Smoothing properties, decay and global existence of solutions to nonlinear coupled systems of thermoelastic type

Jaime E. Muñoz Rivera* Reinhard Racke†

Abstract

We consider nonlinear coupled system of evolution equations, the simplest of which models a thermoelastic plate. Smoothing and decay properties of solutions are investigated as well as the local well-posedness and the global existence of solution. For the system of standard thermoelasticity it is proved that there is no similar smoothing effect.

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1 Introduction

In this paper we mainly consider regularizing properties of systems that are regarded as models for thermoelastic plate equations. We will then show that the vertical deflection of the plate as well as the temperature are arbitrarily smooth for positive times, no matter which regularity the initial vertical deflection and the initial temperature have. This fact we will show in Section 3. This property is not valid for other thermoelastic models, as the thermoelastic bar for example, as we shall see in Section 4. More generally, we consider a nonlinear coupled thermoelastic plate, modelled in a separable Hilbert space $\mathcal{H}$ by

$$u_{tt} + M([u,\theta])A^2u + N([u,\theta])(A+\mu)\theta = 0, \tag{1.1}$$

$$\theta_t + R([u,\theta])(A+\alpha)\theta - Q([u,\theta])(A+\mu)u_t = 0. \tag{1.2}$$

Here $M,\ N,\ R,\ Q : I\!R^5 \longrightarrow I\!R$ are C^2-functions, and M, R and NQ are strictly positive, α, $\mu \in I\!R$. Finally by $[u,\theta]$ we are denoting the following vector field

$$[u,\theta](t) := (||u_t||^2, ||A^{\frac{1}{2}}u||^2, ||A^{\frac{1}{2}}u_t||^2, ||Au||^2, ||A^{\frac{1}{2}}\theta||^2)(t),$$

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where $||\cdot||$ denotes the norm in $\mathcal{H}$, $A : D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is a nonnegative, self-adjoint operator. The solution (u,θ) will satisfy the following initial conditions

$$u(t=0) = u_0, \quad u_t(t=0) = u_1, \quad \theta(t=0) = \theta_0, \tag{1.3}$$

and the abstract "boundary" conditions

$$u(t) \in D(A^2), \quad \theta(t) \in D(A), \quad t \geq 0. \tag{1.4}$$

A simple example is the following system modelling a thermoelastic plate in the linearized version,

$$u_{tt} + \Delta^2 u + \beta\Delta\theta = 0 \quad \text{in} \quad [0,\infty[\times\Omega, \tag{1.5}$$

$$\theta_t - \Delta\theta - \beta\Delta u_t = 0 \quad \text{in} \quad [0,\infty[\times\Omega, \tag{1.6}$$

where $\beta \neq 0$. The boundary $\partial\Omega$ of the open set Ω is assumed to be smooth, u and θ shall satisfy

$$u = \Delta u = 0 \quad \text{on} \quad \partial\Omega, \quad \theta = 0 \quad \text{on} \quad \partial\Omega. \tag{1.7}$$

Kim [6] studied the equations (1.5),(1.6) in a bounded domain with the boundary condition (1.7) for u replaced by $u \in H_0^2(\Omega)$, showing exponential decay of the couple (u,θ).

We are first interested in proving smoothing properties, i. e. the solution (u,θ) is arbitrarily smooth for $t > 0$, no matter which regularity the initial data have. Smoothness for the abstract system (1.1)–(1.4) means that the solution $(u(t),\theta(t))$ belongs to $D(A^m)\times D(A^m)$ for any $m \in \mathbb{N}$ and any $t > 0$. Then we shall investigate the rate of decay for the couple (u,θ) as $t \to +\infty$, depending on A, and in case (1.5)–(1.7) naturally depending on the domain Ω. Finally we show the global existence of solutions (u,θ) if A is strictly positive. These results describe the system (1.1)–(1.2) as parabolic, the similarities to solutions of heat equations will be obvious. In contrast to this we study the system of standard thermoelasticity, cf. [15], [16], which reads as follows in the simplest one-dimensional case:

$$u_{tt} - \tau u_{xx} + \gamma\theta_x = 0, \tag{1.8}$$

$$\theta_t - \kappa\theta_{xx} + \gamma u_{xt} = 0, \tag{1.9}$$

$$u(t=0) = u_0, \quad u_t(t=0) = u_1, \quad \theta(t=0) = \theta_0, \tag{1.10}$$

$$u = \theta_x = 0 \quad on \quad \partial\Omega, \tag{1.11}$$

where, $\Omega =]0,1[$, or $\Omega =]0,+\infty[$, or $\Omega =]-\infty,+\infty[$; (u,θ) is a function of $t \geq 0$ and $x \in \Omega$ and τ, $|\gamma|$, κ are positive constants. It is known, cf. [16], that solutions behave like solutions to the heat equation with respect to the decay behavior; but it is not true for $n-$dimensional thermoelastic systems, if $n \geq 2$. It is well known by now, that in this case for the whole space

$I\!R^n$ the displacement vector field can be decomposed in two parts: the solenoidal part which satisfies the wave equation, and the irrotational part which is a gradient (see [11]). Clearly the solenoidal part propagates singularities. We shall prove that the smoothing property does not hold even for the irrotational part, moreover that it behaves like a wave equation which propagates singularities.

For the formulation of the precise result we introduce the following notation: Ω will denote a domain in $I\!R^n$,

$$H^m(\Omega) = W^{m,2}(\Omega), \quad H_0^m(\Omega) = W_0^{m,2}(\Omega); \quad m \in I\!N,$$

the usual Sobolev spaces based on $L^2(\Omega)$ cf. [1]; ∇: gradient, $< \cdot, \cdot >$: inner product in $L^2(\Omega)$ or in a general separable Hilbert space $\mathcal{H}$; $|\cdot|$ the norm in $L^2(\Omega)$; $C^k(I, E)$, $k \in I\!N$: space of k-times continuously differentiable functions from $I \subset I\!R$ into a Banach space E, analogously: $L^p(I, E), \quad 1 \le p \le \infty$.

The smoothing properties for the systems (1.1),(1.2) and (1.5),(1.6), respectively, is expressed in Theorem 3.1. The local existence of solutions is subject of Theorem 2.4. To describe the decay, we consider the linearized version of (1.1),(1.2) assuming $\alpha = \mu = 0$ (only for simplicity, in the general case α is nonnegative and μ is such that the product $\mu q(t)$ is positive), i.e.

$$u_{tt} + m(t)A^2u + n(t)A\theta = 0, \tag{1.12}$$

$$\theta_t + r(t)A\theta - q(t)Au_t = 0, \tag{1.13}$$

assuming that m, n, r, q are C^1-function of t, satisfying

$$m_0 \le m(t) \le m_1; \quad n_0 \le |n(t)| \le n_1; \quad q_0 \le |q(t)| \le q_1; \quad r_0 \le r(t) \le r_1.$$

with $m_0, \ldots, r_1$ being positive real numbers, $n(t)q(t) > 0 \ \forall t > 0$ and with similar bounds for the derivatives of them m', n', r', q'.

Exponential decay is obtained for coercive operators A (see Theorem 3.5) which implies in particular the decay of solutions for the thermoelastic plates given by (1.5)–(1.7) when Ω is a bounded domain. If the spectrum of A approaches zero, one needs more information on A than given in the general setting. We present in Theorem 3.6 a typical result for the thermoelastic plate equation (1.5),(1.6) if Ω is the whole space $I\!R^n$ or if Ω is an exterior domain, that is L^2- and L^∞- decay rates. By interpolation one also gets decay rates in $L^q(\Omega)$, $2 < q < \infty$.

For the case $A \ge \nu > 0$, $\alpha = \mu = 0$ we shall extend our local existence result to a global existence result (see Theorem 2.8)

We remark that right-hand sides in (1.1) and (1.2), respectively, with appropriate regularity (for Theorem 2.4) and smallness (for Theorem 2.8), easily can be included.

Using Theorem 3.6 it would also be possible to prove a global existence result for (1.5)–(1.7),(1.3) for small data in exterior domains including the whole space $I\!R^n$, cf. [17], we do not go into details here. Finally we turn to the system (1.8)–(1.11) in standard thermoelasticity and related systems as (1.8)–(1.10) with the boundary conditions

$$u_x = \theta = 0 \quad \text{on} \quad \partial\Omega, \tag{1.14}$$

or systems in higher dimensions of the following type

$$u_{tt} - \tau\Delta u + \gamma\nabla\theta = 0, \tag{1.15}$$

$$\theta_t - \kappa\Delta\theta + \gamma div\, u_t = 0, \tag{1.16}$$

$$u(t) \in \overline{\nabla H_0^1(\Omega)} \quad \forall t \geq 0. \tag{1.17}$$

for $\Omega = I\!R^2$ or $\Omega = I\!R^3$. For domains in $I\!R^3$ with smooth boundary we will consider boundary conditions of the form

$$div\, u = \theta = 0 \quad \text{on} \quad \partial\Omega. \tag{1.18}$$

(The initial condition according to (1.10) has to be satisfied in each case.)

The systems (1.8)–(1.10), (1.4) and (1.8)–(1.10), (1.14), respectively, describe the initial boundary value problem for a one-dimensional thermoelastic rod with rigidly clamped and thermally insulated boundary in case of (1.11), and with traction free boundary at constant temperature in case of (1.14). The system (1.15)–(1.17) describes the dissipative part of the solution to the Cauchy problem in $I\!R^2$ or $I\!R^3$, cf. [12], [15]. The system (1.15)–(1.18) are equations for the dissipative part of a thermoelastic problem in $I\!R^3$ with the specific boundary condition given above, cf. [13].

In each case the solution (u, θ) has the same decay rates as solution to the heat equation, see [3], [5], [10], [11], [12], [13], [15], [18], [19]; for a survey cf. [16]. In contrast to this we shall prove that they do not have the same smoothing property: singularities in the initial data are propagated as time increases. This shows that the coupling for thermoelastic plates is much stronger than that in standard thermoelasticity. All problems above can be considered simultaneously, namely, for (1.8),(1.9) or (1.15), (1.16), with boundary conditions given by (1.17), (1.18), it is easy to see that $v := u$ satisfies

$$v_{ttt} - \kappa\Delta v_{tt} - (\gamma^2 + \tau)\Delta v_t + \kappa\tau\Delta^2 v = 0, \tag{1.19}$$

$$v(t=0) = v_0 := u_0, \quad v_t(t=0) = v_1 := u_1, \quad v_{tt}(t=0) = v_2 := \tau\Delta u_0 - \gamma\nabla\theta_0, \tag{1.20}$$

as well as

$$v = \Delta v = 0 \quad \text{on} \quad \partial\Omega \tag{1.21}$$

in case (1.11), and

$$v_x = v_{xxx} = 0 \quad \text{on} \quad \partial\Omega \tag{1.22}$$

in case (1.14). For (1.10), (1.15)–(1.18) $v := div\, u$ satisfies (1.19) too, as well as (1.20) with

$$v_0 = div\, u_0, \quad v_1 := div\, u_1, \quad v_2 := \tau\Delta div\, u_0 - \gamma\Delta\theta_0,$$

and the boundary condition (1.21). We remark that θ satisfies a similar differential equation as v with appropriate boundary conditions. Denoting by A the *vector*-Laplace operator with domain

$$D(A) := H^2(\Omega) \cap H_0^1(\Omega), \quad Av := -\Delta v,$$

respectively

$$D(A) := \Big\{ v \in H^2(\Omega); \forall\varphi \in H^1(\Omega); <\nabla v, \nabla\varphi> = - < v, \Delta\varphi > \Big\}; \quad Av := -\Delta v,$$

we see that v satisfies

$$v_{ttt} + \kappa A v_{tt} + (\gamma^2 + \tau) A v_t + \kappa\tau A^2 v = 0, \tag{1.23}$$

$$v(t=0) = v_0, \quad v_t(t=0) = v_1, \quad v_{tt}(t=0) = v_2, \tag{1.24}$$

$$v(t) \in D(A^2), \quad t \geq 0. \tag{1.25}$$

(1.23)–(1.25) will be considered in a separable Hilbert space $\mathcal{H}$ again, $v : [0,\infty[\to \mathcal{H}$ and our result on propagation of singularities will be proved in Theorem 4.1.

2 Existence results

First we study the linearized problem,

$$u_{tt} + m(t) A^2 u + n(t)(A + \mu)\theta = f_1, \tag{2.1}$$

$$\theta_t + r(t)(A + \alpha)\theta - q(t)(A + \mu)u_t = f_2, \tag{2.2}$$

$$u(t=0) = u_0,\ u_t(t=0) = u_1,\ \theta(t=0) = \theta_0, \tag{2.3}$$

(with more general right-hand sides), and we look for solutions (u, θ), satisfying

$$u \in C^2([0,\infty[, \mathcal{H}) \cap C^1([0,\infty[, D(A)) \cap C^0([0,\infty[, D(A^2)), \tag{2.4}$$

$$\theta \in C^1([0,\infty[, \mathcal{H}) \cap C^0([0,\infty[, D(A)). \tag{2.5}$$

Rewriting (2.1)–(2.3) as a first-order system for

$$V := \begin{pmatrix} u_t \\ Au \\ \theta \end{pmatrix}, \quad V^0 := \begin{pmatrix} u_1 \\ Au_0 \\ \theta_0, \end{pmatrix}$$

we consider

$$V_t + \underbrace{\begin{pmatrix} 0 & m(t)A & n(t)(A+\mu) \\ -A & 0 & 0 \\ -q(t)(A+\mu) & 0 & r(t)(A+\alpha) \end{pmatrix}}_{=:B(t)} V(t) = \underbrace{\begin{pmatrix} f_1 \\ 0 \\ f_2 \end{pmatrix}}_{=:F(t)}, \tag{2.6}$$

$$V(t=0) = V^0. \tag{2.7}$$

The coefficients $m,\ n,\ q,\ r$ are considered to be C^1-functions. For $V = (V^1, V^2, V^3)$, $W = (W^1, W^2, W^3)$, $t \geq 0$, the Hilbert space $\mathcal{H}_t$ is defined by the inner product

$$< V, W >_t := < V^1, W^1 >_{\mathcal{H}} + m(t) < V^2, W^2 >_{\mathcal{H}} + \frac{n(t)}{q(t)} < V^3, W^3 >_{\mathcal{H}} .$$

Observe that the corresponding norm $||\cdot||_t$ is equivalent to the norm $||\cdot||$ in $\mathcal{H}$. Defining the operator $B(t)$ by $D(B(t)) := D(A)$ (in each component), it is not difficult to see that $-B(t)$ generates a C_0-semigroup with constants $M = 1,\ \hat{\beta} = \max\{0, -\alpha\}$ and hence $\{B(t)\}_{t\geq 0}$ is a stable family of negative generators in $\mathcal{X} = H$ with stability constants $(M, \beta),\ \beta$ depending on $m',\ n',\ q'$. With $\mathcal{Y} := D(A)$ we see that for $t,\ s \geq 0,\ v \in \mathcal{Y}$.

$$||B(t)v - B(s)v||_{\mathcal{X}} \leq |t-s|\, ||v||_{\mathcal{Y}}$$

Therefore, $(\{B(t)\}_{t\geq 0}, \mathcal{X}, \mathcal{Y})$ are a CD-system in the terminology of Kato [4]. As a consequence we have

Lemma 2.1 *For $V^0 \in \mathcal{Y},\ F : [0,T] \to \mathcal{X}$ Lipschitz continuous, there is a unique solution $V \in C^1([0,T], \mathcal{X}) \cap C^0([0,T], \mathcal{Y})$ of (2.6), (2.7).*

Corollary 2.2 *For $u_0 \in D(A^2),\ u_1 \in D(A),\ \theta_0 \in D(A),\ f_1, f_2 : [0,T] \to \mathcal{X}$ Lipschitz continuous, there is a unique solution (u, θ) of (2.1)–(2.3) satisfying (2.4), (2.5).*

The higher regularity for more regular data is given by the following Lemma, where we assume that $f_1 = 0, f_2 = 0$ for simplicity.

Lemma 2.3 *Let $k \geq 2,\ u_0 \in D(A^k),\ u_1 \in D(A^{k-1}),\ \theta_0 \in D(A^{k-1}), m,\ n,\ r,\ q, \in\ C^{k-1}$. Then there is a unique solution (u, θ) solution to (2.1)–(2.3) satisfying*

$$u \in \cap_{j=0}^{k} C^j([0,\infty[, D(A^{k-j})),\ \theta \in \cap_{j=0}^{1} C^j([0,\infty[, D(A^{k-j-1}))$$

Proof of Lemma 2.3. $\exists \lambda \geq 0,\ (A+\lambda)^{-1} : \mathcal{H} \to \mathcal{H}$ is bounded, let $w := (A+\lambda)u,\ \psi := (A+\lambda)\theta$. Formally we obtain, assuming $\lambda = 0$ for simplicity,

$$w_{tt} + m(t)A^2 w + n(t)(A+\mu)\psi = 0 \tag{2.8}$$

$$\psi_t + r(t)(A+\alpha)\psi - q(t)(A+\mu)w_t = 0, \tag{2.9}$$

$$w(t=0) = Au_0,\ w_t(t=0) = Au_1,\ \psi(t=0) = A\theta_0. \tag{2.10}$$

Observe that $||A^{-1}w|| \le c||w||$ holds. Assuming $u_0 \in D(A^3),\ u_1 \in D(A^2), \theta_0 \in D(A^2)$ we can solve (2.8)-(2.10) with Lemma 2.2. Then $\hat{u} =: A^{-1}w, \hat{\theta} =: A^{-1}\psi$ solves (2.1)-(2.3), hence is equal to u and θ respectively. We conclude from the regularity of (w,ψ) :

$$u \in \cap_{j=0}^{3} C^j([0,\infty[, D(A^{3-j}),\quad \theta \in \cap_{j=0}^{2} C^j([0,\infty[, D(A^{2-j}))$$

where we used the differential equations, and we needed $m,\ n,\ q,\ r$ to be C^1-functions. The case $k \ge 4$ is obtained taking $w(t=0) = A^k u_0,\ w_t(t=0) = A^k u_1,\ \psi(t=0) = A^k\theta_0$.

Q.E.D.

Our local existence result is summarized in the following theorem.

Theorem 2.4 *Let $k \ge 3$, let $M,\ N,\ Q,\ R\ \in C^{k-1}(\mathbb{R}^5, \mathbb{R})$, with M, R and the product NQ being positive functions, let*

$$(u_0, u_1, \theta_0) \in D(A^k) \times D(A^{k-1}) \times D(A^{k-1}).$$

Then there exists a unique solution (u,θ) to (1.1)–(1.4) satisfying

$$(u,\theta) \in \cap_{j=0}^{k} C^j([0,T], D(A^{k-j})) \times \cap_{j=0}^{k-1} C^j([0,T], D(A^{k-1-j}))$$

for some $T > 0$. T depends only on the initial data, $T = T(\rho)$, where

$$\rho := (||u_0||_{D(A^2)}, ||u_1||_{D(A)}, ||\theta_0||_{D(A)}),$$

and $T \to \infty$ as $\rho \to 0$.

In order to prove Theorem 2.4 we shall use a fixed point argument in appropriate spaces. Let $u_0 \in D(A^3),\ u_1 \in D(A^2),\ \theta_0 \in D(A^2)$. For $N_1 > 0,\ N_2 > 0,\ T > 0$ let

$$\mathcal{X}(N_1, N_2, T) =:$$

$$\left\{(u,\theta) : [0,T] \to \mathcal{H}; \partial_t^j u \in L^\infty([0,T]; D(A^{3-j})), j = 1,2,3.\ \partial_t^k \theta \in L^\infty([0,T]; D(A^{2-k}));\ k = 0,1,2\right\}$$

intersected with the set of couples (u,θ) satisfying

$$u(t=0) = u_0,\ u_t(t=0) = u_1,\ \theta(t=0) = \theta_0,$$

$$\sup_{0\le t\le T} \left\{ \sum_{j=0}^{2} ||\partial_t^j u(t)||^2_{D(A^{2-j})} + \sum_{k=0}^{1} ||\partial_t^k \theta(t)||^2_{D(A^{1-k})} \right\} \le N_1^2,$$

$$\sup_{0\le t\le T} \left\{ \sum_{j=0}^{3} ||\partial_t^j u(t)||^2_{D(A^{3-j})} + \sum_{k=0}^{2} ||\partial_t^k \theta(t)||^2_{D(A^{2-k})} \right\} \le N_2^2.$$

We observe that $\mathcal{X}(N_1, N_2, T) \ne \emptyset$ if $N_j = N_j(||u_0||_{D(A^{1+j})}, ||u_1||_{D(A^j)}, ||\theta_0||_{D(A^j)})\ j = 1,2$, is large enough.

Lemma 2.5 *$\mathcal{X}(N_1, N_2, T)$ is a closed subspace of the complete metric space $\mathcal{Z}$ defined by*

$$\mathcal{Z} := \left\{(u,\theta) : [0,T] \to \mathcal{H}; u_t,\ \theta \in L^\infty([0,T], D(A^{\frac{1}{2}})),\ u \in L^\infty([0,T], D(A))\right\}$$

and the metric

$$d((u,\theta),(v,\eta)) := ||(u_t - v_t, A^{\frac{1}{2}}(u_t - v_t), u - v, Au - Av, \theta - \eta, A^{\frac{1}{2}}\theta - A^{\frac{1}{2}}\eta)||_{L^\infty([0,T],\mathcal{H})}.$$

The standard proof (cf. [18]) of Lemma 2.5 exploits the weak–$*$ compactness of bounded sets in $L^\infty([0,T],\mathcal{H})$ (observe that $\mathcal{H}$ is assumed to be separable).

A mapping $\mathcal{S} : \mathcal{X}(N_1, N_2, T) \subset \mathcal{Z} \to \mathcal{Z}$ is defined by $\mathcal{S}(u,\theta) := (\hat{u}, \hat{\theta}) :=$ solution to

$$\hat{u}_{tt} + M([u,\theta])A^2\hat{u} + N([u,\theta])(A+\mu)\hat{\theta} = 0,$$

$$\hat{\theta}_t + R([u,\theta])(A+\alpha)\hat{\theta} - Q([u,\theta])(A+\mu)\hat{u}_t = 0,$$

$$\hat{u}(t=0) = u_0,\ \hat{u}_t(t=0) = u_1,\ \hat{\theta}(t=0) = \theta_0,$$

which exists according to Lemma 2.3. Observe that $\frac{d}{dt}M([u,\theta](t))$ is bounded since $(u,\theta) \in \mathcal{X}(N_1, N_2, T)$.

Lemma 2.6 *The mapping $\mathcal{S}$ defined above maps $\mathcal{X}$ into itself if T is sufficiently small, depending on N_1.*

Proof of Lemma 2.6. Let us denote by

$$\hat{V} = \begin{pmatrix} \hat{u}_t \\ A\hat{u} \\ \hat{\theta} \end{pmatrix},\ \hat{V}_0 = \begin{pmatrix} \hat{u}_1 \\ A\hat{u}_0 \\ \hat{\theta}_0 \end{pmatrix}.$$

Then $\hat{V}$ satisfies

$$\hat{V}_t(t) + \hat{B}(t)\hat{V}(t) = 0,,\quad \hat{V}(t=0) = V_0,$$

where $\hat{B}$ equals the previously defined $B(t)$ with $m(t) := M([u,\theta](t))$ and so on.

$$||\hat{V}(t)|| \le Me^{\beta t}||\hat{V}_0||$$

with $\beta \le cN_1^2$, cf. [4], since β depends on $m',\ n'\ q'$ essentially, and

$$||A\hat{V}(t)|| + ||\hat{V}_t(t)|| \le \hat{M}e^{\hat{\beta}t}||\hat{V}_0||_{D(A)}$$

with $\hat{\beta} \le cN_1^2$. $\hat{W} := A\hat{V}$ satisfies

$$\hat{W}_t(t) + \hat{B}(t)\hat{W}(t) = 0,\quad \hat{W}(t=0) = A\hat{V}^0$$

and hence we obtain also

$$||A\hat{V}_t(t)|| + ||A^2\hat{V}(t)|| \leq \hat{M}e^{\hat{\beta}t}||A\hat{V}^0||_{D(A)}.$$

This implies $(u,\theta) \in \mathcal{X}(N_1, N_2, T)$ if

$$N_1^2 \geq 4||\hat{V}^0||^2_{D(A)}, \quad N_2^2 \geq 4||\hat{V}^0||^2_{D(A)}, \quad Me^{\beta T} \leq 2$$

which is true if

$$T \leq \frac{1}{2cN_1^2}\log(\frac{2}{N_1}). \tag{2.11}$$

Q.E.D.

Lemma 2.7 *The mapping $\mathcal{S}$ defined above is a contraction mapping if T is sufficiently small, depending on N_1.*

Proof of Lemma 2.7. Let $(\hat{u}^j, \hat{\theta}^j) := \mathcal{S}(u^j, \theta^j)$ $j = 1, 2$, and let (2.11) be satisfied. Then $w := \hat{u}^1 - \hat{u}^2$, $\psi := \hat{\theta}^1 - \hat{\theta}^2$ satisfy

$$w_{tt} + m(t)A^2 w + n(t)(A+\mu)\psi = \hat{m}(t)A^2\hat{u}^2 + \hat{n}(t)(A+\mu)\hat{\theta}^2,$$

$$\psi_t + r(t)(A+\mu)\psi - q(t)(A+\mu)w_t = \hat{r}(t)(A+\mu)\hat{\theta}^2 - \hat{q}(t)(A+\mu)\hat{u}^2_t,$$

$$w(t=0) = 0, \ w_t(t=0) = 0, \psi(t=0) = 0,$$

where $m := M([u^1, \theta^1])$, $\hat{m} := M([\hat{u}^1, \hat{\theta}^1]) - M([\hat{u}^2, \hat{\theta}^2])$, and so on. Let (cf. (3.9))$K_3 = K_3(w, \psi)$ be given by

$$K_3 := \frac{1}{2}\Big\{||w_t||^2 + m||Aw||^2 + \frac{n}{q}||\psi||^2 - \frac{\varepsilon q_0}{4} < w_t, Aw > -\varepsilon < \psi, w_t > \tag{2.12}$$

$$+||Aw_t||^2 + m(t)||A^2 w||^2 + \frac{n}{q}||A\psi||^2 - \frac{\varepsilon q_0}{4} < Aw_t, A^2 w > -\varepsilon < A\psi, Aw_t > \Big\}.$$

Using the multiplicative techniques we obtain for sufficiently small ε

$$\frac{d}{dt}K_3(t) \leq c_1(N_1^2)K_3(t) + c_2 N_1^2\left\{|\hat{m}(t)|^2 + |\hat{n}(t)|^2 + |\hat{r}(t)|^2 + |\hat{q}(t)|^2\right\}$$

$$\leq c_1(N_1^2)K_3(t) + c_3(N_1^2)d^2((u^1, \theta^1), (u^2, \theta^2)),$$

where $c_j \geq 0, \ j = 1, 2, 3$. This implies

$$\sup_{0 \leq t \leq T} K_3(t) \leq c_4(N_1^2)N_1^2 T d^2((u^1, \theta^1), (u^2, \theta^2))$$

if ε is small enough. We obtain

$$d^2((\hat{u}^1, \hat{\theta}^1), (\hat{u}^2, \hat{\theta}^2)) \leq c_5(N_1^2)N_1^2 T d^2((u^1, \theta^1), (u^2, \theta^2))$$

$$\leq \sigma^2 d^2((u^1, \theta^1), (u^2, \theta^2))$$

with $0 < \sigma < 1$ if $T = T(N_1)$ is small enough.

Q.E.D.

The unique fixed point of $\mathcal{S}$ in $\mathcal{X}(N_1, N_2, T)$ is the desired solution (u, θ) in Theorem 2.4 for $k = 3$. The case $k \geq 4$ can either be dealt with stuying a corresponding $\mathcal{X}(N_1, N_2, ..., N_{k-1}, T)$ or inclusively proving the higher regularity of (u, θ) by introducing $w := A^{-1}u$, $\psi := A^{-1}\theta$ as in the proof of Lemma 3.4. (Observe that the new nonlinearities look like $\hat{M}([w, \psi]) = M([A^{-1}w, A^{-1}\psi)])$ and are easier to deal with since they are of lower order. This completes the proof of Theorem 2.4.

Q.E.D.

Theorem 2.8 *Let $A \geq \nu > 0$, $\alpha = \mu = 0$ and $k \geq 3$. Then there is $\delta > 0$ with the following property: For any*

$$(u_0, u_1, \theta_0) \in D(A^k) \times D(A^{k-1}) \times D(A^{k-1})$$

satisfying

$$||u_0||_{D(A^2)} + ||u_1||_{D(A)} + ||\theta_0||_{D(A)} < \delta,$$

there exists a unique global solution (u, θ) of (1.1)–(1.4) satisfying

$$(u, \theta) \in \cap_{j=0}^{k} C^j([0, \infty[, D(A^{k-j})) \times \cap_{j=0}^{k-1} C^j([0, \infty[, D(A^{k-1-j}))$$

Moreover, (u, θ) decays exponentially.

Proof.- Let (u, θ) be a local solution according to Theorem 2.4. Under the assumption of Theorem 2.8 we obtain for

$$K_4 := K_3(u, \theta),$$

K_3 having been defined in (2.12),

$$\frac{d}{dt} K_4(t) \leq c_1(N_1^2) K_4^2(t) - d_1(N_1^2) K_4(t)$$

with $c_1,\ d_1 > 0$ depending on N_1^2. If $N_1^2 \leq 1$, we have

$$c_1(N_1^2) \leq c_0,\ d_1(N_1^2) \geq d_0 > 0$$

with c_0, d_0 being independent of N_1 and also of t. Then

$$\frac{d}{dt} K_4(t) \leq c_0 K_4^2(t) - d_0 K_4(t).$$

this implies by a standard arguments, using Gronwall's inequality, that if $K_4(0)$ is sufficiently small, then

$$K_4(s) \leq e^{-\frac{d_0}{2}s} K_4(0) \tag{2.13}$$

holds on some interval $0 \le s \le t_1 > 0$. This yields an a priori estimate in $s = t_1$ and by a continuation argument the solution exists globally and satisfies (2.13) for all $s \in I\!R$. The smallness of $K_4(0)$ is guaranteed by choosing $||u_0||_{D(A^2)} + ||u_1||_{D(A)} + ||\theta_0||_{D(A)}$ small enough.

Q.E.D.

3 Smoothing effect

The main result of this section is given by

Theorem 3.1 *Let* $(u, \theta) \in \cap_{j=0}^{2} C^j([0,T], D(A^{2-j})) \times \cap_{j=0}^{1} C^j([0,T], D(A^{1-j}))$ *be a solution of (1.1)–(1.4) for some* $T > 0$ *with* $(u_0, u_1, \theta_0) \in D(A^2) \times D(A) \times D(A)$. *Then for any* $t \in]0,T]$ *and for all* $m \in I\!N$ *we have that:* $(u(t), \theta(t)) \in D(A^m) \times D(A^m)$.

Proof.- Let us denote by $m(t) := M([u,\theta](t))$, $n(t) := N([u,\theta](t))$, $r(t) := R([u,\theta](t))$, $q(t) := Q([u,\theta](t))$. Then $m,\ n,\ r,\ q\ \in C^1([0,T])$ since

$$|[u,\theta]| \le c\left\{||u_{tt}||^2 + ||u_t||^2 + ||Au_t||^2 + ||Au||^2 + ||\theta||^2 + ||\theta_t||^2 + ||A\theta||^2\right\}.$$

By the spectral theorem for self-adjoint operators (cf. [2], [8]) there exists a Hilbert space

$$\tilde{\mathcal{H}} = \int_{\oplus} \mathcal{H}(\lambda) d\mu(\lambda),$$

a direct integral of Hilbert spaces $\mathcal{H}(\lambda)$, $\lambda \in I\!R$, with respect to a pointwise measure μ, and a unitary operator $\mathcal{U} : \mathcal{H} \to \tilde{\mathcal{H}}$ such that

$$D(A^m) = \left\{v \in \mathcal{H}; \lambda \mapsto \lambda^m \mathcal{U}v(\lambda) \in \tilde{\mathcal{H}}\right\}, \quad m \in I\!N_0,$$

and

$$\mathcal{U}(A^m v)(\lambda) = \lambda^m \mathcal{U}v(\lambda).$$

Moreover,

$$||A^m v||^2 = \int_0^\infty \lambda^{2m} |\mathcal{U}v(\lambda)|^2\, d\mu(\lambda).$$

Let us denote by $v := \mathcal{U}u$, $\psi := \mathcal{U}\theta$. Then (1.1),(1.2) turn into

$$v_{tt} + m(t)\lambda^2 v + n(t)(\lambda + \mu)\psi = 0, \tag{3.1}$$

$$\psi_t + r(t)(\lambda + \alpha)\psi - q(t)(\lambda + \mu)v_t = 0, \tag{3.2}$$

where we have dropped the parameters t and λ in v and ψ. Let

$$\mathcal{E}(t,\lambda) := \frac{1}{2}|v_t(t,\lambda)|^2 + \frac{m(t)}{2}\lambda^2|v(t,\lambda)|^2 + \frac{n(t)}{2q(t)}|\psi(t,\lambda)|^2$$

where $|\cdot|$ is to be understood in $\mathcal{H}(\lambda)$. Multiplying equation (3.1) by v_t and (3.2) by $\frac{n}{q}\psi$ and summing up we get

$$\frac{d}{dt}\mathcal{E}(t,\lambda) = -\frac{n(t)r(t)}{q(t)}(\lambda+\alpha)|\psi|^2 + \frac{m'(t)}{2}\lambda^2|v|^2 + \frac{d}{dt}\left\{\frac{n}{2q}\right\}|\psi|^2, \tag{3.3}$$

$$\frac{d}{dt}Re\{\lambda v_t v\} \le \lambda|v_t|^2 - m(t)\lambda^3|v|^2 + n(t)\lambda^2|\psi||v| + n(t)\lambda|\mu||\psi||v|. \tag{3.4}$$

We will suppose that $q(t) \ge q_0$. (Otherwise we take $-\psi v_t$ instead of ψv_t). Multiplying (3.2) by v_t we obtain

$$\frac{d}{dt}Re\{\psi v_t\} = -r(t)(\lambda+\alpha)Re\{\psi v_t\} + q(t)\lambda|v_t|^2 \tag{3.5}$$

$$+\mu q(t)|v_t|^2 - m(t)\lambda^2 Re\{v\psi\} - n(t)\lambda|\psi|^2 - n(t)\mu|\psi|^2$$

The inequalities (3.4) and (3.5) imply

$$\frac{d}{dt}Re\{\lambda v_t v\} \le \lambda|v_t|^2 - \frac{m_0}{2}\lambda^3|v|^2 + \frac{n^2(t)}{2m_0}\lambda|\psi|^2 + n(t)\lambda|\mu||\psi||v|, \tag{3.6}$$

$$\frac{d}{dt}Re\{-\psi v_t\} \le -\frac{q_0}{2}\lambda|v_t|^2 + \frac{r^2(t)\lambda}{2q_0}|\psi|^2 + \{r(t)|\alpha| - q(t)\mu\}|v_t|^2 \tag{3.7}$$

$$+\frac{q_0 m_0}{16}\lambda^3|v|^2 + \frac{4m^2(t)}{q_0 m_0}\lambda|\psi|^2 + n(t)\lambda|\psi|^2 + (n(t)|\mu| + r(t)|\alpha|)|\psi|^2,$$

respectively. From (3.6), (3.7) we conclude

$$\frac{d}{dt}Re\left\{\frac{q_0}{4}\lambda v_t v - \psi v_t\right\} \le -\frac{q_0}{4}\lambda|v_t|^2 \tag{3.8}$$

$$-\frac{q_0 m_0 \lambda^3}{16}|v|^2 + c\lambda|\psi||v| + c|v_t|^2 + c\lambda|\psi|^2 + c|\psi|^2$$

with c being a constant depending essentially on T, possibly varying from formula to formula. Combining (3.3) and (3.8) we obtain

$$\frac{d}{dt}\mathcal{K}(t,\lambda) \le -c\lambda\left\{|v_t(t,\lambda)|^2 + \lambda^2|v(t,\lambda)|^2 + |\psi(t,\lambda)|^2\right\} \tag{3.9}$$

$$+c\left\{\lambda^2|v(t,\lambda)|^2 + |\psi(t,\lambda)|^2 + |v_t(t,\lambda)|^2\right\},$$

where

$$\mathcal{K}(t,\lambda) := \mathcal{E}(t,\lambda) + \varepsilon\frac{q_0}{4}\lambda Re\{v_t v\} - \varepsilon Re\{\psi v_t\}.$$

Taking ε small enough, we get

$$\frac{1}{2}\mathcal{E}(t,\lambda) \le \mathcal{K}(t,\lambda) \le 2\mathcal{E}(t,\lambda), \tag{3.10}$$

hence

$$\frac{d}{dt}\mathcal{K}(t,\lambda) \le -c_1\lambda\mathcal{K}(t,\lambda) + c_2\mathcal{K}(t,\lambda)$$

$$\frac{d}{dt}\mathcal{K}(t,\lambda) \leq -c_1\lambda\mathcal{K}(t,\lambda) + c_2\mathcal{K}(t,\lambda)$$

with positive constants $c_1,\ c_2$. We will consider two cases, first when $\lambda \geq \frac{2c_2}{c_1} =: c_3$, then $\lambda \leq c_3$. So for $\lambda \geq c_3$ we get

$$\frac{d}{dt}\mathcal{K}(t,\lambda) \leq -\frac{c_1}{2}\lambda\mathcal{K}(t,\lambda)$$

thus

$$\mathcal{E}(t,\lambda) \leq c\mathcal{E}(0,\lambda)e^{-\frac{c_1}{2}\lambda t}. \tag{3.11}$$

Multiplying by λ^m and integrating for $\lambda \geq c_3$ we get

$$\int_{\lambda\geq c_3} \lambda^m\mathcal{E}(\lambda,t)\,d\mu(\lambda) \leq \int_{\lambda\geq c_3} \lambda^m\mathcal{E}(\lambda,0)e^{-\frac{c_1}{2}\lambda t}\,d\mu(\lambda). \tag{3.12}$$

On the other hand, if $\lambda < c_3$ we get

$$\frac{d}{dt}\mathcal{K}(t,\lambda) \leq c_2\mathcal{K}(t,\lambda) \Rightarrow \mathcal{K}(\lambda,t) \leq e^{c_2t}\mathcal{K}(\lambda,0) \quad \forall\lambda\ \in\ [0,c_3].$$

Using (3.10) we get

$$\mathcal{E}(t,\lambda) \leq c\mathcal{E}(0,\lambda)e^{c_2t} \tag{3.13}$$

Multiplying by λ^m and integrating over $0 \leq \lambda \leq c_3$ we obtain

$$\int_{\lambda\leq c_3} \lambda^m\mathcal{E}(\lambda,t)\,d\mu(\lambda) \leq cc_3^m \int_{\lambda\leq c_3} \mathcal{E}(\lambda,0)e^{c_2t}\,d\mu(\lambda) \tag{3.14}$$

Finally from (3.12) and (3.14) we conclude that for $t > 0$,

$$\int_0^\infty \lambda^m\mathcal{E}(\lambda,t)\,d\mu(\lambda) \leq c(t,m)\int_0^\infty \mathcal{E}(\lambda,0)\,d\mu(\lambda).$$

Using the diagonalization theorem, cf. [2], we get

$$\forall t > 0:\ ||A^m u(t)|| + ||A^m\theta(t)|| \leq c(t,m)\left\{||u_1|| + ||Au_0|| + ||A\theta_0||\right\}. \tag{3.15}$$

Q.E.D.

Remark 3.2 *The constant $c(m,t)$ given in inequality (3.15) is such that $c(m,t) \to \infty$ as $t \to 0$.*

Remark 3.3 *If $M,\ N,\ R$ and Q are C^{k-1}-function then the solution (u,θ) of (1.1),(1.2) satisfies*

$$(u,\theta) \in C^k(]0,T]; \cap_{j\in I\!N} D(A^j))$$

Remark 3.4 *The smoothness effect property does not depend on the largeness of the initial data, because the method we used can be applied for local or global solutions.*

Theorem 3.5 *Let $A \geq \nu > 0$, $\alpha \geq 0$, $\mu \geq 0$, and let $(u,\theta) \in \cap_{j=0}^{2} C^j([0,\infty], D(A^{2-j})) \times \cap_{j=0}^{1} C^j([0,\infty], D(A^{1-j}))$ be a solution to (1.12),(1.13),(1.3),(1.4). Then (u,θ) decays to zero exponentially, i.e.*

$$E(t) \leq Me^{-dt}E(0),$$

for some positive constants M, d, where

$$E(t) := \frac{1}{2}\left\{||u_t(t)||^2 + m(t)||Au(t)||^2 + \frac{n(t)}{q(t)}||\theta(t)||^2\right\}.$$

Proof.- With the same technique as in the proof of Theorem 3.1 — the energy method — we conclude from (3.11) that there are $M > 0$ and $c_1 > 0$ for which we have

$$E(t) = \int_{\nu}^{\infty} \mathcal{E}(t,\lambda)\, d\mu(\lambda) \leq M \int_{\nu}^{\infty} e^{-c_1\lambda t}\mathcal{E}(0,\lambda)\, d\mu(\lambda) \leq Me^{-c_1\nu t}E(0).$$

Q.E.D.

We observe that c_1 depends on the C^1-norm of m, n and q.

When the operator A is not coercive, that is $A \geq 0$ only, the exponential decay is not expected. In the following theorem we will study this case when $\Omega = I\!R^n$ and $\Omega = I\!R^n \setminus B$ where B is a bounded closed set.

Theorem 3.6 *Let $\Omega = I\!R^n$ or let $n \geq 3$ and $\Omega = I\!R^n \setminus B$, where $B \neq \emptyset$ is a bounded closed set with smooth boundary, and let $I\!R^n \setminus \Omega$ be star-shaped. Then we have for the solution (u,θ) of (1.5)–(1.7),(1.3) that*

$$||(u_t, \Delta u, \theta)(t)||_{L^\infty(\Omega)\{L^2(\Omega)\}} \leq ct^{-\frac{n}{2}\{-\frac{n}{4}\}}||(u_1, \Delta u_0, \theta_0)(t)||_{L^1(\Omega)}$$

with a positive constant c neither depending on t nor on the initial data.

Proof.- First let $\Omega = I\!R^n$. Denoting by $\hat{u}(t,\xi)$ and $\hat{\theta}(t,\xi)$ the Fourier transform of u and θ, respectively, we obtain

$$\hat{u}_{tt}(t,\xi) + |\xi|^4\hat{u}(t,\xi) - \beta|\xi|^2\hat{\theta}(t,\xi) = 0, \tag{3.16}$$

$$\hat{\theta}_t(t,\xi) + |\xi|^2\hat{\theta}(t,\xi) + \beta|\xi|^2\hat{u}_t(t,\xi) = 0, \tag{3.17}$$

Combining (3.16), (3.17) with (3.1), (3.2) and defining

$$\hat{\mathcal{E}}(t,\xi) := \frac{1}{2}\left\{|\hat{u}_t(t,\xi)|^2 + |\xi|^4|\hat{u}(t,\xi)|^2 + |\hat{\theta}(t,\xi)|^2\right\}$$

we obtain by the same multiplicative technique as in the proof of Theorem 3.1.

$$\exists M > 0\ \exists d > 0\ \forall t \geq 0\ \forall \xi \in I\!R^n : \hat{\mathcal{E}}(t,\xi) \leq Me^{-d|\xi|^2 t}\hat{\mathcal{E}}(0,\xi)$$

which implies for

$$\mathcal{E}(t) := \frac{1}{2}\left\{|u(t)|^2 + |\Delta u(t)|^2 + |\theta(t)|^2\right\}$$

$$\mathcal{E}(t,\lambda) = \int_{I\!R^n} \hat{\mathcal{E}}(t,\xi)\, d\xi \le M \int_{I\!R^n} e^{-d|\xi|^2 t}\hat{\mathcal{E}}(0,\xi)\, d\xi \tag{3.18}$$

$$\le ct^{-\frac{n}{2}}||\hat{\mathcal{E}}(0,\xi)||_{L^\infty_\xi} \le ct^{-\frac{n}{2}}||(u_1, \Delta u_0, \theta_0)||_{L^1(\Omega)}.$$

Moreover,

$$|u_t(x,t)| \le |\left\{\frac{1}{\sqrt{2\pi}}\right\}^n \int_{I\!R^n} e^{ix\xi}\hat{u}_t(t,\xi)\, d\xi| \tag{3.19}$$

$$\le c\int_{I\!R^n} e^{-\frac{d}{2}|\xi|^2 t}\sqrt{\hat{\mathcal{E}}(0,\xi)}\, d\xi \le ct^{-\frac{n}{2}}||(u_1, \Delta u_0, \theta_0)||_{L^1(\Omega)}.$$

(3.18) and (3.19) prove Theorem 3.6 for $\Omega = I\!R^n$. Now, let $\Omega \neq I\!R^n$ be an exterior domain, $n \ge 3$. There exists a generalized Fourier transform $\mathcal{F} : L^2(\Omega) \to L^2(I\!R^n)$ such that

$$\mathcal{F}(\varphi(A)w)(\xi) = \varphi(|\xi|^2)(\mathcal{F}w)(\xi), \tag{3.20}$$

where A is the Laplace operator defined on $H^1_0(\Omega) \cap H^2(\Omega)$ and $\varphi(A)$ is assumed to be defined via the spectral theorem. $\mathcal{F}$ is represented by

$$(\mathcal{F}w)(\xi) = \int_\Omega \hat{\psi}(x,\xi)w(x) \equiv \hat{w}(t,\xi),$$

$$(\mathcal{F}^{-1}\hat{w})(x) = \int_{I\!R^n} \psi(x,\xi)\hat{w}(\xi)\, d\xi,$$

with a kernel $\psi(x,\xi)$, see [14], [17]. In [17] it is proved, based on results from [9], that

$$\exists m \in N\ \exists c > 0\ \exists x \in \Omega\ \forall \xi \ \in\ I\!R^n \setminus \{0\} : \quad |\psi(x,\xi)| \le c(1 + |\xi|)^m \tag{3.21}$$

holds, provided $I\!R^n \setminus \Omega$ is star-shaped. Using (3.20) we obtain the analogue of (3.16), (3.17). Essentially repeating the calculation following (3.17) we obtain (3.18) again, and using (3.19) we get

$$|u_t(t,x)| \le c\int_{I\!R^n} e^{-\frac{d}{2}|\xi|^2 t}\sqrt{\mathcal{E}(0,\lambda)}(1 + |\xi|)^m \le ct^{-\frac{n}{2}}||(u_1, \Delta u_0, \theta_0)||_{L^1(\Omega)}.$$

Q.E.D.

In one space dimension we can use the Fourier-sine transform [7], for example if $\Omega =]0,\infty[$, to obtain the corresponding result. For $n = 2$ the known estimate for $\psi(x,\xi)$ has a factor $\log|\xi|$, as $|\xi| \to 0$ wich leads to a decay like $c_\varepsilon t^{-\frac{n}{4}+\varepsilon}$ and $c_\varepsilon t^{-\frac{n}{2}+\varepsilon}$, respectively (instead of $ct^{-\frac{1}{2}}$ and ct^{-1} as expected).

4 Propagation of Singularities

Theorem 4.1 *Let $A \geq 0$ be self-adjoint in a separable Hilbert space $\mathcal{H}$, let v be a solution to (1.23)–(1.25). Then we have for $v_0 = v_2 = 0$ that*

$$\forall s \geq 0 : v_1 \notin D(A^{s+2}) \Rightarrow \forall t \geq 0 : \ \lambda \mapsto \lambda^{s+2} \left(\mathcal{U}v_t(t,\lambda), \lambda^{1/2}\mathcal{U}v(t,\lambda) \right) \notin \mathcal{H} \times \mathcal{H}.$$

Remark 4.2 *In terms of the example from one-dimensional thermoelasticity, the non-smoothing of the "hyperbolic" energy $\|u_t(t)\|^2 + \|u_x(t)\|^2$ is proved. The formulation in terms of v_1 is made for simplicity of the exposition, similar results could be obtained in terms of v_0 and v_2.*

Proof of Theorem 4.1.- Using the spectral theorem we conclude from (1.23), (1.24) that $w(t,\lambda) := \mathcal{U}v(t,\lambda)$ satisfies

$$w_{ttt} + \kappa\lambda w_{tt} + (\gamma^2 + \tau)\lambda w_t + \kappa\tau\lambda^2 w = 0, \tag{4.1}$$

$$w(t=0) = w_0 := \mathcal{U}v_0, \ w_t(t=0) = w_1 := \mathcal{U}v_1, \ w_{tt}(t=0) = w_2 := \mathcal{U}v_2. \tag{4.2}$$

The solution w of (4.1), (4.2) is given by

$$w(t,\lambda) = \sum_{j=1}^{3} b_j(\lambda) e^{-\beta_j(\lambda)t},$$

where $\beta_j(\lambda), \ j = 1,2,3$, are the roots of the characteristic equation

$$-\beta^3 + \kappa\lambda\beta^2 - (\gamma^2 + \tau)\lambda\beta + \kappa\tau\lambda^2 = 0,$$

and

$$b_j(\lambda) := \sum_{k=0}^{2} b_j^k(\lambda) w_k(\lambda)$$

with

$$b_j^0 := \frac{\prod_{l \neq j} \beta_l}{\prod_{l \neq j}(\beta_j - \beta_l)}, \quad b_j^1 := \frac{\sum_{l \neq j} \beta_l}{\prod_{l \neq j}(\beta_j - \beta_l)}, \quad b_j^2 := \frac{1}{\prod_{l \neq j}(\beta_j - \beta_l)}.$$

Since $w_0 = w_2 = 0$ we obtain

$$w(t,\lambda) = \sum_{j=1}^{3} b_j^1 e^{-\beta_j(\lambda)t} w_1(\lambda) \equiv \sum_{j=1}^{3} f^j(t,\lambda)$$

The asymptotic behavior of $\beta_j(\lambda)$ is known (see [19], [15]) and given as follows:

Lemma 4.3 *As $\lambda \to 0$:*

$$\beta_1(\lambda) = \frac{\kappa\tau}{\tau + \gamma^2}\lambda + O(\lambda^{\frac{3}{2}}), \quad \beta_{2/3}(\lambda) = \frac{\kappa\gamma^2}{2(\tau + \gamma^2)}\lambda \pm i\sqrt{\tau + \gamma^2}\sqrt{\lambda} + O(\lambda^{\frac{3}{2}}),$$

as $\lambda \to \infty$:

$$\beta_1(\lambda) = \kappa\lambda - \frac{\gamma^2}{\kappa} - \frac{\alpha_1}{\kappa^3}\lambda^{-1} + O(\lambda^{-\frac{3}{2}}),$$

$$\beta_{2/3}(\lambda) = \frac{\gamma^2}{2\kappa} + \frac{\alpha_2}{2\kappa^3}\lambda^{-1} + O(\lambda^{-2}) \mp i\left\{\sqrt{\tau\lambda} + \frac{\alpha_3}{\kappa_2}\lambda^{-\frac{1}{2}} + O(\lambda^{-\frac{3}{2}})\right\},$$

where $\alpha_j = \alpha_j(\gamma, \tau)$ *are constants* $(j = 1, 2)$.

Except for at most two values of $\lambda > 0$ *we get*

$$\beta_j(\lambda) \neq \beta_k(\lambda),\ j \neq k.$$

For any value of $\lambda \neq 0$: $Re\beta_j(\lambda) > 0,\ j = 1, 2, 3.$

There are positive constants r_1 *and* C_j, $j = 1, 2, 3.$ *such that*

$$\lambda \leq r_1^2 \Rightarrow C_1\lambda \leq Re\beta_j(\lambda) \leq C_2\lambda$$

$$\lambda \geq r_1^2 \Rightarrow Re\beta_j(\lambda) \geq C_3\ (j = 1, 2, 3).$$

This implies the following asymptotic behavior for $b_j^1(\lambda)$ $j = 1, 2, 3$.

Lemma 4.4 *As* $\lambda \to 0$:

$$b_1^1(\lambda) = \kappa\gamma^2 + O(\sqrt{\lambda}), \quad b_{2/3}^1(\lambda) = \frac{\pm i}{2\sqrt{\lambda(\tau + \gamma^2)}} + O(1),$$

as $\lambda \to \infty$:

$$b_1^1(\lambda) = O(\lambda^{-2}), \quad b_{2/3}^1(\lambda) = \frac{\mp i}{2\sqrt{\lambda}} + O(\frac{1}{\lambda}).$$

Observe that the leading term for $b_{2/3}^1(\lambda)$ as $\lambda \to 0$ is like $\lambda^{-\frac{1}{2}}$, but still $b_1^1 e^{-\beta_2 t} + b_3^1 e^{-\beta_3 t} = O(1)$ as $\lambda \to 0$, hence the interesting part is $\lambda \to \infty$. Now let $t > 0$. It is easy to see that for any $m \in I\!N$

$$\infty > \int_0^\infty \lambda^{2m}\left\{|f_t^1(t,\lambda)|^2 + \lambda|f^1(t,\lambda)|^2\right\} d\mu(\lambda).$$

Hence the f^1-part is arbitrarily smooth. We will prove now that the remaining part of w it is not smoother than w_1. In fact, let us suppose the contrary, so we have

$$\infty > \int_{r_1}^\infty \lambda^{2s+4}\left\{|f_t^2(t,\lambda) + f_t^3(t,\lambda)|^2 + \lambda|f^2(t,\lambda) + f^3(t,\lambda)|^2\right\} d\mu(\lambda)$$

and we obtain for $r_1 > 0$ sufficiently large, depending on t later on, using Lemma 4.3 and Lemma 4.4,

$$\infty > \int_{r_1}^\infty \lambda^{2s+4}\left\{|\frac{i}{2\sqrt{\lambda}}(\beta_2 e^{-\beta_2 t} - \beta_3 e^{-\beta_3 t})|^2 + O(\frac{1}{\lambda^2}) + \lambda|\frac{i}{2\sqrt{\lambda}}(e^{-\beta_3 t} - e^{-\beta_2 t})|^2 + O(\frac{1}{\lambda})\right\} |w_1(\lambda)|^2\, d\mu(\lambda)$$

$$= \int_{r_1}^{\infty} \lambda^{2s+4} \frac{e^{-2at}}{4} \left\{ |\sqrt{\tau}\cos(bt) + O(\frac{1}{\sqrt{\lambda}})|^2 + |\sin(bt)|^2 + O(\frac{1}{\lambda}) \right\} |w_1(\lambda)|^2 \, d\mu(\lambda),$$

where $a := Re\,\beta$, $b := Im\,\beta$. Thus we obtain for $r_1 = r_1(t)$ sufficiently large

$$\infty > \frac{\min\{\tau, 1\}}{4} \, e^{-\frac{\gamma^2 t}{2\kappa}} \int_{r_1(t)}^{\infty} \lambda^{2s+4} |w_1(\lambda)|^2 \, d\mu(\lambda),$$

which is a contradiction because $v_1 \notin D(A^{s+2})$.

Q.E.D.

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Jaime E. Muñoz Rivera, National Laboratory for Scientific Computation, Department of Research and Development, Rua Lauro Müller 455, Botafogo Cep. 22290, Rio de Janeiro, RJ, Brasil, and IM, Federal University of Rio de Janeiro

Reinhard Racke, Fakultät für Mathematik, Universität Konstanz, Universitätsstr. 10, 7750 Konstanz 1 (from July 1, 1993: 78434 Konstanz), Germany