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Keywords
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Smoothness and Locality for Nonunital Spectral Triples

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Abstract. To deal with technical issues in noncommutative geometry for nonunital algebras, we introduce a useful class of algebras and their modules. These algebras and modules allow us to extend all of the smoothness results for spectral triples to the nonunital case. In addition, we show that smooth spectral triples are closed under the C^∞ functional calculus of self-adjoint elements. In the final section we show that our algebras allow the formulation of Poincaré Duality and that the algebras of smooth spectral triples are H-unital.

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Key words: spectral triples, functional calculus, K-homology, Hochschild homology.

1. Introduction

The recent interest in Connes' noncommutative manifolds within both mathematics and physics has created a need for a broader set of axioms, [3]. One would like to be able to consider noncompact, nonspin, [18], and pseudo-Riemannian examples, [28], as well as 'spectral triples with boundary.' In this paper, and a sequel [24], we address just the first of these issues. We expect this will have bearing on the other generalisations mentioned. We also note that similar proposals have been advanced in [10].

There is inevitably a certain amount of overlap between this paper and [24], since one can not completely divorce analytic, algebraic and homological aspects of noncommutative geometry. Nevertheless, we have tried to make each paper self-contained, and focussed on particular aspects. This paper addresses the analytic issues surrounding smoothness and locality for spectral triples over nonunital algebras. The paper [24] deals with summability and dimension spectrum, and shows how Connes–Moscovici's Local Index Theorem extends to our context.

We begin in Section 2 with a discussion of the smooth algebras we employ, and their ideals. The most important feature of the algebras we employ is their local structure, which mimics the way the algebra of compactly supported functions sits inside its various completions.

In Sections 3 and 4, we look at modules for smooth algebras which are in a suitable sense 'finite projective.' This is necessarily more difficult in the nonunital case, because ultimately we want Hilbert space completions of such modules, and this implies that there must be some kind of decay conditions on elements of these modules if we hope to be able to recover (modules of sections of) smooth vector bundles over infinite volume manifolds.

Section 5 introduces the definitions of spectral triples and smooth spectral triples, $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. We present some technical results before reaching our main result for this section, namely a C^∞-functional calculus for spectral triples.

The final section contains three results. First, locality of the differential forms on a spectral triple is equivalent to a finite propagation property for the operator $\mathcal{D}$. The next result shows that those local algebras $\mathcal{A}$ which have an associated smooth spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ are H-unital, and so obey excision in Hochschild homology. Finally, we define a 'compactly supported' K-homology group for local algebras, and use this to formulate Poincaré Duality in the nonunital case.

The special structure of the algebras employed here is critical to proving most of our important results. The class of algebras we consider is broad enough to include all relevant algebras of smooth functions on manifolds, as well as all smooth unital algebras and products of these two kinds of algebras. Numerous results and examples in this paper and the sequel, [24], show that some restriction of this kind is inevitable.

Nevertheless, it should be noted that the question of how well the theory developed here can deal with other examples of noncompact, noncommutative manifolds remains to be seen. The noncommutative $\mathbf{R}^n$, presented in [7], falls within the framework developed here, and we sketch how this works in an appendix.

2. Smooth Algebras

For compact spaces, the difference between algebras of smooth functions and algebras of continuous functions is just smoothness. When the underlying space is not compact, one must worry about issues such as integrability, and decay at infinity.

In order to be able to formulate our regularity requirements in the nonunital case, we feel it is necessary to begin by restricting to separable and σ-unital C^*-algebras. This will inevitably be a restriction on the *nonunital* algebras we consider, however the various unitizations that arise may not be separable. All our results apply to the unital case also, sometimes trivially, provided we assume separability.

We now discuss smooth versions of these nonunital algebras. We begin with smooth algebras, then introduce our notion of local algebras and smooth ideals. The usual definition of smooth in noncommutative geometry is as follows.

DEFINITION 1. A $*$-algebra $\mathcal{A}$ is smooth if it is Fréchet and $*$-isomorphic to a proper dense subalgebra $i(\mathcal{A})$ of a C^*-algebra A which is stable under the holomorphic functional calculus.

Thus saying that $\mathcal{A}$ is smooth means that $\mathcal{A}$ is Fréchet and a pre-C^*-algebra. Asking for $i(\mathcal{A})$ to be a *proper* dense subalgebra of A immediately implies that the Fréchet topology of $\mathcal{A}$ is finer than the C^*-topology of A (since Fréchet means locally convex, metrizable and complete.) We will sometimes speak of $\overline{\mathcal{A}} = A$, particularly when $\mathcal{A}$ is represented on Hilbert space and the norm closure $\overline{\mathcal{A}}$ is unambiguous. At other times we regard $i\colon \mathcal{A} \hookrightarrow A$ as an embedding of $\mathcal{A}$ in a C^*-algebra. We will use both points of view.

It has been shown that if $\mathcal{A}$ is smooth in A then $M_n(\mathcal{A})$ is smooth in $M_n(A)$, [11, 27]. This ensures that the K-theories of the two algebras are isomorphic, the isomorphism being induced by the inclusion map i. This definition ensures that a smooth algebra is a 'good' algebra, [11], so these algebras have a sensible spectral theory which agrees with that defined using the C^*-closure, and the group of invertibles is open.

We will always suppose that we can define the Fréchet topology of $\mathcal{A}$ using a countable collection of submultiplicative seminorms which includes the C^*-norm of $\overline{\mathcal{A}} = A$, and note that the multiplication is jointly continuous, [19]. By replacing any seminorm q by $\frac{1}{2}(q(a) + q(a^*))$, we may suppose that $q(a) = q(a^*)$ for all $a \in \mathcal{A}$.

Stability under the holomorphic functional calculus extends to nonunital algebras, since the spectrum of an element in a nonunital algebra is defined to be the spectrum of this element in the 'one-point' unitization, though we must of course restrict to functions satisfying $f(0) = 0$. Likewise, the definition of a Fréchet algebra does not require a unit. However, many analytical problems arise because of the lack of a unit (concerning summability), as well as some homological ones (namely excision in Hochschild homology). We make two definitions to address these issues. The first deals with 'compactly supported functions', while the second addresses completeness.

DEFINITION 2. An algebra $\mathcal{A}$ has local units if for every finite subset of elements $\{a_i\}_{i=1}^n \subset \mathcal{A}$, there exists $\phi \in \mathcal{A}$ such that for each i, $\phi a_i = a_i \phi = a_i$.

DEFINITION 3. Let $\mathcal{A}$ be a Fréchet algebra and $\mathcal{A}_c \subseteq \mathcal{A}$ be a dense ideal with local units. Then we call $\mathcal{A}$ a local algebra (when $\mathcal{A}_c$ is understood.)

Remark. The dense ideal $\mathcal{A}_c$ is saturated in the following sense. If $a \in \mathcal{A}$ and $\exists \phi \in \mathcal{A}_c$ such that $\phi a = a\phi = a$, then $a \in \mathcal{A}_c$. This follows because $\mathcal{A}_c$ is an ideal. Note that unital algebras $\mathcal{A}$ are automatically local, with $\mathcal{A}_c = \mathcal{A}$.

MOTIVATING EXAMPLE. If X is a (paracompact, σ-compact) smooth manifold, there are a variety of topological algebras that we can consider. The first algebra we consider is $C_b^\infty(X)$, the smooth functions all of whose derivatives are bounded. To simplify the discussion (and without any loss of generality) we fix an increasing family of compact subsets $\{K_j\}$ of X such that their union is X and K_j is contained in the interior of K_{j+1}. We define the C^∞ topology on this algebra by means of the seminorms

$$q_{mj}(f) = \sup_{|\alpha|\leqslant m} (\sup_{x\in K_j} |D^\alpha f(x)|).$$

Unfortunately, the completion, $C^\infty(X)$, of $C_b^\infty(X)$ in this topology is all smooth functions, bounded or not. So we introduce the uniform C^∞ topology via the seminorms

$$q_m(f) = \sup_{|\alpha|\leqslant m} (\sup_{x\in X} |D^\alpha f(x)|). \tag{1}$$

It is well known that with this topology $C_b^\infty(X)$ is a Fréchet space, and that it is stable under the holomorphic functional calculus. It is important to stress that this is the natural topology to use when we wish to consider these functions as operators on Hilbert space. Essentially, we retain the uniformity of the operator norm, and add only smoothness conditions. This will return to haunt us when we consider questions of summability in [24].

Next we define the nonunital algebra $C_0^\infty(X)$ of smooth functions all of whose derivatives vanish at infinity. The natural topology here is uniform convergence of all derivatives, given by the seminorms in Equation 1 above. The algebra $C_0^\infty(X)$ is a Fréchet algebra, a pre-C^*-algebra in $C_0(X)$, and is a closed ideal in $C_b^\infty(X)$. We also define $C_c^\infty(X)$ to be the smooth compactly supported functions on X. An easy result is

LEMMA 1. *The algebra $C_c^\infty(X)$ is dense in $C_0^\infty(X)$ for the topology of uniform convergence of all derivatives.*

So $C_c^\infty(X)$ appears as a dense ideal with local units in the smooth functions vanishing at infinity. However, in the theory of distributions, one takes a different topology on $C_c^\infty(X)$ which makes it a complete space, [19, 22]. First notice that if we consider $C_c^\infty(K_j)$, the smooth compactly supported functions with support contained in K_j, then the topologies induced on $C_c^\infty(K_j)$ by $C_b^\infty(X)$ and $C_0^\infty(X)$ are the same. Moreover, they make $C_c^\infty(K_j)$ a Fréchet space, and this topology also agrees with that induced by $C_c^\infty(K_{j+1})$. Hence we can consider the (strict) inductive limit topology on $C_c^\infty(X) = \cup_j C_c^\infty(K_j)$. In this topology $C_c^\infty(X)$ is complete, [19, 22].

We will mimic these properties below to establish the good behaviour of spectral triples over smooth local algebras. In [24] we show that this is a useful starting point for discussions of summability, and in the last section of this paper we use the local structure to define compactly supported K-homology, formulate Poincaré Duality for nonunital noncommutative geometries and show that local algebras associated to spectral triples are H-unital.

EXAMPLE. The finite rank operators, F, on a separable Hilbert space $\mathcal{H}$ have local units. If we have $a_1, \ldots, a_n \in F$, let $V \subset \mathcal{H}$ be the union of the ranges of the a_i and the orthogonal complements of their kernels. Then P_V, the projection onto V, serves as a local unit for the a_i. We endow the algebra of matrices of rapid decrease

$$RD(\mathcal{H}) = \left\{(a_{ij})_{i,j\in\mathbf{N}}\colon \sup_{i,j} i^k j^l |a_{ij}| < \infty\ \forall k, l\right\}$$

with the topology determined by the seminorms

$$q_{k,l}(a_{ij}) = \sup_{i,j} i^k j^l |a_{ij}|, \quad k, l > 0,$$

$$q_{0,0} = \|(a_{ij})\| = \text{ operator norm.}$$

Then $RD(\mathcal{H})$ is a Fréchet algebra, and $RD(\mathcal{H})$ is the completion of F in the topology determined by the seminorms above. Moreover, $q_{0,0}$ is a C^*-norm, and the completion of F in this norm is simply $\mathcal{K}$, the compact operators on $\mathcal{H}$. This example is important for a smooth version of stability. If $\mathcal{A}$ is a local algebra, then so is $\mathcal{A} \otimes RD(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$; the tensor product is the projective tensor product.

EXAMPLE. Let $\mathcal{A}_c \subseteq \mathcal{A}$ be a smooth, local algebra and $\mathcal{B}$ a smooth, unital algebra. Then the projective tensor product $\mathcal{A} \otimes \mathcal{B}$ is a smooth, local algebra. The dense ideal with local units consists of finite sums

$$\sum_i a_i \otimes b_i, \quad a_i \in \mathcal{A}_c,\ b_i \in \mathcal{B},$$

with local unit $\phi \otimes 1$, where ϕ is a local unit for each a_i.

LEMMA 2. *If $\mathcal{I}$ is a closed two sided ideal in the local algebra $\mathcal{A}_c \subseteq \mathcal{A}$, then $\mathcal{A}/\mathcal{I}$ is a local algebra.*

Proof. The quotient is Fréchet (by a standard result, [25, p. 120]), and stable under the holomorphic functional calculus (using the fact that $\mathcal{I}$ is a closed ideal in $\mathcal{A}$), so we just need to show that there is a dense ideal in $\mathcal{A}/\mathcal{I}$ with local units. If $a \in \mathcal{A}_c$ and $\phi \in \mathcal{A}_c$ is a local unit for $a \in \mathcal{A}_c$, then $(a + \mathcal{I})(\phi + \mathcal{I}) = (a\phi + \mathcal{I}) = (a + \mathcal{I})$, so $\mathcal{A}_c + \mathcal{I} \subseteq \mathcal{A}/\mathcal{I}$ has local units. It is a straightforward exercise to show that $\mathcal{A}_c + \mathcal{I}$ is dense. □

Remark. It seems reasonable to suppose that closed ideals of local algebras are also local, with the dense ideal given by $\mathcal{I} \cap \mathcal{A}_c$, but (as stated) this is not true. For example, let $\mathcal{A} = C_0^\infty(\mathbf{R})$ with $\mathcal{A}_c = C_c^\infty(\mathbf{R})$ and $\mathcal{I}$ be the closed ideal of smooth functions vanishing outside the open interval (a, b). Then the function

$$f(x) = \begin{cases} e^{-\frac{(x-a)^2}{(x-b)^2}} e^{-\frac{(x-b)^2}{(x-a)^2}}, & x \in (a, b), \\ 0, & \text{otherwise}, \end{cases}$$

is in the ideal $\mathcal{I}$, and indeed is in $\mathcal{I} \cap \mathcal{A}_c$, but has no local unit in $\mathcal{I}$. Nevertheless, this ideal is local, but our most general result in this direction requires commutativity of the ideal $\mathcal{I}$ and extra ingredients provided by a spectral triple associated to the algebra. This will be proved in Section 5 once we have established the C^∞-functional calculus for spectral triples.

The basic properties of these local algebras are summarised in the following lemma.

LEMMA 3. *If $\mathcal{A}_c \subset \mathcal{A}$ is a local algebra, then $\exists \{\phi_n\}_{n \geqslant 1} \subset \mathcal{A}_c$ such that*

(1) *$\{\phi_n\}_{n \geqslant 1}$ is an approximate unit for $\mathcal{A}$, with $\phi_n a \to a$ in the Fréchet topology of $\mathcal{A}$.*
(2) *$\forall a \in \mathcal{A}_c\, \exists i$ such that $\forall n \geqslant i$ $\phi_n a = a\phi_n = a$.*
(3) *For all $i < n$, $\phi_n \phi_i = \phi_i \phi_n = \phi_i$.*
(4) *$\mathcal{A}_c = \cup_n \mathcal{A}_n$, where $\mathcal{A}_n = \{a \in \mathcal{A}\colon \phi_n a = a\phi_n = a\}$.*

Proof. Since we have assumed that the norm closure $\overline{\mathcal{A}}$ is separable, there is a countable dense subset $\{a_i\}_{i=1}^\infty \subset \overline{\mathcal{A}}$. Since we have assumed also that $\mathcal{A}$ is dense in $\overline{\mathcal{A}}$ for the norm topology, we may take $\{a_i\}_{i=1}^\infty \subset \mathcal{A}$. As $\mathcal{A}_c$ is dense in $\mathcal{A}$ for the smooth topology (and so dense in $\overline{\mathcal{A}}$ for the norm topology), we can in fact take $\{a_i\}_{i=1}^\infty \subset \mathcal{A}_c$. Thus every element of $\overline{\mathcal{A}}$ can be approximated in norm by 'compactly supported, smooth elements.'

We begin with the proof of (4). Let Λ be the collection of subalgebras $\mathcal{B} \subseteq \mathcal{A}_c$ constructed as follows. Choose a local unit ϕ_1 for a_1, where $\{a_i\}_{i=1}^\infty \subset \mathcal{A}_c$ is the countable set above. Choose a local unit ϕ_2 for the set $\{a_1, \phi_1, a_2\}$, and then choose a local unit ϕ_3 for the set $\{a_1, \phi_1, a_2, \phi_2, a_3\}$, and so on. Set

$$\mathcal{A}_n^\phi = \{a \in \mathcal{A}\colon \phi_n a = a\phi_n = a\},$$

and finally define $\mathcal{B}^\phi = \cup_n \mathcal{A}_n^\phi$. With the subset $\{a_i\}_{i=1}^\infty$ fixed, this construction depends only on the choice of the ϕ_n.

Define a partial order on Λ by saying $\mathcal{B}^\phi \leqslant \mathcal{B}^\psi$ iff $\forall i\, \exists j$ such that $\psi_j \phi_i = \phi_i \psi_j = \phi_i$. By the construction of the subalgebras $\mathcal{B}^\phi$, this relation is obviously reflexive, and transitivity follows easily from the definition of the relation. To see that it is antisymmetric, simply note that if $\mathcal{B}^\phi \leqslant \mathcal{B}^\psi$ and $\mathcal{B}^\psi \leqslant \mathcal{B}^\phi$ then every $\mathcal{A}_n^\phi$ is contained in some $\mathcal{A}_N^\psi$, and so $\mathcal{B}^\phi \subseteq \mathcal{B}^\psi$. Likewise, every $\mathcal{A}_n^\psi$ is contained in some $\mathcal{A}_M^\phi$ and so $\mathcal{B}^\psi \subseteq \mathcal{B}^\phi$ and so they are equal.

Let $\Phi \subseteq \Lambda$ be a totally ordered subset, and let $\Phi^{\phi^1} \leqslant \Phi^{\phi^2} \leqslant \Phi^{\phi^3} \leqslant \cdots$ be any increasing countable subset of Φ (if Φ is finite one could use a simpler argument than that below) where $\phi^i = \{\phi^i_j\}_{j=1}^{\infty}$ is a choice of local units as above.

Pick any element $\phi^1_{k_1}$ of $\{\phi^1_n\}$. Let $\phi^2_{k_2}$ be an element of $\{\phi^2_n\}$ which is a local unit for $\phi^1_{k_1}$. Continuing in this fashion, choose $\phi^m_{k_m}$ to be a local unit for $\phi^{m-1}_{k_{m-1}}$ (and so for all previously chosen units). Set

$$\mathcal{A}_n = \{a \in \mathcal{A}\colon \phi^n_{k_n} a = a\phi^n_{k_n} = a\}$$

and $\Upsilon = \cup_n \mathcal{A}_n$.

Now if $\Phi^\psi \in \Phi$, then for each ψ_i and all sufficiently large n, $\phi^n_{k_n}\psi_i = \psi_i\phi^n_{k_n} = \psi_i$, since Φ is totally ordered. Hence $\Phi^\psi \leqslant \Upsilon$, and Υ is an upper bound for Φ. Thus, by Zorn's Lemma, there is a maximal element of Λ. Claim this maximal element is $\mathcal{A}_c$.

For if not, let $\mathcal{B}^\psi \subset \mathcal{A}_c$ denote the maximal element, and let $a \in \mathcal{A}_c \backslash \mathcal{B}^\psi$. Let ϕ be a local unit for a and let ϕ_1 be a local unit for the set $\{a, \phi, a_1, \psi_1\}$. Then let ϕ_2 be a local unit for $\{a, \phi, a_1, \psi_1, a_2, \psi_2, a_3\}$, and so on. The algebra $\mathcal{B}^\phi$ so constructed is in Λ, and clearly $\mathcal{B}^\psi < \mathcal{B}^\phi$, contradicting maximality.

Hence, there exists a local unit $\{\phi_n\}$ such that (4) holds. By construction, (2) and (3) also hold. Finally, if $\mathcal{A}_c \ni b_i \to b \in \mathcal{A}$, for each i there is an $n(i)$ such that $\phi_{n(i)} b_i = b_i\phi_{n(i)} = b_i$. Thus for all $m \geqslant 0$ and $\epsilon > 0$ there is a $j = j(m, \epsilon)$ so that for every $i > j$

$$\begin{aligned} q_m(b - \phi_{n(i)} b) &= q_m(b - b_i + b_i - \phi_{n(i)} b) \\ &\leqslant q_m(b - b_i) + q_m(\phi_{n(i)}(b_i - b)) \to 0 \end{aligned}$$

where the last conclusion follows from the continuity of the multiplication. Thus (1) holds and the Lemma is proved. □

We call an approximate unit with the above properties a local approximate unit. We will also assume without loss of generality that $\|\phi_n\| \leqslant 1$ and that $\phi_n^* = \phi_n \geqslant 0$ for all n. Before showing that $\mathcal{A}_c$ is the strict inductive limit of the $\mathcal{A}_n$, and so a complete locally convex algebra for the inductive limit topology, we make an important observation.

PROPOSITION 4. *If $\mathcal{A}_c \subseteq \mathcal{A}$ is a smooth local algebra, $\mathcal{A}_c$ is stable under the holomorphic functional calculus.*

Proof. Let f be a function defined and holomorphic on a neighbourhood of the spectrum of $a \in \mathcal{A}_c$ with $f(0) = 0$. Then $f(a) \in \mathcal{A}$ since $\mathcal{A}$ is stable under the holomorphic functional calculus, and we can write

$$f(a) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(a - \lambda)^{-1}\mathrm{d}\lambda,$$

where Γ winds once around $spec(a)$. Now let $\phi \in \mathcal{A}_c$ be a local unit for a, so $\phi a = a\phi = a$. Then for λ in the resolvent of a,

$$\begin{aligned}(1-\phi) &= (1-\phi)(a-\lambda)(a-\lambda)^{-1} \\ &= -\lambda(1-\phi)(a-\lambda)^{-1} \\ &\Rightarrow (1-\phi)(a-\lambda)^{-1} = \frac{-1}{\lambda}(1-\phi).\end{aligned}$$

So we have

$$\begin{aligned}(1-\phi)f(a) &= \frac{1}{2\pi i}\int_\Gamma f(\lambda)(1-\phi)(a-\lambda)^{-1}\mathrm{d}\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{-f(\lambda)}{\lambda}(1-\phi)\mathrm{d}\lambda \\ &= (1-\phi)f(0) = 0.\end{aligned}$$

Thus $f(a) = \phi f(a)$ and similarly $f(a) = f(a)\phi$. As $\phi \in \mathcal{A}_c$, and $\mathcal{A}_c$ is an ideal, $f(a) \in \mathcal{A}_c$. □

Next suppose that $\mathcal{A}_c \subseteq \mathcal{A} \subseteq \overline{\mathcal{A}} = A$ is a local algebra and $\{\phi_n\}$ is a local approximate unit. Define subalgebras

$$\mathcal{A}_n = \{a \in \mathcal{A}\colon a\phi_n = \phi_n a = a\}.$$

LEMMA 5. *For each n the algebra $\mathcal{A}_n$ is complete in the topology induced by $\mathcal{A}$. Moreover, the topologies on $\mathcal{A}_n$ induced by $\mathcal{A}_{n+1}$ and $\mathcal{A}$ agree.*

Proof. Let $\{a_k\}_{k\geqslant 1} \subset \mathcal{A}_n$ be a sequence converging in the topology of $\mathcal{A}$. Since we are assuming that $\mathcal{A}$ is complete, there exists an $a \in \mathcal{A}$ such that for all $m > 0$ and all $\epsilon > 0$ there is a sufficiently large k so that $q_m(a - a_k) < \epsilon$. So we can estimate, since $\phi_n a_k = a_k \phi_n = a_k$ for all k,

$$\begin{aligned}q_m(a-\phi_n a) &= q_m(a - a_k + a_k - \phi_n a) \\ &\leqslant q_m(a-a_k) + q_m(\phi_n(a_k - a)) \\ &< (1+C_m)\epsilon,\end{aligned}$$

the last line following from the continuity of the multiplication. Since this is true for all seminorms q_m and $\epsilon > 0$, we must have $a - \phi_n a = 0$.

Finally, the compatibility follows from $\phi_{n+1}\phi_n = \phi_n$. □

COROLLARY 6. *The algebra $\mathcal{A}_c = \cup_n \mathcal{A}_n$ is complete in the inductive limit topology defined by the inclusion maps.*

Proof. This follows from standard results. The continuity of the multiplication on the inductive limit comes from [19, Lemma 2.2] and the continuity of the multiplication on each $\mathcal{A}_n$. The inductive limit is locally m-convex by [19, Proposition 3.2], and the completeness follows from [25, Proposition VII.3]. □

Remark. A *sequence* $\{a_i\}_{i=1}^{\infty} \subset \cup_n \mathcal{A}_n$ converges in the inductive limit topology if there is n, N such that for all $i > N$, $a_i \in \mathcal{A}_n$ and $q_m(a_i - a_j) \to 0$ for each seminorm q_m as $i, j \to \infty$.

Since we will be dealing mainly with the nonunital case, we will also require (not necessarily closed) ideals in smooth algebras.

DEFINITION 4. Let $\mathcal{B}$ be a smooth, unital algebra, and $\mathcal{A}$ a not necessarily closed ideal in $\mathcal{B}$. If $\mathcal{A}$ possesses a topology for which $\mathcal{A}$ is a local algebra (in particular, complete) such that the inclusion map $i\colon \mathcal{A} \hookrightarrow \mathcal{B}$ is continuous, we call $\mathcal{A}$ a local ideal. A local ideal $\mathcal{A}$ in a smooth local algebra $\mathcal{B}$ is called essential if whenever $b\mathcal{A} = \{0\}$ for some $b \in \mathcal{B}$, then we have $b = 0$.

Remark. Note that the topology making $\mathcal{A}$ local is necessarily at least as fine as that on $\mathcal{B}$, and we will always suppose that the family of seminorms defining the topology of $\mathcal{B}$ (including the C^*-norm of $\mathcal{B}$) can be included in any family of seminorms defining the topology of $\mathcal{A}$. This ensures that $\overline{\mathcal{A}}$ is a (closed) ideal in $\overline{\mathcal{B}}$.

COROLLARY 7. *If $\mathcal{A}_c \subseteq \mathcal{A}$ is a local algebra and $\mathcal{A} \hookrightarrow \mathcal{B}$ is an essential local ideal, then $\mathcal{A}_c \hookrightarrow \mathcal{B}$ is an essential local ideal, where $\mathcal{A}_c$ carries the inductive limit topology.*

Proof. We have a local approximate unit in $\mathcal{A}$ which converges in the topology of $\mathcal{B}$ to the identity. If for some $b \in \mathcal{B}$ we have $b\mathcal{A}_c = \{0\}$, then $b\phi_n = 0$ for all elements ϕ_n of any local approximate unit. Since $q_m(\phi_n b - b) \to 0$ for all seminorms q_m on $\mathcal{B}$, we have $b = 0$. □

EXAMPLE. Let X be a noncompact Riemannian manifold, and let $C_0^\infty(X)$ be the smooth functions all of whose derivatives vanish at infinity, with the topology of uniform convergence of all derivatives. Let $C_b^\infty(X)$ be the smooth functions all of whose derivatives are bounded, with the same topology of uniform convergence. Then $C_0^\infty(X)$ is a local ideal in $C_b^\infty(X)$. Here the two algebras carry the same topology, so the inclusion map is obviously continuous. Note that $C_0^\infty(x)$ is an essential ideal.

EXAMPLE. Let X be as above and $C_1^\infty(X)$ be the smooth functions all of whose derivatives are integrable. Topologizing this algebra with uniform convergence of all derivatives *and* L^1-convergence of all derivatives makes $C_1^\infty(X)$ a local algebra. Since this topology is stronger than the uniform topology on $C_b^\infty(X)$, the inclusion map is continuous, and $C_1^\infty(X)$ is a local ideal in $C_b^\infty(X)$. Again, $C_1^\infty(X)$ is an essential ideal.

EXAMPLE. Let X be as above and consider $C_c^\infty(X)$ with its inductive limit topology. This topology is stronger than the topology of uniform convergence of all derivatives, so $C_c^\infty(X)$ is a local ideal in $C_b^\infty(X)$. The compactly supported functions form an essential ideal in $C_b^\infty(X)$.

This idea of smooth ideal is quite flexible, as the above examples show. We use it primarily to to be able to control integrability, and have stronger growth conditions on various ideals.

3. Modules for Nonunital C^*-Algebras

Our first task in this section is to obtain a characterisation of the modules of continuous sections of vector bundles tending to zero at infinity on locally compact spaces. We do not constrain ourselves to those bundles trivial at infinity.

To begin, recall that a unitization $A \hookrightarrow A_b$ is an embedding of A as an essential ideal of the unital C^*-algebra A_b. Next, observe that if $i\colon A \hookrightarrow A_b$ is a unitization of A, we can define the pull-back of a right A_b-module E by

$$i^*E = E|_A := Ei(A) = \{ei(a) : e \in E,\ a \in A\}, \tag{2}$$

with the obvious right action of A on $E|_A$. Note that as an A_b-module, $E|_A$ is a submodule of E. In fact, the pullback works for any embedding $i\colon A \hookrightarrow A_b$ of A as an ideal in A_b, even if it is not closed.

EXAMPLE. If $i\colon X \hookrightarrow X^c$ is a compactification of the locally compact Hausdorff space X, then we may define a unitization of $C_0(X)$ as follows. Define, [21, p. 25],

$$i_*\colon C_0(X) \hookrightarrow C(X^c) \tag{3}$$

by

$$(i_*f)(y) = \begin{cases} 0, & \text{if } y \notin i(X), \\ f(x), & \text{if } y = i(x),\ x \in X. \end{cases} \tag{4}$$

EXAMPLE. Let $1 \to X^c$ be the trivial line bundle over X^c, where X^c is a compactification of the locally compact noncompact Hausdorff space X. The space of sections of this line bundle is $E = C(X^c)$, and this is a right $C(X^c)$ module. We have the unitization map, $i\colon C_0(X) \hookrightarrow C(X^c)$, and so we can pull E back to $C_0(X)$. We find

$$i^*E = i^*C(X^c) = C(X^c)i(C_0(X)) = C_0(X). \tag{5}$$

Thus we obtain the space of sections vanishing at infinity.

EXAMPLE. Let X be as above, and consider the embedding $i\colon C_c(X) \hookrightarrow C_0(X)$, where $C_c(X)$ is the algebra of compactly supported functions. If $E = \Gamma_0(X, V)$ is the $C_0(X)$-module of sections vanishing at infinity of the vector bundle V, then we can pull it back to $C_c(X)$ as above. Then

$$i^*E = \Gamma_c(X, V) \tag{6}$$

and we obtain the compactly supported sections. In this case the resulting module is only a pre-C^*-module, because $C_c(X)$ is not a C^*-algebra.

Recall that we assume that any nonunital C^*-algebra is separable and σ-unital. As we will also be discussing multiplier algebras, it is worth noting immediately that the separability requirement will generally not be satisfied by the multiplier algebra.

DEFINITION 5. Let A be a separable σ-unital C^*-algebra. If $i\colon A \hookrightarrow A_b$ is a unitization, we say that an A-module is A_b finite projective if it is of the form i^*E for some finite projective A_b-module E.

THEOREM 8 (Nonunital Serre–Swan). *Let X be a locally compact Hausdorff space, $A = C_0(X)$, and $A_b = C(X^c)$ for some compactification X^c of X. Then a right A-module E is of the form $E = pA^n$, $p \in M_n(A_b)$ a projection, if and only if $E = \Gamma_0(X, V|_X)$, where $V \to X^c$ is a vector bundle and Γ_0 denotes the sections vanishing at infinity.*

Proof. Suppose that $E = \Gamma_0(X, V|_X)$, where $V \to X^c$ is a vector bundle. Then for some n and projection $p \in M_n(A_b)$,

$$\Gamma(X^c, V) \cong p(A_b)^n, \tag{7}$$

by the Serre-Swan theorem, [29]. Note that as $X \subset X^c$ is dense and open, and rank p is locally constant, $p \notin M_n(A)$. Setting $i\colon A \hookrightarrow A_b$ to be the unitization, we have

$$i^*\Gamma(X^c, V) = pA^n = \Gamma_0(X, V|_X). \tag{8}$$

Conversely, let $E = pA^n$ with $p \in M_n(A_b)$ a projection. Then we can define a finite projective A_b-module, $\tilde{E} = p(A_b)^n$, with the obvious right action of A_b. By the Serre-Swan theorem, [29], there is a vector bundle $V \to X^c$ such that

$$\tilde{E} \cong p(A_b)^n = \Gamma(X^c, V). \tag{9}$$

Employing the pull-back by the unitization map as above immediately shows that

$$i^*\tilde{E} = E = \Gamma_0(X, V|_X). \tag{10}$$

□

We now drop the assumption that A is commutative, and describe the endomorphism algebras of A_b finite projective A-modules. By results in [11, Section 2.5], finite projective A_b-modules E can be regarded as C^*-modules once an A_b-Hermitian form is chosen. This form automatically restricts to the A_b-submodule

$E|_A$, since A is an ideal in A_b. It is the endomorphism algebras of these modules that we now describe. This is relevant for the noncommutative geometry of nonunital algebras, as we will identify such modules as the noncommutative analogue of sections of vector bundles vanishing at infinity.

PROPOSITION 9. *Let $E|_A$ be an A_b-finite projective A-module. Then, as a C^*A-module, we have*

$$\mathrm{End}^0_A(E|_A) = pM_n(A)p, \qquad \mathrm{End}_A(E|_A) = pM_n(A_b)p, \tag{11}$$

where $\mathrm{End}^0_A(E|_A)$ *denotes the compact endomorphisms of* $E|_A$ *and* $\mathrm{End}_A(E|_A)$ *denotes the C^*-algebra of adjointable operators on $E|_A$.*

Proof. The A_b-module E is of the form $E = p(A_b)^n$. Writing $M(B)$ for the multiplier algebra of a C^*-algebra B, it is then standard that, [15, Theorem 2.4],

$$\mathrm{End}_{A_b}(E) \cong M(\mathrm{End}^0_{A_b}(E)) = \mathrm{End}^0_{A_b}(E) \cong pM_n(A_b)p. \tag{12}$$

These equalities follow from $\mathrm{End}^0_{A_b}(E)$ being unital, and so its own multiplier algebra.

To prove the above assertions, we begin with a variant on a standard isomorphism, [11, Lemma 2.18]. Denoting the A-finite rank operators on $E|_A$ by $\mathrm{End}^{00}_A(E)$, define θ: $\mathrm{End}^{00}_A(E|_A) \longrightarrow pM_n(A)p$ by setting $\theta(|p\xi)(p\eta|)$ to be the matrix with i,j-th entry $\sum_{k,l} p_{ik}\xi_k\eta_l^* p_{lj}$. One checks that this is a *-homomorphism and an isometry and has dense range. As such, θ extends to a $*$-isomorphism of $\mathrm{End}^0_A(E|_A) \cong pM_n(A)p$.

As i^*E is a right A_b-submodule of E, we see that

$$\mathrm{End}_{A_b}(i^*E) = \mathrm{End}_A(i^*E). \tag{13}$$

Also, i^*E is a left $\mathrm{End}_{A_b}(E)$-submodule, as is easily checked. Consequently,

$$\mathrm{End}_A(i^*E) = \mathrm{End}_{A_b}(E) = pM_n(A_b)p. \tag{14}$$

□

Remark. This result implicitly supposes, and we shall continue to do so, that the A-valued inner product is given by

$$(\xi,\eta) = \sum \xi_i^*\eta_i = \sum \xi_i^* p_{ij}\eta_j.$$

In [12, Appendix A], elements of $\mathrm{End}^0_A(E|_A)$ are called endomorphisms carried by A. Modules of the above type and their endomorphism algebras are used to describe elements of relative K-theory.

4. Smooth Modules

We now turn to smooth modules. We have several aims, based largely on what we have achieved in the preceeding section, and the requirements of spectral triples in the next section. Ultimately we require Hilbert space completions of finite projective modules. The Lebesgue measure on $\mathbf{R}$ and the function $(1+x^2)^{-1/2}$ show that 'vanishing at infinity' is an inadequate starting point.

The Riesz representation theorem, [26, p 40], says that the positive linear functionals on $C_c(X)$, where X is a locally compact Hausdorff space, are precisely the positive outer regular Borel measures on X. If X is in addition σ-compact (and therefore paracompact) then these measures are automatically regular, [26, p. 48].

We take as our starting point the idea that elements of $\mathcal{A}_c \subseteq \mathcal{A}$ should be integrable. To avoid ending up with a theory of distributions, discussed in [24], we consider the continuous positive linear functionals on $\overline{\mathcal{A}_c} = \cup_n \overline{\mathcal{A}_n}$, where we take the (strict) inductive limit topology on the increasing family of C^*-algebras $\overline{\mathcal{A}_n}$.

However, we want to simultaneously deal with smoothness as well as integrability, and also retain some notion of 'finite projective' for our smooth modules. This is to retain our notion of noncommutative vector bundles, and remain close to the C^*-module theory. For this reason we will restrict ourselves to faithful positive linear functionals when we come to deal with the Hilbert space aspects of these modules.

Let $\mathcal{A}$ be a closed essential local ideal in the smooth unital algebra $\mathcal{A}_b$. Let $p \in M_N(\mathcal{A}_b)$ be a projection, and define a nondegenerate (by Corollary 7) right $\mathcal{A}_b$-module $E_c = p\mathcal{A}_c^N$. If we give E_c the inductive limit topology (using any local approximate unit $\{\phi_n\}_{n\geqslant 1}$ acting on the right), then E_c is complete (though not metrizable). Similarly we define $E_0 = p\mathcal{A}^N$ and $E_\infty = p\mathcal{A}_b^N$, both with the Fréchet topology defined using the Hermitian form and the topology on $\mathcal{A}_b$.

DEFINITION 6. *A smooth $\mathcal{A}_b$-module E_2 is a Fréchet space with a continuous action of $\mathcal{A}_b$ such that*

$$E_c \hookrightarrow E_2 \hookrightarrow E_0 \hookrightarrow E_\infty, \tag{15}$$

as linear spaces, where $\hookrightarrow$ denotes continuous inclusion.

Remark. In the unital case, where a local unit for every element is the identity of the algebra, this forces a smooth module to be $p\mathcal{A}^N$, and convergence in $p\mathcal{A}^N$ to be equivalent to 'componentwise' convergence in $\mathcal{A}$.

Remark. The subscripts are intended to have the following 'meaning' by analogy with the commutative case: c denotes compact support, *2* denotes 'square integrable', 0 means vanishes at infinity and ∞ indicates bounded.

Remark. Requiring that $E_c \subseteq E_2 \subseteq E_0$ ensures that the C^*-module closures satisfy $\overline{E_2} = \overline{E_0}$.

EXAMPLE. For the trivial line bundle over $\mathbf{R}^n$ we could take E_2 to be the functions all of whose derivatives are square integrable, or the Schwartz class of functions, or anything in between.

LEMMA 10. *Right multiplication by $b \in \mathcal{A}_b$ is a continuous linear map on E_c.*

Proof. Let $\{\phi_n\}_{n\geqslant 1}$ be any local approximate unit, and $\mathcal{A}_n = \{a \in \mathcal{A}_c : a\phi_n = a\}$. We first show that multiplication by $a \in \mathcal{A}_c$ is continuous on E_c. This is equivalent to multiplication by a being continuous as a map $p\mathcal{A}_n \to E_c$ for all $n \geqslant 1$. Let m be such that $a\phi_m = \phi_m a = a$, and set $k = \max\{m, n\}$. Then if $\{\xi_i\}_{i\geqslant 1} \subseteq p\mathcal{A}_n^N$ converges to zero, then by definition $q((\xi_i, \xi_i)) \to 0$ for all continuous seminorms on $\mathcal{A}$. Then $\{\xi_i a\}_{i\geqslant 1} \subseteq p\mathcal{A}_k^N$, and we have

$$q((\xi_i a, \xi_i a)) = q(a^*(\xi_i, \xi_i)a) \leqslant q(a^*)q(a)q(\xi_i, \xi_i) \to 0.$$

Hence, multiplication by a is continuous at 0, and so is continuous.

Now let $b \in \mathcal{A}_b$. As $\mathcal{A}_c$ is an ideal in $\mathcal{A}_b$, multiplication by b maps E_c to E_c. By the previous argument, multiplication by $\phi_m b\phi_m$ is continuous on E_c, for fixed m. Also, for each $\xi \in E_c$, $\xi\phi_m b\phi_m \to \xi b$, so multiplication by b is the pointwise limit of continuous maps. To see this note that because multiplication by b maps E_c to E_c, for all $\xi \in E_c$ there exists M such that $\xi b\phi_M = \xi b$. Hence, for m sufficiently large, $\xi\phi_m b\phi_m$ is contained in $p\mathcal{A}_M^N$, and $\xi\phi_m b\phi_m = \xi b$.

As E_c is barrelled (being the inductive limit of Fréchet, and so barrelled, spaces, [25, Proposition V.6]), the operators $\phi_m b\phi_m$ are continuous and $\phi_m b\phi_m \to b$ pointwise, a standard equicontinuity result, [25, p 69], shows that multiplication by $b \in \mathcal{A}_b$ is continuous. □

LEMMA 11. *Let $\{\xi_i\} \subseteq p\mathcal{A}_{n+1}^N$ converge in the topology of E_0. Then $\{\xi_i\}$ converges in E_2 with the same limit $\xi \in p\mathcal{A}_{n+1}^N$.*

Proof. For a compactly supported sequence, convergence in E_c is the same as convergence in E_0, by definition. However, the topology on E_c is stronger than that on E_2, so convergence in E_c implies convergence in E_2. □

Thus convergence of compactly supported sequences is the same in all the spaces we consider, so the differing topologies are 'the same' as regards smoothness, differing only in how they control convergence at infinity.

LEMMA 12. *Let $T \in pM_N(\mathcal{A}_b)p$. Then left multiplication by T is a continuous map on E_c, E_2, E_0 and E_∞.*

Proof. For E_0 and E_∞ this is straightforward. For E_c we note that $T\colon p\mathcal{A}_n^N \to p\mathcal{A}_n^N \subseteq E_c$ for all n, so as T is continuous in the topology on E_0, it is continuous on E_c. So we are left with E_2. Let T_n be the operator $T_n\xi = T\xi\phi_n$, where $\{\phi_n\}$ is any local approximate unit. Then $T_n\colon E_2 \to p\mathcal{A}_{n+1}^N$, and is continuous, since $T_n\colon E_2 \to E_0$ is continuous (since E_2 is continuously included in E_0). So, using our favourite equicontinuity result, [25, p. 69], we just need to show that T is the pointwise limit of the T_n. So let $\xi \in E_2$, and observe that $T\xi\phi_n \to T\xi$ since $\phi_n \to 1$ (strongly/point-wise) since $E_2 \subseteq E_0$. Hence multiplication by T is continuous. □

LEMMA 13. *Let E_2 be a smooth $\mathcal{A}_b$-module and let $\mathcal{A}_1 \subseteq \mathcal{A}$ be the completion of the ideal generated by the range of the Hermitian form on E_2 in the topology induced by E_2. Then $\mathcal{A}_1$ is Fréchet and is continuously embedded in $\mathcal{A}$. The closure of the ideal generated by the range of the Hermitian form on E_0 in the topology of $\mathcal{A}$ is $\mathcal{A}$ if and only if $\mathcal{A}_c \subseteq \mathcal{A}_1$. In this case $\mathcal{A}_1$ is an essential, local ideal in $\mathcal{A}$ and the C^*-module closure $\overline{E_2}$ of E_2 is a full right $\overline{\mathcal{A}}$-module.*

Proof. Since E_2 is contained in $p\mathcal{A}^N$, the Hermitian form makes sense on E_2. The range of this Hermitian form is certainly an ideal, since

$$(\xi, \eta)a = (\xi, \eta a), \ a(\xi, \eta) = (\xi a^*, \eta) \ \xi, \eta \in E_2, \ a \in \mathcal{A}_b.$$

Let $C \subseteq E_2$ be a balanced, convex, absorbent neighbourhood of zero. Define $\tilde{C} = (C, C)$, where we mean all convex sums of elements (ξ, η) in the range of the inner product, with $\xi, \eta \in C$. Then it is routine to show that $\tilde{C}$ is a balanced, convex, absorbent set contained in the range of the Hermitian form. By considering all such sets C (which determine the topology of E_2) we obtain a family of neighbourhoods $\{\tilde{C}\}$ of zero in the ideal generated by the range of the Hermitian form, which determine a locally convex topology. As these neighbourhoods are also multiplicative, the multiplication is jointly continuous [19, Proposition 1.6], and extends to a jointly continuous multiplication on the completion.

As E_2 is supposed Hausdorff, so is the completion $\mathcal{A}_1$ of the ideal generated by the Hermitian form, and as E_2 has a countable base of neighbourhoods of zero, so does $\mathcal{A}_1$. Consequently, $\mathcal{A}_1$ is Fréchet, since it is complete, [19, Section I.4]. The topology on $\mathcal{A}_1$ is at least as strong as that on $\mathcal{A}$ since $E_2 \subseteq E_0$, so the inclusion $\mathcal{A}_1 \hookrightarrow \mathcal{A}$ is continuous. The continuity of the action of $\mathcal{A}$ on E_2 shows that the completion $\mathcal{A}_1$ is still an ideal in $\mathcal{A}$.

Let us temporarily denote the ideal generated by the Hermitian form on E_0 by $\mathcal{A}_0$. Now if $\mathcal{A}_c \subseteq \mathcal{A}_1$, then $\mathcal{A}_1$ is dense in $\mathcal{A}$ (in the topology of $\mathcal{A}$), and since $\mathcal{A}_c$ contains an approximate unit, it is essential (cf. Corollary 7). Hence, the closure of $\mathcal{A}_1$ in the topology of $\mathcal{A}$ is equal to $\mathcal{A}$, and this closure necessarily contains the closure of $\mathcal{A}_0$ in the topology of $\mathcal{A}$. Moreover, $\mathcal{A}_c$ is dense in the C^*-closure $\overline{\mathcal{A}}$ in the norm topology, so $\mathcal{A}_1$ is norm dense in $\overline{\mathcal{A}}$. Hence, the C^*-module $\overline{E}$ is full.

Conversely, suppose that the completion of $\mathcal{A}_0$ in the topology of $\mathcal{A}$ is $\mathcal{A}$, and $a \in \mathcal{A}_c$. As $a \in \mathcal{A}$, there is a sequence $\{\sum_{k=1}^{K(i)} (\xi_i^k, \eta_i^k)\}_{i \geqslant 1} \subseteq \mathcal{A}_0$ which converges to a in the topology of $\mathcal{A}$. Let $\{\phi_n\}$ be a local approximate unit, with $a\phi_m = \phi_m a = a$. Then the sequence $\{a_i = (\sum_k \xi_i^k \phi_m, \eta_i^k \phi_m)\}$ is contained in $(p\mathcal{A}_{m+1}^N, p\mathcal{A}_{m+1}^N) \subseteq \mathcal{A}_1$, and $a_i \to a$ in the topology of $\mathcal{A}$. By Lemma 11, $a_i \to a$ in the topology of $\mathcal{A}_1$. As $a \in \mathcal{A}_c$ was arbitrary, $\mathcal{A}_c \subseteq \mathcal{A}_1$. □

Remark. There are several issues here. The ideal $\mathcal{A}_1$ automatically inherits its topology from the module E_2, but it may lack many of the algebraic properties that we find desirable. For instance, the ideal $\mathcal{A}_c p \mathcal{A}_c$ consisting of the range of the

Hermitian form on $p\mathcal{A}_c^N$ is algebraically generated by $\{\sum_{i,j} a_i p_{ij} b_j : a_i, b_j \in \mathcal{A}_c\}$, and may not have local units. Certainly it will have local units in $\mathcal{A}_c$, but $\mathcal{A}_c$ may not be contained in $\mathcal{A}_c p \mathcal{A}_c$. For free modules, $\mathcal{A}_c \subseteq \mathcal{A}_1$ and we conjecture that this remains true for modules with full C^*-module closure. This would provide a stronger version of the above result.

It is difficult to sensibly define endomorphisms of a smooth module in a manner analogous to the C^*-module case. An investigation of endomorphism algebras, and related ideas of Morita equivalence, for smooth modules will not be undertaken here, but we make a couple of points. An endomorphism of a smooth module, $T: E_2 \to E_2$, should be adjointable with respect to the Hermitian form, and be continuous. However, this will not ensure boundedness with respect to the C^*-norm of the module. Whether this be good or bad depends on one's taste, see for instance [10]. In any case, elements of the algebra $pM_N(\mathcal{A}_b)p$ certainly satisfy these properties, by Lemma 12, and we regard it as a good candidate for the smooth endomorphisms.

For the compact operators on a smooth module, we start from the finite rank operators. As usual, the finite rank operators are generated by the rank one operators $\Theta_{\xi,\eta}$, $\xi, \eta \in E_2$, where $\Theta_{\xi,\eta}\rho = \xi(\eta, \rho)$, $\rho \in E_2$.

If $C \subseteq E_2$ is a balanced convex absorbent neighbourhood of zero, define $\tilde{C}_\Theta$ to be the convex hull of the set $\{\Theta_{\xi,\eta} : \xi, \eta \in C\}$. This is in analogy with the definition of the algebra $\mathcal{A}_1$.

DEFINITION 7. We denote by $\mathrm{End}^1_{\mathcal{A}_b}(E_2)$ the completion of the finite rank operators in the locally convex topology determined by the sets $\tilde{C}_\Theta$.

We regard this definition as providing a good candidate for the compact endomorphisms of a smooth module. This point of view is partially supported by the following result.

PROPOSITION 14. *The algebra* $\mathrm{End}^1_{\mathcal{A}_b}(E_2)$ *is an essential local ideal in* $pM_N(\mathcal{A})p$, *and if* $\mathcal{A}_c \subseteq \mathcal{A}_1$, *then* $\mathrm{End}^1_{\mathcal{A}_b}(E_2) = pM_N(\mathcal{A}_1)p$.

Proof. For the first statement, it suffices to show that $pM_N(\mathcal{A}_c)p \subseteq \mathrm{End}^1_{\mathcal{A}_b}(E_2)$, and that the topology is at least as strong as that on $\mathrm{End}_{\mathcal{A}_b}(E_2)$. For $a \in \mathcal{A}_c$, let $A_{ij} \in M_N(\mathcal{A})$ be the matrix with i, j entry equal to a and all others zero. Then

$$pA_{ij}p = \Theta_{\xi,\eta},\ \xi = p(0, \ldots, a, \ldots, 0)^T,\ \eta = p(0, \ldots, \phi, \ldots, 0)^T.$$

Here $\phi a = a\phi = a$ and ϕ is in the jth slot, a in the ith. Hence $pM_N(\mathcal{A}_c)p \subseteq \mathrm{End}^1_{\mathcal{A}_b}(E_2)$. Now $T_i \to 0$ in $\mathrm{End}^1_{\mathcal{A}_b}(E_2)$ if for all sufficiently large i $T_i = \sum_{k=1}^{K(i)} \lambda^i_k \Theta_{\xi^i_k,\eta^i_k}$ with $\sum_k \lambda^i_k = 1$ and $\xi^i_k, \eta^i_k \to 0$ in E_2 as $i \to \infty$. However this means that $\sum_k^{K(i)} \lambda^i_k(\xi^i_k, \eta^i_k) \to 0$ in $\mathcal{A}_1$, and conversely. As the topology on $\mathcal{A}_1$ is at least as strong as that on $\mathcal{A}$, the topology on $\mathrm{End}^1_{\mathcal{A}_b}(E_2)$ is at least as strong as that on $pM_N(\mathcal{A})p$.

Finally, if $\mathcal{A}_c \subseteq \mathcal{A}_1$, then the above argument shows that $\mathrm{End}^1_{\mathcal{A}_b}(E_2) = pM_N(\mathcal{A}_1)p$. □

EXAMPLE. Let X be a noncompact Riemannian manifold, and $C_0^\infty(X) \subset C_b^\infty(X)$ both be given the topology of uniform convergence of all derivatives. We denote the seminorms defining this topology by q_n, and completing $E = pC_c^\infty(X)^N$ using these seminorms and the recipe above, we obtain $E_2 = pC_0^\infty(X)^N = E_0$.

Alternatively, we could employ the collection of seminorms on $\mathcal{A}_c$ given by

$$q_n'(f) = \sup_{|\alpha| \leqslant n} \int_X |\partial^\alpha f| \mathrm{d}^p x, \;\; f \in C_c^\infty(X), \;\; n \geqslant 0.$$

In this case we obtain $E_2 = pC_2^\infty(X)^N$, the smooth sections all of whose derivatives are square integrable. Of course, completing $C_c^\infty(X)$ with respect to the family of seminorms $\{q_n'\}$ above gives $C_1^\infty(X)$, the algebra of smooth functions all of whose derivatives are integrable, and the range of the Hermitian form is contained in this algebra. The 'compact endomorphisms' are given by $pM_N(C_1^\infty(X))p$, since the module is full, and consists of matrices of smooth integrable functions. Thus we can compose the integral over X with the matrix trace on $pM_N(C_1^\infty(X))p$ to obtain a natural integral on the compact endomorphisms. Variations such as graded or super traces in the fibre are obviously possible also.

The final result of this section concerns Hilbert space completions of smooth modules. Let E_2 be a smooth (left) $\mathcal{A}_b$-module. Let $\tau\colon \mathcal{A}_1 \to \mathbf{C}$ be a faithful positive (in the sense of linear maps on $*$-algebras), linear functional which is continuous with respect to the topology on $\mathcal{A}_1$. Then if the Hermitian form on E_2 is $(\cdot, \cdot)$, define a positive definite inner product on E_2 by $\langle \xi, \eta \rangle = \tau((\xi, \eta))$.

Define $\mathcal{H} = L^2(E, \tau)$ to be the Hilbert space completion of $E_c = p\mathcal{A}_c^N$ with respect to the norm defined by the inner product. The following is a consequence of the definitions.

LEMMA 15. *If E_2 is a smooth $\mathcal{A}_b$-module and $\mathcal{H} = L^2(E, \tau)$ is as above, then there is a continuous inclusion $E_2 \hookrightarrow L^2(E, \tau)$.*

We say that $L^2(E, \tau)$ is the completion of a smooth module. Smooth modules are thus the appropriate generalisation of smooth sections of vector bundles which satisfy additional integrability criteria.

5. Nonunital Spectral Triples

In this section we present the definitions of spectral triples and examine technical issues related to smoothness and locality.

DEFINITION 8. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by

(1) A representation $\pi: \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of a local $*$-algebra $\mathcal{A}$ on the Hilbert space $\mathcal{H}$.
(2) A self-adjoint (unbounded, densely defined) operator $\mathcal{D}$: dom$\mathcal{D} \to \mathcal{H}$ such that $[\mathcal{D}, a]$ extends to a bounded operator on $\mathcal{H}$ for all $a \in \mathcal{A}$ and $a(1+\mathcal{D}^2)^{-\frac{1}{2}}$ is compact for all $a \in \mathcal{A}$.

The triple is said to be even if there is an operator $\Gamma = \Gamma^*$ such that $\Gamma^2 = 1$, $[\Gamma, a] = 0$ for all $a \in \mathcal{A}$ and $\Gamma\mathcal{D} + \mathcal{D}\Gamma = 0$ (i.e. Γ is a $\mathbf{Z}_2$-grading such that $\mathcal{D}$ is odd and $\mathcal{A}$ is even.) Otherwise the triple is called odd.

Remark. Since $\mathcal{A}$ is represented on a Hilbert space we may unambiguously speak about the norm on $\mathcal{A}$, and the norm closure $\overline{\mathcal{A}} = A$. The word local is usually not part of this definition, but we will always work in this setting. We will almost always suppress the representation π, regarding $\mathcal{A}$ as a subalgebra of $\mathcal{B}(\mathcal{H})$.

DEFINITION 9. If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple, then we define $\Omega^*_{\mathcal{D}}(\mathcal{A})$ to be the algebra generated by $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$.

Alternatively, this is the representation on $\mathcal{H}$ of the universal differential algebra of $\mathcal{A}$, $\Omega^*(\mathcal{A})$. We will not require this point of view. The following definition will be used at one point in the final section, and repeatedly in [24].

DEFINITION 10. A local spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple with $\Omega^*_{\mathcal{D}}(\mathcal{A}_c) \subseteq \Omega^*_{\mathcal{D}}(\mathcal{A})$ a local algebra.

We may assume without loss of generality that a local spectral triple has a local approximate unit $\{\phi_n\}_{n\geqslant 1} \subset \mathcal{A}_c$ such that $\phi_{n+1}\phi_n = \phi_n$ and $\phi_{n+1}[\mathcal{D}, \phi_n] = [\mathcal{D}, \phi_n]$.

EXAMPLE. Let X be a complete Riemannian spin manifold (geodesically complete with no boundary) with metric g. Let $S \to X$ be the spinor bundle on X and $\mathcal{D}$: $\Gamma_c(S) \to \Gamma_c(S)$ the Dirac operator on the smooth, compactly supported sections of the spinor bundle. The module $\Gamma_c(S)$ is an $\mathcal{A}_b$ finite projective $\mathcal{A}_c$-module, where $\mathcal{A}_c$ is the algebra of smooth compactly supported functions on X and $\mathcal{A}_b = C_b^\infty(X)$ is the algebra of smooth functions all of whose derivatives are bounded.

In dimensions $6, 7, 8$ mod 8, S is a real bundle and complexifying yields a representation space (modulo a small caveat; see below) $\Gamma_c(S_{\mathbf{C}})$ for the algebra

$$\mathbf{C}\mathit{lif}\, f_b(X) := \Gamma_b(\mathbf{C}\mathit{lif}\, f(T^*X \otimes \mathbf{C}, g)),$$

the smooth bounded sections of the Clifford algebra on the complexified cotangent bundle. In other dimensions the spinor bundle is already a complex (or quaternionic) module, and so carries a representation of $\mathbf{C}\mathit{lif}\, f_b(X)$. In all dimensions we will refer to the spinor bundle, despite meaning its complexification in dimensions 6, 7, 8 mod 8.

We let $\Gamma_2(S)$ be the smooth $C_b^\infty(X)$-module of smooth sections all of whose derivatives are square integrable. Since the spinor bundle is an irreducible representation space for the complex Clifford bundle, and fibrewise the Clifford bundle comprises a matrix algebra of full rank (see the caveat below), it is not hard to see that $\mathbf{C}liff_b(X) = \Gamma_b(S \otimes S^*)$, and if we can write $\Gamma_b(S) = pC_b^\infty(X)^N$ then $\mathbf{C}liff_b(X) = pM_N(C_b^\infty(X))p$. Thus the endomorphism algebra of the spinor bundle is precisely the Clifford algebra.

Similarly, the compact endomorphisms of $\Gamma_2(S)$ are the sections of the Clifford bundle all of whose derivatives vanish at infinity. Thus

$$\mathrm{End}^0_{C_b^\infty(X)}(\Gamma_2(S)) = \mathbf{C}liff_0(X) = pM_N(C_0^\infty(X))p.$$

Set $\mathcal{H} = L^2(X, S)$, the square integrable sections of the spinor bundle for the measure determined by the metric g. Since we are assuming that the manifold X is geodesically complete, the operator $\mathcal{D}$ extends to a closed, unbounded, self-adjoint operator, [16, 12], $\mathcal{D}$: dom$\mathcal{D} \to \mathcal{H}$. Here the domain of $\mathcal{D}$ may be taken to be those sections ξ such that $\mathcal{D}\xi$ is square integrable. If $f \in C_b^\infty(X)$ acts by multiplication on $\mathcal{H}$, then

$$[\mathcal{D}, f] = \mathrm{d}f\cdot = \text{Clifford multiplication by } \mathrm{d}f,$$

and so is bounded. As $\mathcal{D}$ preserves supports, a local unit for f is also a local unit for $\mathrm{d}f\cdot$. In [24] we show that, with $p = \dim X$, and any compactly supported function f

$$f(1+\mathcal{D}^2)^{-\frac{1}{2}} \in \mathcal{L}^{(p,\infty)}(\mathcal{H}) \subset \mathcal{K}(\mathcal{H}).$$

In fact, a general result proved below shows that if $f \in C_0^\infty(X)$, then $f(1+\mathcal{D}^2)^{-\frac{p}{2}}$ is a compact operator (though not necessarily in any smaller ideal). This is shown in [12] also.

Thus we know that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a local spectral triple, and we just wish to determine its parity. In even dimensions $S = S^- \oplus S^+$, with the two orthogonal subbundles the ± 1 eigenbundles of the complex volume form $\omega_{\mathbf{C}}$, [16], acting on S (or $S_{\mathbf{C}}$.) The complex volume form satisfies, [16],

$$\mathcal{D}\omega_{\mathbf{C}} + \omega_{\mathbf{C}}\mathcal{D} = 0.$$

In odd dimensions S does not split, and $\omega_{\mathbf{C}}$ is central in $\mathbf{C}liff(X)$. In fact, we do not consider the full Clifford algebra in odd dimensions (this is the caveat) but instead utilise the isomorphism $\mathbf{C}liff_{2n+1} \cong \mathbf{C}liff_{2n} \otimes C_1$, where C_1 is the commutative algebra generated by 1 and $\omega_{\mathbf{C}}$ over $\mathbf{C}$. In this case, the (complexified) spinor bundle provides an irreducible representation of $\mathbf{C}liff_{2n} \otimes \omega_C$, not the full Clifford algebra, and we fix the representation of $\omega_{\mathbf{C}}$ by demanding that it acts as the identity. Thus in odd dimensions the 'endomorphism algebra' is provided by this 'reduced' Clifford algebra. We will be sloppy and simply refer to the Clifford algebra.

Summarising, the tuple $(C_0^\infty(X), L^2(X, S), \mathcal{D}, \omega_{\mathbf{C}})$ comprises a spectral triple. It is even if the dimension of X is even, with grading provided by the complex volume form, and odd if the dimension of X is odd. This is our principal example, and we will henceforth refer to it as the Dirac spectral triple of a complete spin manifold. It is also a smooth spectral triple in the following sense.

For details of many other *unital* examples, such as noncommutative tori, group algebras, finite triples, and products, see [4, 6,11] as well as further references cited therein.

DEFINITION 11. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is smooth if

$$\mathcal{A} \text{ and } [\mathcal{D}, \mathcal{A}] \subseteq \bigcap_{m \geqslant 0} \operatorname{dom} \delta^m,$$

where for $x \in \mathcal{B}(\mathcal{H})$, $\delta(x) = [|\mathcal{D}|, x]$.

Remark. Note the difference between the definitions of smooth for topological algebras and spectral triples. In [11] such triples are called regular. In fact we have the following lemma (this result is also noted in [11]).

LEMMA 16. *If* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *is a smooth spectral triple, then* $(\mathcal{A}_\delta, \mathcal{H}, \mathcal{D})$ *is also a smooth spectral triple, where* $\mathcal{A}_\delta$ *is the completion of* $\mathcal{A}$ *in the locally convex topology determined by the seminorms* $q_{ni}(a) = \|\delta^n d^i(a)\|$, $n \geqslant 0$, $i = 0, 1$, *where* $d(a) = [\mathcal{D}, a]$. *Moreover,* $\mathcal{A}_\delta$ *is a smooth algebra.*

Proof. First, if a is in the completion, then $\delta^n(a) = \lim \delta^n(a_i)$ for some $\{a_i\}_{i=1}^\infty \in \mathcal{A}$ and all n, so $\mathcal{A}_\delta \subseteq \cap_m \operatorname{dom}\delta^m$. Also, $[\mathcal{D}, a] = \lim[\mathcal{D}, a_i]$, so $[\mathcal{D}, a]$ is bounded for all $a \in \mathcal{A}_\delta$. Finally, $\delta^n([\mathcal{D}, a])$ is bounded also, so $\Omega^*_{\mathcal{D}}(\mathcal{A}) \subseteq \cap_m \operatorname{dom}\delta^m$. The only remaining thing to check in order to show that $(\mathcal{A}_\delta, \mathcal{H}, \mathcal{D})$ is a smooth spectral triple is that $a(1 + \mathcal{D}^2)^{-\frac{1}{2}}$ is compact. This follows from the norm convergence of the a_i and the compactness of $a_i(1 + \mathcal{D}^2)^{-\frac{1}{2}}$. It is worth noting that on the algebra $\mathcal{A}_\delta$, the derivations d and δ commute.

To prove the second statement, note that we can also realise $\mathcal{A}_\delta$ as the completion of $\mathcal{A}$ with respect to the seminorms $q'_n(a) = \|\delta^n(a)\|_{\mathcal{D}}$, where $\|a\|_{\mathcal{D}} = \|a\| + \|[\mathcal{D}, a]\|$. Note that $\|\cdot\|_{\mathcal{D}}$ is a submultiplicative norm on $\mathcal{A}$. With either collection of seminorms, $\mathcal{A}_\delta$ is a complete, metrisable locally convex algebra, and so Fréchet, by definition. To see that $\mathcal{A}_\delta$ is stable under the holomorphic functional calculus, we refer to [4, page 247], where it is shown that the domain of any closed derivation $\delta\colon B_1 \to B_2$ from a Banach algebra B_1 to a Banach B_1-bimodule B_2 is stable under the holomorphic functional calculus. We employ this by noting that the closure of $\mathcal{A}_\delta$ in the norm $\sum_{n=0}^m q'_n$ is a Banach algebra, $\mathcal{A}_m$. Now $\delta\colon \mathcal{A}_m \to \mathcal{B}(\mathcal{H})$ is densely defined and closable, so dom $\overline{\delta}$ is stable under the holomorphic functional calculus; call this subalgebra dom $\overline{\delta}_m$. Then

$\mathcal{A}_\delta = \cap_{m \geqslant 1} \mathrm{dom}\, \overline{\delta}_m$. So if $\mathcal{A}_\delta$ is unital and $a \in \mathcal{A}_\delta$ is invertible in the norm closure, then $a^{-1} \in \mathrm{dom}\, \overline{\delta}_m$ for all m and hence $a^{-1} \in \mathcal{A}_\delta$. When $\mathcal{A}_\delta$ is nonunital, and $\lambda - a$ is invertible in the one point unitization of the norm closure, then there is $b \in \overline{\mathcal{A}_\delta}$ such that $(\lambda - a)(\lambda^{-1} - b) = 1$ or $\lambda^{-1}a + \lambda b = ab$. Applying δ repeatedly to this equation, using the Liebniz rule and the fact that $a \in \mathcal{A}_\delta$, we see that $b \in \mathcal{A}_\delta$. Thus $(\lambda - a)^{-1}$ is in all the domain of all powers of δ, and given a function f holomorphic in a neighbourhood of the spectrum of a with $f(0) = 0$, we have $f(a) \in \mathcal{A}_\delta$. □

Remark. It would be preferable to employ the family of seminorms $q_n(a) = \|\delta^n(a)\|$. To do this one would have to show that if $\delta^n(a_i) \to \delta^n(a)$ in norm for all $n \geqslant 0$, then $[\mathcal{D}, a_i]$ converges (necessarily to $[\mathcal{D}, a]$ as $[\mathcal{D}, \cdot]$ is a closed derivation) and the limit is contained in $\cap_m \mathrm{dom}\delta^m$. We regard this as an important open problem. This difficulty is also noted in [11]. We should also remember to include the C^*-norm in this family of seminorms in order to comply with our conventions for smooth algebras. We will henceforth suppose that $\mathcal{A}$ is complete in the topology provided by the seminorms $\|\delta^n(\cdot)\|_{\mathcal{D}}$, and refer to this as the δ-topology.

Remark. To obtain submultiplicative seminorms, we need to employ the directed family of seminorms

$$\tilde{q}_n(a) = \sum_{i=0}^{n} \|\delta^i(a)\| + \|\delta^i([\mathcal{D}, a])\|.$$

For most of our results, it suffices to prove continuity with respect to the seminorms q_n.

COROLLARY 17. *The algebra $\Omega^*_{\mathcal{D}}(\mathcal{A}_\delta)$ is complete in the topology determined by the seminorms $q_{ni}(\omega) = \|\delta^n(d^i\omega)\|$, $i = 0, 1$, where*

$$d(a_0[\mathcal{D}, a_i] \cdots [\mathcal{D}, a_k]) = [\mathcal{D}, a_0][\mathcal{D}, a_1] \cdots [\mathcal{D}, a_k].$$

On a related note, Connes shows ([6, 8]) that if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a smooth spectral triple, then the operators a, $[\mathcal{D}, a]$ for $a \in \mathcal{A}$ are order zero, [6, 8]. That is

$$a, [\mathcal{D}, a]\colon \mathcal{H}_k \to \mathcal{H}_k, \quad \forall a \in \mathcal{A}$$

are continuous, where $\mathcal{H}_k = \cap_{m=1}^{k} \mathrm{dom}|\mathcal{D}|^m$, and $k \in \{1, 2, \ldots, \infty\}$. In particular, these operators preserve the domain of $\mathcal{D}$. The same is true of $\delta^k(a)$ and $\delta^k([\mathcal{D}, a])$. Note that the self-adjointness of $\mathcal{D}$ implies that $\mathcal{H}_\infty$ is dense in $\mathcal{H}$, [22].

We now want to combine our smooth modules with spectral triples. The following is the key result on which we base most of the remaining analysis of spectral triples built from smooth modules.

THEOREM 18. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a smooth spectral triple with $\mathcal{A} \subseteq \mathcal{A}_b$ a closed essential ideal, where both algebras are complete for the δ-topology. Then*

$$\mathcal{H}_\infty = \bigcap_{m \geqslant 1} \mathrm{dom}\mathcal{D}^m = \bigcap_{m \geqslant 1} \mathrm{dom}|\mathcal{D}|^m$$

is a core for $\mathcal{D}$, $|\mathcal{D}|$ *and* $\mathcal{D}^2$. *Furthermore, if* $\mathcal{H} = L^2(E, \tau)$ *is the completion of a smooth (right)* $\mathcal{A}_b$*-module* E_2 *and* $\mathcal{D}$: $E_2 \to E_2$ *is continuous, then* $\mathcal{H}_\infty$ *is a smooth module if and only if the range of the Hermitian form is contained in* $\cap_m \text{dom}\delta^m$ *and*

$$\langle |\mathcal{D}|^m \xi_i, |\mathcal{D}|^m \eta_i\rangle \to 0 \,\forall\, m \;\Rightarrow\; \|\delta^k((\xi_i, \eta_i))\| \to 0 \,\forall\, k, \tag{16}$$

for $\{\xi_i\}, \{\eta_i\} \subseteq \mathcal{H}_\infty$.

Proof. For the first statement, it is necessary and sufficient to show that for all $t \in \mathbf{R}$, $e^{it\mathcal{D}}$, $e^{it|\mathcal{D}|}$, $e^{it\mathcal{D}^2}$: $\mathcal{H}_\infty \to \mathcal{H}_\infty$, [22, Thm VIII.11, p. 269]. So for $n = 0, 1, 2, \ldots$ define seminorms (using the Hilbert space inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{H}$)

$$q_n'(\xi)^2 = \langle \mathcal{D}^n\xi, \mathcal{D}^n\xi\rangle, \;\xi \in \mathcal{H}_\infty.$$

To comply with our definitions for smooth modules we also need to include the C^*-module norm in the defining family as well. We will show that $e^{it\mathcal{D}}\xi \in \mathcal{H}_\infty$ whenever ξ is (the argument for $|\mathcal{D}|$ and $\mathcal{D}^2$ is identical). However, $e^{it\mathcal{D}}$ is unitary for all t, and commutes with $\mathcal{D}$. So

$$\begin{aligned} q_n'(e^{it\mathcal{D}}\xi)^2 &= \langle \mathcal{D}^n e^{it\mathcal{D}}\xi, \mathcal{D}^n e^{it\mathcal{D}}\xi\rangle \\ &= \langle e^{it\mathcal{D}}\mathcal{D}^n\xi, e^{it\mathcal{D}}\mathcal{D}^n\xi\rangle \\ &= \langle \mathcal{D}^n\xi, \mathcal{D}^n\xi\rangle = q_n'(\xi)^2. \end{aligned}$$

The comment preceeding the lemma shows that a, $[\mathcal{D}, a]$ preserve $\mathcal{H}_\infty$ for all $a \in \mathcal{A}$, so $\mathcal{H}_\infty$ is an $\mathcal{A}$-module. We show below that $\mathcal{H}_\infty$ is also an $\mathcal{A}_b$-module.

For the second statement, because we choose to work with right modules, we are really considering a spectral triple $(\mathcal{A}^{\text{op}}, \mathcal{H}, \mathcal{D})$, where $\mathcal{A}^{\text{op}}$ is the opposite algebra of $\mathcal{A}$ and it is represented in $\mathcal{H} = L^2(E, \tau)$ by $a^{op}\xi = \xi a$. We first show that $E_c = p\mathcal{A}_c^N$ is continuously embedded in $\mathcal{H}_\infty$. As $\mathcal{D}$ is continuous as a linear map on E_2, for all seminorms q_k on E_2 there exists a seminorm q_m on E_2 and a constant $C > 0$ such that for all $\xi \in E_2$ $q_k(\mathcal{D}\xi) \leqslant Cq_m(\xi)$. Likewise, the continuous inclusion $E_2 \hookrightarrow L^2(E, \tau)$ implies that for all seminorms q_k on E_2 there is a constant $C_k > 0$ such that for all $\xi \in E_2$ $\|\xi\| \leqslant C_k q_k(\xi)$. So now let $\{\xi_i\} \subseteq E_c$ be a sequence which is Cauchy for all seminorms q_k on E_2, so that $\xi_i \to \xi \in E_2$. Then

$$\begin{aligned} \|\mathcal{D}^n(\xi_i - \xi_j)\| &\leqslant C_k q_k(\mathcal{D}^n(\xi_i - \xi_j)) \\ &\leqslant C_{k,m,n} q_m(\xi_i - \xi_j) \longrightarrow 0. \end{aligned}$$

Thus $E_2 \hookrightarrow \mathcal{H}_\infty$ and as $E_c \hookrightarrow E_2$, we obtain the inclusion $E_c \hookrightarrow \mathcal{H}_\infty$. Observe that only the continuity of $\mathcal{D}$ on E_2 and the inclusion $E_2 \hookrightarrow \mathcal{H}$ are necessary to have $E_c \hookrightarrow \mathcal{H}_\infty$.

We now show that $\mathcal{H}_\infty \hookrightarrow E_0$ when the condition 16 is satisfied. The ideal $\mathcal{A}_1^{\mathcal{D}}$ generated by the range of the Hermitian form on $\mathcal{H}_\infty$ contains $\mathcal{A}_1$ (the ideal generated by the range of the Hermitian form on E_2), and $\mathcal{A}_1$ is dense in $\mathcal{A}_1^{\mathcal{D}}$ for the topology determined by $\mathcal{H}_\infty$. Let $a \in \mathcal{A}_1^{\mathcal{D}}$ and choose $\{a_i\}_{i\geqslant 1} \subseteq \mathcal{A}_1$ with $a_i \to a$ in the topology on $\mathcal{A}_1^{\mathcal{D}}$. The condition 16 then implies that a_i converges in the δ-topology, and as $\{a_i\} \subseteq \mathcal{A}$, $a \in \mathcal{A}$. As $a \in \mathcal{A}_1^{\mathcal{D}}$ was arbitrary, we have $\mathcal{A}_1^{\mathcal{D}} \hookrightarrow \mathcal{A}$. Hence convergence in $\mathcal{H}_\infty$ implies convergence in E_0, and so $\mathcal{H}_\infty \hookrightarrow E_0$.

Conversely, if $\mathcal{H}_\infty \hookrightarrow E_0$, then convergence in $\mathcal{H}_\infty$ implies convergence in E_0, so if the sequence $\langle \mathcal{D}^m\xi_i, \mathcal{D}^m\eta_i\rangle \to 0$ for all m we must have $\|\delta^m((\xi_i,\eta_i))\| \to 0$ for all k. Clearly in this case $(\mathcal{H}_\infty, \mathcal{H}_\infty) \subseteq \cap_m \mathrm{dom}\delta^m$. □

These seminorms are compatible with those on $\mathcal{A}$ in the following sense:

$$\begin{aligned}
\langle \mathcal{D}^n a^{\mathrm{op}}\xi, \mathcal{D}^n a^{\mathrm{op}}\xi\rangle &= \langle |\mathcal{D}|^n a^{\mathrm{op}}\xi, |\mathcal{D}|^n a^{\mathrm{op}}\xi\rangle \\
&= \left\langle \sum_{k=0}^{n} \delta^k(a^{\mathrm{op}})|\mathcal{D}|^{n-k}\xi, \sum_{j=0}^{n} \delta^j(a^{\mathrm{op}})|\mathcal{D}|^{n-j}\xi \right\rangle \\
&\leqslant \sum_{j,k=0}^{n} q_k(a^{\mathrm{op}})q_j(a^{\mathrm{op}})q'_{n-k}(\xi)q'_{n-j}(\xi) \\
&\leqslant C^2\left(\sum_{j=0}^{n} q_j(a)\right)^2 q'_n(\xi)^2,
\end{aligned}$$

where $C = \max_{0\leqslant k\leqslant n}\||\mathcal{D}|^{-k}\|$. If $|\mathcal{D}|$ is not invertible, we may redefine the seminorms using $(1+\mathcal{D}^2)^{1/2}$, and the same results apply since $|\mathcal{D}| - (1+\mathcal{D}^2)^{1/2}$ is bounded, by the functional calculus. Note that in fact the seminorms we have defined on $\mathcal{H}_\infty$ are compatible with the *directed* family of seminorms $\tilde{q}_n(a^{\mathrm{op}}) = \sum_0^n q_k(a^{\mathrm{op}})$. So $q'_n(a^{\mathrm{op}}\xi) \leqslant C\tilde{q}_n(a^{\mathrm{op}})q'_n(\xi)$.

LEMMA 19. *Let* $(\mathcal{A}^{\mathrm{op}}, \mathcal{H}, \mathcal{D})$ *be a smooth spectral triple with* $\mathcal{A}$ *complete in the* δ*-topology, and suppose that* $\mathcal{H} = L^2(E,\tau)$ *is the completion of a smooth* $\mathcal{A}_b$*-module,* $\mathcal{D}$: $E_2 \to E_2$ *is continuous, and condition* 16 *is satisfied. Then* $\mathcal{A}^{\mathrm{op}}\mathcal{H}_\infty$ *and* $\mathcal{A}_c^{\mathrm{op}}\mathcal{H}_\infty$ *are dense in* $\mathcal{H}_\infty$ *(and so* $\mathcal{H}$*). Moreover,* $\mathcal{H}_\infty$ *is a right* $\mathcal{A}_b$*-module, and* $[\mathcal{D}, b] \in \mathcal{B}(\mathcal{H})$ *for all* $b \in \mathcal{A}_b$*, and* $\Omega^*_{\mathcal{D}}(\mathcal{A}_b^{\mathrm{op}}) \subseteq \cap_m\mathrm{dom}\delta^m$.

Proof. From the previous Theorem we know that $E_c = p\mathcal{A}_c^N$ is dense in $\mathcal{H}_\infty$ for the smooth topology determined by the inner product and $\mathcal{D}$. Thus for all $\xi \in \mathcal{H}_\infty$ there exists $\{\xi_i\} \subset E_c$ such that $\xi_i - \xi \to 0$. Let $\phi_i \in \mathcal{A}_c$ be a local unit for ξ_i. Then $\phi_i^{\mathrm{op}}\xi_i - \xi = \xi_i - \xi \to 0$. Thus $\mathcal{A}_c^{\mathrm{op}}\mathcal{H}_\infty$ is dense in $\mathcal{H}_\infty$, and thus so is $\mathcal{A}^{\mathrm{op}}\mathcal{H}_\infty$.

The representation of $\mathcal{A}^{\mathrm{op}}$ is nondegenerate, and so can be extended uniquely to all of $\mathcal{A}_b^{\mathrm{op}}$ by defining $\overline{\pi}(b^{\mathrm{op}})$, $b \in \mathcal{A}_b$, on the dense subset $\mathcal{A}^{\mathrm{op}}\mathcal{H}_\infty$. The usual proof that $*$-homomorphisms are norm decreasing shows that $\overline{\pi}$: $\mathcal{A}_b^{\mathrm{op}} \to \mathcal{B}(\mathcal{H})$. Let $\{\phi_n\}$ be any local approximate unit, and observe that $\phi_n^{\mathrm{op}} \to 1$ strongly on $\mathcal{H}$. So for any $b \in \mathcal{A}_b$, $(\phi_n b)^{\mathrm{op}} \to b^{\mathrm{op}}$ strongly. For each n, the (densely defined) operator $[\mathcal{D}, (\phi_n b)^{\mathrm{op}}]$ extends to a bounded operator on all of $\mathcal{H}$, and converges strongly to $[\mathcal{D}, b^{\mathrm{op}}]$. Thus $[\mathcal{D}, b^{\mathrm{op}}]$ is bounded. Next repeatedly applying the same argument as above for the closed derivation δ, we find that $\Omega^*_{\mathcal{D}}(\mathcal{A}_b^{\mathrm{op}}) \subseteq \cap_m\mathrm{dom}\delta^m$. The form of the seminorms defining the topology of $\mathcal{H}_\infty$, shows that $\mathcal{A}_b^{\mathrm{op}}$ maps $\mathcal{H}_\infty$ to itself, and that right multiplication by $b \in \mathcal{A}_b$ is a continuous map on $\mathcal{H}_\infty$. □

Thus we can assume without loss of generality that the algebra of any smooth spectral triple is smooth, in the algebra sense, and that $\mathcal{D}$ has a core of smooth elements. Moreover, we may assume that when the Hilbert space is the completion of a smooth module, and Condition 16 is satisfied, that $\mathcal{H}_\infty$ is also a smooth module.

We will make these assumptions from now on, though it will have ramifications on the summability of elements of our algebra, [24]. The next result shows that elements of this smooth completion need not be integrable even in the commutative case.

PROPOSITION 20. *If X is a (finite-dimensional) geodesically complete Riemannian spin manifold, and* $(C_0^\infty(X), L^2(X, S), \mathcal{D}, \omega_{\mathbf{C}})$ *is the Dirac spectral triple of X, then the topology on* $C_0^\infty(X)$ *defined by the seminorms*

$$q_n(f) = \|\delta^n(f)\|, \qquad \delta(f) = [|\mathcal{D}|, f],$$

is the topology of uniform convergence of all derivatives.

Remark. It will follow from the proof that if $\delta^n(f_i) \to \delta^n(f)$ in norm for all n, then $[\mathcal{D}, f_i] \to [\mathcal{D}, f]$ in norm, and $[\mathcal{D}, f] \in \cap_m \mathrm{dom}\delta^m$, so we lose nothing by restricting to the above seminorms.

Proof. First of all, $\delta^n(f)$ is a bounded operator for all n and $f \in C_b^\infty(X)$. For a proof of this, see [11, p. 489], or note that $|\mathcal{D}|$ is a classical pseudodifferential operator of order one whose symbol commutes with *f*. Hence, $\delta(f)$ is a pseudodifferential operator of order zero, and so extends to a bounded operator on Hilbert space. Since $\delta(f)$ also has symbol commuting with the symbol of $|\mathcal{D}|$, one can now continue by induction. In addition, smoothness of a spectral triple, $\mathcal{A} \subseteq \cap_{m\geqslant 1}\mathrm{dom}\delta^m$ is equivalent to, [6, 8], $\mathcal{A} \subseteq \cap_{k,l\geqslant 1}\mathrm{dom}L^kR^l$, where L, R are the commuting operators

$$L(a) = (1+\mathcal{D}^2)^{-\frac{1}{2}}[\mathcal{D}^2, a], \qquad R(a) = [\mathcal{D}^2, a](1+\mathcal{D}^2)^{-\frac{1}{2}}.$$

As $|\mathcal{D}| - (1+\mathcal{D}^2)^{1/2}$ is bounded (and commutes with $|\mathcal{D}|$), we will employ a common sloppiness and write $\delta(a)$ for either of $[|\mathcal{D}|, a]$, $[(1+\mathcal{D}^2)^{1/2}, a]$, the difference between the two definitions of the derivation δ being a derivation with domain $\mathcal{B}(\mathcal{H})$. This does not affect the convergence issues addressed here. So with this caveat in place we have

$$L(a) = 2\delta(a) - (1+\mathcal{D}^2)^{-\frac{1}{2}}\delta^2(a), \qquad R(a) = 2\delta(a) + \delta^2(a)(1+\mathcal{D}^2)^{-\frac{1}{2}},$$

so convergence in the topology determined by the collection of seminorms $\{\|L^kR^l(\cdot)\|\}_{k,l\geqslant 0}$ is equivalent to convergence in the topology determined by the collection $\{\|\delta^n(\cdot)\|\}_{n\geqslant 0}$.

We begin by proving the lemma for the special case of $\mathbf{R}^n$. If $f\colon \mathbf{R}^n \to \mathbf{C}$ is smooth and bounded, and $\mathcal{D}$ is the Dirac operator on $\mathbf{R}^n$,

$$(1+\mathcal{D}^2)^{-\frac{1}{2}}[\mathcal{D}^2, f] = -(1+\mathcal{D}^2)^{-\frac{1}{2}} \sum_i \left(\frac{\partial^2 f}{\partial x_i^2} + 2\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_i} \right)$$

as an operator on (a dense subspace of) L^2 sections of the spinor bundle on $\mathbf{R}^n$. The norm of this element can be estimated using

$$\|(1+\mathcal{D}^2)^{-\frac{1}{2}}\| \leqslant 1 \quad \text{and} \quad \left\|(1+\mathcal{D}^2)^{-\frac{1}{2}}\frac{\partial}{\partial x_i}\right\| \leqslant 1,$$

both of which are easy to check using the functional calculus. We obtain

$$\|(1+\mathcal{D}^2)^{-\frac{1}{2}}[\mathcal{D}^2, f]\| \leqslant \sum_i 3\left\|\frac{\partial^2 f}{\partial x_i^2}\right\| + 2\left\|\frac{\partial f}{\partial x_i}\right\|.$$

Thus if $f_i \to f$ in the C^2 uniform topology, $L(f_i) \to L(f)$ in norm. For the converse, observe that $\|L(f_i) - L(f)\| \to 0$ is equivalent to

$$\sup_{\psi\neq 0} \frac{1}{\|\psi\|}\left\|(1+\mathcal{D}^2)^{-1/2}\sum_i \left(\frac{\partial^2(f_i - f)}{\partial x_i^2} + 2\frac{\partial(f_i - f)}{\partial x_i}\frac{\partial}{\partial x_i}\right)\psi\right\| \to 0.$$

There are a number of choices of ψ for which ψ and $\partial_i\psi$ are orthogonal, so these two terms must vanish independently, whence $f_i \to f$ in the uniform C^2 topology. One choice for the ψ would be

$$\psi(x_1, \ldots, x_n) = \prod_{i=1}^{n} H_m(x_i)\mathrm{e}^{-x_i^2/2}, \quad m \geqslant 0,$$

where the H_m are Hermite polynomials.

A similar result holds for $R(f)$. We have shown that $f_i \to f$ in the C^{2k} uniform topology if and only if $L^k(f_i - f) \to 0$ in norm (the above choice of ψ, for m sufficiently large also allows one to prove this in the same way as the $k = 1$ case above). As the analogous statement for R^l follows in exactly the same way, and R, L commute, we can conclude that $f_i \to f$ in the smooth uniform topology if and only if $L^k R^l(f_i - f) \to 0$ in norm.

Note that the result on $\mathbf{R}^n$ still holds if we replace $\mathcal{D}^2$ with an operator of the form

$$\tilde{\mathcal{D}}^2 = \sum_{ij} -g_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + a^i\frac{\partial}{\partial x_i} + b, \tag{17}$$

where the g_{ij} are smooth functions comprising a positive definite matrix at each point of $\mathbf{R}^n$, and the a^i, b are smooth functions. The only change is in the combinatorics of the proof.

So now let f be a smooth function on the manifold X all of whose derivatives are bounded, and suppose $f_i \to f$ in the smooth uniform topology. Thus for all $m \geqslant 0$

$$\sup_{|\alpha| \leqslant m} \sup_{x \in X} |\partial^\alpha (f_i - f)(x)| \to 0$$

as $i \to \infty$. By employing a locally finite partition of unity, we may consider the behaviour on a single chart U with coordinates $x_1, \ldots, x_p$. In these local coordinates, the square of the Dirac operator is a generalised Laplacian of the form 17, and so our results for $\mathbf{R}^n$ carry over to each coordinate chart. □

Thus both $C_0^\infty(X)$ and $C_b^\infty(X)$ are complete in the topology provided by the Dirac operator. With a little more work and applying results in [11, p. 489] one can also show

COROLLARY 21. *With the same hypotheses as the lemma, the topology on* $\mathrm{End}_{C_b^\infty(X)}(\Gamma_2(S))$ *defined by* δ *is the uniform convergence of all derivatives.*

The point is that the symbol of $|\mathcal{D}|$ is central in the endomorphism algebra.

Remark. The function $x \to (1 + x^2)^{-1/2}$ is an element of $C_0^\infty(\mathbf{R})$, but is not integrable. So despite the desirability of completeness of $\mathcal{A}$ in the δ-topology (holomorphic functional calculus, C^∞-functional calculus (see below), well-definedness of topological Hochschild homology (see below) and so on), this completeness is incompatible with integrability of elements of $\mathcal{A}$. We address this issue in [24].

Algebras of smooth spectral triples have a C^∞-functional calculus. This does not follow from the closure of $\mathcal{A}$ under the holomorphic functional calculus, instead being derived from the smoothness condition on the spectral triple and the completeness of $\mathcal{A}$ in the δ-topology.

PROPOSITION 22 (C^∞ Functional Calculus). *Let* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *be a smooth spectral triple, and suppose* $\mathcal{A} = \mathcal{A}_\delta$ *is complete. Let* $f: \mathbf{R} \to \mathbf{C}$ *be a* C^∞ *function in a neighbourhood of the spectrum of* $a = a^* \in \mathcal{A}$, *such that* $f(0) = 0$. *Then defining* $f(a) \in \overline{\mathcal{A}}$ *using the continuous functional calculus, we have* $f(a) \in \mathcal{A}$.

Proof. Without loss of generality, we may take f to be compactly supported, by multiplying f by a smooth compactly supported function which is identically one on a closed set containing the spectrum of a. Furthermore, $f(a) \in \overline{\mathcal{A}}$, since f is continuous and $f(0) = 0$. As $\mathcal{A} = \overline{\mathcal{A}} \cap \mathrm{dom}\delta^m d^i$, by Lemma 16, we need only show that $\|\delta^k d^i(f(a))\| < \infty$ for all k and $i = 0, 1$. We will concentrate on the estimates necessary for δ^k, the extra details necessary to deal with $d = [\mathcal{D}, \cdot]$ being similar.

As a is a self-adjoint operator on $\mathcal{H}$, we can form the operator e^{isa} for any $s \in \mathbf{R}$. If f is a smooth function on the spectrum of a, we have, by the functional calculus,

$$f(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(s) \mathrm{e}^{isa} \mathrm{d}s.$$

As $f^{(n)}$ is smooth and compactly supported for all $n \geqslant 0$, $\widehat{f^{(n)}}$ is of rapid decrease, and for all real s and $N \geqslant 1$,

$$|\widehat{f}(s)| \leqslant D_N(1+|s|)^{-N}, \tag{18}$$

for some positive constant D_N, by the Paley–Wiener Theorem, [22, Theorem IX.11]. A standard result is

$$\delta(\mathrm{e}^{isa}) = s\int_0^1 \mathrm{e}^{itsa} i\delta(a)\mathrm{e}^{i(1-t)sa}\mathrm{d}t. \tag{19}$$

Ignoring the factor of $1/\sqrt{2\pi}$ we employ this by computing

$$\|\delta(f(a))\| = \left\| \int_{-\infty}^{\infty} \widehat{f}(s)\delta(\mathrm{e}^{isa})\mathrm{d}s \right\| \leqslant \|\delta(a)\| \int_{-\infty}^{\infty} |\widehat{f}(s)||s|\mathrm{d}s,$$

and by Equation (18) above, this is finite. Employing Equation (19) again, we see that

$$\|\delta^2(f(a))\| \leqslant 2\|\delta(a)\|^2 \int |\widehat{f}(s)||s|^2\mathrm{d}s + \|\delta^2(a)\| \int |\widehat{f}(s)||s|\mathrm{d}s,$$

and again by Equation (18), this is finite. Iterating shows that for suitable constants C_j

$$\|\delta^k(f(a))\| \leqslant \sum_{j=1}^{k} \sum_{|p|=k,l(p)=j} C_j \|\delta^{p_1}(a)\| \cdots \|\delta^{p_j}(a)\| \int_{-\infty}^{\infty} |\widehat{f}(s)||s|^{k+1-\max p_j}\mathrm{d}s,$$

where $l(p) = j$ denotes a multiindex p with j terms, and $|p| = k$ says $\sum p_i = k$. As this is a finite sum of terms, and all the integrals are finite by Equation 18, we are done. □

Remark. The proof holds for the unital case without the requirement that $f(0) = 0$.

COROLLARY 23. *With f as above and $a \in \mathcal{A}_c$, $f(a) \in \mathcal{A}_c$.*

Proof. There is a $\phi \in \mathcal{A}_c$ such that $\phi a = a\phi = a$, and by the continuous functional calculus we have $\phi f(a) = f(a)\phi$. Let $\epsilon > 0$ and choose a polynomial p_ϵ with $p_\epsilon(0) = 0$ such that $\|f - p_\epsilon\| < \epsilon$, where the norm is the sup norm for functions on the spectrum of a. Then

$$\|f(a)\phi - f(a)\| = \|f(a)\phi - p_\epsilon(a)\phi + p_\epsilon(a) - f(a)\| < 2\epsilon.$$

As ϵ was arbitrary, $f(a) = f(a)\phi$. Finally, $f(a) \in \mathcal{A}$, $\phi \in \mathcal{A}_c$, so $f(a) \in \mathcal{A}_c$. □

LEMMA 24. *Suppose that $\mathcal{I}$ is a closed two-sided* commutative *ideal in the local algebra $\mathcal{A}_c \subseteq \mathcal{A}$, and that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple with $\mathcal{A}$ complete in the δ-topology. Then $\mathcal{I}$ is a smooth local algebra.*

Proof. If $\mathcal{I}_c$ has local units we are done. So suppose that $\mathcal{I}_c := \mathcal{I} \cap \mathcal{A}_c$ does not have local units (i.e. there exists at least one $a \in \mathcal{I}_c$ with no local unit in $\mathcal{I}_c$).

Choose $1/4 > \epsilon > 0$ and a smooth function $f \in C_c^\infty(\mathbf{R})$ such that

$$f(x) = \begin{cases} 0 & x \leqslant \epsilon, \\ 1 & 2 \geqslant x \geqslant 2\epsilon, \\ 0 & x \geqslant 2 + \epsilon. \end{cases}$$

Then by the C^∞-functional calculus, $f(a^*a) \in \mathcal{I}$ for all $a \in \mathcal{I}$.

Since $\mathcal{I}$ is commutative, the C^*-completion $\overline{\mathcal{I}}$ is the algebra of continuous functions vanishing at infinity on a locally compact Hausdorff space X. As we always assume that our C^*-completions are separable, X is metrisable (indeed Polish, [21]) and so paracompact. Consequently there exists a locally finite countable partition of unity, $\{\chi_n\}_{n \geqslant 1} \subseteq \overline{\mathcal{I}}$. By passing to a subsequence, we may suppose that the functions $\phi_n = \sum_{m=1}^n \chi_m$ satisfy $\phi_n \phi_{n+1} = \phi_n$ and for all $a \in \overline{\mathcal{I}}$, $\phi_n a \to a$ in norm as $n \to \infty$. However, while the functions ϕ_n are in the C^*-closure, they may not be in $\mathcal{I}$.

Now we first notice that the function $f(\phi_{n+1})$ is identically 1 on the support of $f(\phi_n)$ so $f(\phi_{n+1})f(\phi_n) = f(\phi_n)$. As $\mathcal{I}$ is dense in $\overline{\mathcal{I}}$, we may choose sequences $\{g_{ni}\}_{i \geqslant 1} \subseteq \mathcal{I}$ for each n with $g_{ni} \to \phi_n$ in the uniform norm. By passing to a subsequence we may also suppose that $\|g_{ni} - \phi_n\| < \epsilon$ for all i and n. Then $\mathrm{supp}(f(g_{ni})) \subseteq \mathrm{supp}(\phi_n)$ and for all $x \in X$, $\phi_n(x) = 1 \Rightarrow f(g_{ni})(x) = 1$. Consequently, $f(g_{n+1,n+1})f(g_{nn}) = f(g_{nn})$. The C^∞-functional calculus tells us that each $f(g_{nn}) \in \mathcal{I}$, and by construction $f(g_{nn})a \to a$ in norm for all $a \in \mathcal{I}$.

Finally, if we define $\mathcal{I}_n = \{a \in I\colon af(g_{nn}) = a\}$ and set $\tilde{\mathcal{I}}_c = \cup_n \mathcal{I}_n$, then $\tilde{\mathcal{I}}_c$ is dense in $\mathcal{I}$ and has local units. Lemma 3 now ensures that we have a local approximate unit $\{\phi_n\}_{n \geqslant 1} \subseteq \tilde{\mathcal{I}}_c$ such that for all $a \in \mathcal{I}$, $\phi_n a \to a$ in the smooth topology of $\mathcal{I}$. □

6. Locality

There are three main consequences of locality that we wish to prove here. First we have a result relating the invariance of the compactly supported elements of the Hilbert space to the locality of the spectral triple.

Given a spectral triple $(\mathcal{A}^{\mathrm{op}}, \mathcal{H}, \mathcal{D})$, we say that $\mathcal{D}$ has locally finite propagation speed if for all local approximate units $\{\phi_n\} \subseteq \mathcal{A}_c$ and all m there exists an M such that

$$\xi\phi_m = \xi \Rightarrow (\mathcal{D}\xi)\phi_M = \mathcal{D}\xi, \ \xi \in \mathcal{H}.$$

This is different to the usual definition of (locally) finite propagation speed for manifolds, [12], and is not uniform, in that M depends on m.

THEOREM 25. *Let* $(\mathcal{A}^{\rm op}, \mathcal{H}, \mathcal{D})$ *be a smooth spectral triple with* $\mathcal{H} = L^2(E, \tau)$, *where* $E_c = p\mathcal{A}_c^N$ *for some projection* $p \in M_N(\mathcal{A}_b)$ *and* $\tau\colon \mathcal{A}_c \to \mathbf{C}$ *a continuous, positive, faithful linear map. Then* $\mathcal{D}$ *has locally finite propagation speed if and only if* $\Omega^*_{\mathcal{D}}(\mathcal{A}_c)$ *has local units.*

Proof. First suppose that $\mathcal{D}$ has locally finite propagation speed. To see that $\Omega^*_{\mathcal{D}}(\mathcal{A}_c)$ has local units, let $\xi \in E_c$ have local unit $\phi_m^{\rm op}$, where $\{\phi_n\}_{n\geqslant 1} \subseteq \mathcal{A}_c$ is a local approximate unit. Then

$$\mathcal{D}\xi = \mathcal{D}\phi_m^{\rm op}\xi = [\mathcal{D}, \phi_m^{\rm op}]\xi + \phi_m^{\rm op}\mathcal{D}\xi,$$

and there exists $M \geqslant m$ such that

$$0 = (1 - \phi_M^{\rm op})\mathcal{D}\xi = (1 - \phi_M^{\rm op})[\mathcal{D}, \phi_m^{\rm op}]\xi.$$

As this is true for all ξ with $\phi_m^{\rm op}\xi = \xi$, the operator

$$(1 - \phi_M^{\rm op})[\mathcal{D}, \phi_m^{\rm op}]\phi_{m-1}^{\rm op} = 0 = (1 - \phi_M^{\rm op})[\mathcal{D}, \phi_{m-1}^{\rm op}]. \tag{20}$$

So now let $a \in \mathcal{A}_c$ have local unit ϕ_{m-1}. Then (dropping the op)

$$\begin{aligned}[\mathcal{D}, a] = [\mathcal{D}, \phi_M a] &= [\mathcal{D}, \phi_M]a + \phi_M[\mathcal{D}, a]\\ &= [\mathcal{D}, \phi_M]\phi_{m-1}a + \phi_M[\mathcal{D}, a]\\ &= [\mathcal{D}, \phi_{m-1}]a - \phi_M[\mathcal{D}, \phi_{m-1}]a + \phi_M[\mathcal{D}, a]\\ &= (1 - \phi_M)[\mathcal{D}, \phi_{m-1}]a + \phi_M[\mathcal{D}, a]\\ &= \phi_M[\mathcal{D}, a].\end{aligned}$$

Taking the adjoint shows that $[\mathcal{D}, a^*] = [\mathcal{D}, a^*]\phi_M$, and as $a^*\phi_{m-1} = a^*$, we may conclude that $[\mathcal{D}, a]\phi_M = [\mathcal{D}, a]$. Finally, if $\omega = \sum a_0^i[\mathcal{D}, a_1^i]\ldots[\mathcal{D}, a_k^i]$, then we choose a local unit for each a_j^i and $[\mathcal{D}, a_j^i]$. The largest of these will be a local unit for ω.

Now suppose that $\Omega^*_{\mathcal{D}}(\mathcal{A}_c)$ has local units, and let $\{\phi_n\} \subseteq \mathcal{A}_c$ be a local approximate unit. Then for all m there exists M such that $\phi_M^{\rm op}[\mathcal{D}, \phi_m^{\rm op}] = [\mathcal{D}, \phi_m^{\rm op}]$. If $\xi \in p\mathcal{A}_c^N$ has local unit $\phi_m^{\rm op}$, then

$$\begin{aligned}\mathcal{D}\xi &= \mathcal{D}\phi_m^{\rm op}\xi\\ &= [\mathcal{D}, \phi_m^{\rm op}]\xi + \phi_m^{\rm op}\mathcal{D}\xi\\ &= \phi_M^{\rm op}[\mathcal{D}, \phi_m^{\rm op}]\xi + \phi_M^{\rm op}\phi_m^{\rm op}\mathcal{D}\xi\\ &= (\mathcal{D}\xi)\phi_M.\end{aligned}$$

□

We now address two homological aspects of locality. The first is the *H*-unitality of the algebras of smooth spectral triples. This is equivalent to excision in Hochschild homology for these algebras, and so important in the general theory of noncommutative differential forms, as well as cyclic cohomology.

Many of the important properties of Chern characters rely on periodicity and Connes' exact sequence, [4], and thus on the Hochschild cohomology of the algebra. In addition, Connes' identification of the Hochschild class of the Chern character (that is the *integral*, [4, IV.2.γ] and [2]) naturally involves the Hochschild theory. So the real problem is that excision holds only for a certain class of algebras in the Hochschild theory. These are called the *H*-unital algebras, and our immediate purpose is to show that smooth local algebras which are complete for the topology induced by a spectral triple are topologically *H*-unital.

If $\mathcal{A}$ is a complete, locally convex, unital algebra, and M is a topological $\mathcal{A}$-bimodule, set

$$C_n(\mathcal{A}, M) = M \otimes \mathcal{A}^{\otimes n},\ \ n \geqslant 0$$

and define the topological Hochschild homology of $\mathcal{A}$ with coefficients in M, which we denote by $HH_*(\mathcal{A}, M)$, to be the homology of the complex $(C_n(\mathcal{A}, M), b)$. The map b: $C_n(\mathcal{A}, M) \to C_{n-1}(\mathcal{A}, M)$ is defined by (writing $a_0 \otimes a_1 \otimes \cdots \otimes a_k = (a_0, a_1, \ldots, a_k)$ and using the projective tensor product throughout, [19, Chapter X])

$$\begin{aligned} b(a_0, \ldots, a_n) \ &= \ (a_0 a_1, a_2, \ldots, a_n) + \sum_{i=1}^{n-1} (-1)^i \times \\ &\quad \times (a_0, a_1, \ldots, a_i a_{i+1}, \ldots, a_n) + (-1)^n (a_n a_0, a_1, \ldots, a_{n-1}) \end{aligned}$$

where $a_0 \in M$ and $a_i \in \mathcal{A}$, $i \geqslant 1$. Continuity of the multiplication shows that b is continuous. When $M = \mathcal{A}$ we write $C_n(\mathcal{A})$ and $HH_n(\mathcal{A})$, while the cycles of degree k are denoted by $Z_k(\mathcal{A}, M)$. It is shown in [17] that the homology of $(\tilde{C}(\mathcal{A}, M), b)$ also yields $H_*(\mathcal{A}, M)$, where

$$\tilde{C}_n(\mathcal{A}, M) = M \otimes \tilde{\mathcal{A}}^{\otimes n},\ \ \tilde{\mathcal{A}} = \mathcal{A}/\mathbf{C}\mathrm{Id}.$$

This is called the normalised Hochschild complex. There is also a reduced complex $\tilde{C}_*(\mathcal{A})_{\mathrm{red}}$ defined by the short exact sequence

$$0 \longrightarrow \mathbf{C}[0] \longrightarrow \tilde{C}_*(\mathcal{A}) \longrightarrow \tilde{C}_*(\mathcal{A})_{\mathrm{red}},$$

where $\mathbf{C}[0]$ is the complex consisting of $\mathbf{C}$ in degree zero only. Note that the reduced complex is the same as the normalised complex except that in degree zero we have $\tilde{\mathcal{A}}$ instead of $\mathcal{A}$. In general we obtain an exact sequence

$$\begin{aligned} 0 \ \longrightarrow \ & HH_1(\mathcal{A}) \longrightarrow HH_1(\mathcal{A})_{\mathrm{red}} \longrightarrow \mathbf{C} \longrightarrow HH_0(\mathcal{A}) \\ & \longrightarrow HH_0(\mathcal{A})_{\mathrm{red}} \longrightarrow 0, \end{aligned}$$

and $HH_n(\mathcal{A}) = HH_n(\mathcal{A})_{\mathrm{red}}$ for $n \geqslant 2$. If $\mathcal{A}$ is nonunital and topologically *H*-unital (see below), then we find that

$$HH_n(\mathcal{A}) \cong HH_n(\mathcal{A}^+)_{\mathrm{red}}\ n \geqslant 0 \tag{21}$$

where on the left we define the Hochschild homology as above.

The obvious way to define Hochschild homology of nonunital algebras is precisely as one would for unital algebras, since the definition makes no use of a unit (this is what we did on the left hand side of Equation 21). The problem is that this definition does not always give rise to a well-behaved homology theory; in particular it does not always agree with other natural definitions or guarantee that excision holds. However, for the following class of algebras, this definition is apppropriate, and agrees with the other possible definitions, [17, Section 1.4].

DEFINITION 12. A topological algebra $\mathcal{A}$ is topologically H-unital if the topological bar complex is acyclic.

The bar complex is the following

$$\cdots \to \mathcal{A}^{\otimes n+1} \xrightarrow{b'} \mathcal{A}^{\otimes n} \xrightarrow{b'} \cdots \xrightarrow{b'} \mathcal{A}^{\otimes 2} \xrightarrow{b'} \mathcal{A}$$

where

$$b'(a_0, a_1, \ldots, a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_n).$$

For any unital algebra $\mathcal{A}$, this complex is acyclic, and the following theorem of Wodzicki shows that this is the correct generalisation of unitality for homological purposes. Note that the proof of Wodzicki's Excision Theorem extends to the topological setting provided we are working with a complete algebra, as all the relevant maps are continuous.

PROPOSITION 26 (Wodzicki [30]). *Let $\mathcal{A}$ be a Fréchet algebra. Then the following are equivalent*:

(1) *$\mathcal{A}$ is H-unital*
(2) *$\mathcal{A}$ satisfies excision in topological Hochschild homology*.

We recall that an approximate unit $\{\phi_n\}$ in a locally convex algebra $\mathcal{A}$ is bounded if for every seminorm q on $\mathcal{A}$ there is a constant $C > 0$ such that $q(\phi_n) \leqslant C$ for all n.

PROPOSITION 27. *Let $\mathcal{A}_c \subseteq \mathcal{A}$ be a smooth*, *local algebra. Then $\mathcal{A}_c$ with the inductive limit topology is H-unital. Moreover*, *if $\mathcal{A}$ has a bounded local approximate unit*, *then $\mathcal{A}$ is topologically H-unital*.

Proof. Consider first $\mathcal{A}_c$ with its inductive limit topology. Since $\mathcal{A}_c$ has local units, it is H-unital, [17, Proposition 1.4.8].

Next consider $\mathcal{A}$. We show that it is H-unital using the local properties of $\mathcal{A}_c$ and the existence of a bounded local approximate unit. To employ the local properties of $\mathcal{A}_c$, observe that the closure of the algebraic tensor product $\mathcal{A}_c^{\odot n}$ in the projective tensor product topology is $\mathcal{A}^{\otimes n}$. As $\mathcal{A}_c$ has local units, it is algebraically H-unital, [17, Proposition 1.4.8, Lemma 1.6.6]. In other words, the algebraic bar complex of $\mathcal{A}_c$ is acyclic.

Since b' is continuous, it extends to the closure of the algebraic bar complex in the projective tensor product topology, and $\ker b'$ is closed and so Fréchet.

If $\omega = \sum a_0^i \otimes \cdots \otimes a_n^i \in \ker b' \cap \mathcal{A}^{\otimes n+1}$ then for any $a \in \mathcal{A}$ we have

$$b'(a \otimes \omega) = a\omega - a \otimes b'\omega = a\omega,$$

where $a\omega := \sum aa_0^i \otimes \cdots \otimes a_n^i$. Our aim is to show that $\omega \in \mathrm{Im} b'$.

As $Z_n = \ker b' \cap \mathcal{A}^{\otimes n+1}$ is Fréchet and $Z_{nc} = \ker b' \cap \mathcal{A}_c^{\odot n+1}$ is dense, there is a sequence $\{c_i\} \subseteq Z_{nc}$ such that $c_i \to 0$ in Z_n as $i \to \infty$, and $\omega = \sum_i \lambda_i c_i$, with $\sum |\lambda_i| < \infty$, [25, p. 133]. The sequence of partial sums, $\omega_N = \sum^N \lambda_i c_i$ converges to ω in the topology of Z_n.

Let $\{\phi_n\}$ be a bounded local approximate unit. Then, by passing to a subsequence if necessary, we may suppose that $\phi_i c_i = c_i$. Define $\rho_N \in \mathcal{A}^{\otimes n+2}$ by $\rho_N = \sum_{i=1}^N \lambda_i \phi_i \otimes c_i$. Then $b'\rho_N = \omega_N$, and our main concern is the convergence of the sequence $\{\rho_N\}$. The topology on $\mathcal{A}^{\otimes n+2}$ is given by the seminorms q_α where α is a multiindex and

$$q_\alpha(c) = \inf \sum_i q_{\alpha_0}(b_0^i) \cdots q_{\alpha_{n+1}}(b_{n+1}^i),$$

where the infimum is taken over all tuples such that $c = \sum b_0^i \otimes \cdots \otimes b_{n+1}^i$ and the q_{α_i} are seminorms on $\mathcal{A}$. So we have

$$\begin{aligned} q_\alpha(\rho_N - \rho_M) &= \inf \sum q_{\alpha_0}(b_0^i) \cdots q_{\alpha_{n+1}}(b_{n+1}^i) \\ &\leqslant \sum_{i=M+1}^{N} |\lambda_i| q_{\alpha_0}(\phi_i) q_{\alpha'}(c_i) \\ &\leqslant C_{q_{\alpha_0}} \sum_{i=M+1}^{N} |\lambda_i| q_{\alpha'}(c_i) \to 0. \end{aligned}$$

Here $(\alpha_0, \alpha') = \alpha$ as multiindices. Hence $\{\rho_N\}$ is Cauchy, and there exists $\rho \in \mathcal{A}^{\otimes n+2}$ such that $\rho_N \to \rho$. Moreover,

$$b'\rho = b' \lim_N \rho_N = \lim_N b'\rho_N = \lim_N \omega_N = \omega.$$

Hence, $\mathrm{Im} b' = \ker b'$ and the bar complex of $\mathcal{A}$ is acyclic, and so $\mathcal{A}$ is H-unital. □

COROLLARY 28. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a smooth spectral triple with $\mathcal{A}$ complete in the δ-topology and such that the representation of $\mathcal{A}$ on $\mathcal{H}$ is nondegenerate. Then $\mathcal{A}$ is topologically H-unital.*

Proof. We show that $\mathcal{A}$ possesses a bounded local approximate unit (in fact many). For any local approximate unit $\{\phi_n\}$ we have $\sup_n \|\phi_n\| = 1$. Moreover, since the representation is nondegenerate, $\phi_n \xi \to Id_{\mathcal{H}} \xi$ for all $\xi \in \mathcal{H}$. Consequently, for $\xi \in \mathrm{dom}|\mathcal{D}|$, $\delta(\phi_n)\xi \to 0$. As $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a smooth spectral triple,

$\delta(\phi_n)$ extends to a bounded operator on $\mathcal{H}$ for all n. Using the density of dom$|\mathcal{D}|$, we see that the sequence $\delta(\phi_n)$ is strongly convergent to zero, and so is bounded. Hence there is a constant C_1 such that $\sup_n \|\delta(\phi_n)\| \leqslant C_1$.

By considering the dense subspace dom$|\mathcal{D}|^m \subseteq \mathcal{H}$, using the boundedness of $\delta^m(\phi_n)$, and the fact that $\delta^m(\phi_n)\xi \to 0$ for all $\xi \in$ dom$|\mathcal{D}|^m$, we obtain constants $C_m > 0$ such that $\sup_n \|\delta^m(\phi_n)\| \leqslant C_m$ for all n. Similar comments apply when we consider $\|\delta^m([\mathcal{D}, \phi_n])\|$.

Finally, as $\mathcal{A}$ has a bounded local approximate unit for the δ-topology, and $\mathcal{A}$ is complete in this topology, the previous result shows that $\mathcal{A}$ is H-unital. □

Thus local algebras with bounded approximate units satisfy excision in Hochschild homology. Since closed *commutative* ideals of local algebras are local, Lemma 24, this shows that every ideal of a commutative local algebra with a bounded approximate unit also satisfies excision in Hochschild homology, allowing the use of long exact sequences. For unital algebras we have no such guarantee. These difficulties were noted by Gorokhovsky, [9], in his review of [23]. We suspect the answer in the unital and *commutative* case is to demand that all maximal ideals are local algebras, thus ensuring that all ideals satisfy excision in Hochschild homology, by Lemma 24.

The other aspect of Hochschild homology relevant to noncommutative geometry is Connes' axiom of orientability for a spectral triple, [3]. A tentative reformulation, consistent with the commutative case, is to ask for a sequence of compactly supported volume forms $\{c_i\}_{i\geqslant 1}$ (i.e. Hochschild cycles for the algebra $\mathcal{A}_c$) such that their representatives, $\pi(c_i)$, on the Hilbert space converge strongly to the identity operator (or the $\mathbf{Z}_2$-grading in the even case). More precisely, orientability for a (p, ∞)-summable spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ (see [24] for this notion in the nonunital case), p integral, is the statement that

$$\exists\{c_i\}_{i\geqslant 1} \subseteq Z_p(\mathcal{A}_c, \mathcal{A}_c \otimes \mathcal{A}_c^{\mathrm{op}}),\ c_i = \sum_{k=1}^{N(i)} a_0^k \otimes b_0^{k\mathrm{op}} \otimes a_1^k \otimes \cdots \otimes a_p^k$$

such that the sequence of operators

$$\sum_{k=1}^{N(i)} \pi(a_0^k)\pi(b_0^{k\mathrm{op}})[\mathcal{D}, \pi(a_1^k)] \cdots [\mathcal{D}, \pi(a_p^k)]$$

converges strongly to the identity (or the $\mathbf{Z}_2$-grading in the even case).

Whilst this formulation should be regarded as provisional, along with all the axioms of noncommutative manifolds, we note that it is satisfied by a complete noncompact Riemannian spin manifold X (take a sequence of representatives of the class generating $H_c^p(X, \mathbf{R})$ and use the usual Clifford representation) and for Connes and Dubois-Violette's noncommutative manifolds (see the Appendix).

Finally, we describe a formulation of Poincaré Duality for local algebras. In the unital case, Poincaré Duality was formulated in [4, 3, 14].

DEFINITION 13. If A is a unital algebra, we say that A satisfies spin^c Poincaré Duality if there is a class $\mu \in K^*(A \otimes A^{\mathrm{op}})$ such that $K_*(A) \bigcap \mu \cong K^*(A)$.

Here $\cap$ is a special instance of the Kasparov product, [4, 12, 13], called the cap product.

The justification for such a formulation of Poincaré Duality in noncommutative geometry, as well as the stronger spin or Real form, is taken up extensively in [3–5]. In the nonunital case we would like an analogous formulation.

The usual statement of Poincaré Duality for an orientable noncompact homology manifold of dimension p is the isomorphism, [20, Section 67],

$$\cdot \bigcap \Gamma\colon H_c^*(X;\mathbf{Z}) \stackrel{\cong}{\to} H_{p-*}(X;\mathbf{Z}) \tag{22}$$

where $\Gamma \in H_p^\infty(X;\mathbf{Z})$ is the fundamental class, $H_c^*(X;\mathbf{Z})$ is the compactly supported cohomology, $H_*(X;\mathbf{Z})$ is the usual singular homology of X defined using finite chains, and $H_*^\infty(X;\mathbf{Z})$ is the homology defined using infinite but locally finite chains.

EXAMPLE. The basic example is $\mathbf{R}^p$. In this case the Poincaré Lemmas give us

$$\begin{array}{ll} H_0(\mathbf{R}^p) = \mathbf{Z} & H_k(\mathbf{R}^p) = \{0\}\ \forall k > 0, \\ H_p^\infty(\mathbf{R}^p) = \mathbf{Z} & H_k^\infty(\mathbf{R}^p) = \{0\}\ \forall k \neq p, \\ H_c^p(\mathbf{R}^p) = \mathbf{Z} & H_c^k(\mathbf{R}^p) = \{0\}\ \forall k \neq p. \end{array}$$

Checking that the cap product with either of the generators $[\mathbf{R}^p] \in H_*^\infty(\mathbf{R}^p)$ yields the isomorphism 22 is straightforward.

To translate this into K-theory, we need analogues of all these groups and the fundamental class. The usual definition of the K-theory of a nonunital algebra (noncompact space) is precisely the analogue of compactly supported cohomology. From results in [1] comparing relative and absolute K-homology of a nonunital algebra, we know that the analogue of locally finite homology is $K^*(A)$, where A is nonunital.

Lastly, we need to make a sensible definition of finitely supported or compactly supported K-homology. This seemingly has no good counterpart in the world of arbitrary σ-unital C^*-algebras, but for those that are the completions of local algebras we have a clean definition.

So suppose that $\mathcal{A}_c \subseteq \mathcal{A} \subseteq \overline{\mathcal{A}} = A$ is a local algebra and $\{\phi_n\}$ is a local approximate unit. Define subalgebras

$$\mathcal{A}_n = \{a \in \mathcal{A}\colon a\phi_n = \phi_n a = a\},\ \ A_n = \overline{\mathcal{A}_n},$$

and define the compactly supported K-homology of A to be

$$K_c^*(A) := \lim_{\leftarrow} K^*(A_n).$$

The inverse limit is defined with respect to the obvious inclusion maps $i_{nm}\colon A_n \hookrightarrow A_m$ defined whenever $n \leqslant m$. Recall that elements of the inverse limits are sequences $(F_1, F_2, \ldots, F_k, F_{k+1}, \ldots)$ with $F_n \in K^*(A_n)$ and such that $F_n = i^*_{n,n+1}F_{n+1}$ for all n. Note that typically $\phi_n \notin A_n$ because $\phi_n^2 \neq \phi_n$.

LEMMA 29. *Let $\mathcal{A}$ be a smooth local algebra with $\overline{\mathcal{A}} = A$ separable. Then $K^*_c(A)$ is independent of the local approximate unit used to define it.*

Proof. Let $\{\phi_n\}$ and $\{\psi_n\}$ be local approximate units. Then, by virtue of the fact that for all $n \in \mathbf{N}$ and $\phi_n, \psi_n \in \mathcal{A}_c$, there is some $m \geqslant n$ so that for all $N > m$ we have

$$\psi_N\phi_n = \phi_n\psi_N = \phi_n \quad \text{and} \quad \psi_n\phi_N = \phi_N\psi_n = \psi_n.$$

So for all n there exists $N > n$ such that $A^\phi_n \subseteq A^\psi_N$, where A^ϕ_n is the subalgebra defined using ϕ_n and A^ψ_N is defined using ψ_N. Clearly this is symmetric.

Next, we notice that the family of inclusion maps $i_n\colon A_n \to A$ provide us with a family of surjective maps $i_n^*\colon K^*(A) \to K^*(A_n)$. The compatability of these inclusions shows that they assemble to give a surjective map $(i_n^*)\colon K^*(A) \to K^*_c(A)$. Thus if $\mathbf{F} = (F_1, F_2, \ldots, F_n, \ldots)$ is an element of $K^*_c(A^\phi)$, the compactly supported K-homology group defined using units ϕ_n and inclusions i_n, there is an $F \in K^*(A)$ such that $F_n = i_n^*F$ for all n. Write $K^*_c(A^\psi)$ for the group defined using units ψ_n and corresponding inclusions j_n. Now define a map

$$\Phi\colon K^*_c(A^\phi) \to K^*_c(A^\psi)$$

by setting

$$\Phi(F_1, F_2, \ldots, F_n, \ldots) = (j_1^*F, j_2^*F, \ldots, j_n^*F, \ldots).$$

This is well-defined, for if F is in the kernel of all the maps j_n^*, then it is in the kernel of all the maps i_n^*. This also shows that the map is injective. The symmetry of the relation between the two local units allows one to easily check that Φ is an isomorphism by constructing the obvious (analogously defined) inverse map. □

The key points that we require are that the K-theory of a local algebra A is actually the direct limit of the K-theory groups $K_*(A_n)$, and that the image of the cap product with μ is consequently contained in $K^*_c(A)$.

LEMMA 30. *Suppose that $\mathcal{A}$ is a smooth local algebra with $\overline{\mathcal{A}} = A$ separable. Then*

$$K_*(A) = \lim_{\rightarrow} K_*(A_n).$$

Furthermore, if $\mu \in K^*(A \otimes A^{\mathrm{op}})$ then

$$K_*(A) \bigcap \mu \subseteq K^*_c(A).$$

Proof. First, the algebras A_n together with the inclusions $i_{nm}\colon A_n \to A_m$ form a directed system of algebras, and clearly $A = \overline{\cup_n A_n}$. Similarly, extending the inclusion maps to unital maps, [12], $A^+ = \overline{\cup_n A_n^+}$. Now K_* is a continuous functor, so

$$K_*(A) := \widetilde{K}_*(A^+) = \lim_{\to} \widetilde{K}_*(A_n^+).$$

So the first statement is proved. For the second statement, we first must explain what we mean. Using the inclusion maps, we have a family of surjective maps, as above,

$$i_n^*\colon K^*(A) \to K^*(A_n).$$

Since $i_{n,n+1}^* i_{n+1}^* = i_n^*$, we have a well-defined surjective map

$$(i_n^*)\colon K^*(A) \to K_c^*(A),$$

which takes $[\kappa] \in K^*(A)$ to $(i_n^*[\kappa])_{n\geqslant 1} \in K_c^*(A)$. To say that $K_*(A) \cap \mu \subseteq K_c^*(A)$ is the same thing as saying that (i_n^*) is an injective map on the range of the cap product.

To prove this, we need to show that if

$$i_n^*([x] \cap \mu) = i_n^*([y] \cap \mu), \quad [x], [y] \in K_*(A),$$

for all $n \geqslant 1$, then $[x] \cap \mu = [y] \cap \mu$. The statement $i_n^*([x] \cap \mu) = i_n^*([y] \cap \mu)$ for all $n \geqslant 1$, means that the restriction of the action of A to A_n on any Fredholm module representing the class of $[x] \cap \mu$ gives a K-homology class for A_n which agrees with that obtained by restricting the action on $[y] \cap \mu$. As $A = \overline{\cup_n A_n}$, we conclude that any representatives of these two classes define equivalent Fredholm modules for A, and so $[x] \cap \mu = [y] \cap \mu$. Hence $K_*(A) \cap \mu$ can be regarded as lying in $K_c^*(A)$. □

Remark. We can not deduce from the above that $[x] = [y]$ unless we know that the cap product with μ is injective on $K_*(A)$.

COROLLARY 31. *When X is a paracompact, complete spin^c manifold, the map*

$$\cdot \bigcap \mu\colon K_*(A) \to K_c^*(A)$$

is an isomorphism, where $A = C_0(X)$ and $\mu \in K^*(A)$ is the K-homology class of any Dirac operator on the fundamental complex spinor bundle.

Proof (Sketch). We have already seen that the smooth functions on X vanishing at infinity form a local algebra with the compactly supported functions forming the dense subalgebra with local units. It is shown in many places, for example [12, Proposition 10.2.10], that the Dirac operator on a complete space defines a K-homology class $[\mathcal{D}] \in K^*(A \otimes A)$. The previous result shows that the cap product with $[\mathcal{D}]$ lies in the compactly supported K-homology, so we need only show that this map is an isomorphism.

To prove this, one notes that the isomorphism is true for $\mathbf{R}^p$, and then uses a suitably chosen partition of unity together with a standard Mayer-Vietoris argument to patch together the various isomorphisms, [12]. □

So now we come to our proposed K-theoretic definition of Poincaré Duality for the nonunital case.

DEFINITION 14. Suppose we have a unitization of a local algebra $\mathcal{A}_c \subseteq \mathcal{A} \subseteq A$ and a class $\mu \in K^*(A \otimes A^{\mathrm{op}})$. Then we say that A satisfies Poincaré Duality and that μ is a fundamental class if

$$\cdot \bigcap \mu\colon K_*(A) \to K_c^*(A) \tag{23}$$

is an isomorphism. If one (and so both) of these groups is finitely generated, we say that A is of finite type.

7. Conclusion

In the context of local algebras we have managed to reproduce most of the important smoothness features of spectral triples over unital algebras. The C^∞-functional calculus is the main result of this kind. Locality allows us to prove H-unitality, and formulate Poincaré Duality. A finite propagation property for the operator $\mathcal{D}$ is shown to be equivalent to the algebra of 'differential forms' being local.

Open problems remaining from this paper are the topological issues concerning completing $\mathcal{A}$ in the topology coming from the seminorms $\|\delta^k(a)\|$, H-unitality for ideals of smooth, unital algebras, and the locality of closed ideals of noncommutative smooth, local algebras.

8. Appendix

Recently, Connes and Dubois-Violette presented a noncommutative $\mathbf{R}^n$, [7], and more generally isometric deformations of geodesically complete spin manifolds by torus actions. We briefly sketch how these examples can naturally be described in terms of the theory presented here.

The special case of $\mathbf{R}^n_\theta$ is simply described using the results of [7], particularly Section 11, where the authors note that their constructions continue to hold for the fixed point algebra $C_c^\infty(\mathbf{R}^n_\theta)$ of the diagonal action of the torus on $C_c^\infty(\mathbf{R}^n) \otimes C^\infty(T^n_\theta)$. This algebra is easily seen to have local units (by taking local units for $C_c^\infty(\mathbf{R}^n)$, tensoring by the identity of $C^\infty(T^n_\theta)$ and averaging by the diagonal action). The C^*-completion is simply the fixed point algebra of the usual C^*-tensor product $C_0(\mathbf{R}^n) \otimes A_\theta$.

The authors built their Hilbert spaces from smooth modules, starting with the Fréchet space $\Gamma_c(\mathbf{R}^n, S) \otimes C^\infty(T^n_\theta)$, where the first factor in the tensor product is the module of compactly supported smooth sections of the spinor bundle of $\mathbf{R}^n$. One takes the subspace $\Gamma(\mathbf{R}^n_\theta, S)$ of elements invariant under the torus action (the action on the left must be a double cover of that on the algebra, see [7, p. 38]). This is a full module, as can be seen using Proposition 13. We complete it using the product of the Lebesgue measure and the normalised trace on A_θ. This is a positive and faithful map on $C_c^\infty(\mathbf{R}^n_\theta)$, and we may use this to define various completions of both modules and ideals, as described in this paper.

The compactly supported volume forms for the manifold may be averaged by the torus action, giving rise to Hochschild cycles for $C_c^\infty(\mathbf{R}^n_\theta)$ which satisfy our proposed version of Connes' orientability axiom. The Dirac operator is just the restriction of the tensor product $\mathcal{D} \otimes 1$ to $\Gamma(\mathbf{R}^n_\theta, S)$, and using this one can show that the algebra $\Omega^*_{\mathcal{D}}(C_c^\infty(\mathbf{R}^n_\theta))$ has local units.

This brief description is easily adapted to the general case of isometric deformations of geodesically complete spin manifolds by torus actions. Thus our general theory is well adapted to discussing a wide range of new examples.

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