the toric surface which fibres over $\mathbb{P}^{1}$ with two degenerate fibres, one of type $(O)$ (where there are $m(-2)$-curves as above) and one of type $(I)$ with self-intersection numbers

$$
-b,-1,-2, \ldots,-2
$$

(where the chain of $(-2)$-curves has length $(b-1)$ ) and two disjoint torus-invariant sections with self-intersection numbers $-a$ and $-(b+3)$ which intersect the first fibre in the two $(-1)$-curves and the second in the end components with self-intersection numbers $-b$ and -2 respectively. Note that the equality $m=a+(b+3)-2$ ensures that this does define a toric surface. The surface $Y$ has singularities an $A_{m}$ singularity and the cyclic quotient singularities obtained by contracting the chains of smooth rational curves with self-intersection numbers $-a,-b$ and $-2, \ldots,-2,-(b+3)$. This last singularity is of type $T_{1}$ by Prop. 2.9. Hence the singularities of $X$ are deformations of the singularities of $Y$ - the first two singularities are not deformed, and the $T_{1}$-singularity is smoothed. Moreover, this deformation does not change the Picard number. Let $X^{\prime}$ be the general fibre of a 1-parameter deformation of $X$ inducing this deformation of the singularities. Let $\hat{Y} \rightarrow Y$ and $\hat{X}^{\prime} \rightarrow X^{\prime}$ be the minimal resolutions of the singularities we do not deform. Thus $\hat{Y}$ has a single $T_{1}$ singularity given by contracting the chain of smooth rational curves with self-intersection numbers $-2, \ldots,-2,-(b+3)$ on $\tilde{Y}$. Let $C_{1}$ and $C_{2}$ be the images of the $(-1)$-curves on $\tilde{Y}$ incident to the ends of this chain, so $C_{1}$ and $C_{2}$ are smooth rational curves meeting in a node at the singular point. Then, as above, $C=C_{1}+C_{2}$ deforms to a smooth ( -1 )-curve on $\hat{X}^{\prime}$. We deduce that, on $\hat{X}^{\prime}$, we have a cycle of smooth rational curves of self-intersections

$$
-a,-b,-1,-2, \ldots,-2,-1
$$

(where the chain of $(-2)$-curves has length $m$ ). Indeed, the chains $-a,-b$ and $-2, \ldots,-2$ are the exceptional loci of the minimal resolutions of the two singular points of $X^{\prime}$, the first $(-1)$-curve is the deformation of $C$, and the last $(-1)$-curve is the deformation of the $(-1)$-curve on $\hat{Y}$. Let $\tilde{X}^{\prime} \rightarrow \hat{X}^{\prime}$ be the minimal resolution. Observe that the chain $-1,-2, \ldots,-2,-1$ defines a ruling of $\tilde{X}^{\prime}$. There are no other degenerate fibres of this ruling because $\rho\left(X^{\prime}\right)=1$. We deduce that $X^{\prime} \simeq X$.

If we are in case (2) of Thm. 7.6, then the non Du Val singularities of $X$ are of type $\frac{1}{4}(1,1)$. In particular, $2 K_{X}$ is Cartier. Similarly, if $X$ has a $D$ or $E$ singularity then $2 K_{X}$ is Cartier by Thm. 5.1. So in these cases we can refer to the classification of log del Pezzo surfaces of Picard rank 1 and index $\leqslant 2$ given by Alexeev and Nikulin [AN06, Thms. 4.2,4.3]. By Prop. 6.1 the only such surfaces which are not $\mathbb{Q}$-Gorenstein deformations of toric surfaces are those which have either a $D$ singularity, an $E$ singularity, or at least 4 Du Val singularities. These are the sporadic surfaces listed in Ex. 8.3. This completes the proof.

Lemma 8.5. Let $\left[a_{1}, \ldots, a_{r}\right]$ and $\left[b_{1}, \ldots, b_{s}\right]$ be conjugate strings.
(1) The conjugate of $\left[a_{r}, \ldots, a_{1}, t+2, b_{1}, \ldots, b_{s}\right]$ is a $T_{t+1}$-string.
(2) Given $b_{s+1} \geqslant 2$, let $\left[d_{1}, \ldots, d_{u}\right]$ be the conjugate of $\left[a_{r}, \ldots, a_{1}, t+2, b_{1}, \ldots, b_{s}, b_{s+1}\right]$. Then $\left[d_{1}, \ldots, d_{u}, b_{s+1}\right]$ is a $T_{t+1}$-string.

Proof. Let an $S_{t}$-string be a string $\left[a_{r}, \ldots, a_{1}, t+2, b_{1}, \ldots, b_{s}\right]$ as above. Then, by Lem. 7.3, we have
(a) $[2, t+2,2]$ is an $S_{t}$-string.
(b) If $\left[e_{1}, \ldots, e_{v}\right]$ is an $S_{t}$-string, then so are $\left[e_{1}+1, \cdots, e_{v}, 2\right]$ and $\left[2, e_{1}, \ldots, e_{v}+1\right]$.
(c) Every $S_{t}$-string is obtained from the example in (a) by iterating the steps in (b).

Now (1) follows from Prop. 2.9 and Lem. 7.3. To deduce (2), let $\left[e_{1}, \ldots, e_{v}\right]$ be the conjugate of $\left[a_{r}, \ldots, a_{1}, t+2, b_{1}, \ldots, b_{s}\right]$. Then

$$
\left[d_{1}, \ldots, d_{u}, b_{s+1}\right]=\left[2, \ldots, 2, e_{1}+1, e_{2}, \ldots, e_{v}, b_{s+1}\right]
$$

(where there are $\left(b_{s+1}-2\right) 2$ 's) by Lem. 7.3. This string is of type $T_{t+1}$ by (1) and Prop. 2.9.
Lemma 8.6. Let $(P \in S, D)$ denote the local pair $\left(\frac{1}{d n^{2}}(1, d n a-1),(u v=0)\right)$. Let $\pi: \tilde{S} \rightarrow S$ be the minimal resolution of $S$ and $D^{\prime}$ the strict transform of $D$. Write $\pi^{*} D=D^{\prime}+F$ where $F$ is $\pi$-exceptional. Then $F^{2}=-1$.

Proof. We may assume $S$ is a projective toric surface, $P \in S$ is the unique singular point, and $D$ is the toric boundary. Then $\tilde{S}$ is toric with boundary $\tilde{D}:=D^{\prime}+\sum E_{i}$, where $E_{1}, \ldots, E_{r}$ are the exceptional divisors of $\pi$. In particular $D \in\left|-K_{S}\right|$ and $\tilde{D} \in\left|-K_{\tilde{S}}\right|$. Since $P \in S$ is a $T_{d}$-singularity, by Prop. 2.6 we have

$$
K_{\tilde{S}}^{2}+\rho(\tilde{S})=K_{S}^{2}+\rho(S)+(d-1)
$$

So $\tilde{D}^{2}+r=D^{2}+(d-1)$. Now $\tilde{D}^{2}=D^{\prime 2}+\sum E_{i}^{2}+2(r+1)$, so

$$
F^{2}=D^{\prime 2}-D^{2}=d-3 r-3-\sum E_{i}^{2}
$$

Finally, $\sum E_{i}^{2}=d-3 r-2$ by the inductive description of resolutions of $T_{d}$-singularities (see Prop. 2.9), so $F^{2}=-1$ as claimed.

Proof of Thm. 1.3. Let $X$ denote the special fibre of $f: V \rightarrow T$. Thus $X$ is a del Pezzo surface with quotient singularities which admits a $\mathbb{Q}$-Gorenstein smoothing. Since $H^{1}\left(\mathcal{O}_{X}\right)=H^{2}\left(\mathcal{O}_{X}\right)=0$ the restriction map Pic $V \rightarrow \operatorname{Pic} X$ is an isomorphism. Hence $\rho(X)=\rho(V / T)=1$.

By Thm. 7.1 there exists a (reduced) curve $D \in\left|-K_{X}\right|$ with only nodal singularities. We have $H^{1}\left(-K_{X}\right)=0$ by Kawamata-Viehweg vanishing. So there exists a lift $S \in\left|-K_{V}\right|$ of $D \in\left|-K_{X}\right|$ such that the general fibre of $S / T$ is smooth. The surface $S$ is normal, so the special fibre of $S / T$ equals $D$ (there are no embedded points). Hence $S / T$ is a smoothing of a nodal curve, and $S$ has only Du Val singularities of type $A$.

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