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Smoothness criteria in surface wave tomography

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SUMMARY

We discuss and develop further the methods of surface wave tomography in the frame of the geometric ray approximation. The general approach for determining the lateral phase or group velocity distribution, which is a standard 2-D tomography problem, involves linearization, representation of the unknown function as a series in some basis functions, and evaluation of the coefficients by the methods of linear algebra. If the wave paths cover the area under investigation non-uniformly, the basis functions should not be chosen *a priori*, but constructed proceeding from the pattern of paths. Different criteria for constructing the basis functions are compared, and a relation between them is considered.

A more preferable approach is joint interpretation of phase and group velocity data for different periods, because it allows the information about phase velocity variations to be enlarged due to the use of the group velocity data. Both the phase and group traveltimes are represented as linear functionals of the unknown phase slowness corrections. A specific form of the data kernels allows the basis functions to be represented as a product of two functions, one depending on the horizontal coordinates, and the other on frequency.

Key words: basis functions, phase and group velocities, surface waves, tomography.

INTRODUCTION

Phase or group surface wave velocities observed along different paths are widely used to study lateral variations and anisotropy of lithospheric structure (see for example Sato & Santo 1969; Avetisyan & Yanovskaya 1973; Berteussen, Levshin & Ratnikova 1982; Nakanishi & Anderson 1982, 1983, 1984; Yanovskaya 1982; Nishimura & Forsyth 1985; Maaz *et al.* 1985; Tanimoto & Anderson 1985; Suetsugu & Nakanishi 1985; Montagner 1986; Sabitova & Yanovskaya 1986; Gobarenko, Nikolova & Yanovskaya 1987; Hadiouche & Jobert 1988; Montagner & Jobert 1988). The final aim of most of these studies is to map local values of the velocities and—if possible—to display azimuthal anisotropy for a set of periods. In this paper we shall ignore anisotropy estimation and concentrate on determination of lateral heterogeneities. For qualitative conclusions on lateral heterogeneities of the lithosphere, or for determination of a 3-D model, the maps of local surface wave velocities are used. The 3-D model may be constructed by solving a set of 1-D inverse problems to determine vertical velocity and density distributions at each point of the area under investigation from corresponding dispersion curves. Montagner (1986) has shown that this approach is equivalent to 3-D inversion, if the vertical and horizontal velocity variations are decoupled (this assumption is practically fulfilled). Such a two-step approach is more preferable than a direct 3-D inversion, because in this case the calculations are much simpler. Thus the 3-D inverse problem using surface wave data may be separated into two independent problems: 2-D inversion for phase or group velocities for a fixed period, resulting in lateral variations of these velocities for a set of periods, and 1-D inversion for the vertical distribution of the elastic parameters, such as P and S velocities as well as density.

However, this approach does not allow the phase and group velocities to be processed jointly, whereas they are related. Besides, it is preferable to use the data for all periods simultaneously to achieve better resolution. It should also be noted that to calculate the vertical velocity distribution the phase velocity dispersion curves are preferable: the non-uniqueness of the solution is much stronger, if we use group velocity dispersion curves, because one and the same group velocity curve corresponds to an infinite set of phase velocity curves and consequently to an infinite set of vertical velocity distributions. However, the phase velocities are obtained from seismological observations with more difficulty, so that usually a poor set of phase velocity data is available. Therefore, it is better to use jointly phase and group velocity data and to combine the data for different periods. Such an approach was proposed by Yanovskaya *et al.* (1988). It was shown by model examples that to reconstruct the main features of the phase velocity distributions within a wide range of periods it is sufficient to use an extremely scanty amount of phase velocity data: a deficiency of them is filled up by the data on group velocity.

Since the 1-D inversion of phase (or group) velocity dispersion curves poses no problems, we shall neglect this last step in solving the 3-D inverse problem and formulate the surface wave tomography problem either as a standard 2-D seismic tomography problem on a plane or on a spherical surface, or as a so-called '3-D' inverse problem in the domain x, y, T (or θ, φ, T) using the phase and/or group velocity data over different paths and corresponding to different periods.

We shall review and compare the methods developed for solving both problems (2-D and 3-D), taking into account the following properties of surface wave observations:

(1) the paths of the waves always cover the area under investigation non-uniformly and the total amount of them is never too large; and

(2) lateral surface wave velocity variations are small enough, so that a starting model can be taken as laterally homogeneous, and the ray paths in the starting model are straight lines or great circles.

GENERAL APPROACH TO THE TOMOGRAPHY PROBLEM

In this paper we shall treat only traveltime inversion. As is well-known, in this case the geometric ray approximation is allowed for determining the relation between the data and the model, if the linear dimensions of heterogeneities under investigation are much larger than the wavelength, and the effect of small-scale heterogeneities on the traveltimes may be neglected. This assumption is practically fulfilled for surface waves, so when we refer below to a 'tomography problem', we shall imply geometric tomography.

For regional studies we may use Cartesian coordinates x, y, because a small part of a spherical surface is easily reduced to a plane by a suitable transformation of the coordinates and velocity (Yanovskaya 1982; Jobert & Jobert 1983). For global studies it is necessary to solve the problem in spherical coordinates θ, φ. The general approach for solving the plane and the spherical problem is the same, but the functions, in which the solution is represented, are different for these two cases. Therefore for brevity we shall treat later the plane case in detail, and the results for the spherical case will be adduced in the points, where a difference exists. We emphasize the 2-D problem here: the 3-D problem in the domain x, y, T may be reduced to the 2-D one, as we shall see in the last section.

The general 2-D tomography problem is formulated as follows. The data set are the traveltimes $t_i (i = 1, 2, \ldots, N)$ along different paths L_i. The data involve experimental errors ε_i described by the covariance matrix $\mathbf{R}_t$. We assume a velocity V_0 in the starting model and calculate traveltimes t_{0i} along the same paths in the starting model.

Since the lateral velocity variations are small enough, we may determine the relative slowness correction

$$m(x, y) = [V(x, y)^{-1} - V_0^{-1}]/V_0^{-1}$$

instead of the unknown velocity distribution $V(x, y)$, and use the traveltime residuals $\delta t_i = t_i - t_{0i}$.

According to the above-mentioned assumption, $|m(x, y)| \ll 1$. Then

$$\delta t_i = \int_{L_{0i}} m(x, y) V_0^{-1}\, ds + \varepsilon_i \tag{1}$$

where L_{0i} is a segment of a straight line.

A general solution of this problem may be written in the form

$$m(x, y) = \sum \alpha_j \varphi_j(x, y)$$

where $\varphi_j(x, y)$ are basis functions, which may be either assumed *a priori* (Nolet 1987) or constructed proceeding from the given data set in explicit form (Tarantola & Nercessian 1984; Ditmar & Yanovskaya 1987) or in implicit form (Suetsugu & Nakanishi 1985).

If the basis functions are assumed *a priori* and their number is less than the number of the data, the unknown coefficients α_i, forming the vector $\mathbf{a}$, are determined by minimizing the functional

$$(\mathbf{Sa} - \boldsymbol{\delta}\mathbf{t})^{\mathrm{T}} \mathbf{R}_t^{-1} (\mathbf{Sa} - \boldsymbol{\delta}\mathbf{t})$$

where

$$S_{ij} = \int \varphi_j(x, y) V_0^{-1}\, ds.$$

The main disadvantage of this approach is that the solution depends seriously on the choice of the set of basis functions. In particular, *a priori* regionalization is an example of this approach, and difficulties connected with it are well-known (Suetsugu & Nakanishi 1985). If the basis functions are spherical harmonics or polynomials, then in the parts of the area covered densely by wave paths the solution becomes too smooth, and on the other hand, in the parts where the number of paths is small, the solution may contain spurious anomalies connected with peculiarities of the basis functions. Therefore, it is more expedient to determine a set of basis functions in concordance with the pattern of paths.

The methods for determining the basis functions fitting this requirement seem to be principally different. But as we show later, they may be reduced to the same principle. The difference between them results from *a priori* assumptions concerning the unknown function. Below we describe the approach proposed by the authors (Ditmar & Yanovskaya 1987) in a form valid for inaccurate data. Then the other criteria for constructing the basis functions will be reviewed and compared.

CRITERION BASED ON AN ASSUMPTION ABOUT THE SMOOTHNESS OF THE SOLUTION

In the above-mentioned paper (Ditmar & Yanovskaya 1987) the authors proposed to determine the unknown function

$m(x, y)$ proceeding from the following conditions:

(i) the solution must be smooth in the following sense

$$\int_S |\nabla m|^2 \, d\mathbf{r} = \min; \tag{2a}$$

(ii) on the contour C_S of the area S

$$(\partial m/\partial n)_{C_S} = 0; \tag{2b}$$

(iii) the solution obeys the constraints corresponding to accurate data

$$\int_S G_i(\mathbf{r}) m(\mathbf{r}) \, d\mathbf{r} = \delta t_i$$

where

$$\int_S G_i(\mathbf{r}) m(\mathbf{r}) \, d\mathbf{r} = \int_{L_i} m(x, y) V_0^{-1} \, ds.$$

This approach can be easily expanded to inaccurate data in the framework of Tikhonov's 'regularization' method (Tikhonov & Arsénine 1976). According to this method we shall look for the solution proceeding from the following assumption on the function $m(\mathbf{r})$:

$$(\boldsymbol{\delta}\mathbf{t} - \mathbf{Gm})^{\mathrm{T}} \mathbf{R}_t^{-1} (\boldsymbol{\delta}\mathbf{t} - \mathbf{Gm}) + \alpha \int_S |\nabla m|^2 \, d\mathbf{r} = \min, \tag{3}$$

where α is a parameter of regularization, the notation

$$(\mathbf{Gm})_i = \int_S G_i(\mathbf{r}) m(\mathbf{r}) \, d\mathbf{r}$$

is implied, and $\mathbf{G}(\mathbf{r})$ is regarded as a vector with components $G_i(\mathbf{r})$. The unknown parameter α must be chosen so that the first term in (3) would be equal to the total number N of data. Later we shall refer to (3) as *criterion 1*.

We assume S to be the whole plane (C_S is removed to infinity) and $m(\mathbf{r})$ to be a constant at infinity. So that $(\partial m/\partial n)_{C_S} = 0$.

Minimizing (3) and taking into account the Green's formula

$$\int_S (\nabla m, \nabla \eta) \, d\mathbf{r} = -\int_S \eta \, \Delta m \, d\mathbf{r} + \int_C \eta (\partial m/\partial n) \, dl$$

and the above-mentioned assumption about $(\partial m/\partial n)_{C_S}$, we obtain the following integro-differential equation:

$$\alpha \, \Delta m = \mathbf{G}^{\mathrm{T}}(\mathbf{r}) \mathbf{R}_t^{-1} (\mathbf{Gm} - \boldsymbol{\delta}\mathbf{t}). \tag{4}$$

The Green's function of the 2-D Laplace equation in the x, y plane is

$$g(\mathbf{r}', \mathbf{r}) = (2\pi)^{-1} \ln |\mathbf{r}' - \mathbf{r}|^{-1} = g(\mathbf{r}, \mathbf{r}').$$

We may look for the solution of (3) in the form

$$m(\mathbf{r}) = \int F(\mathbf{r}') g(\mathbf{r}', \mathbf{r}) \, d\mathbf{r}' + \Phi(\mathbf{r}) \tag{5}$$

where $F(\mathbf{r})$ and $\Phi(\mathbf{r})$ are unknown functions, $\Phi(\mathbf{r})$ being a harmonic function.

Substituting (5) into (4) and using the notations

$$\psi_i(\mathbf{r}) = \int G_i(\mathbf{r}') g(\mathbf{r}', \mathbf{r}) \, d\mathbf{r}',$$

$$C_i = \int G_i(\mathbf{r}) \Phi(\mathbf{r}) \, d\mathbf{r},$$

$$U_i = \int \psi_i(\mathbf{r}) F(\mathbf{r}) \, d\mathbf{r},$$

we obtain

$$\alpha F(\mathbf{r}) = \mathbf{G}^{\mathrm{T}}(\mathbf{r}) \mathbf{R}_t^{-1} (\boldsymbol{\delta}\mathbf{t} - \mathbf{U} - \mathbf{C}) \tag{6}$$

Multiplying (6) by $\psi_i(\mathbf{r})$ and integrating over the area we obtain an algebraic system of equations with respect to $\mathbf{U}$ and $\mathbf{C}$:

$$\alpha \mathbf{U} = \mathbf{S} \mathbf{R}_t^{-1} (\boldsymbol{\delta}\mathbf{t} - \mathbf{U} - \mathbf{C}), \tag{7}$$

where

$$S_{ij} = \iint G_i(\mathbf{r}) G_j(\mathbf{r}') g(\mathbf{r}, \mathbf{r}') \, d\mathbf{r} \, d\mathbf{r}'$$

$$= \int_{L_i} \int_{L_j} \ln |\mathbf{r}' - \mathbf{r}|^{-1} V_0^{-1} \, ds_i V_0^{-1} \, ds_j.$$

Solving (7) for $\mathbf{U}$ and substituting the solution into (6) we obtain

$$F(\mathbf{r}) = \mathbf{G}^{\mathrm{T}}(\mathbf{r}) \mathbf{R}_t^{-1} (\mathbf{S} \mathbf{R}_t^{-1} + \alpha \mathbf{I})^{-1} (\boldsymbol{\delta}\mathbf{t} - \mathbf{C})$$

and

$$m(\mathbf{r}) = \boldsymbol{\psi}^{\mathrm{T}}(\mathbf{r}) \mathbf{R}_t^{-1} (\mathbf{S} \mathbf{R}_t^{-1} + \alpha \mathbf{I})^{-1} (\boldsymbol{\delta}\mathbf{t} - \mathbf{C}) + \Phi(\mathbf{r}). \tag{8}$$

Now let us recall the assumption about the behaviour of $m(\mathbf{r})$ at infinity. The first term of the right-hand side of (8) increases at infinity like $\ln r$ (since $\psi_u(\mathbf{r}) \to t_{0i} \ln r^{-1}$, $r \to \infty$), and the second is a harmonic function, which either is a constant or increases like a polynomial of x, y at $r \to \infty$. Therefore, in order that $m(\mathbf{r})$ is constant at infinity, $\Phi(\mathbf{r})$ must be a constant (c) and the coefficient of $\ln r$ ($r \to \infty$) in the first term must be equal to zero. Consequently

$$\mathbf{C} = c \mathbf{t}_0 \tag{9a}$$

and

$$\mathbf{t}_0^{\mathrm{T}} \mathbf{R}_t^{-1} (\mathbf{S} \mathbf{R}_t^{-1} + \alpha \mathbf{I})^{-1} (\boldsymbol{\delta}\mathbf{t} - c \mathbf{t}_0) = 0,$$

or

$$\mathbf{t}_0^{\mathrm{T}} (\mathbf{S} + \alpha \mathbf{R}_t)^{-1} (\boldsymbol{\delta}\mathbf{t} - c \mathbf{t}_0) = 0. \tag{9b}$$

Hence

$$c = \frac{\mathbf{t}_0^{\mathrm{T}} (\mathbf{S} + \alpha \mathbf{R}_t)^{-1} \, \boldsymbol{\delta}\mathbf{t}}{\mathbf{t}_0^{\mathrm{T}} (\mathbf{S} + \alpha \mathbf{R}_t)^{-1} \mathbf{t}_0}$$

and

$$m(\mathbf{r}) = \boldsymbol{\psi}^{\mathrm{T}}(\mathbf{r}) (\mathbf{S} + \alpha \mathbf{R}_t)^{-1} \, \boldsymbol{\delta}\mathbf{t} + \frac{1 - \boldsymbol{\psi}^{\mathrm{T}} (\mathbf{S} + \alpha \mathbf{R}_t)^{-1} \mathbf{t}_0}{\mathbf{t}_0^{\mathrm{T}} (\mathbf{S} + \alpha \mathbf{R}_t)^{-1} \mathbf{t}_0} \mathbf{t}_0^{\mathrm{T}} (\mathbf{S} + \alpha \mathbf{R}_t)^{-1} \, \boldsymbol{\delta}\mathbf{t}. \tag{10}$$

For accurate data $\mathbf{R}_t = 0$, and we obtain the solution proposed earlier (Ditmar & Yanovskaya 1987).

Although in the spherical case it is unnecessary to assume a boundary condition similar to that in the plane case, the final solution will also be of the form (10), where the Green's function $g(\mathbf{r}', \mathbf{r})$ must be substituted by the function

$$\bar{g}(\theta', \varphi'; \theta, \varphi) = -(4\pi)^{-1} \ln [2(1-\cos\Delta)], \tag{11}$$

where Δ is the great circle distance between the points (θ, φ) and (θ', φ'). This result follows from the fact that the function (11) is not a real Green's function on the sphere, so that to express a function $m(\theta, \varphi)$, which fits the Poisson equation

$$\Delta m = f(\theta, \varphi)$$

in the form

$$m(\theta, \varphi) = \iint f(\theta', \varphi')\bar{g}(\theta', \varphi'; \theta, \varphi) \sin\theta \, d\theta \, d\varphi$$

we must assume

$$\iint f(\theta, \varphi) \sin\theta \, d\theta \, d\varphi = 0,$$

which is analogous to (9b).

The system of basis functions in this case is following:

$$\varphi_i(r) = \begin{cases} \psi_i(\mathbf{r}), & 1 \le i \le N, \\ 1, & i = N+1. \end{cases} \tag{12}$$

The function $\psi_i(\mathbf{r})$ has a maximum along the ray, the derivative being discontinuous on the ray. The behaviour of these functions is illustrated in Fig. 1.

It should be noted that in general we may require any rate of smoothness, so that the second term in the left-hand side of (3) will be as follows:

$$\int \| \underbrace{\nabla\nabla \cdots \nabla}_{n} m(\mathbf{r}) \|^2 \, d\mathbf{r}$$

where $\underbrace{\nabla\nabla \cdots \nabla}_{n}$ is the tensor of nth derivatives of the

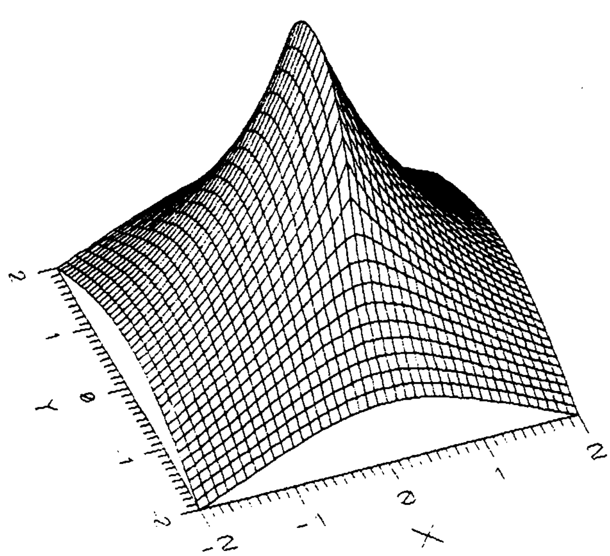

Figure 1. Example of the basis function $\psi(x, y)$ for a ray along the y-axis between the points $y = -1$ and $y = +1$. It is clear that the derivative $d\psi/dx$ is discontinuous on the ray.

function $m(x, y)$, and $\| \cdots \|$ denotes its norm:

$$\| \underbrace{\nabla\nabla \cdots \nabla}_{n} m \|^2 = \sum_{k=0}^{n} C_n^k \left(\frac{\partial^n m(x, y)}{\partial x^k \, \partial y^{n-k}} \right)^2 .$$

To obtain a unique solution for $m(\mathbf{r})$ it is necessary to add some conditions on the contour.

OTHER CRITERIA FOR CONSTRUCTING THE BASIS FUNCTIONS

Criterion 2. Tarantola & Nersessian (1984) proposed to solve the tomography problem proceeding from a Bayesian approach. If the *a priori* covariance function of the model $R_{m0}(\mathbf{r}, \mathbf{r}')$ is assumed, $m(\mathbf{r})$ is determined by minimizing the functional

$$\mathbf{m}^T \mathbf{R}_{m0}^{-1} \mathbf{m} + (\mathbf{Gm} - \boldsymbol{\delta}\mathbf{t})^T \mathbf{R}_t^{-1} (\mathbf{Gm} - \boldsymbol{\delta}\mathbf{t}) \tag{13}$$

where $\mathbf{m} = m(\mathbf{r})$ and $\mathbf{R}_{m0} = R_{m0}(\mathbf{r}, \mathbf{r}')$ are regarded as a vector and a matrix in Hilbert space, so that

$$\mathbf{m}^T \mathbf{R}_{m0}^{-1} \mathbf{m} = \iint m(\mathbf{r}) R_{m0}^{-1}(\mathbf{r}, \mathbf{r}') m(\mathbf{r}') \, d\mathbf{r} \, d\mathbf{r}'. \tag{14}$$

The solution is

$$\mathbf{m} = \mathbf{R}_{m0} \mathbf{G}^T (\mathbf{R}_t + \mathbf{G}\mathbf{R}_{m0}\mathbf{G}^T)^{-1} \boldsymbol{\delta}\mathbf{t} \tag{15}$$

and the basis functions are

$$\varphi_i(\mathbf{r}) = \int R_{m0}(\mathbf{r}, \mathbf{r}') G_i(\mathbf{r}') \, d\mathbf{r}' = \int R_{m0}(\mathbf{r}, \mathbf{r}') V_0^{-1} \, ds. \tag{16}$$

A delicate point of this approach is how to choose the *a priori* covariance function. Tarantola & Nersessian (1984) proposed different functions to describe the covariance between the points $\mathbf{r}$ and $\mathbf{r}'$. The simplest analytical form of the function is Gaussian:

$$R_{m0}(\mathbf{r}, \mathbf{r}') = \sigma^2 \exp(-|\mathbf{r} - \mathbf{r}'|^2 / 2L^2) \tag{17}$$

where L is a correlation length. The behaviour of the basis function (16) with $\mathbf{R}_{m0}$ defined by (16) is shown in Fig. 2.

Montagner (1986) proposed a covariance function on a sphere:

$$R_{m0}(M, M') = \sigma(M)\sigma(M') \exp[(\cos\Delta - 1)/L^2] \tag{18}$$

where Δ is the angular distance between the points M and M', and L is also a correlation length.

The correlation length L is a kind of smoothing parameter: if it is too large, the solution will be very smooth, and if it is too small, the solution will be concentrated along rays. Montagner (1986) discussed the problem how to choose L: it must be larger than the wavelength, and if we assume the effective path width to be equal L, the paths must ensure a good coverage of the area. However, a difficulty arises in those cases when the coverage of the area by the paths is non-uniform: in principle it is possible to assume L as a function of $\mathbf{r}$, but it would complicate calculations too much.

The ART solution (Suetsugu & Nakanishi 1985) may be regarded as a particular case corresponding to *criterion 2.* In this case the unknown function is replaced by a set of parameters representing the values of m in blocks. As was shown by Herman (1980), the iterative procedure ART

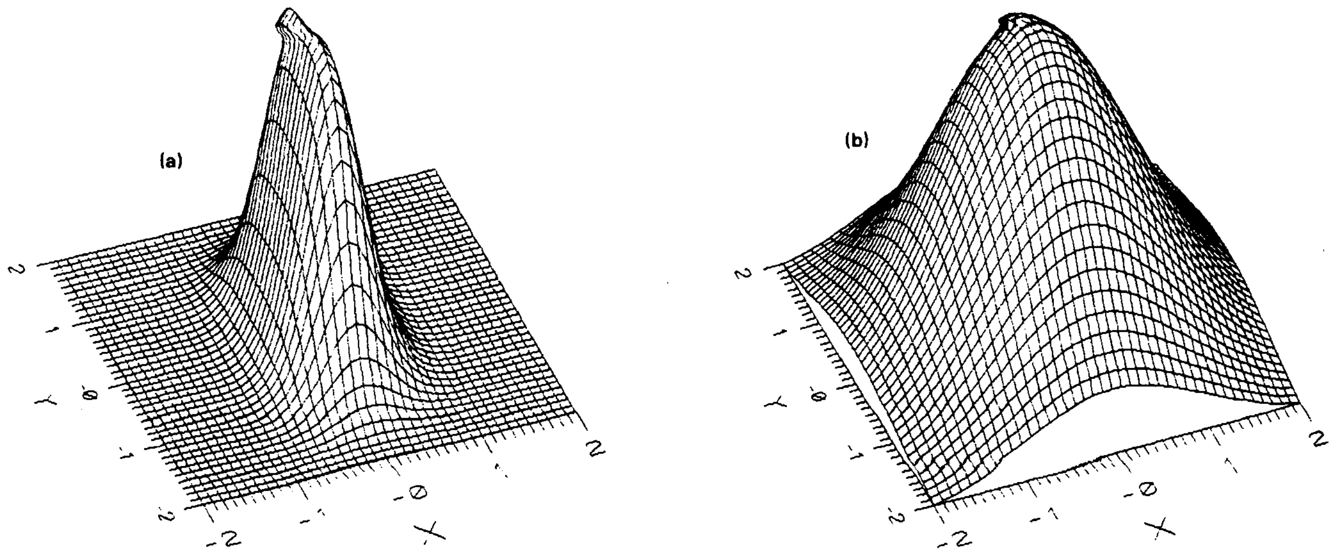

Figure 2. Examples of the basis functions (16) for the same ray as in Fig. 1 for two different values of L: (a), $L = 0.333$; (b), $L = 1.0$.

leads to a solution with minimum norm, so that it is the same as that obtained by minimizing the functional (13), in which $\mathbf{R}_{m0} = \sigma^2\mathbf{I}$ where $\mathbf{I}$ is a unit matrix.

Criterion 3. According to the approach developed by Backus & Gilbert (1968, 1970), Chou & Booker (1979) and Yanovskaya (1980) proposed to construct a solution as a 'local average':

$$\hat{m}(\mathbf{r}) = \int A(\mathbf{r}, \mathbf{r}')m(\mathbf{r}')\, d\mathbf{r}' \tag{19}$$

and determine the averaging (or resolving) kernel $A(\mathbf{r}, \mathbf{r}')$ proceeding from a certain deltaness criterion. Ditmar & Yanovskaya (1987) have shown that a solution fitting the criteria (2a), (2b) may be regarded as a local average, the averaging kernel $A(\mathbf{r}, \mathbf{r}')$ satisfying the following deltaness criterion:

$$s(\mathbf{r}) = \int |\mathbf{E}(\mathbf{r}, \mathbf{r}') - \mathbf{H}(\mathbf{r}, \mathbf{r}')|^2\, d\mathbf{r}' = \min, \tag{20a}$$

$$\int A(\mathbf{r}, \mathbf{r}')\, d\mathbf{r}' = 1, \tag{20b}$$

where

$$\operatorname{div} \mathbf{E}(\mathbf{r}, \mathbf{r}') = A(\mathbf{r}, \mathbf{r}'),$$
$$\mathbf{H}(\mathbf{r}, \mathbf{r}') = (2\pi)^{-1}V' \ln |\mathbf{r} - \mathbf{r}'|\,.$$

This criterion is an extension to the 2-D case of that proposed by Johnson & Gilbert (1972) for the 1-D inverse ray problem.

For inaccurate data the criterion (20a) turns into the following:

$$s(\mathbf{r}) + \alpha \operatorname{Var}[\hat{m}(\mathbf{r})] = \min \tag{21}$$

where the trade-off parameter α is the same as in (3).

An alternative criterion is obtained if the 'spread' $s(\mathbf{r})$ is defined as follows

$$s(\mathbf{r}) = \int \left[\int A(\mathbf{r}, \mathbf{r}')f(\mathbf{r}', \boldsymbol{\rho})\, d\mathbf{r}' - f(\mathbf{r}, \boldsymbol{\rho})\right]^2 d\boldsymbol{\rho} \tag{22}$$

where $f(\mathbf{r}, \mathbf{r}')$ is a particular function. In principle this function may be chosen arbitrarily, so that different $f(\mathbf{r}, \mathbf{r}')$ result in different criteria. But to ensure better 'deltaness' this function should have some extremal points. For instance, it may have a maximum in $\mathbf{r} = \mathbf{r}'$.

It is easy to show that the criterion (21) with $s(\mathbf{r})$ determined by (22) results in the following form of the basis functions:

$$\varphi_j(\mathbf{r}) = \int_{L_{0j}} F(\mathbf{r}, \mathbf{r}_j)V_0^{-1}\, ds_j \tag{23}$$

where

$$F(\mathbf{r}, \mathbf{r}') = \int f(\mathbf{r}, \boldsymbol{\rho})f(\mathbf{r}', \boldsymbol{\rho})\, d\boldsymbol{\rho}. \tag{24}$$

RELATION BETWEEN THE METHODS FOR DETERMINING THE BASIS FUNCTIONS

The most natural *a priori* assumption about the unknown function seems to be its smoothness of a certain degree.* Therefore it is important to realize whether the three above-mentioned criteria are principally different, or if they may be deduced from each other.

At first we consider the relation between the criteria 1 and 2. Assuming the *a priori* covariance function (17) it can be shown that (14) with (17) is reduced to the following expression:

$$\begin{aligned} S = (2\pi)^{-1}(\sigma L)^{-2} \sum_{n=0}^{\infty} \frac{1}{n!}(L^2/2)^n \Bigg[&\int \|\underbrace{\nabla\nabla\cdots\nabla}_{n} m\|^2\, d\mathbf{r} \\ &+ \frac{(-1)^{n+1}}{n+1}(L^2/2)\int_C m \frac{\partial \Delta^n m}{\partial x_i} n_i\, dl \\ &+ \frac{(-1)^{n+1}}{(n+1)(n+2)}(L^2/2)^2 \int_C \frac{\partial m}{\partial x_i}\frac{\partial^2 \Delta^n m}{\partial x_i\, \partial x_j} n_j\, dl + \cdots \Bigg] \end{aligned} \tag{25}$$

* There is an analogy with spline-approximation of a function known in some discrete points: it is well-known that the power of the spline depends on the required degree of smoothness.

where C is the contour of the area, Δ the Laplacian, $x_1 = x$, $x_2 = y$, and n the normal to the contour C.

If we assume the conditions on the contour

$$m = \partial m/\partial n = \partial^2 m/\partial n^2 = \cdots = 0,$$

which are equivalent to $\partial^k m/\partial x_q^1 \, \partial x_2^{k-q} = 0$ for all k and $q \le k$, then the integrals over the contour in (25) vanish and $m(\mathbf{r})$ will be determined by minimizing the functional

$$\alpha(2\pi)^{-1}(\sigma L)^{-2} \sum_{n=0}^{\infty} \frac{1}{n!}(L^2/2)^n \int \| \underbrace{\nabla\nabla\cdots\nabla}_{n} m \|^2 \, d\mathbf{r} + (\mathbf{Gm} - \boldsymbol{\delta}\mathbf{t})^T \mathbf{R}_t (\mathbf{Gm} - \boldsymbol{\delta}\mathbf{t}). \tag{26}$$

The same functional is obtained for the spherical case on a global scale, if the covariance function is assumed in the form (17) with σ = constant (see Appendix).

Thus the Bayesian approach for determining the unknown function is reduced to a certain criterion for its smoothness, involving some conditions on the contour of the area. The method, outlined in the Appendix, is valid for determining the criterion of smoothness corresponding to a wide class of correlation functions.

The relation between criteria 1 and 3 was mentioned above (see equation 20a,b). Similar deltaness criteria may be derived also for the cases when a higher degree of smoothness is required.

Now we consider the relation between criteria 2 and 3. Tarantola & Valette (1982) and Montagner (1986) discussed this problem proceeding from analysis of the *a posteriori* covariance function $R_m(\mathbf{r}, \mathbf{r}')$ for the solution satisfying criterion 2, which is expressed as follows:

$$R_m(\mathbf{r}, \mathbf{r}') = \int [\delta(\mathbf{r} - \mathbf{r}'') - A(\mathbf{r}, \mathbf{r}'')] R_{m0}(\mathbf{r}'', \mathbf{r}') \, d\mathbf{r}' \tag{27}$$

or in matrix notation,

$$\mathbf{R}_m = (\mathbf{I} - \mathbf{A})\mathbf{R}_{m0}$$

where $\mathbf{I}$ is identity operator, and

$$\mathbf{A} = \mathbf{R}_{m0}\mathbf{G}^T(\mathbf{R}_t + \mathbf{G}\mathbf{R}_{m0}\mathbf{G}^T)^{-1}\mathbf{G}$$

is a generalization of the resolving kernel. It is obvious that the closer $A(\mathbf{r}, \mathbf{r}')$ approaches the δ-function, the better the solution is resolved, and the *a posteriori* covariance function tends to vanish.

Now let us put the following question: if the solution is determined proceeding from criterion 2, in what sense tends the resolving kernel to the δ-function? In other words, what deltaness criterion for the resolving kernel corresponds to the solution (15)? It turns out that this criterion is (21) with $s(\mathbf{r})$ determined by (22), the function $f(\mathbf{r}, \mathbf{r}')$ being related to the *a priori* covariance function $R_{m0}(\mathbf{r}, \mathbf{r}')$. According to the Backus–Gilbert approach the resolving kernel is expressed as a linear combination of the data kernels:

$$A(\mathbf{r}, \mathbf{r}') = \sum \lambda_i(\mathbf{r}) G_i(\mathbf{r}'). \tag{28}$$

Using the notation $\boldsymbol{\Lambda}^T = [\lambda_1(\mathbf{r}), \ldots, \lambda_N(\mathbf{r})]$, we may write (28) in the form

$$\mathbf{A} = \boldsymbol{\Lambda}^T\mathbf{G}. \tag{28a}$$

Inserting (28a) into (19) and (22) we obtain

$$\operatorname{var}[\hat{m}(\mathbf{r})] = \boldsymbol{\Lambda}^T\mathbf{R}_t\boldsymbol{\Lambda},$$

$$s(\mathbf{r}) = \boldsymbol{\Lambda}^T\mathbf{G}\mathbf{f}\mathbf{f}^T\mathbf{G}^T\boldsymbol{\Lambda} - 2\boldsymbol{\Lambda}^T\mathbf{G}\mathbf{f}\mathbf{f}^T + g(\mathbf{r}),$$

where $\mathbf{f}$ denotes the function $f(\mathbf{r}, \mathbf{r}')$ and $\mathbf{f}\mathbf{f}^T = \mathbf{F}$ is defined by the right-hand side of (24),

$$(\mathbf{G}\mathbf{f}\mathbf{f}^T\mathbf{G}^T)_{ij} = \iint G_i(\mathbf{r}')F(\mathbf{r}', \mathbf{r}'')G_j(\mathbf{r}'') \, d\mathbf{r}' \, d\mathbf{r}'',$$

$$g(\mathbf{r}) = \int [f(\mathbf{r}, \mathbf{r}')]^2 \, d\mathbf{r}'.$$

Minimizing (21) with respect to λ_i we obtain the following solution:

$$\boldsymbol{\Lambda} = (\alpha\mathbf{R}_t + \mathbf{G}\mathbf{f}\mathbf{f}^T\mathbf{G}^T)^{-1}\mathbf{G}\mathbf{f}\mathbf{f}^T$$

and

$$\hat{m}(\mathbf{r}) = \boldsymbol{\Lambda}^T \, \boldsymbol{\delta}\mathbf{t} = \mathbf{f}\mathbf{f}^T\mathbf{G}^T(\alpha\mathbf{R}_t + \mathbf{G}\mathbf{f}\mathbf{f}^T\mathbf{G}^T)^{-1} \, \boldsymbol{\delta}\mathbf{t}.$$

Comparing this expression with the solution (15) we immediately obtain that they coincide, if $\alpha = 1$ and $\mathbf{f}\mathbf{f}^T = \mathbf{R}_{m0}$.

If $\mathbf{R}_{m0}$ is the Gaussian function (17), then

$$f(\mathbf{r}, \mathbf{r}') = \sqrt{2/\pi}L^{-1}\sigma \exp(|\mathbf{r} - \mathbf{r}'|^2/L^2).$$

If the criterion (21) is supplemented with the normalization condition (20a), then interpretation of the solution from the standpoint of criterion 2 yields

$$m(\mathbf{r}) = \mu(\mathbf{r}) + m_0,$$

where

$$\int \mu(\mathbf{r}) \, d\mathbf{r} = 0,$$

the functional (14) is changed by replacing $m(r)$ to $\mu(r)$, and the additional normalization condition allows the additional unknown m_0 to be determined.

Thus the *a priori* assumption, which is used for constructing a solution of the tomography problem, may be interpreted differently: either in the frame of a Bayesian approach, the *a priori* covariance function being assumed; or as a certain degree of smoothness of the unknown function; or in the frame of Backus–Gilbert technique by approaching the resolving kernel to the δ-function. Certainly, the choice of a specific form of the criterion (for example, an analytical form and parameters of the covariance function) depends on our assumptions and the *a priori* knowledge about the unknown function. However, it turns out that if the data set is complete enough, the solution will depend mainly on the data set rather than on the *a priori* assumption. This fact is demonstrated by the following model example.

The initial velocity distribution has been taken in the following form:

$$V(x, y) = 3.0 + 1.5 \exp[-\alpha(x^2 + y^2)]$$

with $\alpha = 4.5 \times 10^{-4}$. The traveltimes have been calculated by integration along linear paths, in order to eliminate the effect of non-linearity of the problem. The paths are shown in Fig. 3. A different density of the paths in different parts

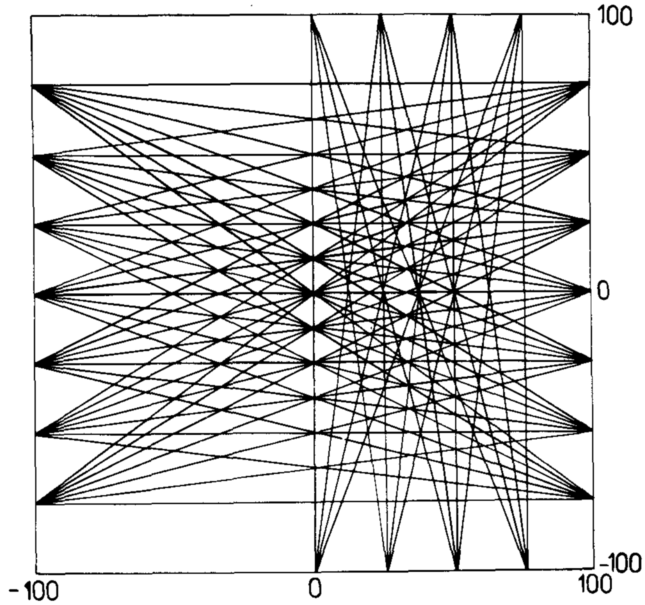

Figure 3. A pattern of paths for testing the solutions corresponding to different criteria.

of the area was taken deliberately, to demonstrate how the smoothness of the solution is affected by the data set.

Figs 4(a) and (b) show the solutions obtained proceeding from the criteria 1 and 2, respectively, the *a priori* covariance function being assumed to be (17) with $L = 12.5$. The main features of the velocity distributions, as well as the rate of smoothness in the left and in the right parts of the area are practically the same for both solutions.

So we may conclude that if our assumption about linearity of paths is valid, the basis functions can be taken in the general form (23) with an arbitrary function $F(\mathbf{r}, \mathbf{r}')$: the solutions will differ in small details. In reality the paths are not linear—they are bent due to lateral heterogeneity, and traveltimes along such paths differ from those corresponding to linear paths. When the traveltimes calculated along real paths were taken as the initial data for inversion, (certainly with greater variance than in the preceding case), and the problem was solved under the assumption on linearity, we obtained the solutions having the same main features as in Figs 4(a) and (b), but more smoothed (e.g. maximum velocity in the centre of the area was about 4.1 instead of 4.3–4.35), but the difference between the two solutions corresponding to criteria 1 and 2 became even less due to smoothing.

Since errors in real seismological data are usually larger than the traveltime residuals resulting from non-linearity of paths, it is convenient for processing real data to choose a function $F(\mathbf{r}, \mathbf{r}')$, which yields simple calculations and stability of the solution. For example, if $F(\mathbf{r}, \mathbf{r}') = \ln|\mathbf{r} - \mathbf{r}'|$, which corresponds to the criterion 1, the basis functions and the elements of the matrix $\mathbf{S}$ may be written in closed form and calculated without numerical integration, which is necessary when the solution is constructed proceeding from the criterion 2.

3-D PROBLEM

As was mentioned above, the so-called 3-D tomography problem for surface wave data is estimating the phase velocity corrections as a function of the variables x, y, T (or x, y, ω) from the phase and/or group traveltimes along different paths and corresponding to different periods. A special aspect of this problem is caused by the fact that the data are expressed as integrals along lines in the plane x, y, the integrands involving the derivatives of the unknown function with respect to ω.

If we denote

$$\mu(x, y, \omega) = V^{-1}(x, y, \omega) - V_0^{-1}(\omega)$$

where $V(x, t, \omega)$ is the unknown phase velocity and $V_0(\omega)$ is

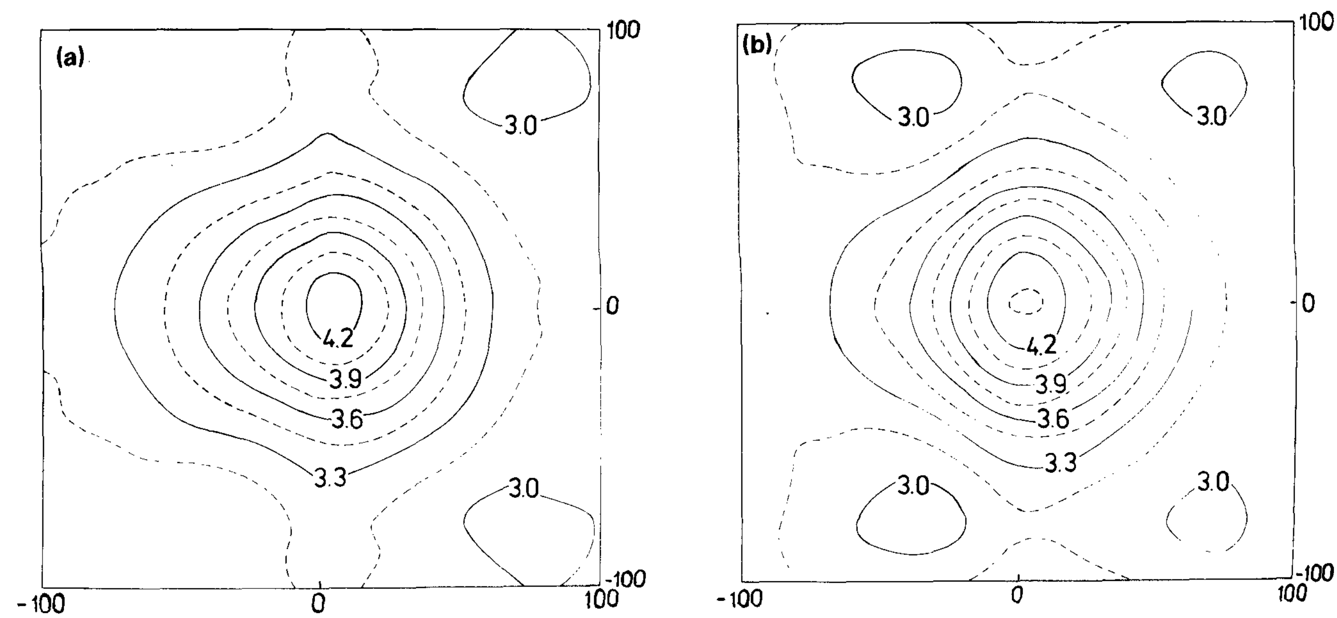

Figure 4. The isolines of velocity: (a), the solution corresponding to the *criterion 1*; (b), the solution corresponding to the *criterion 2*.

the phase velocity in a starting model, then

$$\int_{L_{0j}} \mu(x, y, \omega_{kj})\, ds = \delta t^{\mathrm{ph}}(\omega_{kj}),$$
$$\int_{L_{0j}} \left[\frac{d(\omega\mu)}{d\omega}\right]_{\omega=\omega_{qj}} ds = \delta t^{\mathrm{gr}}(\omega_{qj}). \tag{29}$$

In (29) ω_{kj} $(k = 1, 2, \ldots, K_j)$ and ω_{qj} $(q = 1, 2, \ldots, Q_j)$ are the frequencies, for which the phase and group velocities (or traveltimes) are available for the jth path.

The relationship (29) may be written in the form of linear functionals of the unknown function $\mu(x, y, \omega)$:

$$t^{\mathrm{ph}}(\omega_{kj}) = \iiint \mu(x, y, \omega)[H_j(x, y)\delta(\omega - \omega_{kj})]\, dx\, dy\, d\omega,$$
$$t^{\mathrm{gr}}(\omega_{qj}) = \iiint \mu(x, y, \omega)\left[\omega H_j(x, y)\frac{d\delta(\omega - \omega_{qj})}{d\omega}\right] dx\, dy\, d\omega, \tag{30}$$

where

$$\iint H_j(x, y)\, dx\, dy = \int_{L_{0j}} ds = l_{0j}.$$

In order to determine the function $\mu(x, y, \omega)$ fitting the constraints (30), we use one of the approaches described above, and each of them is reduced to a representation of the basis functions as the functionals of the data kernels. Then we may conclude that any basis function must be represented as the product of a function of x, y and a function of ω. To construct such functions we may use any of the criteria mentioned in the preceding sections.

Representation of the basis functions in the form of the product $F_1(x, y)F_2(\omega)$ implies that if the data correspond to a uniform set of frequencies the functions $F_2(\omega)$ may be chosen arbitrarily, whereas the functions $F_1(x, t)$ are determined as in the 2-D case, i.e. as functionals of the data kernels $H_j(x, y)$. Such a method has been developed by Yanovskaya *et al.* (1988), who proposed to represent the unknown function $\mu(x, y, \omega)$ as a polynomial of a certain degree in ω and to determine the coefficients of the polynomial, which are functions of x, y, proceeding from criterion 1. In this case the basis functions are

$$\phi_{jq}(x, y, \omega) = \varphi_j(x, y)\omega^q, \qquad q = 0, 1, \ldots, n,$$

where $\varphi_j(x, y)$ is defined by (12) and n is the degree of the polynomial. This approach was applied by Nesterov & Yanovskaya (1988) for determination of Rayleigh wave phase velocity patterns in southeastern Europe.

Certainly, this method is not the only possible one, but it is simple to achieve.

CONCLUSIONS

Solving the 2-D surface wave tomography problem we should choose the basis functions in concordance with the data, i.e. proceeding from the set of wave paths. Criteria for constructing the basis functions are different from the standpoint of *a priori* assumptions concerning the unknown function, but formally they are equivalent, and the solutions obtained proceeding from the different criteria are similar. Therefore it is convenient to choose that criterion, which simplifies the computational procedure.

For the 3-D problem, which is the determination of the phase velocity as a function of the horizontal coordinates and frequency, the basis functions turn out to be representable as a product of two functions, one depending on x, y, and the other on ω. Since practically always the data correspond to a uniform set of frequencies, the functions of ω may be chosen *a priori*, and the functions of x, y should be constructed as for the 2-D problem, i.e. proceeding from the wave paths corresponding to the data set.

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APPENDIX: RELATION BETWEEN CRITERIA 1 AND 2

Plane case

The *a priori* covariance function (17) may be represented in the form of the following Fourier integral:

$$R_{m0}(\mathbf{r}, \mathbf{r}') = (2\pi)^{-1}\sigma^2 L^2 \int \exp\left[-L^2 |\mathbf{k}|^2/2 + i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')\right] d\mathbf{k}. \tag{A1}$$

It is easy to see that

$$R_{m0}^{-1}(\mathbf{r}, \mathbf{r}') = (2\pi)^{-3}\sigma^{-2}L^{-2} \times \int \exp\left[L^2 |\mathbf{k}|^2/2 + i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')\right] d\mathbf{k}. \tag{A2}$$

Consequently the functional (14) may be written in the form

$$(2\pi)^{-3}\sigma^{-2}L^{-2}\iint\left\{\int \exp\left[L^2|\mathbf{k}|^2/2 + i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')\right] d\mathbf{k}\right\} \times m(\mathbf{r})m(\mathbf{r}')\, d\mathbf{r}\, d\mathbf{r}' \equiv S. \tag{A3}$$

Represent $\exp(L^2|\mathbf{k}|^2/2)$ in the form of a Taylor series:

$$\exp(L^2|\mathbf{k}|^2/2) = \sum_{n=0}^{\infty}\frac{1}{n!}(L^2/2)^n |\mathbf{k}|^{2n}. \tag{A4}$$

Insert (A4) into (A3) and examine the nth term of the series:

$$S_n = \frac{1}{n!}(L^2/2)^n(2\pi)^{-3}(\sigma L)^{-2}\iint\left\{\int |\mathbf{k}|^{2n}\exp\left[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')\right] d\mathbf{k}\right\} \times m(\mathbf{r})m(\mathbf{r}')\, d\mathbf{r}\, d\mathbf{r}'. \tag{A5}$$

Taking into account the identity

$$\int |\mathbf{k}|^{2n}\exp\left[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')\right] d\mathbf{k} = 4\pi^2(-1)^n\Delta^n\delta(\mathbf{r}-\mathbf{r}') \tag{A6}$$

as well as that

$$\int \Delta^n\delta(\mathbf{r}-\mathbf{r}')m(\mathbf{r}')\, d\mathbf{r}' = \Delta^n m(\mathbf{r}) \tag{A7}$$

we obtain

$$S_n = \frac{1}{n!}(L^2/2)^n(2\pi)^{-1}(-1)^n(\sigma L)^{-2}\int m(\mathbf{r})\,\Delta^n m(\mathbf{r})\, d\mathbf{r}. \tag{A8}$$

Applying Green's formula to the integral on the right-hand side of (A8), we obtain

$$\int m(\mathbf{r})\,\Delta^n m(\mathbf{r})\, d\mathbf{r} = -\int (\nabla m\cdot\nabla\Delta^{n-1}m(\mathbf{r}))\, d\mathbf{r} + \int_C m\frac{\partial\Delta^{n-1}m}{\partial n}\, dl. \tag{A9}$$

It can be shown that

$$\int [\nabla m(r)\cdot\nabla\Delta\varphi]\, d\mathbf{r} = -\int (\nabla\nabla m\cdot\nabla\nabla\varphi)\, d\mathbf{r} + \int_C \frac{\partial m}{\partial x_i}\frac{\partial^2\varphi}{\partial x_i\,\partial x_j}n_j\, dl \tag{A10}$$

where $\nabla\nabla$ is a tensor of second derivatives, and the product of the tensors is defined as

$$(\nabla\nabla m\cdot\nabla\nabla\varphi) = \frac{\partial^2 m}{\partial x_i\,\partial x_j}\frac{\partial^2\varphi}{\partial x_i\,\partial x_j}, \tag{A11}$$

summation over the repeated subscripts being implied.

So we have

$$-\int (\nabla m\cdot\nabla\Delta^{n-1}m)\, d\mathbf{r} = \int (\nabla\nabla m\cdot\nabla\nabla\Delta^{n-2}m)\, d\mathbf{r} - \int_C \frac{\partial m}{\partial x_i}\frac{\partial^2\,\Delta^{n-2}m}{\partial x_i\,\partial x_j}n_j\, dl. \tag{A12}$$

Reiterating the transformation (A10) we obtain the following expression for the nth term of the series:

$$S_n = \frac{1}{n!}(L^2/2)^n(2\pi)^{-1}(\sigma L)^{-2}\Bigg[\int \|\underbrace{\nabla\nabla\cdots\nabla}_{n} m\|^2\, d\mathbf{r} + (-1)^n\int_C m\frac{\partial\Delta^{n-1}m}{\partial x_i}n_i\, dl + (-1)^{n+1}\int \frac{\partial m}{\partial x_i}\frac{\partial^2\,\Delta^{n-2}m}{\partial x_i\,\partial x_j}n_j\, dl + (-1)^{n+2} \times \int \frac{\partial^2 m}{\partial x_i\,\partial x_j}\frac{\partial^3\,\Delta^{n-3}m}{\partial x_i\,\partial x_j\,\partial x_k}n_k\, dl + \cdots\Bigg]. \tag{A13}$$

Summation of all functionals S_n results in the following expression for the functional S:

$$S = (2\pi)^{-1}(\sigma L)^{-2} \sum_{n=0}^{\infty} \frac{1}{n!}(L^2/2)^n \left[\int \| \underbrace{\nabla\nabla\cdots\nabla}_{n} m \|^2 \, d\mathbf{r} + \frac{(-1)^{n+1}}{n+1}(L^2/2) \int_C m \frac{\partial \Delta^n m}{\partial x_i} n_i \, dl + \frac{(-1)^{n+1}}{(n+1)(n+2)}(L^2/2)^2 \int_C \frac{\partial m}{\partial x_i} \frac{\partial^2 \Delta^n m}{\partial x_i \partial x_j} n_j \, dl + \cdots \right]. \quad \text{(A14)}$$

A similar approach for transforming the functional (14) may be applied for any covariance function $R_{m0}(\mathbf{r}, \mathbf{r}')$, if a function inversely proportional to the Fourier transform of $R_{m0}(\mathbf{r}, \mathbf{r}')$ may be expanded in a Taylor series around $\mathbf{k} = 0$.

Spherical case

The *a priori* covariance function (18) with $\sigma(M) = \sigma(M') \equiv \sigma$ is represented as a series of Legendre functions

$$R_{m0}(\theta', \varphi', \theta, \varphi) = \sigma^2 \exp(-L^{-2}) \sum \alpha_n P_n(\cos \Delta) \quad \text{(A15)}$$

where

$$\alpha_n = (2n+1)\sqrt{2\pi L}/2I_{n+1/2}(L^{-2}).$$

Using the summation theorem for associated Legendre functions we obtain that $R_{m0}^{-1}(\theta', \varphi', \theta, \varphi)$ may be also written in the form of a series of Legendre functions:

$$R_{m0}^{-1}(\theta', \varphi', \theta', \varphi) = \sigma^{-2} \exp(L^{-2}) \sum \beta_n P_n(\cos \Delta)$$

where

$$\beta_n = \frac{2n+1}{2 \times 4\pi^2 L \sqrt{2\pi}\, I_{n+1/2}(L^{-2})}.$$

If we expand the function $\Phi(n) = \dfrac{1}{L\sqrt{2\pi}\, I_{n+1/2}(L^{-2})}$ in a series in powers of $N^2 = n(n+1)$:

$$\Phi(n) = \sum_k q_k(L^2) N^{2k}$$

and take into account that

$$\Delta P_n(\cos \Delta) = (-1)^k N^{2k} P_n(\cos \Delta)$$

we easily obtain

$$R_m^{-1}(\theta', \varphi', \theta, \varphi) = (2\pi)^{-1}\sigma^{-2} \exp(L^{-2}) \sum_k q_k(L^2) \Delta^k \, \delta(\theta' - \theta, \varphi' - \varphi),$$

where $\delta(\theta - \theta', \varphi - \varphi')$ is the δ-function on a sphere: $\delta(\Delta)/(2\pi \sin \Delta)$ and Δ is the distance between the points (θ, φ) and (θ', φ').

The subsequent derivation is analogous to that in the plane case, but the integrals along the contour vanish. Finally we obtain

$$S = (2\pi)^{-1}\sigma^{-2} \sum_k \exp(L^{-2}) q_k(L^2) \times \int_0^{2\pi} \int_0^{\pi} \| \underbrace{\nabla\nabla\cdots\nabla}_{k} m \|^2 \sin\theta \, d\theta \, d\varphi. \quad \text{(A16)}$$